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# Asymptotically Optimal and Near-optimal Aperiodic Quasi-Complementary Sequence Sets Based on Florentine Rectangles

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#### Abstract

Quasi-complementary sequence sets (QCSSs) can be seen as a generalized version of complete complementary codes (CCCs), which enables multicarrier communication systems to support more users. The contribution of this work is two-fold. First, we propose a systematic construction of Florentine rectangles. Secondly, we propose several sets of CCCs and QCSS, using Florentine rectangles. The CCCs and QCSS are constructed over  $\mathbb{Z}_N$ , where  $N \ge 2$  is any integer. The cross-correlation magnitude of any two of the constructed CCCs is upper bounded by N. By combining the proposed CCCs, we propose asymptotically optimal and near-optimal QCSSs with new parameters. This solves a longstanding problem, of designing asymptotically optimal aperiodic QCSS over  $\mathbb{Z}_N$ , where N is any integer.

#### **Index Terms**

Asymptotically optimal quasi-complementary sequence set (QCSSs), Complete complementary codes (CCCs), Florentine rectangles.

### I. INTRODUCTION

Golay complementary pairs (GCPs) were proposed by M. J. Golay in 1951 in his work on multislit spectrometry [1]. Golay complementary pairs are a pair of sequences whose aperiodic

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autocorrelations sum up to zero at each non-zero time shift [1], [2]. In 1972, Tseng and Liu extended the concept from complementary pairs to complementary sets (CS) [3]. In a remarkable work in 1988, Suehiro and Hatori proposed N-shift cross orthogonal sequences [4], which were later termed as perfect complementary sequence sets (PCSS). A set of K mutually orthogonal CS, where each CS consists of M sequences (also known as sub-carriers), each of length N, is called a (K, M, N)- PCSS. PCSSs are also known as mutually orthogonal Golay complementary sets (MOGCS) [5]. Owing to their ideal correlation properties, PCSSs have been widely used in multi-carrier code division multiple access (MC-CDMA) systems for the reduction of the peak-to-average power ratio (PAPR) [6], channel estimation [7], [8], etc. One of the main drawbacks of PCSS with M sub-carriers is that, when used in MC-CDMA systems, it can support at most M users [10], [11].

Working towards the goal of enabling MC-CDMA systems to support more users, Liu *et al.* [15] designed low correlation zone complementary sequence sets (LCZ-CSSs) in 2011. Later in 2013, Liu *et al.* designed quasi-complementary sequence sets (QCSSs) [16] by generalizing the concept of LCZ-CSSs. The concept of QCSS also includes Z-complementary sequence sets (ZCSSs) [17]–[23]. A QCSS of set size K, flock size M, sequence length N and maximum aperiodic or periodic correlation tolerance  $\delta_{max}$ , is written as  $(K, M, N, \delta_{max})$ - QCSS. For QCSS, the set size K denotes the number of users it can support, the flock size M denotes the number of sub-carriers. When  $K \leq M$  and the periodic or aperiodic correlation tolerance  $\delta_{max} = 0$ , QCSS becomes PCSS. When K = M, a PCSS is called a complete complementary code (CCC) [16].

The first correlation lower bound of sequences was given by Welch [24] in 1974. Later, in a series of works in 2011, 2014 and 2017 [25]–[27], Liu *et al.* derived some special conditions for aperiodic QCSSs and further tightened the lower bound. We call a QCSS optimal if  $\delta_{\text{max}}$  achieves these lower bounds.

Systematic constructions of optimal QCSS, both periodic and aperiodic, for various parameters remains very challenging till date. In [16], Liu *et al.* designed periodic optimal QCSS by using Signer difference sets. Utilizing the properties of the difference sets and almost difference sets Li *et al.* gave systematic framework to construct periodic optimal and near-optimal QCSSs in [28], [29]. In [30] and [31], Li *et al.* proposed periodic QCSSs using characters over finite fields. Recently, in 2019, asymptotically optimal aperiodic QCSSs were first proposed by Li *et al.* [32], based on low-correlation CSSs over the alphabet  $\mathbb{Z}_N$ , where N is a prime integer or power

References	Set Size	Flock Size	Sequence Length	$\delta_{\max}$	Alphabet	Parameter constraint(s)
[33]	t(t-1)	t	t	t	$\mathbb{Z}_t$	t is an odd prime.
Theorem 1 [32]	u(u+1)	u	u	u	$\mathbb{Z}_{u}$	u is power of a prime.
Theorem 3 [32]	$u^2$	u	u-1	u	$\mathbb{Z}_{u}$	$u$ is power of a prime, and $u \ge 5$ .
[34]	$N(t_0 - 1)$	Ν	Ν	N	$\mathbb{Z}_N$	$N$ is odd, $N \ge 5$ , and $t_0$ is
[34]						the smallest prime factor of $\boldsymbol{N}$
Proposed	$N \times F(N)$	N	N	N	$\mathbb{Z}_N$	$N \ge 2$ is any integer.

TABLE I: Asymptotically optimal aperiodic QCSSs.

of a prime integer. In [33], Li *et al.* designed a systematic framework to construct aperiodic asymptotically optimal QCSSs using several sets of CCCs having prime length sequences over the alphabet  $\mathbb{Z}_N$ , where N is a prime integer. Recently in 2020, Zhou *et al.* [34] proposed a general construction of QCSS over  $\mathbb{Z}_N$ , for any odd integer  $N \ge 3$ . Asymptotically optimal aperiodic QCSSs with corresponding parameters, reported till date, are given in Table I.

Analysing closely the results of [32], [33] and [34] it is being observed that the number of CCC's and eventually the set size of the QCSS are small when N is 3 or have the smallest prime factor 3. Also it has been observed in all the previous constructions [32]–[34] that the optimal QCSS are designed over  $\mathbb{Z}_N$ , where N always a prime, power of prime or an odd integer, depending on the constructions. To overcome these problems, in search of new approaches to design QCSSs over any alphabet size N, we propose several sets of CCCs and eventually QCSSs using Florentine rectangles.

