

Coded Caching Schemes with Linear Subpacketizations

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Abstract

In coded caching system we prefer to design a coded caching scheme with low subpacketization and small transmission rate (i.e., the low implementation complexity and the efficient transmission during the peak traffic times). Placement delivery arrays (PDA) can be used to design code caching schemes. In this paper we propose a framework of constructing PDAs via Hamming distance. As an application, two classes of coded caching schemes with linear subpacketizations and small transmission rates are obtained.

Index Terms

Coded caching scheme, Placement delivery array, Hamming distance.

I. INTRODUCTION

THE immense growth in wireless data traffic leads to an enormous pressure on wireless network especially the high temporal variability of network traffic results in congestion during the peak traffic times while underutilization during the off-peak times. In order to make fully use of the local caching to solve this problem, coded caching system was proposed in [11] which can significantly reduce the amount of transmission during the peak traffic time.

In an F -division (K, M, N) centralized coded caching system, a server containing N files with equal size connects to K users, each of which has memory of size M files, through an error-free shared link. This system consists of two phases, i.e., the placement phase during the off-peak traffic times and the delivery phase during the peak traffic times. In the placement phase, without knowledge of later demands, the server divides each file into F packets with equal size where F is referred as subpacketization, and then places some contents generated by the packets of all the files to each user's cache memory. In the delivery phase, each user requires one file randomly. Then the server sends some coded signals with the size of at most R files (R is referred as transmission), which satisfies various demands from users.

The first coded caching scheme proposed by Maddah-Ali and Niesen in [11], which is referred as MN scheme in this paper, achieves the approximatively minimum transmission rate when $N > K$. The MN scheme has been extended to numerous models, such as Device-to-Device (D2D) caching systems [8], online caching [13], hierarchical caching [10], secure caching [16] and so on. While the subpacketization of the MN scheme increases exponentially with the number of users K , which leads to high implementing complexity and infeasibility in reality.

A. Previously known results

It is well known that there exists a tradeoff between subpacketization and transmission rate. Hence reducing the subpacketization must be at the cost of increasing the transmission rate compared with MN scheme.

The authors in [20] proposed an $F \times K$ array called placement delivery array (PDA) to generate an F -division coded caching scheme with K users. By constructing PDAs, they obtained two classes of schemes with lower subpacketizations compared with the MN scheme. Apart from PDA, there are many other characterizations of coded caching schemes such as hypergraphs [18], strong edge coloring of bipartite graphs [21], Ruzsa-Szemerédi graphs [17], combinatorial design theory [1], [19], line graphs [9] and so on. We list most of the previously known deterministic schemes, which have advantages on the subpacketization or the transmission rate, in Table I.

In [17], it was pointed out that all the deterministic coded caching schemes introduced above can be represented by PDAs. Hence constructing appropriate PDAs makes great sense to coded caching. There are some known constructions of PDAs from view points of combinatorial designs [4]–[7], bipartite graphs [12] and concatenating construction [14], [22] so on. It is worth noting that the framework of constructing coded caching schemes proposed in [7] can include most of the previously

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TABLE I: Summary of some known coded caching schemes where all the variables are positive integers unless otherwise stated.

References	Number of Users K	Cache Fraction $\frac{M}{N}$	Rate R	Subpacketization F
[11]: $\frac{KM}{N} \in \mathbb{Z}^+$	K	$\frac{M}{N}$	$\frac{K(1-\frac{M}{N})}{K\frac{M}{N}+1}$	$\binom{K}{K\frac{M}{N}}$
[21]: $a, b < m$ $\lambda < \min\{a, b\}$	$\binom{m}{a}$	$\frac{\binom{a}{\lambda}\binom{m-a}{b-\lambda}}{\binom{m}{a}}$	$\frac{\binom{m}{a+b-2\lambda}\binom{a+b-2\lambda}{a-\lambda}}{\binom{m}{b}}$	$\binom{m}{b}$
[2]: $n + m + 1 \leq z = k - t$, prime power q	$\frac{q^{\frac{n(n+1)}{2}} \prod_{i=0}^n \begin{bmatrix} z+1-i \\ 1 \end{bmatrix}_q}{(n+1)!}$	$1 - q^{\frac{(m+1)(n+1)}{2}} \prod_{i=0}^n \frac{\begin{bmatrix} z-m-i \\ 1 \end{bmatrix}_q}{\begin{bmatrix} z+1-i \\ 1 \end{bmatrix}_q}$	$\frac{(m+1)! q^{\frac{(n+1)(\frac{m}{2}+m+1)}{2}}}{(m+n+2)!} \prod_{i=m+1}^{m+n+1} \begin{bmatrix} z+1-i \\ 1 \end{bmatrix}_q$	$\frac{q^{\frac{m(m+1)}{2}} \prod_{i=0}^m \begin{bmatrix} z+1-i \\ 1 \end{bmatrix}_q}{(m+1)!}$
[1]: $(v, k, 2)$ -SBIBD	v	$1 - \frac{k-1}{v}$	1	kv
[7]: $t \leq m$	$\binom{m}{t} q^t$	$1 - (\frac{q-1}{q})^t$	$(q-1)^t$	q^{m-1}
[7]: $t \leq m$, $[m, m-t]_q$ maximum distance separable code	$\binom{m}{t} q^t$	$1 - (\frac{q-1}{q})^t$	$q^t - 1$	q^{m-t}
[15]: $\frac{KM}{N}, \frac{K}{g+1} \in \mathbb{Z}^+$	K	$\frac{M}{N}$	$\frac{K}{g+1} (1 - \frac{1}{\lceil \frac{M}{N} \rceil})$	$\mathcal{O}(e^g)$
[4]: $t < k$	$\binom{k}{t+1}$	$1 - \frac{t+1}{k}$	$\frac{k}{\binom{k}{t}}$	$\binom{k}{t}$

known schemes. Furthermore based on some special PDAs, the authors in [23] generated some improved schemes with smaller subpacketizations and memory sizes compared with the scheme generated by the method in [20].

B. Contributions and arrangement of this paper

In this paper we focus on linear subpacketization schemes with small transmission rates when $N > K$, where linear means linear to the number of users. Firstly we propose a framework of constructing PDAs using the concept of Hamming distance. Secondly, we obtain two classes of coded caching schemes with linear subpacketizations. Our new schemes have advantages on number of users, subpacketizations, memory size and transmission rates compared with several previously known deterministic schemes with linear subpacketizations.

The rest of the paper is organized as follows. In Section II we state the preliminaries about coded caching, placement delivery array and their relationship. In Section III we introduce the framework of constructing PDAs via Hamming distance. Two classes of schemes are obtained in Sections IV and V respectively. The performance analysis of our new schemes is proposed in Section VI. Finally conclusion is drawn in Section VII.

II. PRELIMINARIES

In this paper, we use bold capital letter, bold lower case letter and curlicue letter to denote array, vector and set respectively. For any positive integers m and t with $t < m$, let $[0, m) = \{0, 1, \dots, m-1\}$.

A. Centralized coded caching system

In a centralized coded caching system, a sever containing N files, denoted by $\mathcal{W} = \{W_n \mid n \in [0, N)\}$, links to K users, denoted by $\mathcal{K} = [0, K)$ with $K < N$ through an error-free shared link. Assume that each user has a memory of size M files with $M < N$. An F -division (K, M, N) coded caching scheme operates in two phases which can be sketched as follows:

- 1) **Placement Phase:** All the files are divided into F equal packets where F is referred as subpacketization¹, i.e., $\mathcal{W} = \{W_{n,j} \mid j \in [0, F), n \in [0, N)\}$. Each user caches some coded packets or uncoded packets from \mathcal{W} . \mathcal{Z}_k denotes the content cached by user k . The size of \mathcal{Z}_k is the capacity of each user's cache memory size M .
- 2) **Delivery Phase:** Each user requests one file from \mathcal{W} randomly. Denote the requested file numbers by $\mathbf{d} = (d_0, d_1, \dots, d_{K-1})$, i.e., user k requests file W_{d_k} , where $k \in \mathcal{K}, d_k \in [0, N)$. The server broadcasts coded signals of size at most R files to users, so that each user is able to recover his requested file with help of its caching contents. R is called the transmission rate.

Clearly the efficiency of the data transmission in the delivery phase increases with decreasing transmission rate. Furthermore, the complexity of the implementing a coded caching scheme increases as subpacketization F . So we prefer to design a scheme with both subpacketization and transmission rate as small as possible.

¹Memory sharing technique may lead to non equally divided packets [11], in this paper, we will not discuss this case.

B. Placement delivery array and two realization strategies of coded caching scheme

Yan et al. in [20] first proposed the concept of placement delivery array and a realization strategy of characterizing the placement phase and delivery phase simultaneously.

Definition 1: ([20]) For positive integers K and F , an $F \times K$ array $\mathbf{P} = (p_{i,j})$, $i \in [0, F)$, $j \in [0, K)$, composed of a specific symbol “*” called star and S symbols $\{0, 1, \dots, S-1\}$, is called a (K, F, S) placement delivery array (PDA) if it satisfies C1 in the following conditions:

- C1. For any two distinct entries p_{i_1, j_1} and p_{i_2, j_2} , $p_{i_1, j_1} = p_{i_2, j_2} = s \in \{0, 1, \dots, S-1\}$ only if
- $i_1 \neq i_2$, $j_1 \neq j_2$, i.e., they lie in distinct rows and distinct columns;
 - $p_{i_1, j_2} = p_{i_2, j_1} = *$, i.e., the corresponding 2×2 subarray formed by rows i_1, i_2 and columns j_1, j_2 must be one of the following forms

$$\begin{pmatrix} s & * \\ * & s \end{pmatrix} \text{ or } \begin{pmatrix} * & s \\ s & * \end{pmatrix}.$$

For any positive integer $Z \leq F$, \mathbf{P} is denoted by (K, F, Z, S) PDA if

- C2. each column has exactly Z stars.

Lemma 1: ([20]) An F -division coded caching scheme for (K, M, N) caching system can be realized by a (K, F, Z, S) PDA with memory fraction $\frac{M}{N} = \frac{Z}{F}$ and transmission rate $R = \frac{S}{F}$.

