Characterization of SINR Region for Multiple Interfering Multicast in Power-Controlled Systems

Yi Chen and Chi Wan Sung

Abstract

This paper considers a wireless communication network consisting of multiple interfering multicast sessions. Different from a unicast system where each transmitter has only one receiver, in a multicast system, each transmitter has multiple receivers. It is a well known result for wireless unicast systems that the feasibility of an signal-to-interference-plus-noise power ratio (SINR) without power constraint is decided by the Perron-Frobenius eigenvalue of a nonnegative matrix. We generalize this result and propose necessary and sufficient conditions for the feasibility of an SINR in a wireless multicast system with and without power constraint. The feasible SINR region as well as its geometric properties are studied. Besides, an iterative algorithm is proposed which can efficiently check the feasibility condition and compute the boundary points of the feasible SINR region.

Index Terms

Wireless multicast system, power control, signal-to-interference-plus-noise power ratio (SINR), SINR feasibility, SINR region, Perron-Frobenius eigenvalue.

I. INTRODUCTION

In wireless communication systems, interference is an inherent phenomenon. Due to the broadcast nature of wireless channels, interference arises whenever multiple transmitter-receiver

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pairs are active concurrently in the same frequency band, and each receiver is only interested in retrieving information from its own transmitter. For a particular receiver, the received signal is a superposition of its desired signal, interfering signals and background noise. SINR, defined as the power of desired signal divided by the sum of the power of interfering signals and the power of noise, is a widely used performance measure for wireless communication systems. It is analogous to signal-to-noise ratio (SNR) used for single user communication, which has clearly understood implication on the bit error rate (BER) and capacity for additive white Gaussian noise (AWGN) channels. Using SINR as a surrogate for BER and capacity implicitly assumes that the interference is an AWGN. Although there are limitations of this assumption, as reported in [1], [2], the importance of SINR has never been doubted.

For a system consisting of multiple point-to-point communication sessions, also referred to as *unicast system*, the SINRs of all receivers form a vector. The feasible SINR region includes all the SINR vectors that can be achieved by some transmission powers. The geometric properties of feasible SINR region has been studied in [3]–[5]. Reference [3] proves that in the case of unlimited transmission power, the feasible SINR region is log-convex. In [4], it is shown that under a total power constraint, the infeasible SINR region is not convex. Reference [5] considers a system with only three transmitter-receiver pairs without power constraint, and shows that the feasible SINR region is concave. It also provides certain technical conditions under which a concavity result for systems with a general number of users is established. In [6], for the cases that the transmission powers are subject to arbitrary linear constraints, a mathematical expression for the boundary points of the SINR region is obtained.

In this paper, we consider the feasible SINR region for systems consisting of multiple pointto-multipoint communication sessions, also referred to as *multicast system*. Multicast enables data to be delivered from a source node to multiple destination nodes. Practical examples of such configurations include cellular networks and two-way relay networks. In cellular networks, a base station multicasts a file to multiple mobile devices that request the file at the same time [7]. In two-way relay networks, when network coding is applied, a relay multicasts the coded packets to two sink nodes [8]. The power control and scheduling for wireless multicast systems have been studied in [9]–[12] and the references therein. All these works aim to either minimize the system power or maximize the system throughput, subject to the constraint that the SINR of all receivers are larger than a given threshold. The feasibilities of the problems, however, are unknown.

For a wireless unicast system, the feasibility of an SINR vector without power constraint is determined by the Perron-Frobenius eigenvalue of a nonnegative matrix [13], [14]. In this paper, we generalize this result to a wireless multicast system. We first propose a necessary and sufficient condition under which an SINR is achievable without power limitation. Based on this condition, we figure out the feasible SINR region by giving its boundary points. The approach is to find the farthest point of the feasible SINR region from the origin in a given direction. An iterative algorithm is proposed to find the farthest point, which is also a distributed power control algorithm to solve the power balancing problem [13] aimed to maximize the minimal SINR of all receivers.

Then we analyse the geometric properties of the feasible and infeasible SINR regions. It is found that the feasible SINR region of a multicast system is in fact the intersection of the feasible SINR regions of all its embedded unicast systems. Based on the results in [3]–[5] for unicast systems, we show that the feasible SINR region of a multicast system is log-convex, and the infeasible SINR region of a multicast system with two multicast sessions is convex. We also show by an example that, the convexity property of the infeasible SINR region does not hold for a general multicast system with more than two multicast sessions.

Later, the necessary and sufficient condition for the feasibility of an SINR in a multicast system is extended to include linear constraints on the power. This result generalizes the results in [6] for unicast systems. Besides, in [6], the zero-outage SINR region for a time-varying system is also considered, where the channel gains are selected from a finite set. Suppose the transmission powers are not allowed to vary with the channel gains, we establish a reduction that maps any instance of the zero-outage SINR problem in a time-varying unicast system to a corresponding instance of the feasible SINR problem in a time-invariant multicast system. The idea is to regard the multiple receivers in a multicast session as an identical receiver that can experience a finite set of channel gains.

The rest of the paper is organized as follows: In Section II, the system model and problem formulation are presented. The necessary and sufficient condition on the feasibility of an SINR vector is provided in Section III. Section IV gives the characterizations of the SINR region and proposes an iterative algorithm. Section V studies the geometric properties of the feasible SINR region. Section VI extends the study to include power constraints. Finally, the paper is concluded



Fig. 1: Example of a multicast network consisting of two multicast sessions. Transmitter T_1 wants to transmit data to both R_1^1 and R_1^2 . Transmitter T_2 wants to transmit data to both R_2^1 and R_2^2 . Their transmitted signals interfere with each other. The solid lines represent intended links and the dashed lines represent interfering links.

in Section VII.

Notation: The following notations are used throughout this paper. Vectors are denoted in bold small letter, e.g., x, with their *i*th entry denoted by x_i . They are regarded as column vectors unless stated otherwise. Matrices are denoted by bold capitalized letters, e.g., X, with X_{ij} denoting the (i, j)th entry. Vector and matrix inequalities are component-wise inequalities, e.g., $x \ge y$ if $x_i \ge y_i$ for all i; $X \ge Y$ if $X_{ij} \ge Y_{ij}$ for all i and j. The cardinality of a set is denoted by " $|\cdot|$ ". The Euclidean norm of a vector is denote by " $||\cdot||$ ". The transpose of a vector or matrix is denoted by $(\cdot)^T$. I represents an identity matrix with compatible size. 0 represents a vector with compatible size whose entries are all zero.

II. SYSTEM MODEL

Consider a general wireless communication network consisting of N multicast sessions. The N transmitters are denoted by T_i for i = 1, ..., N. Each T_i wants to multicast common data packets to K_i receivers, denoted by $R_i^{k_i}$ for $k_i = 1, ..., K_i$. Without loss of generality assume $K_i \ge 1$. If $K_i = 1$ for all *i*, the scenario reduces to the unicast case. The total number of receivers in the system is $K = \sum_i^N K_i$. Define $\mathcal{K}_i = \{1, 2, ..., K_i\}$ for i = 1, ..., N. Fig. 1 illustrates an example of such network. Let p_i be the transmission power of transmitter T_i and $\mathbf{p} = [p_1, ..., p_N]^T$. The channel gain between T_j and $R_i^{k_i}$ is denoted by $g_{r_i^{k_i}, t_j}$. All the multicast sessions share the same channel and thus interfere with each other. We assume that interference caused by simultaneous transmissions is treated as AWGN with variance identical to the received

power. The SINR of receiver $R_i^{k_i}$ is given by

$$\gamma_{i}^{k_{i}}(\mathbf{p}) = \frac{g_{r_{i}^{k_{i}}, t_{i}} p_{i}}{\sum_{j \neq i} g_{r_{i}^{k_{i}}, t_{j}} p_{j} + \sigma^{2}},$$
(1)

where σ^2 is the variance of the background noise and without loss of generality, it is assumed to be identical for all receivers. We define the SINR of the *i*-th multicast session as

$$\gamma_i(\mathbf{p}) = \min_{k_i \in \mathcal{K}_i} \left\{ \gamma_i^{k_i}(\mathbf{p}) \right\}.$$

The SINR vector of the system is

$$\Gamma(\mathbf{p}) = [\gamma_1(\mathbf{p}), \gamma_2(\mathbf{p}), \dots, \gamma_N(\mathbf{p})]$$

In this paper, we analyze the feasible SINR region of a multicast system, that is,

$$\Upsilon = \left\{ \Gamma(\mathbf{p}) \in \mathbb{R}^N : \mathbf{p} \ge \mathbf{0}, \mathbf{p} \in \mathbb{R}^N
ight\}.$$

Proposition 1. Given a vector $\mu \in \mathbb{R}^N$. There exists a power vector $\mathbf{p}^* \ge \mathbf{0}$ such that $\Gamma(\mathbf{p}^*) = \mu$, if and only if there exits a power vector $\mathbf{p}' \ge \mathbf{0}$ such that $\Gamma(\mathbf{p}') \ge \mu$.