Florentine rectangles are extensively studied since 1989 [12]–[14]. Almost all of the studies are focused on searching the existence of Florentine rectangles of given orders. Also, most of the available examples are based on computer search results. To the best of the authors' knowledge systematic constructions of Florentine rectangles are available for few particular orders of Florentine rectangles only. To construct the CCCs and eventually the QCSS, in this paper, we give a systematic construction of Florentine rectangle having a flexible order. The proposed method drastically improves the set size of the QCSS, including the cases when N have the smallest prime factor 3. The proposed construction generates multiple CCCs and eventually asymptotically optimal QCSSs over  $\mathbb{Z}_N$ , where N is any even integer, like N = 6, 10, etc. To the best of the authors knowledge, QCSSs over any alphabet size N are not reported before. The construction uses the intrinsic structural properties of Florentine rectangles to construct these CCCs and QCSSs. Using the proposed framework, several sets of CCCs having parameters (N, N, N) are proposed. Further, utilizing the constructed CCCs, we propose  $(N \times F(N), N, N, N)$ -QCSS, where  $F(N) \times N$  Florentine rectangles exist. Since for  $F(N) \ge 3$  (except when N = 2 and 3), the proposed QCSS have set size  $(K) \ge 3M$ , flock size  $(M) \ge 2$  and sequence length (N) $\ge 2$ , we will check the optimality condition using the the correlation lower bound given by Liu *et al.* in [26]. The cross-correlation magnitude among the CCCs is upper bounded by N. The optimality factor  $\rho$  of the proposed QCSSs, obtained by combining the CCCs, are approximately equal to 1, hence resulting asymptotically optimal QCSSs. For the cases when N = 2 and 3, we use the Welch bound [24] to calculate the optimality factor  $\rho$ .

The rest of this paper is organized as follows. In Section II, we recall some definitions and correlation bounds related to QCSS. In Section III, we recall the definitions of Florentine rectangles and Vatican squares. We also give some systematic constructions of Florentine rectangles and Vatican squares in this section. In Section IV, we have utilised the permutations, obtained from the Florentine rectangles, to construct several sets of CCCs. In Section V, we construct QCSS by combining the several sets of CCCs. In Section VI, we have made a comparison of our construction with the previous constructions reported in the literature. Finally, we conclude our paper in Section VII.

### II. PRELIMINARIES

Before we begin, let us define the notations that we will be using in the paper.

- N is an integer.
- Let the ring of integers modulo N be denoted by  $\mathbb{Z}_N$ .
- $\omega_N = e^{\frac{2\pi i}{N}}$  is a primitive *N*-th root of unity.
- Let a set of sequence sets be denoted by C.
- A sequence set be denoted by C.
- A sequence be denoted by C.
- The complex conjugate of x is denoted by  $x^*$ .

Definition 1: Let  $C = (c_0, c_1, \dots, c_{N-1})$  and  $D = (d_0, d_1, \dots, d_{N-1})$  be two length N complex-valued sequences. The aperiodic correlation function (ACF) between C and D is defined as

$$\tilde{R}_{C,D}(\tau) = \begin{cases} \sum_{t=0}^{N-1-\tau} c_t d_{t+\tau}^*, & 0 \le \tau \le N-1\\ \sum_{t=0}^{N-1+\tau} c_{t-\tau} d_t^*, & -N+1 \le \tau < 0 \end{cases}$$
(1)

Definition 2: Consider  $\mathfrak{C} = \{ \mathcal{C}^0, \mathcal{C}^1, \cdots, \mathcal{C}^{K-1} \}$ , consisting K sequence sets, each having M sequences of length N, i.e.,

$$\mathcal{C}^{k} = \begin{bmatrix} C_{0}^{k} \\ C_{1}^{k} \\ \vdots \\ C_{M-1}^{k} \end{bmatrix}_{M \times N}, 0 \le k \le K - 1,$$
(2)

where  $C_m^k$  is the *m*-th sequence of length N and is expressed as  $C_m^k = (c_{m,0}^k, c_{m,1}^k, \cdots, c_{m,N-1}^k)$ ,  $0 \le m \le M-1$ . The set  $\mathfrak{C}$  is called a  $(K, M, N, \delta_{\max})$  quasi-complementary sequence set (QCSS) if for any  $\mathcal{C}^{k_1}, \mathcal{C}^{k_2} \in \mathfrak{C}, \ 0 \le k_1, k_2 \le K-1, \ 0 \le \tau \le N-1, k_1 \ne k_2$  or  $0 < \tau \le N-1, k_1 = k_2$ ,

$$|\tilde{R}_{\mathcal{C}^{k_1}, \mathcal{C}^{k_2}}(\tau)| = \left|\sum_{m=0}^{M-1} \tilde{R}_{C_m^{k_1}, C_m^{k_2}}(\tau)\right| \le \delta_{\max},\tag{3}$$

where K, M, N and  $\delta_{\max}$  denotes the set size, the number of sequences in each sequence set, the length of constituent sequences, and the maximum aperiodic cross-correlation magnitude of  $\mathfrak{C}$ , respectively. When K = M and  $\delta_{\max} = 0$ ,  $(K, M, N, \delta_{\max})$ -QCSS transforms into (M, M, N)-CCC.

We now discuss the lower bound of  $\delta_{\max}$ .

Lemma 1: [24] Considering aperiodic correlation, for a QCSS with set size K, flock size M, sequence length N and aperiodic correlation tolerance  $\delta_{\text{max}}$ , the following inequality holds

$$\delta_{\max} \ge MN \cdot \sqrt{\frac{\left(\frac{K}{M} - 1\right)}{K(2N - 1) - 1}}.$$
(4)

In 2014, Liu *et al.* [26] proposed a tighter lower bound of  $\delta_{\text{max}}$  for aperiodic QCSS by imposing certain restrictions on the values of K, M and N.

Lemma 2: [26] For an aperiodic QCSS with set size K, flock size M, sequence length N and aperiodic correlation tolerance  $\delta_{\max}$ , the following inequality holds

$$\delta_{\max} \ge \sqrt{MN\left(1 - 2\sqrt{\frac{M}{3K}}\right)},\tag{5}$$

when  $K \ge 3M$ ,  $M \ge 2$  and  $N \ge 2$ .

In this work, when  $K \ge 3M$ , i.e., for  $F(N) \ge 4$ , a QCSS is optimal if  $\delta_{\max}$  satisfies (5) with equality. Therefore, when  $K \ge 3M$ , the optimality factor  $\rho$  is defined as follows

$$\rho = \frac{\delta_{\max}}{\sqrt{MN\left(1 - 2\sqrt{\frac{M}{3K}}\right)}}.$$
(6)

When  $K \not\geq 3M$ , i.e., when F(N) < 4, we define the optimality factor  $\rho$  as

$$\rho = \frac{\delta_{\max}}{MN \cdot \sqrt{\frac{\binom{K}{M} - 1}{K(2N - 1) - 1}}}.$$
(7)

In general,  $\rho \ge 1$ . When  $\rho = 1$ , a QCSS is said to be optimal. A QCSS is near-optimal when  $1 < \rho \le 2$ .

# III. FLORENTINE RECTANGLES AND VATICAN SQUARES. ITS EXISTENCE AND CONSTRUCTIONS

In this section we will revisit the definition and existence of Florentine rectangles and Vatican squares. We will also go through some of the constructions of Florentine rectangles and Vatican squares.

Definition 3: A Tuscan-k rectangle of order  $r \times N$  has r rows and N columns such that

- C1: each row is a permutation of the N symbols and
- C2: for any two distinct symbols a and b and for each  $1 \le m \le k$ , there is at most one row in which b is m steps to the right of a.