For the detailed realization method, the interested reader is referred to [20]. In a PDA, a star not contained in any subarray showed as C1-b of Definition 1, is called useless. The authors in [23] pointed out that the useless stars not only make no contribution to reducing the transmission rate of a coded caching scheme realized by that PDA, but also result in a high subpacketization level and a large memory fraction. If each column of a (K, F, Z, S) PDA has Z' useless stars, the authors in [23] improved the realization method in [20] by deleting all the useless stars and using an $[F, F - Z']_q$ maximum distance separable code for some prime power q , and came up with a new coded caching scheme with smaller transmission rate, memory fraction and subpacketization. That is the following result.

Lemma 2: ([23]) For any (K, F, Z, S) PDA if there exist Z' useless stars in each column, then we can obtain $(F - Z')$ -division (K, M, N) coded caching scheme with memory fraction $\frac{M}{N} = \frac{Z - Z'}{F - Z'}$ and transmission rate $R = \frac{S}{F - Z'}$.

From Lemma 1 and Lemma 2, we can obtain a coded caching scheme with small transmission rate and low subpacketization by constructing an appropriate PDA. Clearly given a (K, F, Z, S) PDA, $\frac{Z}{F} > \frac{Z - Z'}{F - Z'}$ and $F > F - Z'$ always hold for any positive integer. So the scheme in Lemma 2 has smaller memory fraction and subpacketization than that of the scheme from Lemma 1. However the operation field of the scheme from Lemma 2 is larger than or equal to the scheme from Lemma 1.

III. NEW CONSTRUCTION VIA HAMMING DISTANCE

In this section, we propose a new construction framework via Hamming distance to generate arrays which satisfy some of the conditions of PDA, and then obtain new PDAs through partitioning the entries of these arrays.

A. The framework of constructing via Hamming distance

Let \mathbf{x} and \mathbf{y} be vectors of length m . The Hamming distance from \mathbf{x} to \mathbf{y} , denoted by $d(\mathbf{x}, \mathbf{y})$, is defined to be the number of coordinates at which \mathbf{x} and \mathbf{y} differ. The Hamming weight of \mathbf{x} , denoted by $wt(\mathbf{x})$, is defined to be the number of nonzero coordinates in \mathbf{x} .

Construction 1: For any positive integers m , ω , F , K and $q \geq 2$ with $\omega < m$, given two subsets \mathcal{A} and \mathcal{B} of $[0, q)^m$ where $|\mathcal{A}| = F$ and $|\mathcal{B}| = K$, an $F \times K$ array $\mathbf{P} = (p_{\mathbf{a}, \mathbf{b}})$, $\mathbf{a} \in \mathcal{A}$, $\mathbf{b} \in \mathcal{B}$ is obtained as follows:

$$p_{\mathbf{a}, \mathbf{b}} = \begin{cases} \mathbf{a} + \mathbf{b} & \text{if } d(\mathbf{a}, \mathbf{b}) = \omega \\ * & \text{otherwise} \end{cases} \quad (1)$$

where $\mathbf{a} = (a_0, \dots, a_{m-1})$, $\mathbf{b} = (b_0, \dots, b_{m-1})$. Here $\mathbf{a} \pm \mathbf{b} = (a_0 \pm b_0, a_1 \pm b_1, \dots, a_{m-1} \pm b_{m-1})$ and all the operations are carried under mod q in this paper.

Example 1: When $m = 3$, $\omega = 2$, and $\mathcal{A} = \mathcal{B} = [0, 2]^m$, the following 8×8 array \mathbf{P} can be obtained by Construction 1.

$\mathbf{a} \backslash \mathbf{b}$	000	100	010	110	001	101	011	111
000	*	*	*	110	*	101	011	*
100	*	*	110	*	101	*	*	011
010	*	110	*	*	011	*	*	101
110	110	*	*	*	*	011	101	*
001	*	101	011	*	*	*	*	110
101	101	*	*	011	*	*	110	*
011	011	*	*	101	*	110	*	*
111	*	011	101	*	110	*	*	*

(2)

In this paper, all the vectors in examples are always written as a string, e.g., $(1, 1, 0, 0)$ is written as 1100.

From Example 1, we can see that no vector occurs more than once in each row and each column. Furthermore when a vector \mathbf{e} occurs in two distinct entries, i.e., $p_{\mathbf{a}_1, \mathbf{b}_1} = p_{\mathbf{a}_2, \mathbf{b}_2} = \mathbf{e}$, we have $p_{\mathbf{a}_1, \mathbf{b}_2} = p_{\mathbf{a}_2, \mathbf{b}_1} = *$ if $d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$, otherwise $p_{\mathbf{a}_1, \mathbf{b}_2} \neq *$ and $p_{\mathbf{a}_2, \mathbf{b}_1} \neq *$. For instance, since vector 110 occurs in entries $(110, 000)$ and $(010, 100)$ with $d(110, 100) \neq 2$, we have $p_{\mathbf{a}_1, \mathbf{b}_2} = p_{\mathbf{a}_1, \mathbf{b}_2} = *$ by (1). However since it also occurs in entries $(110, 000)$ and $(011, 101)$ with $d(110, 101) = 2$, we have $p_{\mathbf{a}_1, \mathbf{b}_2} = p_{\mathbf{a}_1, \mathbf{b}_2} \neq *$ by (1). In fact this is not accidental. In general, we have the following proposition.

Proposition 1: Let \mathbf{P} be the array generated by Construction 1, if there are two distinct entries being the same vector, say $p_{\mathbf{a}_1, \mathbf{b}_1} = p_{\mathbf{a}_2, \mathbf{b}_2} = \mathbf{e}$, then the following two statements hold:

- 1) The vector \mathbf{e} occurs in different columns and different rows, i.e., the condition C1-a in Definition 1 holds.
- 2) The subarray formed by rows $\mathbf{a}_1, \mathbf{a}_2$ and columns $\mathbf{b}_1, \mathbf{b}_2$ satisfies the condition C1-b in Definition 1 if and only if $d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$.

Proof. Suppose that a vector $\mathbf{e} = (e_0, e_1, \dots, e_{m-1}) \in [0, q]^m$ occurs in two distinct entries, say $(\mathbf{a}_1, \mathbf{b}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$ where

$$\begin{aligned} \mathbf{a}_1 &= (a_{1,0}, a_{1,1}, \dots, a_{1,m-1}), & \mathbf{a}_2 &= (a_{2,0}, a_{2,1}, \dots, a_{2,m-1}), \\ \mathbf{b}_1 &= (b_{1,0}, b_{1,1}, \dots, b_{1,m-1}), & \mathbf{b}_2 &= (b_{2,0}, b_{2,1}, \dots, b_{2,m-1}). \end{aligned}$$

From Construction 1 we have

$$\mathbf{e} = \mathbf{a}_1 + \mathbf{b}_1 = \mathbf{a}_2 + \mathbf{b}_2. \quad (3)$$

That is, $\mathbf{a}_1 - \mathbf{a}_2 = \mathbf{b}_2 - \mathbf{b}_1$. Clearly $\mathbf{a}_1 = \mathbf{a}_2$ if and only if $\mathbf{b}_1 = \mathbf{b}_2$. So vector \mathbf{e} occurs in the different columns and different rows. If $d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$, then $p_{\mathbf{a}_1, \mathbf{b}_2} = *$ from Construction 1. Since $d(\mathbf{a}_1, \mathbf{b}_2) = wt(\mathbf{a}_1 - \mathbf{b}_2) \neq \omega$, by (3) we have $\mathbf{a}_2 - \mathbf{b}_1 = \mathbf{a}_1 - \mathbf{b}_2$, so $d(\mathbf{a}_2, \mathbf{b}_1) = d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$ holds. This implies that $p_{\mathbf{a}_2, \mathbf{b}_1} = *$. Conversely if $p_{\mathbf{a}_1, \mathbf{b}_2} = p_{\mathbf{a}_2, \mathbf{b}_1} = *$, we also have $d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$ similarly. \square

From Proposition 1, the array \mathbf{P} generated by Construction 1 has satisfied the Condition C1-a in Definition 1. In order to construct PDAs we only need to make any two distinct entries having the same vectors satisfy Proposition 1-2). So we should make a partition for each collection of entries having the same vector, thereby part it into several non-intersection subsets such that any two different entries in the same subset satisfy Proposition 1-2). That is the main discussion in the following subsection.

B. The partitions of the entries in \mathbf{P}

For any positive integers m, q, ω and the given subsets $\mathcal{A}, \mathcal{B} \in [0, q]^m$ with $\omega < m$ and $q \geq 2$, we can obtain an array \mathbf{P} by Construction 1. Assume that vector \mathbf{e} occurs g_e times in \mathbf{P} , say $p_{\mathbf{a}_1, \mathbf{b}_1} = \dots = p_{\mathbf{a}_{g_e}, \mathbf{b}_{g_e}} = \mathbf{e}$. Let $\mathcal{E}_e = \{(\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2), \dots, (\mathbf{a}_{g_e}, \mathbf{b}_{g_e})\}$. We claim that for some integer $1 \leq h_e \leq g_e$ there always exists a partition $\mathcal{X}_e = \{\mathcal{X}_{e,0}, \mathcal{X}_{e,1}, \dots, \mathcal{X}_{e,h_e-1}\}$ for \mathcal{E}_e , satisfying

Property 1: For any two different entries $(\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2) \in \mathcal{X}_{e,i}, i \in [0, h_e), d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$ always holds.

Extremely, when $h_e = g_e$ there exists an trivial partition where $\mathcal{X}_{e,i} = \{(\mathbf{a}_i, \mathbf{b}_i)\}$ for $i \in h_e$.

Construction 2: Given an $F \times K$ array \mathbf{P} generated by Construction 1, we can obtain a new array $\mathbf{P}' = (p'_{\mathbf{a}, \mathbf{b}})$, $\mathbf{a} \in \mathcal{A}$ and $\mathbf{b} \in \mathcal{B}$, where

$$p'_{\mathbf{a}, \mathbf{b}} = \begin{cases} (\mathbf{e}, i) & \text{if } p_{\mathbf{a}, \mathbf{b}} = \mathbf{e}, (\mathbf{a}, \mathbf{b}) \in \mathcal{X}_{e,i}, i \in [0, h_e) \\ * & \text{otherwise} \end{cases}$$

where $\mathcal{X}_e = \{\mathcal{X}_{e,0}, \dots, \mathcal{X}_{e,h_e-1}\}$ is a partition satisfying **Property 1**. Clearly any two entries $p_{\mathbf{a}_1, \mathbf{b}_1} = p_{\mathbf{a}_2, \mathbf{b}_2} = (\mathbf{e}, i)$ in \mathbf{P}' satisfy $d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$. From Proposition 1, \mathbf{P}' is a PDA.