Proof: The "only if" part is trivial and we show the "if" part. Suppose $\gamma_i(\mathbf{p}') > \mu_i$ for some *i*. Fix such an *i*. Since $\gamma_i(\mathbf{p})$ is monotonically decreasing as p_i is decreasing, we can find a $0 < p_i^{(1)} < p_i'$ and let $\mathbf{p}^{(1)} = [p_1', \ldots, p_{i-1}', p_i^{(1)}, p_{i+1}', \ldots, p_N']^T$ such that $\gamma_i(\mathbf{p}^{(1)}) = \mu_i$. On the other hand, since $\gamma_j(\mathbf{p})$ for $j \neq i$ is monotonically increasing as p_i is decreasing, we have $\gamma_j(\mathbf{p}^{(1)}) \ge \mu_j$. By keeping on decreasing the power of transmitters that achieve higer SINR than μ , we obtain a sequence $p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(t)}, \ldots$ for each $i = 1, \ldots, N$. It can be seen that these sequences are monotonically decreasing and lower bounded by zero, so they are convergent. Denote the limit point by \mathbf{p}^* . For any arbitrarily small $\delta > 0$ and for all *i*, since $\gamma_i(\mathbf{p}) | < \delta$. Meanwhile, since $\gamma_i(\mathbf{p}^{(t')}) = \mu_i$ for some $t' \ge T$, we have $|\gamma_i(\mathbf{p}^*) - \mu_i| < \delta$. Therefore $\gamma_i(\mathbf{p}^*) = \mu_i$ for all *i*.

By Proposition 1, we say that an SINR vector $\boldsymbol{\mu} = [\mu_1, \dots, \mu_N]$ is *feasible* if and only if there exists a power vector $\mathbf{p} \ge \mathbf{0}$ such that

$$p_i - \sum_{j \neq i} \mu_i \frac{g_{r_i^{k_i}, t_j}}{g_{r_i^{k_i}, t_i}} p_j \ge \mu_i \frac{\sigma^2}{g_{r_i^{k_i}, t_i}}, \forall k_i \in \mathcal{K}_i, \forall i.$$

$$(2)$$

In matrix form, it is

$$\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \ge \mathbf{n}(\boldsymbol{\mu}),\tag{3}$$

where

$$\mathbf{A}(\boldsymbol{\mu}) = \begin{bmatrix} 1 & -\mu_{1} \frac{g_{r_{1}^{1},t_{2}}}{g_{r_{1}^{1},t_{1}}} & \cdots & -\mu_{1} \frac{g_{r_{1}^{1},t_{N}}}{g_{r_{1}^{1},t_{1}}} \\ 1 & -\mu_{1} \frac{g_{r_{1}^{2},t_{2}}}{g_{r_{1}^{2},t_{1}}} & \cdots & -\mu_{1} \frac{g_{r_{2}^{2},t_{N}}}{g_{r_{2}^{2},t_{1}}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -\mu_{1} \frac{g_{r_{1}^{1},t_{2}}}{g_{r_{1}^{1},t_{1}}} & \cdots & -\mu_{1} \frac{g_{r_{1}^{1},t_{N}}}{g_{r_{1}^{1},t_{1}}} \\ -\mu_{2} \frac{g_{r_{2}^{1},t_{1}}}{g_{r_{2}^{1},t_{2}}} & 1 & \cdots & -\mu_{2} \frac{g_{r_{2}^{1},t_{N}}}{g_{r_{2}^{1},t_{2}}} \\ \vdots & \vdots & \ddots & \vdots \\ -\mu_{2} \frac{g_{r_{2}^{2},t_{1}}}{g_{r_{2}^{N},t_{2}}} & 1 & \cdots & -\mu_{2} \frac{g_{r_{2}^{2},t_{N}}}{g_{r_{2}^{N},t_{2}}} \\ \vdots & \vdots & \ddots & \vdots \\ -\mu_{N} \frac{g_{r_{N}^{N},t_{1}}}{g_{r_{N}^{N},t_{N}}} & -\mu_{N} \frac{g_{r_{N}^{N},t_{2}}}{g_{r_{N}^{N},t_{N}}}} & \cdots & 1 \\ \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{1}^{1} \\ \mathbf{a}_{1}^{2} \\ \vdots \\ \mathbf{a}_{1}^{K} \\ \mathbf{a}_{2}^{1} \\ \vdots \\ \mathbf{a}_{2}^{K} \\ \vdots \\ \mathbf{a}_{2}^{K} \\ \mathbf{a}_{2}^{K} \\ \vdots \\ \mathbf{a}_{2}^{K} \\$$

and

$$\mathbf{n}(\boldsymbol{\mu}) = \left[\overbrace{g_{r_1^1, t_1}^{1}, \frac{\mu_1 \sigma^2}{g_{r_1^2, t_1}^{1}}, \dots, \frac{\mu_1 \sigma^2}{g_{r_1^{K_1}, t_1}^{K_1}}, \dots, \overbrace{g_{r_N^{K_N}, t_N}^{K_N}}^{K_N}, \dots, \frac{\mu_N \sigma^2}{g_{r_N^{K_N}, t_N}^{K_N}}\right]^T \\ = [n_1^1, n_1^2, \dots, n_1^{K_1}, \dots, n_N^1, \dots, n_N^{K_N}]^T \in \mathbb{R}^{K \times 1}.$$

Each row of $\mathbf{A}(\boldsymbol{\mu})$ corresponds to a receiver. For the convenience of discussion, we use $\mathbf{a}_i^{k_i} \in \mathbb{R}^N$ to denote the row of $\mathbf{A}(\boldsymbol{\mu})$ that corresponds to receiver $R_i^{k_i}$. As in the form (3), the feasibility of $\boldsymbol{\mu}$ can be checked through linear programming [15]. However, in a different way, we propose a necessary and sufficient condition on the feasibility, which generalizes the Perron-Frobenius eigenvalue criteria for unicast systems (square matrices). This condition is used to explicitly characterize the feasible SINR region Υ and to prove some geometric properties of it. Before further discussion, we given some definitions.

Define set

$$\mathcal{G}(\boldsymbol{\mu}) = \left\{ \mathbf{G} = \begin{bmatrix} \mathbf{a}_1^{k_1} \\ \mathbf{a}_2^{k_2} \\ \vdots \\ \mathbf{a}_N^{k_N} \end{bmatrix} \in \mathbb{R}^{N \times N} : k_i \in \mathcal{K}_i \text{ for } i = 1, \dots, N \right\}.$$

Notice that, for each $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$, only one receiver is involved for each transmitter, which is a unicast scenario. So $\mathcal{G}(\boldsymbol{\mu})$ is the set including all the embedded unicast systems and its size is $\prod_{i=1}^{N} K_i$. Considering the example in Fig. 1, the four embedded unicast systems are

$$\mathcal{G}(\boldsymbol{\mu}) = \left\{ \begin{bmatrix} \mathbf{a}_1^1 \\ \mathbf{a}_2^1 \end{bmatrix}, \begin{bmatrix} \mathbf{a}_1^2 \\ \mathbf{a}_2^1 \end{bmatrix}, \begin{bmatrix} \mathbf{a}_1^1 \\ \mathbf{a}_2^2 \end{bmatrix}, \begin{bmatrix} \mathbf{a}_1^2 \\ \mathbf{a}_2^2 \end{bmatrix}, \begin{bmatrix} \mathbf{a}_1^2 \\ \mathbf{a}_2^2 \end{bmatrix} \right\}.$$

In subsequent discussion, we also use $\mathbf{k} = (k_1, k_2, \dots, k_N)$ to specify a $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$. Let $\mathbf{n}_{\mathbf{G}} = [n_1^{k_1}, n_2^{k_2}, \dots, n_N^{k_N}]^T$ denote the noise vector with entries of $\mathbf{n}(\boldsymbol{\mu})$ that correspond to the receivers in **G**. For the simplicity of notation, we sometimes drop the argument $\boldsymbol{\mu}$ of **A**, **n** and \mathcal{G} when the context is clear.

Definition 1. [16] A matrix **X** is called nonnegative if $\mathbf{X} \ge \mathbf{0}$. A nonnegative square matrix **X** is irreducible if for every pair (i, j) of its index set, there exists a positive integer $n \equiv n(i, j)$ such that $X_{ij}^{(n)} > 0$, where $X_{ij}^{(n)}$ is the (i, j)th entry of \mathbf{X}^n .

Definition 2. [16] Let X be an irreducible nonnegative square matrix. The Perron-Frobenius eigenvalue of X is the maximum of the absolute value of eigenvalues of X, and is denoted by $\lambda(\mathbf{X})$.

Let $\mathbf{1} = [1, ..., 1]$ be a vector whose components are all 1. For each $\mathbf{G} \in \mathcal{G}(\mathbf{1})$, $\mathbf{I} - \mathbf{G}$ is the normalized interference link gain matrix of the corresponding embedded unicast system.

Definition 3. A multicast system is called irreducible if and only if the matrices I - G for $G \in \mathcal{G}(1)$ are all irreducible.

It needs to be mentioned that if a multicast system is irreducible, as long as $\mu > 0$, the matrices I - G for $G \in \mathcal{G}(\mu)$ are all irreducible. Throughout the paper, we assume that the multicast system is irreducible.

III. FEASIBILITY CONDITION FOR SINR

In this section, we give a necessary and sufficient condition for the feasibility of an SINR vector in a wireless multicast system. Recall that in a wireless unicast system, the following theorem from [13], [17] is the fundamental results that characterize the feasibility.

Theorem 1. [17] Consider a unicast network setting G and assume I - G is irreducible. The following statements are equivalent:

- 1) There exists a power vector $\mathbf{p} \ge \mathbf{0}$ such that $\mathbf{G}\mathbf{p} \ge \mathbf{0}$.
- 2) $\lambda(I G) < 1.$

3) $\mathbf{G}^{-1} = \sum_{k=0}^{\infty} (\mathbf{I} - \mathbf{G})^k$ exists and is positive component-wise, with $\lim_{k\to\infty} (\mathbf{I} - \mathbf{G})^k = 0$. Moreover, there exists $\mathbf{p} \ge \mathbf{0}$ such that $\mathbf{G}\mathbf{p} = \mathbf{0}$ if and only if $\lambda(\mathbf{I} - \mathbf{G}) = 1$.