When k = N - 1, it is called Tuscan-(N - 1) rectangle or Florentine rectangle. When r = N and k = N - 1, it is called Tuscan-(N - 1) squares or Florentine squares.

Definition 4 (Latin Square): A matrix  $\mathcal{A}$  of size  $N \times N$  is called a Latin square if each row and each column of  $\mathcal{A}$  contains the N symbols, say  $0, 1, \ldots, N - 1$ , exactly once.

Definition 5: An  $N \times N$  Florentine square is called Vatican square if it is also Latin.

*Example 1:* Examples of  $4 \times 4$  Vatican square and  $6 \times 7$  Florentine rectangle are given below in the form of matrices, respectively.

$$\begin{bmatrix} 0, 1, 3, 2\\ 1, 2, 0, 3\\ 2, 3, 1, 0\\ 3, 0, 2, 1 \end{bmatrix}, \begin{bmatrix} 0, 1, 2, 3, 4, 5, 6\\ 0, 2, 4, 6, 1, 3, 5\\ 0, 3, 6, 2, 5, 1, 4\\ 0, 4, 1, 5, 2, 6, 3\\ 0, 5, 3, 1, 6, 4, 2\\ 0, 6, 5, 4, 3, 2, 1 \end{bmatrix}.$$

$$(8)$$

As we see, C1 and C2 of *Definition* 3 hold here.

For each positive integer N let F(N) denote the largest integer such that Florentine rectangle of order  $F(N) \times N$  exists.

	(.	() 101 0 ( 11 <u>-</u>		
Possible value	N	Possible value		
of $F(N)$	11	of $F(N)$		
1	17	16, 17		
2	18	18		
2	19	18, 19		
4	20	6,, 20		
4	21	6,, 21		
6	22	22		
6	23	22, 23		
7	24	6,, 24		
8	25	6,, 25		
10	26	6,, 26		
10	27	6,, 27		

28, 29

30, 31

 $6, \ldots, 32$ 

TABLE II: Possible values of F(N) for 0 < N < 32 [13].

12,13

6,..., 14

 $6, \ldots, 15$ 

N

There are very few constructions of Florentine rectangles reported in literature. All the reported constructions of the Florentine rectangles are mostly based on extensive computer search. Much of the research reported in the literature is on finding the existence of an  $F(N) \times N$  Florentine rectangle for a given N. Based on that, for a given value of  $0 < N \leq 32$ , we have given the probable value of F(N), in Table II, for which an  $F(N) \times N$  Florentine rectangle can exist.

*Lemma 3 (Construction of Vatican squares):* Let p be an odd prime integer. Then the multiplication table of  $\mathbb{Z}_p$ , without the border consisting of all-zero row and column, is a  $(p-1) \times (p-1)$ Vatican square. This also implies that for N = p - 1, where p is an odd prime, F(N) = p - 1.

Theorem 1: Let  $\mathcal{M}$  be a multiplication table of the elements of  $\mathbb{Z}_N$  with the first row  $M_{0,j} = 0$ for all  $0 \le j < N$  and the first column  $M_{i,0} = 0$  for all  $0 \le i < N$ .  $M_{1,j} = j$  for all  $0 \le j < N$ and  $M_{i,1} = i$  for all  $0 \le i < N$ . Let 0 < i < p,  $\mathcal{A} = [M_{i,.}]$ , where p is the smallest prime factor of N. Then A is a  $(p-1) \times N$  Florentine rectangle.

*Proof:* Let the element b be m steps right to a in two rows  $r_1$  and  $r_2$  of the matrix A,

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for some  $r_1 \neq r_2$ ,  $a \neq b$ . Let us assume that a is in the  $A_{r_1,c_1}$  and  $A_{r_2,c_2}$  positions, for some columns  $c_1$  and  $c_2$  of A. Therefore, according to our assumptions b will be in positions  $A_{r_1,c_1+m}$  and  $A_{r_2,c_2+m}$ , respectively. Since A is a multiplication table of the residue of  $\mathbb{Z}_N$ , so we have the following:

$$a \equiv r_1 \cdot c_1 \equiv r_2 \cdot c_2 \pmod{N}. \tag{9}$$

Similarly, we also have

$$b \equiv r_1 \cdot (c_1 + m) \equiv r_2 \cdot (c_2 + m) \pmod{N}.$$
(10)

As per our construction  $A_{r_1,1} = r_1$  and  $A_{r_2,1} = r_2$ . Since  $r_1$  and  $r_2$  are non-zero residues and prime to N therefore their inverses exist in  $\mathbb{Z}_N$ . From (9) we have

$$c_1 \equiv r_1^{-1} \cdot a \pmod{N},$$

$$d \ c_2 \equiv r_2^{-1} \cdot a \pmod{N}.$$
(11)

Replacing the value of  $c_1$  and  $c_2$  in (10) we have

an

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$$b \equiv r_1 \cdot (r_1^{-1} \cdot a + m) \pmod{N},$$
  
also,  $b \equiv r_2 \cdot (r_2^{-1} \cdot a + m) \pmod{N}.$  (12)

Since  $m \not\equiv 0 \pmod{N}$ , therefore from (12) we have,

$$r_1 m \equiv r_2 m \pmod{N}$$

$$\implies r_1 \equiv r_2 \pmod{N}.$$
(13)

Since  $r_1$  and  $r_2$  are less than p, the smallest prime factor of N, therefore (13) implies  $r_1 = r_2$ , or in other words,  $r_1$  and  $r_2$  are same row, which contradicts our assumption. Hence A is a  $(p-1) \times N$  Florentine rectangle.

Theorem 2: Let  $N \ge 4$ , N = 2m and  $m \not\equiv 1 \pmod{3}$ . Let  $\mathcal{A}$  be a  $4 \times N$  matrix and  $A_{i,j}$  denotes the *j*-th element of the *i*-th row. Define  $A_{i,j}$  as follows:

$$A_{0,j} = \begin{cases} j \pmod{N+1}, & \text{for } 1 \le j \le m; \\ (N+1) - A_{0,N+1-j} \pmod{N+1} & \text{(14)} \\ & \text{for } m+1 \le j \le N; \end{cases}$$

$$A_{1,j} = \begin{cases} 2j \pmod{N+1}, & \text{for } 1 \le j \le m; \\ (N+1) - A_{1,N+1-j} \pmod{N+1} & \text{(15)} \\ & \text{for } m+1 \le j \le N; \end{cases}$$

for i = 2 and i = 3,

$$A_{i,j} = A_{3-i,N+1-j}, \text{ for } 1 \le j \le N.$$
 (16)

Then  $\mathcal{A}$  is a  $4 \times N$  Florentine rectangle.