Example 2: Let us consider the parameters in Example 1 and the 8×8 array in (2) again. For \mathcal{E}_{110} , \mathcal{E}_{101} , \mathcal{E}_{011} we make their partitions satisfying **Property 1** as follow.

$$\begin{aligned}\mathcal{E}_{110} &= \mathcal{X}_{110,0} \cup \mathcal{X}_{110,1} = \{(110, 000), (000, 110), (010, 100), (100, 010)\} \\ &\quad \cup \{(111, 001), (001, 111), (011, 101), (101, 011)\} \\ \mathcal{E}_{101} &= \mathcal{X}_{101,0} \cup \mathcal{X}_{101,1} = \{(101, 000), (000, 101), (001, 100), (100, 001)\} \\ &\quad \cup \{(111, 010), (010, 111), (011, 110), (110, 011)\} \\ \mathcal{E}_{011} &= \mathcal{X}_{011,0} \cup \mathcal{X}_{011,1} = \{(011, 000), (000, 011), (001, 010), (010, 001)\} \\ &\quad \cup \{(111, 100), (100, 111), (101, 110), (110, 101)\}\end{aligned}$$

Based on the above partitions and Construction 2 a $(8, 8, 5, 6)$ PDA \mathbf{P}' can be obtained.

$\mathbf{a} \backslash \mathbf{b}$	000	100	010	110	001	101	011	111
000	*	*	*	110, 0	*	101, 0	011, 0	*
100	*	*	110, 0	*	101, 0	*	*	011, 1
010	*	110, 0	*	*	011, 0	*	*	101, 1
110	110, 0	*	*	*	*	011, 1	101, 1	*
001	*	101, 0	011, 0	*	*	*	*	110, 1
101	101, 0	*	*	011, 1	*	*	110, 1	*
011	011, 0	*	*	101, 1	*	110, 1	*	*
111	*	011, 1	101, 1	*	110, 1	*	*	*

(4)

Generally, from Construction 1 and Construction 2 we have the following result.

Theorem 1: The $F \times K$ array \mathbf{P}' generated from Construction 2 is a (K, F, S) PDA with $S = \sum_{\mathbf{e} \in \mathbf{P}} h_{\mathbf{e}}$ where $h_{\mathbf{e}}$ is the cardinality of partition $\mathcal{X}_{\mathbf{e}}$ satisfying **Property 1**.

From the above introductions, we only need to consider designing the appropriate partition $\mathcal{X}_{\mathbf{e}}$ for each set $\mathcal{E}_{\mathbf{e}}$ in \mathbf{P} generated by Construction 1. In the following sections we will obtain several classes of PDAs for the parameter $q = 2$ and $q = 3$ by constructing partitions.

IV. THE NEW SCHEMES WITH $q = 2$

For any sets $\mathcal{A}, \mathcal{B} \subseteq [0, 2)^m$ and all \mathbf{e} in array \mathbf{P} obtained by Construction 1, when m and ω are any integers with $\omega < m$ we first propose a partition $\mathcal{X}_{\mathbf{e}}$, and when $m \geq 2\omega + 1$ we improve this partition.

A. The primary partition for any integer m and ω

When $q = 2$ for any vector $\mathbf{e} = (e_0, e_1, \dots, e_{m-1})$ occurring in \mathbf{P} obtained by Construction 1, let $\mathcal{C}_{\mathbf{e}} = \{i \in [0, m) | e_i = 0\}$. Clearly $|\mathcal{C}_{\mathbf{e}}| = m - \omega$ always holds.

Partition 1: For any vector \mathbf{e} in \mathbf{P} obtained by Construction 1, and for each vector $\mathbf{t} \in [0, 2)^{m-\omega}$ we define

$$\mathcal{X}_{\mathbf{e}, \mathbf{t}} = \{(\mathbf{a}, \mathbf{b}) | (\mathbf{a}, \mathbf{b}) \in \mathcal{E}_{\mathbf{e}}, \mathbf{a}|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{b}|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{t}\}. \quad (5)$$

Here for any m length vector \mathbf{a} and a set $\mathcal{T} \subseteq [0, m)$, $\mathbf{a}|_{\mathcal{T}}$ is a vector obtained by deleting the coordinates $j \in [0, m) \setminus \mathcal{T}$. Let $\mathcal{X}_{\mathbf{e}} = \{\mathcal{X}_{\mathbf{e}, \mathbf{t}} | \mathcal{X}_{\mathbf{e}, \mathbf{t}} \neq \emptyset, \mathbf{t} \in [0, 2)^{m-\omega}\}$. It is easy to check that $\mathcal{X}_{\mathbf{e}}$ is a partition for $\mathcal{E}_{\mathbf{e}}$.

Proposition 2: Partition 1 satisfies **Property 1**. Furthermore, for any two different entries $(\mathbf{a}_1, \mathbf{b}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$ in common partition of Partition 1, $d(\mathbf{a}_1, \mathbf{b}_2) < \omega$ always holds.

Proof. For any vector $\mathbf{e} = (e_0, e_1, \dots, e_{m-1})$ in \mathbf{P} , let us consider the $\mathcal{X}_{\mathbf{e}}$ in Partition 1. For any $\mathcal{X}_{\mathbf{e}, \mathbf{t}} \in \mathcal{X}_{\mathbf{e}}$, $\mathbf{t} \in [0, 2)^{m-\omega}$, our statement always holds if $|\mathcal{X}_{\mathbf{e}, \mathbf{t}}| = 1$. If $|\mathcal{X}_{\mathbf{e}, \mathbf{t}}| \geq 2$, for any two different entries $(\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2) \in \mathcal{X}_{\mathbf{e}, \mathbf{t}}$, let

$$\begin{aligned}\mathbf{a}_1 &= (a_{1,0}, a_{1,1}, \dots, a_{1,m-1}), & \mathbf{a}_2 &= (a_{2,0}, a_{2,1}, \dots, a_{2,m-1}), \\ \mathbf{b}_1 &= (b_{1,0}, b_{1,1}, \dots, b_{1,m-1}), & \mathbf{b}_2 &= (b_{2,0}, b_{2,1}, \dots, b_{2,m-1}).\end{aligned}$$

For any $j \in \mathcal{C}_{\mathbf{e}}$ we have $a_{1,j} = b_{1,j}$, $a_{2,j} = b_{2,j}$ since $q = 2$. By (5) we have

$$\mathbf{a}_1|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{b}_1|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{t} = \mathbf{a}_2|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{b}_2|_{\mathcal{C}_{\mathbf{e}}}$$

i.e., for any $j \in \mathcal{C}_e$, $a_{1,j} = a_{2,j}$ and $a_{1,j} = b_{2,j}$ always hold. Due to $|\mathcal{C}_e| = m - \omega$, we have $d(\mathbf{a}_1, \mathbf{b}_2) \leq \omega$. From Proposition 1-1) we have $\mathbf{b}_1 \neq \mathbf{b}_2$. So $d(\mathbf{a}_1, \mathbf{b}_2) = d(\mathbf{a}_2, \mathbf{b}_1) < \omega$ always holds. Hence, $d(\mathbf{a}_1, \mathbf{b}_2) < \omega$ holds and partition \mathcal{X}_e satisfies the **Property 1**. The proof is completed. \square

We also consider the parameters in Example 1 and the 8×8 array in (2). By Partition 1, the partitions for \mathcal{E}_{110} , \mathcal{E}_{101} and \mathcal{E}_{011} are exactly showed in Example 2. Then from Theorem 1 an $(8, 8, 5, 6)$ PDA can be obtained in (4). Furthermore, we can see that the stars in entries $(111, 000)$, $(011, 100)$, $(101, 010)$, $(001, 110)$, $(110, 001)$, $(010, 101)$, $(100, 011)$ and $(000, 111)$ are useless. In fact, when $\mathcal{A} = \mathcal{B} = [0, 2)^m$, the array generated by Construction 2 based on Partition 1 is the PDA where some stars in each column are useless. So from Theorem 1 and Lemma 2 the following result is obtained.

Theorem 2: For any positive integers m, ω with $\omega < m$, there exists a $(2^m, 2^m, 2^m - \binom{m}{\omega}, \binom{m}{\omega} 2^{m-\omega})$ PDA which can realize a $\sum_{i=0}^{\omega} \binom{m}{i}$ -division $(2^m, M, N)$ coded caching scheme with memory fraction $\frac{M}{N} = 1 - \binom{m}{\omega} / \sum_{i=0}^{\omega} \binom{m}{i}$ and transmission rate $R = \frac{\binom{m}{\omega} 2^{m-\omega}}{\sum_{i=0}^{\omega} \binom{m}{i}}$.