For a multicast system, the main result of this paper is the following theorem.

Theorem 2. Consider a multicast network setting $\mathbf{A}(\boldsymbol{\mu})$ and assume it is irreducible, i.e., the matrices $\mathbf{I} - \mathbf{G}$ for $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$ are all irreducible. There exists a power vector $\mathbf{p} \ge \mathbf{0}$ such that $\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \ge \mathbf{n}(\boldsymbol{\mu})$ if and only if $\max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})}\{\lambda(\mathbf{I} - \mathbf{G})\} < 1$.

Theorem 2 basically says that for a wireless multcast system, an SINR vector μ is feasible if and only if μ is feasible to any of its embedded unicast system.

Corollary 1. When there are only two multcast sessions, i.e., N = 2, the feasibility of μ is determined by the unicast system specified by

$$\mathbf{G}^{*} = \begin{bmatrix} \mathbf{a}_{1}^{k_{1}^{*}} \\ \mathbf{a}_{2}^{k_{2}^{*}} \end{bmatrix} \text{ where } k_{i}^{*} = \arg \max_{k_{i} \in \mathcal{K}_{i}} \{ \mu_{i} \frac{g_{r_{i}^{k_{i}}, t_{j}}}{g_{r_{i}^{k_{i}}, t_{i}}} \}, i = 1, 2, j \neq i.$$

That is, μ is feasible if and only if $\lambda(\mathbf{I} - \mathbf{G}^*) < 1$.

Corollary 1 follows straightforwardly from Theorem 2. Note that when N = 2, for any $\mathbf{G} \in \mathcal{G}$, we have

$$\lambda(\mathbf{I} - \mathbf{G}) = \sqrt{\mu_1 \frac{g_{r_1^{k_1}, t_2}}{g_{r_1^{k_1}, t_1}} \times \mu_2 \frac{g_{r_2^{k_2}, t_1}}{g_{r_2^{k_2}, t_2}}}.$$
(4)

So $\max_{\mathbf{G}\in\mathcal{G}}\{\lambda(\mathbf{I}-\mathbf{G})\}=\lambda(\mathbf{I}-\mathbf{G}^*).$

The proof of the necessary condition for Theorem 2 is straightforward. Suppose there exists a power vector $\mathbf{p} \ge \mathbf{0}$ such that $\mathbf{Ap} \ge \mathbf{n}$. Then for any $\mathbf{G} \in \mathcal{G}$, we have $\mathbf{Gp} \ge \mathbf{n}_{\mathbf{G}} \ge \mathbf{0}$. By Theorem 1, $\lambda(\mathbf{I} - \mathbf{G}) < 1$ for all \mathbf{G} , which implies $\max_{\mathbf{G} \in \mathcal{G}} \{\lambda(\mathbf{I} - \mathbf{G})\} < 1$.

In the rest of this section, we prove the sufficient condition and assume that $\max_{\mathbf{G}\in\mathcal{G}}\{\lambda(\mathbf{I}-\mathbf{G})\} < 1$. By Theorem 1, it indicates that for each $\mathbf{G}\in\mathcal{G}$, $\mathbf{G}^{-1}\geq\mathbf{0}$ exists, and thus $\mathbf{p} = \mathbf{G}^{-1}\mathbf{n}_{\mathbf{G}} \geq \mathbf{0}$ exists. For each receiver, define

$$\mathcal{A}_i^{k_i} = \left\{ \mathbf{p} \in \mathbb{R}^N : \mathbf{a}_i^{k_i} \mathbf{p} \ge n_i^{k_i}, \mathbf{p} \ge 0 \right\}.$$

Note that $a_i^{k_i}$ is a row vector as defined before. $\mathcal{A}_i^{k_i}$ is an intersection of half-spaces and thus is convex. Our proof is based on Helly's theorem given below.

Theorem 3. (Helly's theorem) [18]. Let \mathcal{F} be a finite collection of convex sets in \mathbb{R}^N . The intersection of all the sets of \mathcal{F} is non-empty if and only if any N + 1 of them has non-empty intersection.

In our case,

$$\mathcal{F} = \left\{ \mathcal{A}_i^{k_i} : i = 1, \dots, N, k_i \in \mathcal{K}_i \right\}.$$
(5)

There are in total K convex sets in \mathcal{F} . If any N+1 of them have non-empty intersection, then all of them have non-empty intersection, i.e., the SINR is feasible. The number of all combinations of such N+1 sets is $\binom{K}{N+1}$. We first show the proof for N=2. Then we use the mathematical induction to prove the general case.

Lemma 2. Suppose $\mathbf{X} = (X_{ij})$ is an $N \times N$ matrix satisfying $X_{ij} = 1$ for i = j and $X_{ij} \leq 0$ for $i \neq j$. Let S be a subset of $\{1, \ldots, N\}$ and \mathbf{X}' be the matrix by removing the *i*-th row and *i*-th column of \mathbf{X} for all $i \in S$. If $\lambda(\mathbf{I} - \mathbf{X}) < 1$, then $\lambda(\mathbf{I} - \mathbf{X}') < 1$.

Proof: Since $\lambda(\mathbf{I} - \mathbf{X}) < 1$, by Theorem 1, there exists a vector $\mathbf{p} \ge \mathbf{0}$ such that $\mathbf{X}\mathbf{p} \ge \mathbf{0}$. Let $\mathbf{p}' \in \mathbb{R}^{N-|\mathcal{S}|}$ be the vector constructed by removing the *i*-th entry in \mathbf{p} for all $i \in \mathcal{S}$. Since $X_{ij} \le 0$ for $i \ne j$, it can be verified that $\mathbf{X}'\mathbf{p}' \ge \mathbf{0}$, which implies $\lambda(\mathbf{I} - \mathbf{X}') < 1$.

Lemma 3. Consider $\hat{\mathbf{G}}, \tilde{\mathbf{G}} \in \mathcal{G}$ such that $\hat{\mathbf{G}}$ differs from $\tilde{\mathbf{G}}$ only in one row. i.e., $\hat{k}_i \neq \tilde{k}_i$ for one $i \in \{1, \ldots, N\}$ and $\hat{k}_j = \tilde{k}_j$ for $j \neq i$. Let $\hat{\mathbf{p}} = \hat{\mathbf{G}}^{-1}\mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{G}}^{-1}\mathbf{n}_{\tilde{\mathbf{G}}}$. There exists $\mathbf{p} \in \{\hat{\mathbf{p}}, \tilde{\mathbf{p}}\}$ such that $\tilde{\mathbf{G}}\mathbf{p} \ge \mathbf{n}_{\tilde{\mathbf{G}}}$ and $\hat{\mathbf{G}}\mathbf{p} \ge \mathbf{n}_{\hat{\mathbf{G}}}$.

Proof: Since $\hat{\mathbf{p}} = \hat{\mathbf{G}}^{-1}\mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{G}}^{-1}\mathbf{n}_{\tilde{\mathbf{G}}}$, we have $\hat{\mathbf{G}}\hat{\mathbf{p}} = \mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{p}} = \mathbf{n}_{\tilde{\mathbf{G}}}$. If $\hat{\mathbf{p}} = \tilde{\mathbf{p}}$, automatically we have $\hat{\mathbf{G}}\tilde{\mathbf{p}} = \mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{G}}\hat{\mathbf{p}} = \mathbf{n}_{\tilde{\mathbf{G}}}$. In the following discussion, we consider the case when $\hat{\mathbf{p}} \neq \tilde{\mathbf{p}}$. Without loss of generality, assume that $\hat{\mathbf{G}}$ and $\tilde{\mathbf{G}}$ differ in the first row, that

is, $\hat{k}_1 \neq \tilde{k}_1$ and $\hat{k}_j = \tilde{k}_j$ for $j \neq 1$. Let us partition $\hat{\mathbf{G}}$ into four blocks as follows.

$$\hat{\mathbf{G}} = \begin{bmatrix} 1 & -\mu_1 \frac{g_{\hat{k}_1, t_2}}{g_{\hat{k}_1, t_1}} & \cdots & -\mu_1 \frac{g_{\hat{k}_1, t_N}}{g_{\hat{k}_1, t_1}} \\ -\mu_2 \frac{g_{\hat{k}_2, t_1}}{g_{\hat{k}_2, t_2}} & 1 & \cdots & -\mu_2 \frac{g_{\hat{k}_2, t_N}}{g_{\hat{k}_2, t_2}} \\ \vdots & \vdots & \ddots & \vdots \\ g_{\hat{k}_N, t_1} & g_{\hat{k}_N, t_2} & \cdots & 1 \\ -\mu_N \frac{g_{\hat{k}_N, t_N}}{g_{\hat{k}_N, t_N}} & -\mu_N \frac{g_{\hat{k}_N, t_2}}{g_{\hat{k}_N, t_N}} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & A \\ \hline C & D \end{bmatrix}.$$

Similarly, $\tilde{\mathbf{G}}$ is partitioned into four blocks as $\tilde{\mathbf{G}} = \begin{bmatrix} 1 & B \\ \hline C & D \end{bmatrix}$. Note that $\hat{\mathbf{G}}$ and $\tilde{\mathbf{G}}$ share the same three blocks as $\hat{\mathbf{G}} = \begin{bmatrix} 1 & B \\ \hline C & D \end{bmatrix}$.