*Proof:* Let the element b be k steps right to a in two rows  $r_1$  and  $r_2$  of the matrix  $\mathcal{A}$ , for some  $r_1 \neq r_2$ ,  $a \neq b$ . Let us assume that a is in the  $A_{r_1,c_1}$  and  $A_{r_2,c_2}$  positions, for some columns  $c_1$  and  $c_2$  of  $\mathcal{A}$ . Therefore, according to our assumptions b will be in positions  $A_{r_1,c_1+k}$  and  $A_{r_2,c_2+k}$ , respectively.

Let us consider  $r_1 = 0$  and  $r_2 = 1$ , i.e., consider the positions of a and b in the first and second row. When  $r_1 = 0$ , if  $1 \le c_1 \le m$  then  $a = c_1 \pmod{N+1}$ , if  $m < c_1 \le N$  then  $a = (N+1) - (N+1) + c_1 \pmod{N+1} = c_1 \pmod{N+1}$ . According to our assumption in row  $r_1 = 0$ , b will be in position  $A_{0,c_1+k}$ . For  $r_1 = 0$ , when  $1 < c_1 + k \le m$  then  $b = c_1 + k$  $(\mod N+1)$ , when  $m < c_1 + k \le N$  then  $b = (N+1) - (N+1) + (c_1+k) \pmod{N+1} = c_1 + k$  $(\mod N+1)$ .

When  $r_2 = 1$ , if  $1 \le c_2 \le m$  then  $a = 2c_2 \pmod{N+1}$ , if  $m < c_2 \le N$  then  $a = (N+1) - 2(N+1) + 2c_2 \pmod{N+1} = 2c_2 \pmod{N+1}$ . According to our assumption in row  $r_2 = 1$ , b will be in position  $A_{1,c_2+k}$ . For  $r_2 = 1$ , when  $1 < c_2 + k \le m$  then  $b = 2(c_2+k) \pmod{N+1}$ , when  $m < c_2 + k \le N$  then  $b = (N+1) - 2(N+1) + 2(c_1+k) \pmod{N+1} = 2(c_1+k) \pmod{N+1}$ .

Following the above detail explanation there will be sixteen sub-cases. The first sub-case is when  $c_1 \le m$ ,  $c_1 + k \le m$ ,  $c_2 \le m$ , and  $c_2 + k \le m$ . In this case, we have

$$a \equiv c_1 \equiv 2c_2 \pmod{N+1},\tag{17}$$

and

$$b \equiv c_1 + k \equiv 2(c_2 + k) \pmod{N+1}.$$
 (18)

Combining (17) and (18), we get

$$k \equiv 2k \pmod{N+1},\tag{19}$$

which is impossible, since  $N \ge 4$  and  $k \ge 1$ . Similarly we can prove the other sub-cases. Also similar proof can be given for other values of  $r_1$  and  $r_2$ . Hence A is a  $4 \times N$  Florentine rectangle.

Theorem 3: Let  $N \ge 4$ , N = 2m and  $m \not\equiv 0 \pmod{3}$ . Let  $\mathcal{A}$  be a  $4 \times N$  matrix and  $A_{i,j}$  denotes the *j*-th element of the *i*-th row. Define  $A_{i,j}$  as follows:

$$A_{0,j} = \begin{cases} j, & \text{for } 1 \le j \le m; \\ (N+1) - A_{0,N+1-j} & \text{for } m+1 \le j \le N; \end{cases}$$

$$A_{1,j} = \begin{cases} 1 + \{ [2(i-1) + m - 1] \pmod{N} \}, \\ & \text{for } 1 \le j \le m; \\ (N+1) - A_{1,N+1-j}, \\ & \text{for } m+1 \le j \le N; \end{cases}$$
(20)

for i = 2 and i = 3,

$$A_{i,j} = A_{3-i,N+1-j}, \text{ for } 1 \le j \le N.$$
 (22)

Then  $\mathcal{A}$  is a  $4 \times N$  Florentine rectangle.

*Proof:* Let the element b be k steps right to a in two rows  $r_1$  and  $r_2$  of the matrix  $\mathcal{A}$ , for some  $r_1 \neq r_2$ ,  $a \neq b$ . Let us assume that a is in the  $A_{r_1,c_1}$  and  $A_{r_2,c_2}$  positions, for some columns  $c_1$  and  $c_2$  of  $\mathcal{A}$ . Therefore, according to our assumptions b will be in positions  $A_{r_1,c_1+k}$  and  $A_{r_2,c_2+k}$ , respectively.

Let us consider  $r_1 = 0$  and  $r_2 = 1$ , i.e., consider the positions of a and b in the first and second row. When  $r_1 = 0$ , if  $1 \le c_1 \le m$  then  $a = c_1$ , if  $m < c_1 \le N$  then  $a = (N+1) - (N+1) + c_1 = c_1$ . According to our assumption in row  $r_1 = 0$ , b will be in position  $A_{0,c_1+k}$ . For  $r_1 = 0$ , when  $1 < c_1 + k \le m$  then  $b = c_1 + k$ , when  $m < c_1 + k \le N$  then  $b = (N+1) - (N+1) + (c_1 + k) = c_1 + k$ .

When  $r_2 = 1$ , if  $1 \le c_2 \le m$  then  $a = 1 + \{[2(c_2 - 1) + m - 1] \pmod{N}\}$ , if  $m < c_2 \le N$  then  $a = (N + 1) - 1 - \{[2(N + 1 - c_2 - 1) + m - 1] \pmod{N}\} = N - \{[2(N - c_2) + m - 1] \pmod{N}\}$ . According to our assumption in row  $r_2 = 1$ , b will be in position  $A_{1,c_2+k}$ . For  $r_2 = 1$ , when  $1 < c_2 + k \le m$  then  $b = 1 + \{[2(c_2 + k - 1) + m - 1] \pmod{N}\}$ , when  $m < c_2 + k \le N$  then  $b = (N+1) - 1 - \{[2(N+1-c_2-k-1)+m-1] \pmod{N}\} = N - \{[2(N-c_2-k)+m-1] \pmod{N}\}$ .

Following the above detail explanation there will be sixteen sub-cases. The first sub-case is when  $c_1 \le m$ ,  $c_1 + k \le m$ ,  $c_2 \le m$ , and  $c_2 + k \le m$ . In this case, we have

$$a \equiv c_1 \equiv 1 + \{ [2(c_2 - 1) + m - 1] \pmod{N} \}, \tag{23}$$

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and

$$b \equiv c_1 + k \equiv 1 + \{ [2(c_2 + k - 1) + m - 1] \pmod{N} \}$$
  

$$\implies c_1 + k - 1 \equiv \{ \{ [2(c_2 - 1) + m - 1] \pmod{N} + 2k \pmod{N} \} \pmod{N} \}$$
(24)

From (23) we can write (24) as

$$c_1 + k - 1 \equiv \{ [(c_1 - 1) + 2k \pmod{N}] \pmod{N} \}.$$
(25)

Since  $c_1 + k \leq m$ , therefore  $2k \pmod{N} = 2k$ , and hence (25) can be written as

$$c_1 + k - 1 \equiv \{ [(c_1 - 1) + 2k] \pmod{N} \}$$
  
$$\implies c_1 - 1 + 2k = NQ + c_1 + k - 1,$$
(26)

For some integer Q, and hence

$$k = NQ, \tag{27}$$

which is impossible, since k < N. Hence our assumption is impossible. Similarly we can prove the other sub-cases. Also similar proof can be given for other values of  $r_1$  and  $r_2$ . Hence  $\mathcal{A}$  is a  $4 \times N$  Florentine rectangle.