Proof. Let $\mathcal{A} = \mathcal{B} = [0, 2)^m$. From Construction 1 a $2^m \times 2^m$ array \mathbf{P} is obtained. From (1) the number of no-star entries in each column is $\binom{m}{\omega}$ so that $Z = 2^m - \binom{m}{\omega}$. Since $\mathcal{A} = \mathcal{B} = [0, 2)^m$, the collection of all vectors occurring in \mathbf{P} is exactly the collection of all binary vectors with Hamming weight of ω , i.e., the number of vectors occurring in \mathbf{P} is $S' = \binom{m}{\omega}$. By Partition 1, for any vector \mathbf{e} we have $h_e = |\mathcal{X}_e| = |\{\mathcal{X}_{e,t} \mid \mathcal{X}_{e,t} \neq \emptyset\}| = |\{t \mid \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}, (\mathbf{a}, \mathbf{b}) \in \mathcal{E}_e, \mathbf{a}|_{\mathcal{C}_e} = \mathbf{b}|_{\mathcal{C}_e} = t\}| = |[0, 2)^{m-\omega}| = 2^{m-\omega}$. From Theorem 1, a $(2^m, 2^m, 2^m - \binom{m}{\omega}, \binom{m}{\omega} 2^{m-\omega})$ PDA is obtained where $S = S' h_e = \binom{m}{\omega} 2^{m-\omega}$. From Proposition 2, for any two different entries $(\mathbf{a}_1, \mathbf{b}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$ in common partition of Partition 1, we have $p_{\mathbf{a}_1, \mathbf{b}_2} = *$ and $d(\mathbf{a}_1, \mathbf{b}_2) < \omega$. This means that for any entry $p_{\mathbf{a}, \mathbf{b}} = *$ where $\mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}$ with $d(\mathbf{a}, \mathbf{b}) > \omega$, the star isn't contained by any subarray showed as C1-b of Definition 1, i.e., it's useless star. Then the number of useless stars in each column is $Z' = \sum_{i=\omega+1}^m \binom{m}{i}$. From Lemma 2, the proof of Theorem 2 is completed. \square

In Lemma 2, the authors improved the coded caching scheme by deleting the useless stars of the PDA. This implies that we prefer to get a PDA with the number of useless stars as small as possible. So we can also reduce the number of useless stars by improving the Partition 1.

B. The improve partition for $m \geq 2\omega + 1$

When $m \geq 2\omega + 1$ we further merge some elements of Partition 1 to obtain an improved partition with smaller cardinality. Then from Theorem 1 we can get a new PDA with smaller value S than that of the PDA in Theorem 2.

Partition 2: When $m \geq 2\omega + 1$, for any vector \mathbf{e} in \mathbf{P} obtained by Construction 1, there always exists an l_e -partition $\mathcal{D} = \{\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{l_e-1}\}$ for $[0, 2)^{m-\omega}$ such that the distance of any two different vectors in $\mathcal{D}_i, i \in [0, l_e)$, is at least $\omega + 1$. For \mathcal{X}_e in Partition 1 and for each \mathcal{D}_i , let

$$\mathcal{Y}_{e, \mathcal{D}_i} = \bigcup_{t \in \mathcal{D}_i} \mathcal{X}_{e,t}, \quad \mathcal{X}_{e,t} \in \mathcal{X}_e.$$

Then partition $\mathcal{Y}_e = \{\mathcal{Y}_{e, \mathcal{D}_0}, \mathcal{Y}_{e, \mathcal{D}_1}, \dots, \mathcal{Y}_{e, \mathcal{D}_{l_e-1}}\}$ is an l_e -partition for \mathcal{E}_e for some integer l_e .

Proposition 3: The partition 2 satisfies **Property 1**.

Proof. From Proposition 2, it is sufficient to consider any two different entries $(\mathbf{a}_1, \mathbf{b}_1) \in \mathcal{X}_{e, \mathbf{t}_{j_1}}$ and $(\mathbf{a}_2, \mathbf{b}_2) \in \mathcal{X}_{e, \mathbf{t}_{j_2}}, \mathbf{t}_{j_1}, \mathbf{t}_{j_2} \in \mathcal{D}_i, i \in [0, l_e), j_1 \neq j_2$. By (5) we have $\mathbf{t}_{j_1} = \mathbf{a}_1|_{\mathcal{C}_e}, \mathbf{t}_{j_2} = \mathbf{b}_2|_{\mathcal{C}_e}$. From Partition 2 we have $d(\mathbf{a}_1, \mathbf{b}_2) \geq d(\mathbf{a}_1|_{\mathcal{C}_e}, \mathbf{b}_2|_{\mathcal{C}_e}) = d(\mathbf{t}_{j_1}, \mathbf{t}_{j_2}) \geq \omega + 1$. Clearly $d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$ always holds. And the star $p_{\mathbf{a}_1, \mathbf{b}_2} = *$ and $p_{\mathbf{a}_2, \mathbf{b}_1} = *$ are both used. \square

From Theorem 1, the PDA based on Partition 2 has parameter $S = \sum_{e \in \mathbf{P}} l_e$. Given $\mathcal{A}, \mathcal{B} \subseteq [0, 2)^m$ we only need to consider the value of l_e in partition \mathcal{D} for each \mathbf{e} . By means of the result on the vertex coloring, the following result can be obtained.

Lemma 3: When $m \geq 2\omega + 1$ and $\mathcal{A} = \mathcal{B} = [0, 2)^m$, there exists a $(2^m, 2^m, 2^m - \binom{m}{\omega}, S)$ PDA where

- $S = \binom{m}{\omega} 2^{m-\omega-1}$ when $m = 2\omega + 1$.
- $S \leq \binom{m}{\omega} \sum_{i=0}^{\omega} \binom{m-\omega}{i}$ when $m > 2\omega + 1$.

The proof is referred to Appendix A. We can see that the value of S of the PDA in Theorem 2 is larger than that of the PDA in Lemma 3 for the same $K = F = 2^m$ and $Z = 2^m - \binom{m}{\omega}$. It is worth noting that there are also some stars of the PDA in Lemma 3 are useless. This implies that we can also get a coded caching scheme with small memory fraction from Lemma 2. Now let us take an example to verify our claim.

Example 3: When $m = 4, \omega = 1$ and $\mathcal{A} = \mathcal{B} = [0, 2]^4$, from Construction 1, Partition 1 and Construction 2, a $(16, 16, 12, 32)$ PDA can be obtained. For each vector \mathbf{e} we have

$$\mathcal{X}_{\mathbf{e}} = \{\mathcal{X}_{\mathbf{e},000}, \mathcal{X}_{\mathbf{e},100}, \mathcal{X}_{\mathbf{e},010}, \mathcal{X}_{\mathbf{e},110}, \mathcal{X}_{\mathbf{e},001}, \mathcal{X}_{\mathbf{e},101}, \mathcal{X}_{\mathbf{e},011}, \mathcal{X}_{\mathbf{e},111}\}.$$

Define a 4-partition

$$\mathcal{D} = \{\mathcal{D}_0 = \{000, 111\}, \mathcal{D}_1 = \{100, 011\}, \mathcal{D}_2 = \{010, 101\}, \mathcal{D}_3 = \{110, 001\}\}$$

for $[0, 2]^3$ such that the distance of any two different vectors in one subset of \mathcal{D} is exactly 3. Then from Partition 2, we have

$$\begin{aligned} \mathcal{V}_{\mathbf{e}} &= \{\mathcal{V}_{\mathbf{e},\mathcal{D}_0}, \mathcal{V}_{\mathbf{e},\mathcal{D}_1}, \mathcal{V}_{\mathbf{e},\mathcal{D}_2}, \mathcal{V}_{\mathbf{e},\mathcal{D}_3}\} \\ &= \{\mathcal{X}_{\mathbf{e},000} \cup \mathcal{X}_{\mathbf{e},111}, \mathcal{X}_{\mathbf{e},100} \cup \mathcal{X}_{\mathbf{e},011}, \mathcal{X}_{\mathbf{e},010} \cup \mathcal{X}_{\mathbf{e},101}, \mathcal{X}_{\mathbf{e},110} \cup \mathcal{X}_{\mathbf{e},001}\}. \end{aligned}$$

From Proposition 3 and Theorem 1, we can obtain a $(16, 16, 12, 16)$ PDA listed in Table II. We can check that $S = 16$ reaches the upper bound in Lemma 3 and each column has $Z' = 6$ useless stars. From Lemma 2, we can get a 10-division $(16, M, N)$ coded caching scheme with memory fraction $\frac{M}{N} = \frac{3}{5}$ and transmission rate $R = \frac{6}{5}$.

TABLE II: The $(16, 16, 12, 16)$ PDA based on Lemma 2

$\mathbf{a} \backslash \mathbf{b}$	0000	1000	0100	1100	0010	1010	0110	1110	0001	1001	0101	1101	0011	1011	0111	1111
0000	*	1000 \mathcal{D}_0	0100 \mathcal{D}_0	*	0010 \mathcal{D}_0	*	*	*	0001 \mathcal{D}_0	*	*	*	*	*	*	*
1000	1000 \mathcal{D}_0	*	*	0100 \mathcal{D}_1	*	0010 \mathcal{D}_1	*	*	*	0001 \mathcal{D}_1	*	*	*	*	*	*
0100	0100 \mathcal{D}_0	*	*	1000 \mathcal{D}_1	*	*	0010 \mathcal{D}_2	*	*	*	0001 \mathcal{D}_2	*	*	*	*	*
1100	*	0100 \mathcal{D}_1	1000 \mathcal{D}_1	*	*	*	*	0010 \mathcal{D}_3	*	*	*	0001 \mathcal{D}_3	*	*	*	*
0010	0010 \mathcal{D}_0	*	*	*	*	1000 \mathcal{D}_2	0100 \mathcal{D}_2	*	*	*	*	*	0001 \mathcal{D}_3	*	*	*
1010	*	0010 \mathcal{D}_1	*	*	1000 \mathcal{D}_2	*	*	0100 \mathcal{D}_3	*	*	*	*	*	0001 \mathcal{D}_2	*	*
0110	*	*	0010 \mathcal{D}_2	*	0100 \mathcal{D}_2	*	*	1000 \mathcal{D}_3	*	*	*	*	*	*	0001 \mathcal{D}_1	*
1110	*	*	*	0010 \mathcal{D}_3	*	0100 \mathcal{D}_3	1000 \mathcal{D}_3	*	*	*	*	*	*	*	*	0001 \mathcal{D}_0
0001	0001 \mathcal{D}_0	*	*	*	*	*	*	*	*	1000 \mathcal{D}_3	0100 \mathcal{D}_3	*	0010 \mathcal{D}_3	*	*	*
1001	*	0001 \mathcal{D}_1	*	*	*	*	*	*	1000 \mathcal{D}_3	*	*	0100 \mathcal{D}_2	*	0010 \mathcal{D}_2	*	*
0101	*	*	0001 \mathcal{D}_2	*	*	*	*	*	0100 \mathcal{D}_3	*	*	1000 \mathcal{D}_2	*	*	0010 \mathcal{D}_1	*
1101	*	*	*	0001 \mathcal{D}_3	*	*	*	*	*	0100 \mathcal{D}_2	1000 \mathcal{D}_2	*	*	*	*	0010 \mathcal{D}_0
0011	*	*	*	*	0001 \mathcal{D}_3	*	*	*	0010 \mathcal{D}_3	*	*	*	*	1000 \mathcal{D}_1	0100 \mathcal{D}_1	*
1011	*	*	*	*	*	0001 \mathcal{D}_2	*	*	*	0010 \mathcal{D}_2	*	*	1000 \mathcal{D}_1	*	*	0100 \mathcal{D}_0
0111	*	*	*	*	*	*	0001 \mathcal{D}_1	*	*	*	0010 \mathcal{D}_1	*	0100 \mathcal{D}_1	*	*	1000 \mathcal{D}_0
1111	*	*	*	*	*	*	*	0001 \mathcal{D}_0	*	*	*	0010 \mathcal{D}_0	*	0100 \mathcal{D}_0	1000 \mathcal{D}_0	*

Unfortunately due to nondeterminacy of Partition 2 it is hard to propose an uniform function of the number of useless stars in each column of the PDA from Lemma 3 for any m and ω . By Lemma 1, the following result can be obtained.