same three blocks: 1, *C* and *D*. We consider $\hat{\mathbf{G}}\tilde{\mathbf{p}} - \mathbf{n}_{\hat{\mathbf{G}}}$ and $\tilde{\mathbf{G}}\hat{\mathbf{p}} - \mathbf{n}_{\tilde{\mathbf{G}}}$. Since $\mathbf{a}_{i}^{\tilde{k}_{i}} = \mathbf{a}_{i}^{\tilde{k}_{i}}$ and $n_{\tilde{k}_{i}} = n_{\hat{k}_{i}}$ for $i = 2, \ldots, N$, $\mathbf{a}_{i}^{\hat{k}_{i}}\tilde{\mathbf{p}} = \mathbf{a}_{i}^{\tilde{k}_{i}}\tilde{\mathbf{p}} = n_{\tilde{k}_{i}} = n_{\hat{k}_{i}}$ and $\mathbf{a}_{i}^{\tilde{k}_{i}}\hat{\mathbf{p}} = \mathbf{a}_{i}^{\hat{k}_{i}}\hat{\mathbf{p}} = n_{\tilde{k}_{i}}$ for $i = 2, \ldots, N$. If $\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} = n_{\hat{k}_{1}}$ or $\mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} = n_{\tilde{k}_{1}}$, then $\hat{\mathbf{G}}\tilde{\mathbf{p}} = \mathbf{n}_{\hat{\mathbf{G}}}$ or $\tilde{\mathbf{G}}\hat{\mathbf{p}} = \mathbf{n}_{\tilde{\mathbf{G}}}$, which implies $\hat{\mathbf{p}} = \tilde{\mathbf{p}}$. Therefore when $\hat{\mathbf{p}} \neq \tilde{\mathbf{p}}$, we must have $\mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} \neq n_{\tilde{k}_{1}}$ and $\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} \neq n_{\hat{k}_{1}}$. In the following we prove that, either $\mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} > n_{\tilde{k}_{1}}$ or $\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} > n_{\hat{k}_{1}}$ but not both, that is, $(\mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} - n_{\tilde{k}_{1}})(\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} - n_{\hat{k}_{1}}) < 0$.

Since $\hat{\mathbf{G}}^{-1} > 0$ exists, by Theorem 1 and Lemma 2, D^{-1} exists. By block-wise inversion [19], the inverse of $\hat{\mathbf{G}}$ can be written as

$$\hat{\mathbf{G}}^{-1} = \begin{bmatrix} a & -aAD^{-1} \\ -D^{-1}Ca & D^{-1} + D^{-1}CaAD^{-1} \end{bmatrix},$$

where $a = (1 - AD^{-1}C)^{-1} > 0$. $\tilde{\mathbf{G}}^{-1}$ is in the same form by replacing A with B and replacing a with $b = (1 - BD^{-1}C)^{-1} > 0$. Denote $\mathbf{n}_{\hat{\mathbf{G}}} = \begin{bmatrix} n_{\hat{k}_1} \\ \mathbf{n}' \end{bmatrix}$ and $\mathbf{n}_{\tilde{\mathbf{G}}} = \begin{bmatrix} n_{\tilde{k}_1} \\ \mathbf{n}' \end{bmatrix}$ where $\mathbf{n}' \in \mathbb{R}^{N-1}$. We have

$$\begin{aligned} \mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} &= n_{\tilde{k}_{1}} \\ &= \begin{bmatrix} 1 & B \end{bmatrix} \hat{\mathbf{G}}^{-1}\mathbf{n}_{\hat{\mathbf{G}}} - n_{\tilde{k}_{1}} \\ &= \begin{bmatrix} 1 & B \end{bmatrix} \begin{bmatrix} a & -aAD^{-1} \\ -D^{-1}Ca & D^{-1} + D^{-1}CaAD^{-1} \end{bmatrix} \begin{bmatrix} n_{\tilde{k}_{1}} \\ \mathbf{n}' \end{bmatrix} - n_{\tilde{k}_{1}} \\ &= an_{\tilde{k}_{1}} - BD^{-1}Can_{\tilde{k}_{1}} - aAD^{-1}\mathbf{n}' + BD^{-1}\mathbf{n}' + BD^{-1}CaAD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}} \\ &= an_{\tilde{k}_{1}}(1 - BD^{-1}C) - (1 - BD^{-1}C)aAD^{-1}\mathbf{n}' + BD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}} \\ &= -ab^{-1}(AD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}}) + (BD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}}). \end{aligned}$$

Similarly, we have

$$\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} - n_{\hat{k}_{1}} = -ba^{-1}(BD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}}) + (AD^{-1}\mathbf{n}' - n_{\hat{k}_{1}})$$

Then

$$\begin{aligned} (\mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} - n_{\tilde{k}_{1}})(\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} - n_{\hat{k}_{1}}) \\ &= -ab^{-1}(AD^{-1}\mathbf{n}' - n_{\hat{k}_{1}})^{2} - ba^{-1}(BD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}})^{2} + 2(AD^{-1}\mathbf{n}' - n_{\hat{k}_{1}})(BD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}}) \\ &= -\left[\sqrt{ab^{-1}}(AD^{-1}\mathbf{n}' - n_{\hat{k}_{1}}) - \sqrt{ba^{-1}}(BD^{-1}\mathbf{n}' - n_{\tilde{k}_{1}})\right]^{2} \\ &\leq 0. \end{aligned}$$

Further, since $\mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} \neq n_{\tilde{k}_{1}}$ and $\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} \neq n_{\hat{k}_{1}}$, $(\mathbf{a}_{1}^{\tilde{k}_{1}}\hat{\mathbf{p}} - n_{\tilde{k}_{1}})(\mathbf{a}_{1}^{\hat{k}_{1}}\tilde{\mathbf{p}} - n_{\hat{k}_{1}}) < 0$. In summary, there exists $\mathbf{p} \in \{\hat{\mathbf{p}}, \tilde{\mathbf{p}}\}$ such that $\tilde{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\tilde{\mathbf{G}}}$ and $\hat{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\hat{\mathbf{G}}}$.

A. Two Multicast Sessions N = 2

If K = 2, i.e., $K_1 = K_2 = 1$, it is the unicast scenario and Theorem 2 is true straightforwardly. If $K_1 = 1, K_2 = 2$ or $K_1 = 2, K_2 = 1$ or $K_1 = K_2 = 2$, then any three subsets of \mathcal{F} must be $\mathcal{A}_i^1, \mathcal{A}_i^2, \mathcal{A}_j^{k_j}$ for i = 1 or 2 and $j \neq i$. Let

$$\hat{\mathbf{G}} = egin{bmatrix} \mathbf{a}_i^1 \ \mathbf{a}_j^{k_j} \end{bmatrix} ext{ and } ilde{\mathbf{G}} = egin{bmatrix} \mathbf{a}_i^2 \ \mathbf{a}_j^{k_j} \end{bmatrix}.$$

By Lemma 3, there exists \mathbf{p} such that $\tilde{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\tilde{\mathbf{G}}}$ and $\hat{\mathbf{G}}\mathbf{p} \geq \mathbf{n}_{\tilde{\mathbf{G}}}$, which implies $\mathbf{p} \in (\mathcal{A}_i^1 \cap \mathcal{A}_i^2 \cap \mathcal{A}_j^{k_j})$. Further by Helly's theorem, the intersection of all sets in \mathcal{F} is non-empty. For other values of K_1 and K_2 , we divide the $\binom{K_1+K_2}{3}$ combinations of three sets of \mathcal{F} into two parts: 1) two sets belong to transmitter T_i and one set belongs to transmitter T_j where $j \neq i$; 2) three sets belong to the same transmitter T_i for i = 1 or 2. In the first case, the three sets could be $\mathcal{A}_i^{k_i}, \mathcal{A}_i^{k_i'}, \mathcal{A}_j^{k_j}$. We use the same argument as before, and conclude that the three sets have a non-empty intersection. In the second case, the three sets could be $\mathcal{A}_i^{k_i}, \mathcal{A}_i^{k_i'}, \mathcal{A}_i^{k_i''}$. It is easy to verified that $p_i = \max\{n_i^{k_i}, n_i^{k_i'}, n_i^{k_i''}\}$ and $p_j = 0$ is one of their intersection points. Overall, we prove that any three sets of \mathcal{F} have a non-empty intersection, and thus the intersection of all sets is non-empty.

B. Multicast Sessions with general N

We use mathematical induction to prove Theorem 2. We already show that it is true when N = 2. Assume that the theorem holds for all numbers less than or equal to N - 1 and now we prove that it also holds for N. If $K_i = 1$ for all i, it is the unicast scenario and Theorem 2 is true. Otherwise, we categorize the combinations of N + 1 sets of \mathcal{F} into N parts: 1) Receivers of N transmitters are involved: $\mathcal{A}_1^{k_1}, \mathcal{A}_2^{k_2}, \ldots, \mathcal{A}_N^{k_N}, \mathcal{A}_i^{k'_i}$. 2) Receivers of N - 1 transmitters are involved: $\mathcal{A}_1^{k_1}, \ldots, \mathcal{A}_{j-1}^{k_{j-1}}, \mathcal{A}_{j+1}^{k_{j+1}}, \ldots, \mathcal{A}_N^{k_N}, \mathcal{A}_i^{k'_i}, \mathcal{A}_l^{k'_i}$ where $i, l \neq j. \cdots$ D) Receivers of N - D + 1 transmitters are involved. \cdots N) Receivers of 1 transmitter is involved.