Corollary 1: Let N = 2m + 1. Using Theorem 2 and Theorem 3, we can construct a  $4 \times N$ Florentine rectangle for this case also, just by adding a column of all N's at the end of the rectangle.

Remark 1: For a given N, the values given in Table II are higher. However, systematic construction of Florentine rectangles of order  $F(N) \times N$ , when F(N) achieves the maximum value, is only available when N = p or N = p - 1, where p is prime. Computer search results of Florentine rectangles of order  $F(N) \times N$ , are available for small values of N. Although *Theorem* 1, *Theorem 2, Theorem 3* or *Corollary 1* do not always results to maximum number of rows for a given N which one can obtain by a computer search, we will use these results to construct the Florentine rectangles, since these are systematic constructions.

*Remark 2:* For a given N, in this paper, we consider the value of F(N) as the largest integer for which we can systematically construct a Florentine rectangle of size  $F(N) \times N$ . The Florentine rectangles are constructed as per following:

1) For N = p - 1, where p is prime, we construct the permutations as per *Lemma* 3, and then subtracting 1 from each of the elements. Therefore F(p - 1) = p - 1.

- 2) For N = p, where p is prime, we first construct the Vatican square as per *Lemma* 3 and then add zero column in the left side of the Vatican square. Therefore F(p) = p 1.
- 3) Let N > 5 be any number where N = p<sub>0</sub><sup>r<sub>0</sub></sup> p<sub>1</sub><sup>r<sub>1</sub></sup> ... p<sub>n-1</sub><sup>r<sub>n-1</sub></sup> and p<sub>0</sub> be the least prime factor of N. We also consider N+1, where N+1 = e<sub>0</sub><sup>s<sub>0</sub></sup> e<sub>1</sub><sup>s<sub>1</sub></sub> ... e<sub>m-1</sub><sup>s<sub>m-1</sub> and e<sub>0</sub> is the least prime factor of N+1. Assume p<sub>0</sub> ≠ 2, 3 and e<sub>0</sub> ≠ 2, 3. If p<sub>0</sub> > e<sub>0</sub>, we follow *Theorem* 1 to construct the Florentine rectangle and eventually the permutations over N. Therefore F(N) = p<sub>0</sub> 1. If p<sub>0</sub> < e<sub>0</sub>, we follow *Theorem* 1 to construct the Florentine rectangle and eventually the permutations over N. Therefore for the rectangle and eventually the subtract 1 from each element. Therefore, F(N) = e<sub>0</sub> 1.
  </sup></sup>
- 4) For any positive integer N having the least prime factor 2 or 3 and does not fall under above three cases, we follow *Theorem 2*, *Theorem 3* or *Corollary 1*. Therefore, F(N) = 4.

The value of F(N) considered in this paper is strictly based on the above set of rules.

# IV. MULTIPLE CCCs FROM FLORENTINE RECTANGLES

In this section, first we analyse some intrinsic properties of the Florentine rectangles and then we will utilise those properties to construct several sets of CCCs with low inter-set crosscorrelation magnitude. Let us begin by recalling the following property of the Florentine rectangles.

Property 1: Let  $\mathcal{A}$  be a  $F(N) \times N$  Florentine rectangle over  $\mathbb{Z}_N$ , denoted as follows

$$\mathcal{A} = \begin{bmatrix} A_{0,0}, \dots, A_{0,N-1} \\ A_{1,0}, \dots, A_{1,N-1} \\ \vdots \\ A_{F(N)-1,0}, \dots, A_{F(N)-1,N-1} \end{bmatrix}_{F(N) \times N},$$
(28)

where  $A_{i,j}$  denotes the *j*-th element in the *i*-th row. According to Definition 3, each row of  $\mathcal{A}$ , i.e.,  $A_i$ , is a permutation on  $\mathbb{Z}_N$ . Also, for each  $0 < \alpha < N$ ,  $(A_{i,j}, A_{i,j+\alpha}) \neq (A_{m,n}, A_{m,n+\alpha})$ unless i = m and j = n, where  $0 \le i, m \le F(N) - 1$ ,  $0 \le j, n \le N - 1$ ,  $0 < j + \alpha < N$  and  $0 < n + \alpha < N$ . In other words, if  $\pi_i : \mathbb{Z}_N \to \mathbb{Z}_N$  be a permutation on  $\mathbb{Z}_N$ , where  $\pi_i$  is equivalent to the *i*-th row of  $\mathcal{A}$ , i.e.  $A_i$ , then for each  $0 < \alpha < N$   $(\pi_i(j), \pi_i(j+\alpha)) = (\pi_m(n), \pi_m(n+\alpha))$ if and only if i = m and j = n, where  $0 < j + \alpha < N$  and  $0 < n + \alpha < N$ .

Lemma 4: Let  $\pi_i : \mathbb{Z}_N \to \mathbb{Z}_N$  be a permutation over  $\mathbb{Z}_N$  as defined in Property 1. For  $0 \le i \ne m \le F(N) - 1$ ,  $\pi_i(j) = \pi_m(j + \tau)$ , where  $0 \le j + \tau < N$ , has at most one solution for  $0 \le \tau < N$ .

*Proof:* Let us assume that for  $0 \le i \ne m \le F(N) - 1$ ,  $\pi_i(j) = \pi_m(j + \tau)$ , where  $0 \le j + \tau < N$ , has more than one solution for  $0 \le \tau < N$ . Let two of the solutions be  $j_1$  and  $j_2$ . Then  $\pi_i(j_1) = \pi_m(j_1 + \tau)$  and  $\pi_i(j_2) = \pi_m(j_2 + \tau)$ . Therefore, we have  $(\pi_i(j_1), \pi_i(j_2)) = (\pi_m(j_1 + \tau), \pi_m(j_2 + \tau))$ . This contradicts the definition of Florentine rectangles. Hence,  $\pi_i(j) = \pi_m(j + \tau)$  has at most one solution for each  $0 < \tau < N$  for  $0 \le i \ne m \le F(N) - 1$ .

The following example illustrates the set of permutations on  $\mathbb{Z}_N$  defined by the Florentine rectangles.