Theorem 3: For any positive integers m, ω with $m \geq 2\omega + 1$, there exists a 2^m -division $(2^m, M, N)$ coded caching scheme with memory fraction $\frac{M}{N} = 1 - \frac{\binom{m}{\omega}}{2^m}$ and transmission rate

- $R = \frac{\binom{m}{\omega}}{2^{\omega+1}}$ if $m = 2\omega + 1$
- $R \leq \frac{\binom{m}{\omega} \sum_{i=0}^{\omega} \binom{m-i}{i}}{2^m}$ if $m > 2\omega + 1$.

Finally we should point out that our proposed framework consisting Construction 1 and Construction 2 are also useful to construct the PDAs for any positive integer q . We will take $q = 3$ as another example in the following section.

V. NEW SCHEMES WITH PARAMETER $q = 3$

Similar to Section IV, in this section we also put forward two partitions for the case of $q = 3$. That is, a primary partition for any integers m and ω with $\omega < m$ and an improved partition when $m > \frac{3\omega}{2}$ are proposed.

A. The primary partition for parameters m and ω

When $q = 3$, define $\mathcal{C}_{\mathbf{a}-\mathbf{b}} = \{i \in [0, m) \mid a_i = b_i\}$ for each element (\mathbf{a}, \mathbf{b}) in \mathcal{E}_e . Then $|\mathcal{C}_{\mathbf{a}-\mathbf{b}}| = m - \omega$ always holds.

Partition 3: For any vector \mathbf{e} in \mathbf{P} obtained by Construction 1 and for each element $\mathcal{T} \in \binom{[0, m)}{m-\omega} = \{\mathcal{T} \mid \mathcal{T} \subseteq [0, m), |\mathcal{T}| = t\}$, define

$$\mathcal{X}_{\mathbf{e}, \mathcal{T}} = \{(\mathbf{a}, \mathbf{b}) \mid (\mathbf{a}, \mathbf{b}) \in \mathcal{E}_e, \mathcal{C}_{\mathbf{a}-\mathbf{b}} = \mathcal{T}\}. \quad (6)$$

Let $\mathcal{X}_e = \{\mathcal{X}_{\mathbf{e}, \mathcal{T}} \mid \mathcal{X}_{\mathbf{e}, \mathcal{T}} \neq \emptyset, \mathcal{T} \in \binom{[0, m)}{m-\omega}\}$. Then \mathcal{X}_e is a partition of \mathcal{E}_e .

Proposition 4: Partition 3 satisfies **Property 1**. Furthermore, for any two different entries $(\mathbf{a}_1, \mathbf{b}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$ in common partition of Partition 3, $d(\mathbf{a}_1, \mathbf{b}_2) < \omega$ always holds.

Proof. In Partition 3, for any $\mathcal{X}_{\mathbf{e}, \mathcal{T}} \in \mathcal{X}_e$, $\mathcal{T} \in \binom{[0, m)}{m-\omega}$, our statement always holds if $|\mathcal{X}_{\mathbf{e}, \mathcal{T}}| = 1$. If $|\mathcal{X}_{\mathbf{e}, \mathcal{T}}| \geq 2$, let us consider any two different entries $(\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2) \in \mathcal{X}_{\mathbf{e}, \mathcal{T}}$. By (3), we have $\mathbf{a}_1|_{\mathcal{T}} + \mathbf{b}_1|_{\mathcal{T}} = \mathbf{a}_2|_{\mathcal{T}} + \mathbf{b}_2|_{\mathcal{T}}$. In addition, by (6), $\mathbf{a}_1|_{\mathcal{T}} = \mathbf{b}_1|_{\mathcal{T}}$ and $\mathbf{a}_2|_{\mathcal{T}} = \mathbf{b}_2|_{\mathcal{T}}$ always hold. Then we have

$$\mathbf{a}_1|_{\mathcal{T}} = \mathbf{b}_1|_{\mathcal{T}} = \mathbf{a}_2|_{\mathcal{T}} = \mathbf{b}_2|_{\mathcal{T}} \quad (7)$$

since all the operations are under mod $q = 3$. For any $j \in [0, m) \setminus \mathcal{T}$

$$a_{1,j} \neq b_{1,j}, \quad a_{2,j} \neq b_{2,j}, \quad (8)$$

holds due to $d(\mathbf{a}_1, \mathbf{b}_1) = d(\mathbf{a}_2, \mathbf{b}_2) = \omega$. Then by (3) and (8) we can get $a_{1,j} = b_{2,j}$ or $a_{1,j} = a_{2,j}$. If $a_{1,j} = a_{2,j}$ holds for each $j \in [0, m) \setminus \mathcal{T}$, we have $\mathbf{a}_1 = \mathbf{a}_2$ by (7). Then $\mathbf{b}_1 = \mathbf{b}_2$ by (3) which contradicts our hypothesis $(\mathbf{a}_1, \mathbf{b}_1) \neq (\mathbf{a}_2, \mathbf{b}_2)$. So there are at least one coordinate, say $j' \in [0, m) \setminus \mathcal{T}$, such that $a_{1,j'} = b_{2,j'}$ holds. Then we have $d(\mathbf{a}_1, \mathbf{b}_2) < \omega$. So the partition \mathcal{X}_e satisfies the **Property 1**. \square

Example 4: When $m = 3$, $\omega = 2$, $\mathcal{A} = \mathcal{B} = [0, 3)^m$, for any vector \mathbf{e} in \mathbf{P} obtained by Construction 1, we have a 3-partition $\mathcal{X}_e = \{\mathcal{X}_{\mathbf{e}, 0}, \mathcal{X}_{\mathbf{e}, 1}, \mathcal{X}_{\mathbf{e}, 2}\}$ from Partition 3. From Theorem 1 a $(27, 27, 15, 81)$ PDA \mathbf{P}' can be obtained. Let us consider vector $\mathbf{e} = 110$. The rows and the columns from \mathcal{E}_{110} in \mathbf{P}' form the following 12×12 subarray.

$\mathbf{a} \backslash \mathbf{b}$	000	100	010	110	201	211	202	212	021	121	022	122
000	*	*	*	110, 2	201, 1	*	202, 1	*	021, 0	*	022, 0	*
100	*	*	110, 2	*	001, 1	*	002, 1	*	*	221, 0	*	222, 0
010	*	110, 2	*	*	*	221, 1	*	222, 1	001, 0	*	002, 0	*
110	110, 2	*	*	*	*	021, 1	*	022, 1	*	201, 0	*	202, 0
201	201, 1	001, 1	*	*	*	*	*	110, 0	222, 2	022, 2	*	*
211	*	*	221, 1	021, 1	*	*	110, 0	*	201, 2	002, 2	*	*
202	202, 1	002, 1	*	*	*	110, 0	*	*	*	*	221, 2	021, 2
212	*	*	222, 1	022, 1	110, 0	*	*	*	*	*	201, 2	001, 2
021	021, 0	*	001, 0	*	222, 2	202, 2	*	*	*	*	*	110, 1
121	*	221, 0	*	201, 0	022, 2	002, 2	*	*	*	*	110, 1	*
022	022, 0	*	000, 0	*	*	*	221, 2	201, 2	*	110, 1	*	*
122	*	222, 0	*	202, 0	*	*	021, 2	001, 2	110, 1	*	*	*

From Proposition 4, there are also some useless stars in PDA obtained by Partition 3. Similar to Theorem 2, from Theorem 1 and Lemma 2 the following result can be obtained.

Theorem 4: For any positive integers m, ω with $\omega < m$, there exists a $(3^m, 3^m, 3^m - \binom{m}{\omega} 2^\omega, \binom{m}{\omega} 3^m)$ PDA which gives a $\sum_{i=0}^{\omega} \binom{m}{i} 2^i$ -division $(3^m, M, N)$ coded caching scheme with memory fraction $\frac{M}{N} = 1 - \frac{\binom{m}{\omega} 2^\omega}{\sum_{i=0}^{\omega} \binom{m}{i} 2^i}$ and transmission rate $R = \frac{\binom{m}{\omega} 3^m}{\sum_{i=0}^{\omega} \binom{m}{i} 2^i}$.

Proof. Let $\mathcal{A} = \mathcal{B} = [0, 3)^m$. From Construction 1 we can obtain a $3^m \times 3^m$ array \mathbf{P} . From (1) the number of no-star entries in each column is $\binom{m}{\omega} 2^\omega$ so that $Z = 3^m - \binom{m}{\omega} 2^\omega$. Since for any vector $\mathbf{e} \in [0, 3)^m$ there always exists vectors $\mathbf{a} \in \mathcal{A}$ and $\mathbf{b} \in \mathcal{B}$ where $\mathbf{e} = \mathbf{a} + \mathbf{b}$, $d(\mathbf{a}, \mathbf{b}) = \omega$, the number of vectors occurring in \mathbf{P} is $S' = 3^m$. By Partition 3, for any vector \mathbf{e} , we have $h_e = |\mathcal{X}_e| = |\{\mathcal{T} \in \binom{[0, m)}{m-\omega} \mid \mathcal{X}_{\mathbf{e}, \mathcal{T}} \neq \emptyset\}| = |\{\mathcal{C}_{\mathbf{a}-\mathbf{b}} \mid \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}, (\mathbf{a}, \mathbf{b}) \in \mathcal{E}_e\}| = \binom{m}{\omega}$. From Theorem 1, we have a $(3^m, 3^m, 3^m - \binom{m}{\omega} 2^\omega, \binom{m}{\omega} 3^m)$ PDA where $S = S' h_e = \binom{m}{\omega} 3^m$.