We prove the first part. Let

$$\hat{\mathbf{G}} = egin{bmatrix} \mathbf{a}_{1}^{k_{1}} \\ dots \\ \mathbf{a}_{i}^{k_{i}} \\ dots \\ \mathbf{a}_{N}^{N_{j}} \end{bmatrix} ext{ and } ilde{\mathbf{G}} = egin{bmatrix} \mathbf{a}_{1}^{k_{1}} \\ dots \\ \mathbf{a}_{i}^{k'_{i}} \\ dots \\ \mathbf{a}_{N}^{N_{j}} \end{bmatrix}$$

By Lemma 3, there exists \mathbf{p} such that $\tilde{\mathbf{G}}\mathbf{p} \ge \mathbf{n}_{\tilde{G}}$ and $\hat{\mathbf{G}}\mathbf{p} \ge \mathbf{n}_{\hat{G}}$, which implies $\mathbf{p} \in (\bigcap_{j=1}^{N} \mathcal{A}_{j}^{k_{j}} \cap \mathcal{A}_{j}^{k_{j}'})$.

We prove the D) part for D = 2, ..., N. Suppose the D - 1 transmitters whose receivers are not involved in the N + 1 sets, are $d_1, d_2, ..., d_{D-1} \in \{1, ..., N\}$. We simply let $p_{d_1} = p_{d_2} = \cdots = p_{d_{D-1}} = 0$. The resulting system is equivalent to having N - D + 1 multicast sections characterized by matrix A', which is constructed by removing the rows in A that corresponds to the receivers of transmitter d and the d-th column of A, for $d = d_1, ..., d_{D-1}$. Define $\mathcal{G}' \subset \mathbb{R}^{(N-D+1)\times(N-D+1)}$ for A'. For any $\mathbf{G}' \in \mathcal{G}'$, we can find a $\mathbf{G} \in \mathcal{G}$ such that, \mathbf{G}' is constructed by removing the d-th row and d-th column of G for all $d = d_1, ..., d_{D-1}$. Since $\lambda(\mathbf{I}-\mathbf{G}) < 1$ for all $\mathbf{G} \in \mathcal{G}$, by Lemma 2, $\lambda(\mathbf{I}-\mathbf{G}') < 1$, and therefore $\max_{\mathbf{G}' \in \mathcal{G}'} \{\lambda(\mathbf{I}-\mathbf{G}')\} < 1$. By the inductive hypothesis, we can apply Theorem 2 when N - D + 1 < N, and thus there exists $\mathbf{p}' \ge \mathbf{0}$ such that $\mathbf{A'p'} \ge \mathbf{n'}$, where $\mathbf{n'}$ is obtained by removing the entries that correspond to the receivers of transmitter T_d for $d = d_1, ..., d_{D-1}$. By inserting 0 back into $\mathbf{p'}$ at the position of transmitter T_d for all $d = d_1, ..., d_{D-1}$, we get a power $\mathbf{p} \ge \mathbf{0}$ which is in the N + 1 subsets.

Overall, we have proved that any N+1 subsets of \mathcal{F} has a non-empty intersection. By Helly's theorem, all subsets in \mathcal{F} has an intersection. This completes the proof of Theorem 2.

IV. FEASIBLE SINR REGION AND ALGORITHM

In this section, we characterize the feasible SINR region of a wireless multicast system by analytically obtaining its boundary points. By Proposition 1, we know that the feasible SINR region is downward comprehensive. That is, if μ is feasible, then any μ' satisfying $0 \le \mu' \le \mu$ is also feasible. Therefore, finding the boundary points is enough to figure out the feasible SINR region. Our approach is to find the farthest point from the origin in a given direction. In mathematics, the problem is formulated as

$$\begin{aligned} \sup_{\mathbf{p}} & \beta \\ s.t. & \mathbf{A}(\beta \boldsymbol{\mu}) \mathbf{p} \geq \mathbf{n}(\beta \boldsymbol{\mu}) \\ & \mathbf{p} \geq \mathbf{0}, \end{aligned}$$

where μ is a given direction. By Theorem 2, there is a feasible solution to the above problem if and only if

$$\max_{\mathbf{G}\in\mathcal{G}(\beta\boldsymbol{\mu})} \{\lambda(\mathbf{I}-\mathbf{G})\} = \beta \cdot \max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})} \{\lambda(\mathbf{I}-\mathbf{G})\} < 1.$$

That is

$$\beta < \frac{1}{\max_{\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})} \{\lambda(\mathbf{I} - \mathbf{G})\}}$$

Therefore, the optimal value is

$$\beta^*(\boldsymbol{\mu}) = \frac{1}{\max_{\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})} \{\lambda(\mathbf{I} - \mathbf{G})\}}$$

 $\beta^*(\mu)\mu$ is a boundary point of the SINR region. The open line segment defined by $\{\alpha \mu : 0 < \alpha < \beta^*(\mu)\}$ is in the feasible SINR region Υ , but $\alpha \mu$ is not in the feasible region if $\alpha > \beta^*(\mu)$.

We note that the size of \mathcal{G} is $\prod_{i=1}^{N} K_i$, which grows exponentially with N. It is not an efficient method to calculate the Perron-Frobenius eigenvalue of all the embedded unicast systems and find out the maximum one. Next, we propose an iterative algorithm to compute $\beta^*(\mu)$. For $i = 1, 2, \ldots, N$, let \mathbf{e}_i denote the N-dimensional column vector such that the *i*-th component of \mathbf{e}_i is 1 while the others are 0. The algorithm is described in Algorithm 1.

For receiver $R_i^{k_i}$, $(\mathbf{e}_i^T - \mathbf{a}_i^{k_i})\mathbf{p}^{(k)}$ is the sum of the interference power. The power of transmitter T_i is updated by the maximum interference power experienced by the receivers in its multicast session. This idea is similar to the distributed power control algorithm for unicast systems [20] to solve the power balancing problem. Recall that in [20], given a normalized interference link

Algorithm 1 Iterative algorithm

1: Choose $\mathbf{p}^{(0)} \in \mathbb{R}^N > \mathbf{0}$ and $k \leftarrow 0$ 2: **repeat** 3: **for** i = 1 to N **do** 4: $y_i^{(k)} \leftarrow \max_{k_i \in \mathcal{K}_i} \left\{ \left(\mathbf{e}_i^T - \mathbf{a}_i^{k_i} \right) \mathbf{p}^{(k)} \right\}$ 5: **end for** 6: $\beta^{(k)} \leftarrow \min_{i=1}^N \left\{ \frac{p_i^{(k)}}{y_i^{(k)}} \right\}$ 7: $\mathbf{p}^{(k+1)} \leftarrow \frac{\mathbf{y}^{(k)}}{||\mathbf{y}^{(k)}||}$ 8: $k \leftarrow k + 1$ 9: **until** convergence 10: **return** $\beta^{(k)}$

gain matrix $\mathbf{I} - \mathbf{G}$, the algorithm works as $\mathbf{p}^{(k+1)} = \frac{(\mathbf{I} - \mathbf{G})\mathbf{p}^{(k)}}{||(\mathbf{I} - \mathbf{G})\mathbf{p}^{(k)}||}$, where k is the iteration index. It is well known that when $\mathbf{I} - \mathbf{G}$ is primitive (to be defined later), $||(\mathbf{I} - \mathbf{G})\mathbf{p}^{(k)}||$ converges to the Perron-Frobenius eigenvalue of $\mathbf{I} - \mathbf{G}$, and $\mathbf{p}^{(k)}$ converges to the corresponding eigenvector. In our proposed algorithm, we are dealing with multicast systems. For notation simplicity, define

$$\mathcal{Z}(\boldsymbol{\mu}) = \{ \mathbf{Z} = \mathbf{I} - \mathbf{G} : \mathbf{G} \in \mathcal{G}(\boldsymbol{\mu}) \}.$$
(6)

 \mathcal{Z} includes the normalized interference link gain matrices of all the embedded unicast systems and $\mathbf{Z} \geq \mathbf{0}$ for all $\mathbf{Z} \in \mathcal{Z}$. Given any $\mathbf{p} > 0$, due to the structure of $\mathcal{G}(\boldsymbol{\mu})$, there always exists $\hat{\mathbf{Z}} \in \mathcal{Z}$ such that $\hat{\mathbf{Z}}\mathbf{p} \geq \mathbf{Z}\mathbf{p}$ for all $\mathbf{Z} \in \mathcal{Z}$. Our algorithm works as $\mathbf{p}^{(k+1)} = \frac{\mathbf{Z}^{(k)}\mathbf{p}^{(k)}}{||\mathbf{Z}^{(k)}\mathbf{p}^{(k)}||}$, where $\mathbf{Z}^{(k)} \in \mathcal{Z}$ is chosen such that $\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \geq \mathbf{Z}\mathbf{p}^{(k)}$ for all $\mathbf{Z} \in \mathcal{Z}$. In the rest of this section, we show the convergence of the algorithm.

Lemma 4. The sequence $\{\beta^{(k)}\}$ generated by Algorithm 1 is monotonically increasing and bounded above by $\frac{1}{\max_{\mathbf{G}\in\mathcal{G}(\mu)}\{\lambda(\mathbf{I}-\mathbf{G})\}} = \frac{1}{\max_{\mathbf{Z}\in\mathcal{Z}(\mu)}\{\lambda(\mathbf{Z})\}}$, and thus is convergent.