*Example 2:* Consider N = 4. From Example 1, we have  $\pi_0(\mathbb{Z}_4) = \{0, 1, 3, 2\}, \pi_1(\mathbb{Z}_4) = \{1, 2, 0, 3\}, \pi_2(\mathbb{Z}_4) = \{2, 3, 1, 0\}, \pi_3(\mathbb{Z}_4) = \{3, 0, 2, 1\}.$ 

Utilizing the set of permutations defined in Property 1, we design several sets of CCCs using the construction framework given below.

Construction 1: Consider any positive integer  $N \ge 2$ , for which an  $F(N) \times N$  Florentine rectangle  $\mathcal{A}$  exists over  $\mathbb{Z}_N$ . Also let  $\pi_k$  be the permutation over  $\mathbb{Z}_N$  for  $0 \le k < F(N)$ , defined as above, which satisfies Lemma 4. Then for each  $m \in \mathbb{Z}_N$ , and  $s \in \mathbb{Z}_N$ , define  $f_s^{(k,m)} : \mathbb{Z}_N \to \mathbb{Z}_N$ as follows

$$f_s^{(k,m)}(t) = s \cdot \pi_k(t) + mt \pmod{N}.$$
 (29)

For each  $0 \le k < F(N)$ , define a set

$$\mathfrak{C}^{k} = \{ \mathcal{C}^{(k,0)}, \mathcal{C}^{(k,1)}, \cdots, \mathcal{C}^{(k,N-1)} \},$$
(30)

where

$$\mathcal{C}^{(k,m)} = \begin{bmatrix} C_0^{(k,m)} \\ C_1^{(k,m)} \\ \vdots \\ C_{N-1}^{(k,m)} \end{bmatrix} = \begin{bmatrix} C_{0,0}^{(k,m)}, C_{0,1}^{(k,m)}, \cdots, C_{0,N-1}^{(k,m)} \\ C_{1,0}^{(k,m)}, C_{1,1}^{(k,m)}, \cdots, C_{1,N-1}^{(k,m)} \\ \vdots \\ C_{N-1,0}^{(k,m)}, C_{N-1,1}^{(k,m)}, \cdots, C_{N-1,N-1}^{(k,m)} \end{bmatrix}$$

and

$$C_{s,t}^{(k,m)} = \omega_N^{f_s^{(k,m)}(t)}$$
 for each  $0 \le t \le N - 1$ .

For the sequence sets obtained using Construction 1, we have the following theorem.

Theorem 4: For  $0 \le k < F(N)$ , let  $\mathfrak{C}^k$  be the multiple sequence sets obtained from Construction 1 based on the function  $f_s^{(k,m)} : \mathbb{Z}_N \to \mathbb{Z}_N$ , as given in (29). Then,

1) For each  $0 \le k < F(N)$ ,  $\mathfrak{C}^k$  is an (N, N, N)- CCC.

 For any two different CCCs C<sup>k1</sup> and C<sup>k2</sup>, the inter-set cross-correlation is upper bounded by N, i.e.,

$$\left|\sum_{s=0}^{N-1} \tilde{R}_{C_{s}^{(k_{1},m_{1})},C_{s}^{(k_{2},m_{2})}}(\tau)\right| \le N,$$
(31)

for all  $0 \le k_1 \ne k_2 < F(N)$ ,  $0 \le \tau \le N - 1$ , and  $0 \le m_1, m_2 \le N - 1$ .

Proof: The proof is given in Appendix A.

*Example 3:* Let N = 10, The family of permutations being defined as in the first rule of *Remark 2.* Then we have

$$\pi_{0}(\mathbb{Z}_{10}) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},\$$

$$\pi_{1}(\mathbb{Z}_{10}) = \{1, 3, 5, 7, 9, 0, 2, 4, 6, 8\},\$$

$$\pi_{2}(\mathbb{Z}_{10}) = \{2, 5, 8, 0, 3, 6, 9, 1, 4, 7\},\$$

$$\pi_{3}(\mathbb{Z}_{10}) = \{3, 7, 0, 4, 8, 1, 5, 9, 2, 6\},\$$

$$\pi_{4}(\mathbb{Z}_{10}) = \{4, 9, 3, 8, 2, 7, 1, 6, 0, 5\},\$$

$$\pi_{5}(\mathbb{Z}_{10}) = \{5, 0, 6, 1, 7, 2, 8, 3, 9, 4\},\$$

$$\pi_{6}(\mathbb{Z}_{10}) = \{6, 2, 9, 5, 1, 8, 4, 0, 7, 3\},\$$

$$\pi_{7}(\mathbb{Z}_{10}) = \{7, 4, 1, 9, 6, 3, 0, 8, 5, 2\},\$$

$$\pi_{8}(\mathbb{Z}_{10}) = \{9, 8, 7, 6, 5, 4, 3, 2, 1, 0\}.$$
(32)

By Theorem 4, we get ten (10, 10, 10)-CCCs,  $\mathfrak{C}^0$ ,  $\mathfrak{C}^1$ ,  $\mathfrak{C}^2$ ,  $\mathfrak{C}^3$ ,  $\mathfrak{C}^4$ ,  $\mathfrak{C}^5$ ,  $\mathfrak{C}^6$ ,  $\mathfrak{C}^7$ ,  $\mathfrak{C}^8$ ,  $\mathfrak{C}^9$ . The two CCCs  $\mathfrak{C}^2$  and  $\mathfrak{C}^3$  are shown in Table III.

We propose a framework to systematically construct asymptotically optimal aperiodic QCSSs, in the following section.

# V. PROPOSED CONSTRUCTION OF ASYMPTOTICALLY OPTIMAL QCSSS

Theorem 5: Consider N to be any positive integer greater than 3, for which F(N) > 4. Also, let  $\mathfrak{C}^k$  for  $0 \le k < F(N)$  be obtained using Theorem 4 and  $\mathfrak{C} = \mathfrak{C}^0 \cup \mathfrak{C}^1 \cup \cdots \cup \mathfrak{C}^{F(N)-1}$ , then the sequence set  $\mathfrak{C}$  is an asymptotically optimal aperiodic  $(N \times F(N), N, N, N)$ -QCSS.

*Proof:* Based on Theorem 4, for each  $0 \le k < F(N)$ ,  $\mathfrak{C}^k$  is an (N, N, N)-CCC. Also, for any two  $k_1$ ,  $k_2$ , where  $0 \le k_1, k_2 < F(N)$  and  $k_1 \ne k_2$ , the the inter-set cross-correlation

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TABLE III: The two (10, 10, 10)- CCCs  $\mathfrak{C}^2$  and  $\mathfrak{C}^3$  of *Example 3*.