From Proposition 4, for any two different entries $(\mathbf{a}_1, \mathbf{b}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$ in common partition of Partition 1, we have $p_{\mathbf{a}_1, \mathbf{b}_2} = *$ and $d(\mathbf{a}_1, \mathbf{b}_2) < \omega$. This means that for any entry $p_{\mathbf{a}, \mathbf{b}} = *$ where $\mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}$ with $d(\mathbf{a}, \mathbf{b}) > \omega$, the star isn't contained by any subarray showed as C1-b of Definition 1, i.e., it's useless star. Then the number of useless stars in each column is $Z' = \sum_{i=\omega+1}^m \binom{m}{i} 2^i$. From Lemma 2, the proof of Theorem 4 is completed. \square

We can reduce the number of useless stars by modifying the Partition 3 to improve coded caching schemes.

B. The improved partition when $m > \frac{3\omega}{2}$

When $m > \frac{3\omega}{2}$, we further merge some elements of Partition 3 to obtain an improved partition with smaller cardinality.

Partition 4: When $m > \frac{3\omega}{2}$, for any vector \mathbf{e} in \mathbf{P} obtained by Construction 1, there always exists an $l_{\mathbf{e}}$ -partition $\mathcal{D} = \{\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{l_{\mathbf{e}}-1}\}$ for $\binom{[0, m]}{m-\omega}$, such that the cardinality of the intersection of any two elements from \mathcal{D}_i , $i \in [0, l_{\mathbf{e}})$ is less than $m - \frac{3\omega}{2}$. For $\mathcal{X}_{\mathbf{e}}$ in Partition 3 and for each \mathcal{D}_i , let

$$\mathcal{Y}_{\mathbf{e}, \mathcal{D}_i} = \bigcup_{\mathcal{T} \in \mathcal{D}_i} \mathcal{X}_{\mathbf{e}, \mathcal{T}}, \quad \mathcal{X}_{\mathbf{e}, \mathcal{T}} \in \mathcal{X}_{\mathbf{e}}.$$

Then $\mathcal{Y}_{\mathbf{e}} = \{\mathcal{Y}_{\mathbf{e}, \mathcal{D}_0}, \mathcal{Y}_{\mathbf{e}, \mathcal{D}_1}, \dots, \mathcal{Y}_{\mathbf{e}, \mathcal{D}_{l_{\mathbf{e}}-1}}\}$ is an $l_{\mathbf{e}}$ -partition for $\mathcal{E}_{\mathbf{e}}$.

Proposition 5: Partition 4 satisfies **Property 1**.

Proof. From Proposition 2, we only need to consider any two different entries $(\mathbf{a}_1, \mathbf{b}_1) \in \mathcal{X}_{\mathbf{e}, \mathcal{T}_{j_1}}$ and $(\mathbf{a}_2, \mathbf{b}_2) \in \mathcal{X}_{\mathbf{e}, \mathcal{T}_{j_2}}$, $\mathcal{T}_{j_1}, \mathcal{T}_{j_2} \in \mathcal{D}_i$, $i \in [0, l_{\mathbf{e}})$, $j_1 \neq j_2$. We have $\mathcal{T}_{j_1} = \mathcal{C}_{\mathbf{a}_1 - \mathbf{b}_1}$ and $\mathcal{T}_{j_2} = \mathcal{C}_{\mathbf{a}_2 - \mathbf{b}_2}$ due to Partition 3. From Partition 4 we have $|\mathcal{C}_{\mathbf{a}_1 - \mathbf{b}_1} \cap \mathcal{C}_{\mathbf{a}_2 - \mathbf{b}_2}| = |\mathcal{T}_{j_1} \cap \mathcal{T}_{j_2}| < m - \frac{3\omega}{2}$. Since (3) and that for any $s \in \mathcal{C}_{\mathbf{a}_1 - \mathbf{b}_1} \setminus (\mathcal{C}_{\mathbf{a}_1 - \mathbf{b}_1} \cap \mathcal{C}_{\mathbf{a}_2 - \mathbf{b}_2})$, $a_{1,s} = b_{1,s}$ and $a_{2,s} \neq b_{2,s}$ always hold, we have $2a_{1,s} = a_{2,s} + b_{2,s}$. Hence $a_{1,s} \neq b_{2,s}$. And for any $s \in \mathcal{C}_{\mathbf{a}_2 - \mathbf{b}_2} \setminus (\mathcal{C}_{\mathbf{a}_1 - \mathbf{b}_1} \cap \mathcal{C}_{\mathbf{a}_2 - \mathbf{b}_2})$, $a_{2,s} = b_{2,s}$ and $a_{1,s} \neq b_{1,s}$ always hold, we have $2a_{2,s} = a_{1,s} + b_{1,s}$. Hence $a_{1,s} \neq b_{2,s}$. So $d(\mathbf{a}_1, \mathbf{b}_2) \geq 2(m - \omega - |\mathcal{C}_{\mathbf{a}_1 - \mathbf{b}_1} \cap \mathcal{C}_{\mathbf{a}_2 - \mathbf{b}_2}|) = 2(m - \omega - |\mathcal{T}_{j_1} \cap \mathcal{T}_{j_2}|) > \omega$. Hence $d(\mathbf{a}_1, \mathbf{b}_2) \neq \omega$ always holds. Then $\mathcal{Y}_{\mathbf{e}}$ satisfies the **Property 1**. The proof is completed. \square

Similar to the proof of Lemma 3, using vertex coloring the following result can be obtained.

Lemma 4: When $m > \frac{3\omega}{2}$ and $\mathcal{A} = \mathcal{B} = [0, 3)^m$, there exists a $(3^m, 3^m, 3^m - \binom{m}{\omega} 2^\omega, S)$ where

$$S \leq 3^m \cdot \left(1 + \sum_{i=\lceil m-\frac{3}{2}\omega \rceil}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{m-\omega-i} \right).$$

The proof of Lemma 4 is presented in Appendix B. Similar to the PDA in Lemma 3, there are also some useless stars in the PDA from Lemma 4. We can not find the exactly number of useless stars either. Applying Lemma 1 to the PDA in Lemma 4, the following result can be obtained.

Theorem 5: For any positive integers m, ω with $m > \frac{3\omega}{2}$, there exists a 3^m -division $(3^m, M, N)$ coded caching scheme with memory fraction $\frac{M}{N} = 1 - \frac{\binom{m}{\omega} 2^\omega}{3^m}$ and transmission rate $R \leq 1 + \sum_{i=\lceil m-\frac{3}{2}\omega \rceil}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{m-\omega-i}$.

From Section IV and Section V, we use the same method to construct the two partitions for $q = 2, 3$ respectively. However the partition for $q = 3$ is more complicated than the partition for $q = 2$. In fact, the complexity of constructing partition increases with the growth of q .

VI. THE PERFORMANCE ANALYSIS OF NEW SCHEMES

In this section several comparisons are proposed. First we claim that our schemes from Theorem 2 and Theorem 4 can achieve small memory fractions. Now we take the memory fraction $\frac{M}{N} = 1 - \frac{\binom{m}{\omega}}{\sum_{i=0}^{\omega} \binom{m}{i}}$ of the scheme from Theorem 2 as an example. We have

$$\begin{aligned} \frac{M}{N} &= 1 - \frac{\binom{m}{\omega}}{\sum_{i=0}^{\omega} \binom{m}{i}} = 1 - \frac{1}{\frac{\binom{m}{0}}{\binom{m}{\omega}} + \frac{\binom{m}{1}}{\binom{m}{\omega}} + \dots + \frac{\binom{m}{\omega-1}}{\binom{m}{\omega}} + \frac{\binom{m}{\omega}}{\binom{m}{\omega}}} \\ &= 1 - \frac{1}{\frac{\omega}{m-\omega+1} \frac{\omega-1}{m-\omega+2} \dots \frac{1}{m} + \frac{\omega}{m-\omega+1} \frac{\omega-1}{m-\omega+2} \dots \frac{2}{m-1} + \dots + \frac{\omega}{m-\omega+1} + 1} \\ &< 1 - \frac{1}{\left(\frac{\omega}{m-\omega+1}\right)^\omega + \left(\frac{\omega}{m-\omega+1}\right)^{\omega-1} + \dots + \frac{\omega}{m-\omega+1} + 1} \\ &= 1 - \frac{1 - \frac{\omega}{m-\omega+1}}{1 - \left(\frac{\omega}{m-\omega+1}\right)^{\omega+1}} \end{aligned}$$

For any $\lambda \in [0, 0.5]$ let $\omega = \lambda m$. Clearly

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1 - \frac{\omega}{m-\omega+1}}{1 - \left(\frac{\omega}{m-\omega+1}\right)^{\omega+1}} &= \lim_{m \rightarrow \infty} \frac{1 - \frac{\lambda m}{(1-\lambda)m+1}}{1 - \left(\frac{\lambda m}{(1-\lambda)m+1}\right)^{\lambda m+1}} \\ &= \lim_{m \rightarrow \infty} \frac{1 - \frac{\lambda}{1-\lambda}}{1 - \left(\frac{\lambda}{1-\lambda}\right)^{\lambda m+1}} \\ &= 1 - \frac{\lambda}{1-\lambda} = \frac{1-2\lambda}{1-\lambda} \end{aligned}$$

So we have

$$\lim_{m \rightarrow \infty} \frac{M}{N} = 1 - \frac{1-2\lambda}{1-\lambda} = \frac{\lambda}{1-\lambda}$$

If $\lambda < \frac{1}{3}$, the memory fraction $\frac{M}{N}$ of Theorem 2 is less than $\frac{1}{2}$ even when m is large. Similarly we can also show that the scheme from Theorem 4 can achieve small memory fraction too.