Proof: By Algorithm 1, we have $\mathbf{y}^{(k)} \ge \mathbf{Z}\mathbf{p}^{(k)}$ for all $\mathbf{Z} \in \mathcal{Z}$, and $\mathbf{p}^{(k)} \ge \beta^{(k)}\mathbf{y}^{(k)}$. Then $\mathbf{Z}\mathbf{p}^{(k+1)} = \mathbf{Z}\frac{\mathbf{y}^{(k)}}{||\mathbf{y}^{(k)}||} \le \mathbf{Z}\frac{\mathbf{p}^{(k)}}{\beta^{(k)}||\mathbf{y}^{(k)}||} \le \frac{\mathbf{y}^{(k)}}{\beta^{(k)}||\mathbf{y}^{(k)}||} = \frac{\mathbf{p}^{(k+1)}}{\beta^{(k)}}.$ Since the above inequality holds for all $\mathbf{Z} \in \mathcal{Z}$, we have

$$\beta^{(k)} \le \min_{\mathbf{Z}\in\mathcal{Z}} \left\{ \min_{i=1}^{N} \left\{ \frac{p_i^{(k+1)}}{[\mathbf{Z}\mathbf{p}^{(k+1)^T}]_i} \right\} \right\} = \min_{i=1}^{N} \left\{ \frac{p_i^{(k+1)}}{y_i^{(k+1)}} \right\} = \beta^{(k+1)}.$$

That is, $\{\beta^{(k)}\}\$ is monotonically increasing. On the other hand, since $\mathbf{p}^{(k)} \ge \beta^{(k)} \mathbf{y}^{(k)} \ge \beta^{(k)} \mathbf{Z} \mathbf{p}^{(k)}$ for all $\mathbf{Z} \in \mathcal{Z}$, that is $(\mathbf{I} - \beta^{(k)} \mathbf{Z}) \mathbf{p}^{(k)} \ge \mathbf{0}$, we have $\lambda(\beta^{(k)} \mathbf{Z}) \le 1$ by Theorem 1. Therefore $\beta^{(k)} \le \frac{1}{\max_{\mathbf{Z} \in \mathcal{Z}(\mu)} \{\lambda(\mathbf{Z})\}}$, and thus $\{\beta^{(k)}\}\$ is convergent.

Denote $\lim_{k\to\infty} \beta^{(k)} = \beta^*$. Before we proceed to show that $\beta^* = \frac{1}{\max_{\mathbf{Z} \in \mathcal{Z}(\boldsymbol{\mu})} \{\lambda(\mathbf{Z})\}}$, we introduce the concept of *primitive* matrix and *primitive* set.

Definition 4. [16] A square nonnegative matrix \mathbf{X} is called primitive if there exists a positive integer n such that $\mathbf{X}^n > 0$.

The class of primitive matrices is a subclass of irreducible matrices. If \mathbf{X} is primitive, then its Perron-Frobenius eigenvalue is strictly greater than all other eigenvalues in absolute value. The primitive condition guarantees the convergence of the aforementioned distributed power control algorithm for unicast systems. In our multicast case, we need to use the concept of primitive set, which replaces a single matrix and powers of that matrix with a set of matrices and inhomogeneous products of matrices from the set.

Definition 5. [21] Let Z be a set of $N \times N$ nonnegative matrices. For a positive integer n, let $\Theta(n)$ be an arbitrary product of n matrices from Z, with any ordering and with repetitions permitted. Define Z to be a primitive set if there is a positive integer n such that every $\Theta(n)$ is positive.

It can be seen that a necessary condition for \mathcal{Z} to be primitive is that \mathbf{Z} is primitive for all $\mathbf{Z} \in \mathcal{Z}$. One of the sufficient conditions for \mathcal{Z} to be primitive is that for any $\mathbf{Z} \in \mathcal{Z}$, in each row and each column of \mathbf{Z} , there are more than half of the entries that are positive [21]. Interested readers can refer to [21] for more information of the primitive set. It needs to be mentioned that when the system is composed of two multicast sessions, $\mathbf{Z} \in \mathcal{Z}$ are always non-primitive, and therefore \mathcal{Z} cannot be primitive. However, for this case, Corollary 1 already gives an explicit and simple solution to the feasible SINR region. Algorithm 1 works for systems with more than two multicast sessions and with \mathcal{Z} being primitive.



Fig. 2: Convergence of Algorithm 1. In this example, there are four multicast sessions and each has three receivers. The link gains are randomly drawn from a uniform distribution on [0, 1).

Theorem 4. If the matrix set $\mathcal{Z}(\boldsymbol{\mu})$ defined in (6) is primitive, then $\beta^{(k)}$ converges to $\beta^* = \frac{1}{\max_{\mathbf{Z} \in \mathcal{Z}(\boldsymbol{\mu})} \{\lambda(\mathbf{Z})\}}$ for an arbitrary initial value $\mathbf{p}^{(0)} > 0$. Moreover, $\mathbf{p}^{(k)}$ converges to a power vector \mathbf{p}^* such that $\lim_{\alpha \to \infty} \Gamma(\alpha \mathbf{p}^*)$ achieves the boundary point $\beta^* \boldsymbol{\mu}$.

The proof is provided in Appendix A. Fig. 2 illustrates the typical behavior of Iterative Algorithm 1. In this example, there are four multicast sessions and each has three receivers. The link gains are randomly drawn from a uniform distribution on [0, 1). As we can see, $\beta^{(k)}$ converges within a small number of iterations. By using Algorithm 1, we can efficiently check the feasibility of an SINR vector μ by checking the value of $\beta^*(\mu)$. If $\beta^*(\mu) < 1$, μ is infeasible, and vice versa. Besides, Algorithm 1 can be used to find the optimal solution of the classic power balancing problem for multicast systems, which is in the following form

$$\sup_{\mathbf{p}} \min_{i=1}^{N} \quad \gamma_i(\mathbf{p})$$

s.t. $\mathbf{p} \ge \mathbf{0}$

and the solution is $\beta^*(1)$.

V. GEOMETRIC PROPERTIES OF THE FEASIBLE SINR REGION

In this section, we discuss the geometric properties of the feasible SINR region. Let $D(\mu)$ denote the diagonal matrix constructed by

$$D(\boldsymbol{\mu}) = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_N \end{bmatrix}.$$

By Theorem 2, the feasible SINR region is equivalent to

$$\begin{split} \Upsilon &= \left\{ \boldsymbol{\mu} \in \mathbb{R}^{N} : \max_{\mathbf{Z} \in \mathcal{Z}(\boldsymbol{\mu})} \{\lambda(\mathbf{Z})\} < 1 \right\} \\ &= \left\{ \boldsymbol{\mu} \in \mathbb{R}^{N} : \max_{\mathbf{Z} \in \mathcal{Z}(\mathbf{1})} \left\{\lambda(D(\boldsymbol{\mu})\mathbf{Z})\right\} < 1 \right\} \\ &= \bigcap_{\mathbf{Z} \in \mathcal{Z}(\mathbf{1})} \left\{\boldsymbol{\mu} \in \mathbb{R}^{N} : \lambda(D(\boldsymbol{\mu})\mathbf{Z}) < 1 \right\}. \end{split}$$

That is, the feasible SINR region of a multicast system is the intersection of the feasible SINR regions of all its embedded unicast systems. Let $\Upsilon^c = \mathbb{R}^N_+ \setminus \Upsilon$ denote the complement of Υ in \mathbb{R}^N_+ , i.e., the infeasible SINR region. Next, we investigate the convexity of Υ^c and the log-convexity of Υ .

A. Convexity of Υ^c

For unicast systems, it has been proved in [5] that the infeasible SINR regions of a general two user system and a general three user system are convex. It is also shown in [4] that the convexity of the infeasible SINR region does not hold for a general four user system. For multicast systems, we have the following observation.

Theorem 5. The infeasible SINR region of a general system consisting of two multicast sessions is convex. The convexity property does not hold for a general system consisting of more than two multicast sessions.

When there are two multicast sessions, by Corollary 1, the feasible SINR region is

$$\Upsilon = \left\{ [\mu_1, \mu_2] \in \mathbb{R}^2_+ : \mu_1 \mu_2 < \frac{g_{r_1^{k_1^*}, t_1}}{g_{r_1^{k_1^*}, t_2}} \cdot \frac{g_{r_2^{k_2^*}, t_2}}{g_{r_2^{k_2^*}, t_1}} \right\}$$

It is ready to verify that Υ^c is convex.

When there are more than two multicast sessions, Υ^c is the union of the infeasible SINR regions of all the embedded unicast systems and is in general non-convex. Fig. 3 illustrates the Υ^c for a system consisting of three multicast sessions, where the link gain matrix is given by

	T_1	T_2	T_3
R_1^1	1	0.5	0.1
R_1^2	1	0.1	0.5
R_2^1	0.5	1	0.1
R_2^2	0.1	1	0.5
R_3^1	0.5	0.1	1
R_3^2	0.1	0.5	1

It can be seen that its Υ^c is non-convex.

B. Log-convexity of Υ

We first introduce the notion of log-convexity. Let $\log(\mu) = [\log \mu_1, \log \mu_2, \dots, \log \mu_N]$ and $\log(\Upsilon) = \{\log(\mu) : \mu \in \Upsilon\}$. We say a set Υ is log-convex if $\log(\Upsilon)$ is convex. Since $\log(\cdot) : \Upsilon \to \log(\Upsilon)$ is a bijective mapping, we have

$$\log(\Upsilon) = \bigcap_{\mathbf{Z} \in \mathcal{Z}(\mathbf{1})} \Big\{ \log(\boldsymbol{\mu}) \in \mathbb{R}^N : \lambda \big(D(\boldsymbol{\mu}) \mathbf{Z} \big) < 1 \Big\}.$$

It has been proved in [3] that the feasible SINR region of a unicast system is log-convex. So $\log(\Upsilon)$, the intersection of the SINR regions of all its embedded unicast, is also log-convex. We conclude this by the following theorem.