$\mathcal{C}^{(2,0)}$	$\mathcal{C}^{(2,1)}$	$\mathcal{C}^{(2,2)}$	$\mathcal{C}^{(2,3)}$	$\mathcal{C}^{(2,4)}$	$\mathcal{C}^{(2,5)}$	$\mathcal{C}^{(2,6)}$	$\mathcal{C}^{(2,7)}$	$\mathcal{C}^{(2,8)}$	$\mathcal{C}^{(2,9)}$
0000000000	0123456789	0246802468	0369258147	0482604826	0505050505	0628406284	0741852963	0864208642	0987654321
2580369147	2603715826	2726161505	2849517284	2962963963	2085319642	2108765321	2221111000	2344567789	2467913468
4060628284	4183074963	4206420642	4329876321	4442222000	4565678789	4688024468	4701470147	4824826826	4947272505
6540987321	6663333000	6786789789	6809135468	6922581147	6045937826	6168383505	6281739284	6304185963	6427531642
8020246468	8143692147	8266048826	8389494505	8402840284	8525296963	8648642642	8761098321	8884444000	8907890789
0500505505	0623951284	0746307963	0869753642	0982109321	0005555000	0128901789	0241357468	0364703147	0487159826
2080864642	2103210321	2226666000	2349012789	2462468468	2585814147	2608260826	2721616505	2844062284	2967418963
4560123789	4683579468	4706925147	4829371826	4942727505	4065173284	4188529963	4201975642	4324321321	4447777000
6040482826	6163838505	6286284284	6309630963	6422086642	6545432321	6668888000	6781234789	6804680468	6927036147
8520741963	8643197642	8766543321	8889999000	8902345789	8025791468	8148147147	8261593826	8384949505	8407395284
$\mathcal{C}^{(3,0)}$	$\mathcal{C}^{(3,1)}$	$\mathcal{C}^{(3,2)}$	$\mathcal{C}^{(3,3)}$	$\mathcal{C}^{(3,4)}$	$\mathcal{C}^{(3,5)}$	$\mathcal{C}^{(3,6)}$	$\mathcal{C}^{(3,7)}$	$\mathcal{C}^{(3,8)}$	$\mathcal{C}^{(3,9)}$
0000000000	0123456789	0246802468	0369258147	0482604826	0505050505	0628406284	0741852963	0864208642	0987654321
3704815926	3827261605	3940617384	3063063063	3186419742	3209865421	3322211100	3445667889	3568013568	3681469247
6408620842	6521076521	6644422200	6767878989	6880224668	6903670347	6026026026	6149472705	6262828484	6385274163
9102435768	9225881447	9348237126	9461683805	9584039584	9607485263	9720831942	9843287621	9966633300	9089089089
2806240684	2929696363	2042042042	2165498721	2288844400	2301290189	2424646868	2547092547	2660448226	2783894905
5500055500	5623401289	5746857968	5869203647	5982659326	5005005005	5128451784	5241807463	5364253142	5487609821
8204860426	8327216105	8440662884	8563018563	8686464242	8709810921	8822266600	8945612389	8068068068	8181414747
1908675342	1021021021	1144477700	1267823489	1380279168	1403625847	1526071526	1649427205	1762873984	1885229663
4602480268	4725836947	4848282626	4961638305	4084084084	4107430763	4220886442	4343232121	4466688800	4589034589
7306295184	7429641863	7542097542	7665443221	7788899900	7801245689	7924691368	7047047047	7160493726	7283849405

magnitude between  $\mathfrak{C}^{k_1}$  and  $\mathfrak{C}^{k_2}$  is upper bounded by N. Then  $\mathfrak{C}$  is an aperiodic  $(K, M, N, \delta_{\max})$ -QCSS, where  $K = N \times F(N)$ , M = N, N = N,  $\delta_{\max} = max\{0, N\} = N$ .

The optimality factor of  $(N \times F(N), N, N, N)$ -QCSS is

$$\rho = \frac{N}{\sqrt{N^2 (1 - 2\sqrt{\frac{N}{3N \times F(N)}})}}.$$
(33)

When  $N \to +\infty$  then  $F(N) \to +\infty$ . Therefore from (33),

$$\lim_{F(N)\to+\infty} \rho = \lim_{F(N)\to+\infty} \frac{N}{\sqrt{N^2(1 - 2\sqrt{\frac{N}{3N \times F(N)}})}}$$
$$= \lim_{F(N)\to+\infty} \frac{1}{\sqrt{1 - \frac{2}{\sqrt{3 \times F(N)}}}}$$
$$= 1.$$
(34)

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Therefore,  $\mathfrak{C}$  is an asymptotically optimal aperiodic QCSS.

*Example 4:* The ten (10, 10, 10)-CCCs,  $\mathfrak{C}^0$ ,  $\mathfrak{C}^1$ ,  $\mathfrak{C}^2$ ,  $\mathfrak{C}^3$ ,  $\mathfrak{C}^4$ ,  $\mathfrak{C}^5$ ,  $\mathfrak{C}^6$ ,  $\mathfrak{C}^7$ ,  $\mathfrak{C}^8$ ,  $\mathfrak{C}^9$ , generated in *Example 3* can be used to construct asymptotically optimal (100,10,10,10)- QCSS  $\mathfrak{C} = \mathfrak{C}^0 \cup \mathfrak{C}^1 \cup \mathfrak{C}^2 \cup \mathfrak{C}^3 \cup \mathfrak{C}^4 \cup \mathfrak{C}^5 \cup \mathfrak{C}^6 \cup \mathfrak{C}^7 \cup \mathfrak{C}^8 \cup \mathfrak{C}^9$ .

The asymptotically optimal aperiodic QCSSs obtained using Theorem 5 for some N, are shown in Table IV, with corresponding parameters.

Alphabet	K	M	N	ρ	
$\mathbb{Z}_6$	36	6	6	1.3754	
$\mathbb{Z}_{10}$	100	10	10	1.2551	
$\mathbb{Z}_{12}$	144	12	12	1.2247	
$\mathbb{Z}_{18}$	324	18	18	1.1722	
$\mathbb{Z}_{22}$	484	22	22	1.1518	
$\mathbb{Z}_{28}$	784	28	28	1.1310	
$\mathbb{Z}_{30}$	900	30	30	1.1257	
$\mathbb{Z}_{36}$	1296	36	36	1.1128	
$\mathbb{Z}_{40}$	1600	40	40	1.1061	
$\mathbb{Z}_{42}$	1764	42	42	1.1031	
$\mathbb{Z}_{46}$	2116	46	46	1.0978	
$\mathbb{Z}_{48}$	288	48	48	1.3754	
$\mathbb{Z}_{52}$	2704	52	52	1.0912	
$\mathbb{Z}_{58}$	3364	58	58	1.0857	
$\mathbb{Z}_{60}$	3600	60	60	1.0841	
$\mathbb{Z}_{66}$	4356	66	66	1.0797	
$\mathbb{Z}_{70}$	4900	70	70	1.0771	
$\mathbb{Z}_{72}$	5184	72	72	1.0759	
$\mathbb{Z}_{76}$	456	76	76	1.3754	
$\mathbb{Z}_{78}$	6084	78	78	1.0726	
$\mathbb{Z}_{82}$	6724	82	82	1.0706	
$\mathbb{Z}_{88}$	7744	88	88	1.0679	
$\mathbb{Z}_{90}$	540	90	90	1.3754	
$\mathbb{Z}_{96}$	9216	96	96	1.0647	
$\mathbb{Z}_{100}$	10000	100	100	1.0633	

TABLE IV: Asymptotically optimal aperiodic QCSSs, when N is even number.