A. The comparison between new scheme in Theorem 2 and MN Scheme in [11]

From Theorem 2 we have a coded caching scheme with

$$K = 2^m, \quad \frac{M}{N} = 1 - \frac{\binom{m}{\omega}}{\sum_{i=0}^{\omega} \binom{m}{i}}, \quad F_1 = \sum_{i=0}^{\omega} \binom{m}{i}, \quad R_1 = \frac{\binom{m}{\omega} 2^{m-\omega}}{\sum_{i=0}^{\omega} \binom{m}{i}}.$$

When $t = 2^m - \frac{2^m \binom{m}{\omega}}{\sum_{i=0}^{\omega} \binom{m}{i}}$, from the first row of Table I, we have an MN scheme with the $K = 2^m$ and $\frac{M}{N} = 1 - \frac{\binom{m}{\omega}}{\sum_{i=0}^{\omega} \binom{m}{i}}$ where the subpacketization and transmission rate are respectively

$$F_{MN} = \left(2^m \binom{m}{\omega} / \sum_{i=0}^{\omega} \binom{m}{i} \right), \quad R_{MN} = \frac{2^m \binom{m}{\omega}}{\sum_{i=0}^{\omega} \binom{m}{i} (1 + 2^m) - 2^m \binom{m}{\omega}}.$$

Then we have the following ratios.

$$\frac{F_1}{F_{MN}} = \frac{\sum_{i=0}^{\omega} \binom{m}{i}}{\left(2^m \binom{m}{\omega} / \sum_{i=0}^{\omega} \binom{m}{i} \right)}, \quad \frac{R_1}{R_{MN}} = \frac{1 + 2^m - 2^m \binom{m}{\omega} / \sum_{i=0}^{\omega} \binom{m}{i}}{2^{\omega}} \quad (9)$$

Since it's difficult to estimate an approximate ratio for any m and ω , we consider taking specific parameters that would lead to exact ratios. When $\omega = \frac{m}{2}$,

$$\sum_{i=0}^{\omega} \binom{m}{i} = \sum_{i=0}^{\frac{m}{2}} \binom{m}{i} = 2^{m-1} + \frac{1}{2} \binom{m}{\frac{m}{2}} \approx 2^{m-1} \left(1 + \frac{1}{\sqrt{\frac{\pi m}{2}}} \right) \quad (10)$$

where $\binom{m}{\frac{m}{2}} \approx \frac{2^m}{\sqrt{\frac{\pi m}{2}}}$. (9) can be written as

$$\frac{F_1}{F_{MN}} \approx \frac{2^{m-1} \left(1 + \frac{1}{\sqrt{\frac{\pi m}{2}}} \right)}{\left(2^{m+1} / (1 + \sqrt{\frac{\pi m}{2}}) \right)} < \frac{2^{m-1} \left(1 + \frac{1}{\sqrt{m}} \right)}{\left(2^m / (1 + \sqrt{m}) \right)} \quad (11)$$

$$< \frac{2^{m-1} \left(1 + \frac{1}{\sqrt{m}} \right)}{\left(1 + \sqrt{m} \right) \frac{2^m}{1 + \sqrt{m}}} \quad (12)$$

$$\approx \frac{K}{2} \cdot \frac{1}{(1 + \sqrt{m})^{\frac{K}{1 + \sqrt{m}}}}. \quad (13)$$

by (10), where (11) is result of $\frac{2^{m+1}}{1 + \sqrt{\frac{\pi m}{2}}} < \frac{2^{m+1}}{2}$ and $\frac{2^{m+1}}{1 + \sqrt{\frac{\pi m}{2}}} > \frac{2^m}{1 + \sqrt{m}}$, and (12) holds due to

$$\left(\frac{2^m}{1 + \sqrt{m}} \right) = \frac{(2^m)(2^m - 1)(2^m - 2) \cdots \left(\frac{2^m}{1 + \sqrt{m}} + 1 \right)}{\left(\frac{2^m}{1 + \sqrt{m}} \right) \left(\frac{2^m}{1 + \sqrt{m}} - 1 \right) \left(\frac{2^m}{1 + \sqrt{m}} - 2 \right) \cdots 1} > (1 + \sqrt{m})^{\frac{2^m}{1 + \sqrt{m}}}.$$

Meanwhile,

$$\frac{R_1}{R_{MN}} \approx \left(\frac{1}{2^m} + 1 - \frac{2}{1 + \sqrt{\frac{\pi m}{2}}} \right) \cdot 2^{\frac{m}{2}} \approx \sqrt{K} \cdot \left(1 - \frac{1}{1 + \sqrt{m}} \right) \quad (14)$$

We can see that for the same number of users and memory fraction, subpacketization F_1 of our scheme is at least $\frac{K}{2} \cdot \frac{1}{(1 + \sqrt{m})^{\frac{K}{1 + \sqrt{m}}}}$ times smaller than F_{MN} of MN scheme from (13), meanwhile the transmission rate R_1 is at most $\sqrt{K} \cdot \left(1 - \frac{1}{1 + \sqrt{m}} \right)$ times larger than R_{MN} from (14). Finally we propose the following example to further verify our claim.

Example 5: When $m = 4, 6, 8$ and 10 , let $\omega = 2, 3, 4$ and 5 respectively. For the same number of users K and memory fraction $\frac{M}{N}$, the values of $\frac{F_1}{F_{MN}}$ in (11) and the values of $\frac{R_1}{R_{MN}}$ in (14) are listed in Table III. Clearly the amount of the reducing subpacketizations is much larger than the amount of the increasing transmission rate.

TABLE III: The comparison between scheme in Theorem 2 and MN Scheme in [11]

m	ω	K	$\frac{M}{N}$	$\frac{F_1}{F_{MN}}$	$\frac{R_1}{R_{MN}}$
4	2	2^4	0.545	9×10^{-4}	1.940
6	3	2^6	0.531	2.625×10^{-19}	4.445
8	4	2^8	0.570	3.546×10^{-48}	9.186
10	5	2^{10}	0.605	2.178×10^{-298}	19.415

B. The comparison between new schemes and known linear subpacketization schemes

In this section, we further discuss the performance of our schemes from Theorem 2 and Theorem 4 by comparing known schemes with linear subpacketizations proposed in [2] and [21] respectively. Unfortunately, the theoretical analyses of the comparisons become quite messy and do not yield much intuition. Instead, we illustrate the advantages on the parameters of the user number, memory fraction, subpacketization and transmission rate by numerical comparisons. For the sake of clarity, we mark the parameters of scheme in [2] as (k, n, m, t, q) , the parameters of scheme in [21] as (m, a, b, λ) , the parameters of new scheme from Theorem 2 as $(m, \omega, 2)$, and the parameters of our new scheme from Theorem 4 as $(m, \omega, 3)$. In the following we always use the above parameters to denote their related scheme. For instance, parameters $(6, 2, 2, 1, 2)$ denotes the scheme in [2] with $k = 6, n = m = 2, t = 1$ and $q = 2$. Parameter $(10, 3, 3, 2)$ denotes the scheme in [21] with $m = 10, a = b = 3$ and $\lambda = 2$. Parameters $(16, 6, 2)$ denotes the scheme from Theorem 2 with $m = 16, \omega = 6$ and $q = 2$. And parameters $(10, 6, 3)$ denotes the scheme from Theorem 4 with $m = 10, \omega = 3$ and $q = 3$.

Firstly let us see the comparisons of our two new schemes from Theorem 2 and Theorem 4 and the scheme in [2] in Table IV. We can see that our two new schemes both have lower subpacketizations, memory fractions and observably smaller transmission rate meanwhile can serve more users.

TABLE IV: The comparison between schemes in Theorem 2, Theorem 4 and scheme in [2]

schemes	parameters	K	F	$\frac{M}{N}$	R
(k, n, m, t, q) in [2]	$(6, 2, 2, 1, 2)$	39060	39060	0.6330	716.8000
$(m, \omega, 2)$ in Theorem 2	$(16, 6, 2)$	65536	14893	0.4623	550.6071
$(m, \omega, 3)$ in Theorem 4	$(10, 6, 3)$	59049	26025	0.4836	476.4761
(k, n, m, t, q) in [2]	$(6, 2, 2, 1, 3)$	7927920	7927920	0.4190	230291.1000
$(m, \omega, 2)$ in Theorem 2	$(23, 7, 2)$	8388608	390656	0.3724	41127.2556
$(m, \omega, 3)$ in Theorem 4	$(15, 7, 3)$	14348907	1266027	0.3494	72933.0548
(k, n, m, t, q) in [2]	$(6, 2, 2, 1, 4)$	422021600	422021600	0.3043	14680064.0000
$(m, \omega, 2)$ in Theorem 2	$(19, 7, 2)$	536870912	2182396	0.2848	2999632.4210
$(m, \omega, 3)$ in Theorem 4	$(19, 7, 3)$	1162261467	8628699	0.2525	6787121.7665
(k, n, m, t, q) in [2]	$(6, 2, 2, 1, 5)$	9914404500	9914404500	0.2366	378417968.7500
$(m, \omega, 2)$ in Theorem 2	$(34, 6, 2)$	17179869184	1676116	0.1976	215390771.5911
$(m, \omega, 3)$ in Theorem 4	$(21, 7, 3)$	10460353203	10460353203	0.2214	63631596.7218
(k, n, m, t, q) in [2]	$(6, 2, 2, 1, 6)$	135288489420	135288489420	0.1928	5460095692.8000
$(m, \omega, 2)$ in Theorem 2	$(37, 5, 2)$	137438953472	510416	0.1460	3667916678.6004
$(m, \omega, 3)$ in Theorem 4	$(24, 6, 3)$	282429536481	10161633	0.1523	3740922929.6312
(k, n, m, t, q) in [2]	$(6, 2, 2, 1, 7)$	1255883249600	1255883249600	0.1624	52596891363.8000
$(m, \omega, 2)$ in Theorem 2	$(41, 6, 2)$	2199023255552	5358578	0.1609	28831289808.7916
$(m, \omega, 3)$ in Theorem 4	$(26, 6, 3)$	2541865828329	17101033	0.1384	34220960199.0819

Now let us compare our two new schemes from Theorem 2 and Theorem 4 with the scheme from [21] in Table V. We can see that our new scheme from Theorem 2 has smaller or same subpacketization, lower memory fraction and smaller transmission rate meanwhile is able to serve more users. The our new scheme in Theorem 4 has advantages on the user number, memory fraction and subpacketization at the cost of some transmission rate.