Theorem 6. The feasible SINR region of a multicast system is log-convex. In other words, the feasible SINR, expressed in decibels, is a convex set.

VI. FEASIBILITY OF SINR WITH POWER CONSTRAINTS

So far, we have discussed the feasibility of SINR for a multicast system in the case of unlimited power. In this section, we consider that besides $\mathbf{p} \ge \mathbf{0}$, the power vector are also subject to the linear constraints

$$\sum_{i\in\Omega_m} p_i \le \bar{p}_{\Omega_m}, m = 1, \dots, M,$$



Fig. 3: Infeasible SINR region of a three-multicast-session system.

where $\Omega_m \subseteq \{1, \ldots, N\}$ and M is the number of constraints. When $\Omega_m = \{1, \ldots, N\}$, it is a constraint on the total power. When $\Omega_m = \{i\}$, it is a constraint on the individual power of transmitter T_i . Define the power set by

$$\mathcal{P} = \{\mathbf{p} \ge \mathbf{0} \text{ and } \sum_{i \in \Omega_m} p_i \le \bar{p}_{\Omega_m}, m = 1, \dots, M\}.$$

Now the feasibility of SINR vector μ is decided by whether there exists $\mathbf{p} \in \mathcal{P}$ such that $\mathbf{A}(\mu)\mathbf{p} = \mathbf{n}(\mu)$. Note that the power vectors in \mathcal{P} are downward comprehensive. That is, if $\mathbf{p}' \in \mathcal{P}$, then $\mathbf{p} \in \mathcal{P}$ if $\mathbf{0} \leq \mathbf{p} \leq \mathbf{p}'$. Hence using the same argument as in Proposition 1, we know that μ is feasible if and only if there exists $\mathbf{p} \in \mathcal{P}$ such that $\mathbf{A}(\mu)\mathbf{p} \geq \mathbf{n}(\mu)$. Our results generalize the feasibility condition derived in [6] for a unicast system to a multicast system.

Definition 6. For a matrix $\mathbf{X} \in \mathbb{R}^{K \times N}$, a vector $\mathbf{y} \in \mathbb{R}^{K}$ and a set $\Omega \subseteq \{1, \dots, N\}$, $\psi(\mathbf{X}, \mathbf{y}, \Omega)$ is the operation to add \mathbf{y} to the *j*-th column of \mathbf{X} , for all $j \in \Omega$. That is, $\mathbf{Z} = \psi(\mathbf{X}, \mathbf{y}, \Omega)$, where $Z_{ij} = X_{ij} + y_i$ for all $i \in \{1, \dots, K\}$ and $j \in \Omega$, and $Z_{ij} = X_{ij}$ for the else.

Theorem 7. Consider a multicast network setting $\mathbf{A}(\boldsymbol{\mu})$ and assume the matrices $\mathbf{I} - \mathbf{G}$ for $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$ are all irreducible. There exists a power vector $\mathbf{p} \in \mathcal{P}$ such that $\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \ge \mathbf{n}(\boldsymbol{\mu})$ if

and only if

$$\max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})}\max_{m\in\{1,\dots,M\}}\left\{\lambda\left(\psi\left(\mathbf{I}-\mathbf{G},\frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_{m}}},\Omega_{m}\right)\right)\right\}\leq 1.$$

Proof: It is already known from [6] that, for a unicast system **G**, there exists $\mathbf{p} \in \mathcal{P}$ such that $\mathbf{Gp} \geq \mathbf{n}_{\mathbf{G}}$ if and only if $\max_{m \in \{1,...,M\}} \{\lambda(\psi(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m))\} \leq 1$. We first prove the necessary condition. Suppose there exists $\mathbf{p} \in \mathcal{P}$ such that $\mathbf{A}(\boldsymbol{\mu})\mathbf{p} \geq \mathbf{n}(\boldsymbol{\mu})$. Then for any $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$, $\mathbf{Gp} \geq \mathbf{n}_{\mathbf{G}}$, which implies $\max_{m \in \{1,...,M\}} \{\lambda(\psi(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m))\} \leq 1$. Regarding all $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$, we have $\max_{\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})} \max_{m \in \{1,...,M\}} \{\lambda(\psi(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m))\} leq 1$.

Next we prove the sufficient condition. For any $\mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})$, since $\mathbf{0} \leq \mathbf{I} - \mathbf{G} < \psi(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m)$ for all m, by the Perron-Frobenius Theorem for irreducible matrices [16], $\lambda(\mathbf{I} - \mathbf{G}) < \lambda(\psi(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m)) \leq 1$. This implies that \mathbf{G}^{-1} exists and $\mathbf{p} = \mathbf{G}^{-1}\mathbf{n}_{\mathbf{G}} \in \mathcal{P}$. The rest of the proof follows the same argument as the proof of Theorem 2.

Note that

$$\max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})} \max_{m\in\{1,\dots,M\}} \left\{ \lambda \left(\psi \left(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m \right) \right) \right\} = \max_{m\in\{1,\dots,M\}} \max_{\mathbf{G}\in\mathcal{G}(\boldsymbol{\mu})} \left\{ \lambda \left(\psi \left(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m \right) \right) \right\}.$$

Similar to (6), for each of the M linear constraints, define

$$\mathcal{Z}_{\Omega_m}(\boldsymbol{\mu}) = \left\{ \psi \left(\mathbf{I} - \mathbf{G}, \frac{\mathbf{n}_{\mathbf{G}}}{\bar{p}_{\Omega_m}}, \Omega_m \right) : \mathbf{G} \in \mathcal{G}(\boldsymbol{\mu})
ight\}.$$

By using Algorithm 1 with $\mathcal{Z}_{\Omega_m}(\boldsymbol{\mu})$, we can find a supremum $\beta^*_{\Omega_m}(\boldsymbol{\mu})$. The farthest point of the SINR region in direction $\boldsymbol{\mu}$ is then $\min_{m=1}^M \{\beta^*_{\Omega_m}(\boldsymbol{\mu})\}\boldsymbol{\mu}$. By this approach, the feasible SINR region is characterized. On the other hand, if $\min_{m=1}^M \{\beta^*_{\Omega_m}(\boldsymbol{\mu})\} \ge 1$, $\boldsymbol{\mu}$ is feasible. Fig. 4 plots the feasible SINR region of the network example in Fig. 1, with a power constraint on the total power. In this example, the link gain matrix is

$$\begin{array}{ccc} T_1 & T_2 \\ R_1^1 & \begin{bmatrix} 0.5326 & 0.6801 \\ 0.5539 & 0.3672 \\ R_2^1 & 0.2393 & 0.8669 \\ R_2^2 & 0.5789 & 0.4068 \\ \end{bmatrix}$$

and the power constraint is $p_1 + p_2 \le 2$. The four dashed lines are the boundary of the feasible SINR regions of four embedded unicast systems and the solid line is the boundary of the multicast system. It can be seen that under power contraint, the infeasible SINR region is not necessary to be convex even for a multicast system with two multicast sessions.



Fig. 4: Feasible SINR region for a multcast system with two multicast sessions, under a total power constraint. The dashed lines correspond to the four embedded unicast systems and the solid line corresponds to the multicast system.

In the end of this section, we introduce an application of our multicast model to a time varying unicast system. Consider a unicast system consisting of N transmitter-receiver pairs, where the channel gains among them vary with time due to the mobility of the receivers. Let $\mathbf{h}_i(t)$ for i = 1, ..., N denote the link gain vector from N transmitters to the *i*-th receiver at time *t*. As argued in [6], $\mathbf{h}_i(t)$ can be modeled with discrete states, that is, $\mathbf{h}_i(t)$ is randomly selected from a finte set $\{\mathbf{h}_i^1, \mathbf{h}_i^2, ..., \mathbf{h}_i^{K_i}\}$ for all *i*. An SINR vector $\boldsymbol{\mu}$ is said to be zero-outage if there exists a power such that no matter what the link gain realization is, the SINR is achievable all the time. Such a zero-outage SINR problem can be mapped to a feasible SINR problem of a multicast system. The idea is to let one receiver R_i pretend to be K_i receivers, i.e., $R_i^1, \ldots, R_i^{K_i}$, and $R_i^{k_i}$ only experiences the link gain $\mathbf{h}_i^{k_i}$. The feasible SINR region of this artificial multicast system is exactly the zero-outage SINR region of the original time-varying unicast system. Theorem 2 and Theorem 7 can be applied. It needs to be mentioned that the scenario considered here is different from that in [6]. In [6], the power can change with the channel states and is subject to an average power constraint. In our model, the power is universal for all possible channel states.

VII. CONCLUSION

In this paper, we characterize the feasibility condition of an SINR vector for a multicast system, which generalizes the Perron-Frobenius Theorem for a unicast system. We also propose an iterative algorithm which can efficiently check the condition and compute the boundary points of the feasible SINR region. According to the earlier mentioned Gaussian interference assumption, by describing the feasible SINR region, we directly obtain the feasible rate region by applying the Shannon Capacity formula for AWGN channels that maps the SINR to the rate.