TABLE V: The comparison between schemes in Theorem 2, Theorem 4 and scheme in [21]

schemes	parameters	K	F	$\frac{M}{N}$	R
(m, a, b, λ) in [21]	$(10, 3, 3, 2)$	120	120	0.8250	0.7500
$(m, \omega, 2)$ in Theorem 2	$(7, 3, 2)$	128	120	0.8250	0.7000
(m, a, b, λ) in [21]	$(16, 4, 5, 2)$	1820	4368	0.6978	10.0000
$(m, \omega, 2)$ in Theorem 2	$(11, 6, 2)$	2048	1486	0.6891	9.9489
(m, a, b, λ) in [21]	$(20, 5, 5, 2)$	15504	15504	0.7065	50.0000
$(m, \omega, 2)$ in Theorem 2	$(14, 7, 2)$	16384	9908	0.6236	44.3375
(m, a, b, λ) in [21]	$(20, 8, 7, 3)$	125970	77520	0.6424	273.0000
$(m, \omega, 2)$ in Theorem 2	$(17, 8, 2)$	131072	65536	0.6291	189.9219
(m, a, b, λ) in [21]	$(16, 4, 5, 2)$	1820	4368	0.6978	10.0000
$(m, \omega, 3)$ in Theorem 4	$(7, 5, 3)$	2187	1611	0.5829	28.5084
(m, a, b, λ) in [21]	$(16, 5, 6, 1)$	4368	8008	0.7115	10.0000
$(m, \omega, 3)$ in Theorem 4	$(8, 6, 3)$	6561	5281	0.6607	34.7866
(m, a, b, λ) in [21]	$(17, 10, 7, 5)$	19448	19448	0.7279	21.0000
$(m, \omega, 3)$ in Theorem 4	$(9, 7, 3)$	19683	16867	0.7268	42.0103
(m, a, b, λ) in [21]	$(20, 5, 6, 3)$	15504	38760	0.8826	4.0000
$(m, \omega, 3)$ in Theorem 4	$(9, 8, 3)$	19683	19171	0.8798	9.2404

VII. CONCLUSION

In this paper, we proposed a framework of constructing PDAs via Hamming distance. Consequently the problem of constructing PDAs is equivalent to constructing appropriate partitions. According to the structure of obtained PDAs, we obtained two classes of coded caching schemes with linear subpacketizations. Finally theoretic and numerical comparisons showed that our new schemes have good performance.

In this paper, we pointed out that under the framework of Hamming distance, constructing coded caching scheme with small memory fraction, low subpacketization and small transmission rate depends on designing a partition such that 1) the cardinality of this partition is as small as possible and 2) the number of useless stars in each column of obtained PDA is numerable. So it is interesting to design the partitions satisfying the above two conditions.

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APPENDIX A: THE PROOF OF LEMMA 3

First the following notations are useful. A graph G consists of a set $V(G)$ of vertexes and a set $E(G) \subset \{(u, v) : u, v \in V(G)\}$ of edges. The degree of a vertex v in a graph G is the number of vertices in G that are adjacent to v . The largest degree among the vertices of G is called the maximum degree of G is denoted by $\Delta(G)$. A vertex k -coloring of a graph G is an assignment of k colors to the vertices of G , one color to each vertex, so that adjacent vertices are colored differently. A graph G is k -colorable if there exists a coloring of G from a set of k colors. The minimum positive integer k for which G is k -colorable is the chromatic number of G and is denoted by $\chi(G)$.

Lemma 5: [3] For every graph G , $\chi(G) \leq 1 + \Delta(G)$.

Now let us give the proof of Lemma 3.

Proof. For any \mathcal{A} and $\mathcal{B} \subseteq [0, 2)^m$, from Theorem 1, the PDA obtained by Construction 1, Partition 2 and Construction 2 has $S = S'l_e$, where S' is the number of vectors in array obtained from Construction 1. Given \mathcal{A} and \mathcal{B} we only need to analyze the value of l_e .

- When $m = 2\omega + 1$, i.e., $m - \omega = \omega + 1$, for any vector e occurring in \mathbf{P} obtained by Construction 1, and for any two vectors $t_j, t_k \in [0, 2)^{m-\omega}$, from Partition 2 we can see that \mathcal{X}_{e, t_j} can merge with \mathcal{X}_{e, t_k} if and only if $d(t_j, t_k) = \omega + 1 = m - \omega$, i.e., $t_j + t_k = 1$. Then every two vectors is a element by Partition 2. Hence we have $l_e = 2^{m-\omega-1}$.
- When $m > 2\omega + 1$, we turn the partition problem for $[0, 2)^{m-\omega}$ to a vertex coloring problem. Define a graph G with vertex set $V(G) = [0, 2)^{m-\omega}$ such that there exists an edge connecting any two different vertices t_1 and t_2 in $V(G)$ if and only if $d(t_1, t_2) \leq \omega$. For any vertex $t \in V(G)$, the number of vertices in G that are adjacent to t is $\sum_{i=1}^{\omega} \binom{m-\omega}{i}$, i.e., the degree $\rho(t) \leq \sum_{i=1}^{\omega} \binom{m-\omega}{i}$. Then the maximum degree $\Delta(G) \leq \sum_{i=1}^{\omega} \binom{m-\omega}{i}$. From Lemma 5, we have

$$\chi(G) \leq 1 + \Delta(G) \leq 1 + \sum_{i=1}^{\omega} \binom{m-\omega}{i}. \quad (15)$$

From the definition of $\chi(G)$, there exists a $\chi(G)$ -coloring of G . In fact the vertex $\chi(G)$ -coloring of G corresponds to a $\chi(G)$ -partition for $[0, 2)^{m-\omega}$ in Partition 2. In the vertex $\chi(G)$ -coloring of graph G , we make each collection of vertexes having same color as a subset of vertex set $[0, 2)^{m-\omega}$. Then there exists $\chi(G)$ subsets $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{\chi(G)-1}$. Since each vertex in G has exactly one color, each element in $[0, 2)^{m-\omega}$ is exactly contained in one subset. For any two vertexes $t_1, t_2 \in \mathcal{T}_i, i \in [0, \chi(G))$, there exist no edge (t_1, t_2) , i.e., $d(t_1, t_2) \geq \omega + 1$ which satisfying Partition 2. Hence $\{\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{\chi(G)-1}\}$ is a $\chi(G)$ -partition for $[0, 2)^{m-\omega}$.

From the above discussion, for the vertex $\chi(G)$ -coloring of G , we always have a $l_e = \chi(G)$ -partition for \mathcal{T} . From (15) we have $l_e \leq 1 + \sum_{i=1}^{\omega} \binom{m-\omega}{i} = \sum_{i=0}^{\omega} \binom{m-\omega}{i}$.

Specially, when $\mathcal{A} = \mathcal{B} = [0, 2)^m$, from the proof of Theorem 2 we have $S' = \binom{m}{\omega}$. Due to $S = S'l_e$ the the proof of Lemma 3 is completed. \square

APPENDIX B: THE PROOF OF LEMMA 4

Proof. Similar to the proof of Lemma 3, given $\mathcal{A}, \mathcal{B} \subseteq [0, 3)^m$, from Theorem 1, the PDA obtained by Construction 1, Partition 4 and Construction 2 has $S = S'l_e$, where S' is the number of vectors in array obtained from Construction 1. We tend to analyze the value of l_e and turn it to a vertex coloring problem.

Given set $\binom{[0, m)}{m-\omega}$ in Partition 4, define a graph G with vertex set $V(G) = \binom{[0, m)}{m-\omega}$ such that there exists an edge connecting any two different vertices \mathcal{T}_1 and \mathcal{T}_2 in $V(G)$ if and only if $|\mathcal{T}_1 \cap \mathcal{T}_2| \geq m - \frac{3\omega}{2}$. For any vertex $\mathcal{T} \in V(G)$, the number of vertices in G that are adjacent to \mathcal{T} is $\sum_{i=\lceil m-\frac{3\omega}{2} \rceil}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{m-\omega-i}$, i.e., the degree $\rho(\mathcal{T}) \leq \sum_{i=\lceil m-\frac{3\omega}{2} \rceil}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{m-\omega-i}$. Then the maximum degree $\Delta(G) \leq \sum_{i=\lceil m-\frac{3\omega}{2} \rceil}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{m-\omega-i}$. From Lemma 5, we have

$$\chi(G) \leq 1 + \Delta(G) \leq 1 + \sum_{i=\lceil m-\frac{3\omega}{2} \rceil}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{m-\omega-i}. \quad (16)$$

From the definition of $\chi(G)$, there exists a $\chi(G)$ -coloring of G . Similar to the proof of Lemma 3 in Appendix A, the vertex $\chi(G)$ -coloring of G corresponds to a $\chi(G)$ -partition for $\binom{[0, m)}{m-\omega}$.

In the vertex $\chi(G)$ -coloring of graph G , we make each collection of vertexes having same color as a subset of vertex set. Then there exists $\chi(G)$ subsets $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{\chi(G)-1}$ of $\binom{[0,m]}{m-\omega}$. Since each vertex in G has exactly one color, each element in $\binom{[0,m]}{m-\omega}$ is exactly contained in one subset. For any two vertexes $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{D}_i, i \in [0, \chi(G))$, there exist no edge $(\mathcal{T}_1, \mathcal{T}_2)$, i.e., $|\mathcal{T}_1 \cap \mathcal{T}_2| < m - \frac{3\omega}{2}$ which satisfying Partition 2. Hence $\{\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{\chi(G)-1}\}$ is a $\chi(G)$ -partition for $\binom{[0,m]}{m-\omega}$. For the vertex $\chi(G)$ -coloring of G , we always have a $l_e = \chi(G)$ -partition for $\binom{[0,m]}{m-\omega}$. From (16) we have $l_e \leq 1 + \sum_{i=\lceil m-\frac{3\omega}{2} \rceil}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{m-\omega-i}$. When $\mathcal{A} = \mathcal{B} = [0, 3)^m$, from the proof of Theorem 4 we have $S' = 3^m$. Due to $S = S'l_e$ the proof is completed. \square