APPENDIX A

PROOF OF THEOREM 4

Proof: Let $\mathbf{Z}^{(k)}$ denote one of the matrices at the k-th iteration such that $\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \ge \mathbf{Z}\mathbf{p}^{(k)}$ for all $\mathbf{Z} \in \mathcal{Z}$. From the construction of the algorithm, we have

$$\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \le \frac{1}{\beta^{(k)}}\mathbf{p}^{(k)} \text{ for all } k \in \mathbb{N}.$$
(7)

Moreover, there exists $1 \leq i \leq N$ such that $[\mathbf{Z}^{(k)}\mathbf{p}^{(k)}]_i = \frac{1}{\beta^{(k)}}[\mathbf{p}^{(k)}]_i$. We note that each vector $\mathbf{p}^{(k)}$ is a unit vector, as $||\mathbf{p}^{(k)}|| = 1$. By the Bolzano-Weierstrass Theorem and the compactness of the unit ball in \mathbb{R}^N , there exists a convergent subsequence, that is, $\mathbf{p}^{(k_j)} \to \mathbf{p}^*$. By Lemma 4, $\beta^{(k_j)} \to \beta^*$. Suppose at \mathbf{p}^* , $\mathbf{Z}^* \in \mathcal{Z}$ is one of the matrices that satisfy $\mathbf{Z}^*\mathbf{p}^* \geq \mathbf{Z}\mathbf{p}^*$ for all $\mathbf{Z} \in \mathcal{Z}$. Taking the limit of (7) with respect to the subsequence indexed by k_j , we have $\mathbf{Z}^*\mathbf{p}^* \leq \frac{1}{\beta^*}\mathbf{p}^*$. If $\mathbf{Z}^*\mathbf{p}^* = \frac{1}{\beta^*}\mathbf{p}^*$, since \mathbf{Z}^* is irreducible, $\beta^* = \frac{1}{\lambda(\mathbf{Z}^*)} \geq \frac{1}{\max_{\mathbf{Z} \in \mathcal{Z}(\mu)}\{\lambda(\mathbf{Z})\}}$. On the other hand, $\beta^* \leq \frac{1}{\max_{\mathbf{Z} \in \mathcal{Z}(\mu)}\{\lambda(\mathbf{Z})\}}$ by Lemma 4. Therefore, $\beta^* = \frac{1}{\max_{\mathbf{Z} \in \mathcal{Z}(\mu)}\{\lambda(\mathbf{Z})\}}$.

If $\mathbf{Z}^*\mathbf{p}^* \neq \frac{1}{\beta^*}\mathbf{p}^*$, since \mathcal{Z} is primitive, there exists integer n such that an arbitrary product of n matrices from \mathcal{Z} is positive, i.e., $\Theta(n) > \mathbf{0}$, and therefore $\Theta(n)\mathbf{Z}^*\mathbf{p}^* < \Theta(n)\frac{1}{\beta^*}\mathbf{p}^*$. By the continuity of the mapping, there exists $\mathbf{p}^{(k)}$ close enough to \mathbf{p}^* such that $\Theta(n)\mathbf{Z}^*\mathbf{p}^{(k)} < \Theta(n)\frac{1}{\beta^*}\mathbf{p}^{(k)}$ and $\mathbf{Z}^{(k)} = \mathbf{Z}^*$. We now apply the algorithm for n more iterations from $\mathbf{p}^{(k)}$. For $i = 0, 1, \dots, n-1$ we have the following inequalities

$$\mathbf{Z}^{(k+i+1)}\mathbf{p}^{(k+i)} \le \mathbf{Z}^{(k+i)}\mathbf{p}^{(k+i)}$$
(8)

due to that the selection matrix satisfies $\mathbf{Z}^{(k+i)}\mathbf{p}^{(k+i)} \ge \mathbf{Z}\mathbf{p}^{(k+i)}$ for all $\mathbf{Z} \in \mathcal{Z}$. Meanwhile by Algorithm 1,

$$\mathbf{p}^{(k+i)} = \frac{\mathbf{y}^{(k+i-1)}}{||\mathbf{y}^{(k+i-1)}||} = \frac{\mathbf{Z}^{(k+i-1)}\mathbf{p}^{(k+i-1)}}{||\mathbf{Z}^{(k+i-1)}\mathbf{p}^{(k+i-1)}||} = \frac{\mathbf{Z}^{(k+i-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}}{||\mathbf{Z}^{(k+i-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}||}$$

By substituting $p^{(k+i)}$ into (8), we get

$$\mathbf{Z}^{(k+i+1)}\mathbf{Z}^{(k+i-1)}\mathbf{Z}^{(k+i-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \le \mathbf{Z}^{(k+i)}\mathbf{Z}^{(k+i-1)}\mathbf{Z}^{(k+i-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}.$$
 (9)

Let us take a look at these inequalities step by step. By (8) for i = 0, $\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. By multiplying $\mathbf{Z}^{(k+2)}$ on both side of the inequality, we have

$$\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \le \mathbf{Z}^{(k+2)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}.$$
(10)

By (9) for i = 1, $\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. Along with (10), we have

$$\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$$

By multiplying $\mathbf{Z}^{(k+3)}$ on both side of the above inequality, we have $\mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. By (9) for i = 2, $\mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$. So

$$\mathbf{Z}^{(k+3)}\mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+2)}\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}$$

By repeating this procedure for n-1 times, we can finally get

$$\mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)} \leq \mathbf{Z}^{(k+n-1)}\mathbf{Z}^{(k+n-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{(k)}\mathbf{p}^{(k)}.$$
 (11)

Since $\Theta(n)\mathbf{Z}^*\mathbf{p}^{(k)} < \Theta(n)\frac{1}{\beta^*}\mathbf{p}^{(k)}$ holds for arbitrary $\Theta(n)$, we let $\Theta(n) = \mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}$. Along with (11), we have

$$\mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{*}\mathbf{p}^{(k)} = \Theta(n)\mathbf{Z}^{*}\mathbf{p}^{(k)}$$

$$<\Theta(n)\frac{1}{\beta^{*}}\mathbf{p}^{(k)}$$

$$=\frac{1}{\beta^{*}}\mathbf{Z}^{(k+n)}\mathbf{Z}^{(k+n-1)}\cdots\mathbf{Z}^{(k+1)}\mathbf{p}^{(k)}$$

$$\leq\frac{1}{\beta^{*}}\mathbf{Z}^{(k+n-1)}\mathbf{Z}^{(k+n-2)}\cdots\mathbf{Z}^{(k+1)}\mathbf{Z}^{*}\mathbf{p}^{(k)}.$$

By multiplying $\frac{1}{||\mathbf{Z}^{(k+n-1)}...\mathbf{Z}^{(k+1)}\mathbf{Z}^*\mathbf{p}^{(k)}||}$ on both side of the inequality, we have

$$\mathbf{Z}^{(k+n)}\mathbf{p}^{(k+n)} = \mathbf{Z}^{(k+n)} \frac{\mathbf{Z}^{(k+n-1)} \cdots \mathbf{Z}^{(k+1)} \mathbf{Z}^* \mathbf{p}^{(k)}}{||\mathbf{Z}^{(k+n-1)} \cdots \mathbf{Z}^{(k+1)} \mathbf{Z}^* \mathbf{p}^{(k)}||} < \frac{1}{\beta^*} \frac{\mathbf{Z}^{(k+n-1)} \cdots \mathbf{Z}^{(k+1)} \mathbf{Z}^* \mathbf{p}^{(k)}}{||\mathbf{Z}^{(k+n-1)} \cdots \mathbf{Z}^{(k+1)} \mathbf{Z}^* \mathbf{p}^{(k)}||} = \frac{1}{\beta^*} \mathbf{p}^{(k+n)}.$$

This implies $\beta^{(k+n)} > \beta^*$, which contradicts with that β^* is the limit. Hence there must have $\mathbf{Z}^* \mathbf{p}^* = \frac{1}{\beta^*} \mathbf{p}^*$.

We prove $\mathbf{p}^{(k)} \to \mathbf{p}^*$ by contradiction. Suppose there exists another subsequence such that $\mathbf{p}^{k'_j} \to \mathbf{p}'$ and $\mathbf{p}^* \neq \mathbf{p}'$. Then $\mathbf{Z}^*\mathbf{p}' \leq \frac{1}{\beta^*}\mathbf{p}'$. Meanwhile we already have $\mathbf{Z}^*\mathbf{p}^* = \frac{1}{\beta^*}\mathbf{p}^*$. By the Subinvariance Theorem in [16] (pp. 23), $\mathbf{p}^* = \mathbf{p}'$, which contradicts with the assumption that $\mathbf{p}^* \neq \mathbf{p}'$. Therefore $\mathbf{p}^{(k)}$ converges to \mathbf{p}^* .

Since $\mathbf{Z}^* \mathbf{p}^* = \frac{1}{\beta^*} \mathbf{p}^*$ and $\mathbf{Z} \mathbf{p}^* \leq \frac{1}{\beta^*} \mathbf{p}^*$ for all $\mathbf{Z} \in \mathcal{Z}(\boldsymbol{\mu})$, it is ready to see that $\lim_{\alpha \to \infty} \gamma_i(\alpha \mathbf{p}^*) = \min_{\mathbf{Z} \in \mathcal{Z}(\mathbf{1})} \{ \frac{p_i^*}{[\mathbf{Z} \mathbf{p}^*]_i} \} = \beta^* \mu_i$ for all $i = 1, \dots, N$. So $\lim_{\alpha \to \infty} \Gamma(\alpha \mathbf{p}^*) = \beta^* \boldsymbol{\mu}$.

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