Alphabet	K	M	N	ρ	
$\mathbb{Z}_{14}$	56	14	14	1.5382	
$\mathbb{Z}_{20}$	80	20	20	1.5382	
$\mathbb{Z}_{24}$	96	24	24	1.5382	
$\mathbb{Z}_{26}$	104	26	26	1.5382	
$\mathbb{Z}_{36}$	144	36	36	1.5382	
$\mathbb{Z}_{38}$	152	38	38	1.5382	
$\mathbb{Z}_{44}$	176	44	44	1.5382	
$\mathbb{Z}_{50}$	200	50	50	1.5382	

TABLE V: Near-optimal aperiodic QCSSs, when N is even number.

TABLE VI: Comparison of the parameters of QCSS when the smallest prime factor of N is 3.

Alphabet	K	$K\_prev$	M	N	ρ	$\rho\_prev$
$\mathbb{Z}_{3*5}$	60	30	15	15	1.5382	1.9653
$\mathbb{Z}_{3*7}$	84	42	21	21	1.5382	1.9755
$\mathbb{Z}_{3*11}$	132	66	33	33	1.5382	1.9846
$\mathbb{Z}_{3*5*7}$	420	210	105	105	1.5382	1.9952
$\mathbb{Z}_{3*5*11}$	660	330	165	165	1.5382	1.9970
$\mathbb{Z}_{3*5*7*11}$	4620	2310	1155	1155	1.5382	1.9996
$\mathbb{Z}_{3*5*7*11*13}$	60060	30030	15015	15015	1.5382	2.0000
$\mathbb{Z}_{3*5*7*11*13*17}$	1021020	510510	255255	255255	1.5382	2.0000

Corollary 2: For the cases when F(N) = 4,  $\mathfrak{C}$  is an aperiodic near-optimal (4N, N, N, N)-QCSS.

Proof: From (33), we get

$$\rho = \frac{N}{\sqrt{N^2 (1 - 2\sqrt{\frac{N}{3 \times N \times 4}})}} = 1.5382.$$
(35)

Corollary 3: Let N = 2 or 3. Also, let  $\mathfrak{C}^0$ ,  $\mathfrak{C}^1$  be obtained from Theorem 4 and  $\mathfrak{C} = \mathfrak{C}^0 \cup \mathfrak{C}^1$ . Then  $\mathfrak{C}$  is a near-optimal aperiodic (4, 2, 2, 2)-QCSS or (6, 3, 3, 3)-QCSS, with optimality factor  $\rho = 1.6584$  or 1.7950, respectively. Since  $K \ngeq 3M$ , we have used the Welch bound, discussed in Lemma 1.

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The results of the proposed systematic construction is compared with the results of the existing constructions in the following section.

## VI. COMPARISON WITH PREVIOUS WORKS

The main difference of this construction with all the previous constructions is that here we can construct asymptotically optimal and near-optimal QCSS over  $\mathbb{Z}_N$  for any N > 3, whereas previously  $N \ge 5$  was only odd integer [34], prime [32], [33] or power of prime [32]. For comparing the parameters with the existing results, in Table VI,  $K_{prev}$  and  $\rho_{prev}$  denote the previous set size and the previously reported optimality factor, respectively, of the QCSS over  $\mathbb{Z}_N$ , for some values of N, when N has a prime factor 3. The values of  $K_{prev}$  and  $\rho_{prev}$  are obtained from [34]. Compared with [32], [33] and [34], the uniqueness of our construction can be listed down as follows:

- 1) We can obtain asymptotically optimal QCSS with more flexible parameters using the proposed framework, as compared to the constructions proposed in [32], [33] and [34]. For instance, the asymptotically optimal and near-optimal QCSS over  $\mathbb{Z}_N$ , where N is an even composite number and not a prime power, can only be obtained by the proposed construction till date. The parameters and the corresponding optimality factor of the asymptotically optimal QCSS for some of these N can be seen in Table IV. Also, the parameters and the corresponding optimality factors of some near-optimal QCSSs are shown in Table V, where N is even.
- 2) When N has the smallest prime factor 3, near-optimal aperiodic QCSS can be obtained. Compared to the the results reported in [34], the obtained QCSSs display a lower optimality factor. Table VI compares the parameters for some of these values of N with the QCSS designed in [34].

# VII. CONCLUSION

In this paper, we have presented a systematic construction of QCSSs with new flexible parameters based on Florentine rectangles. In the proposed construction, N > 3 can take any positive value. This construction not only fills the gap left by all the previous constructions, where N was considered odd, prime or prime powers, but also improves the optimality factor as compared with the previous constructions. We first proposed a new set of permutations on  $\mathbb{Z}_N$ based on Florentine rectangles and utilized those permutations to construct (N, N, N)- CCCs. Combining the newly constructed CCCs we have designed new sets of  $(N \times F(N), N, N, N)$ -QCSS, where F(N) is the maximum number of rows for which an  $F(N) \times N$  Florentine rectangle exists. The designed QCSSs are asymptotically optimal and near-optimal with respect to the correlation bound in [26]. For N = 2 and N = 3, our construction results to near-optimal (2N, N, N, N)-QCSS, with respect to the Welch bound [24]. We have also compared our proposed construction with the previous constructions reported in the literature. The proposed construction results to new QCSSs when over  $\mathbb{Z}_N$ , when N > 3 is an even integer. When N > 3 is an odd integer, the set size increases and eventually the value of the optimality factor decreases, as compared to the previous constructions.

# APPENDIX A

# **PROOF OF THEOREM 4**

First, let us prove that, for  $0 \le k < F(N)$ ,  $\mathfrak{C}^k$  is an (N, N, N)- CCC. Let  $\mathcal{C}^{(k,m_1)}, \mathcal{C}^{(k,m_2)} \in \mathfrak{C}^k$ , where  $0 \le k < F(N)$ ,  $0 \le m_1, m_2 \le N - 1$  and  $f_s^{(k,m)}(t)$  is as given in (29). Then

$$\sum_{s=0}^{N-1} \tilde{R}_{C_{s}^{(k,m_{1})},C_{s}^{(k,m_{2})}}(\tau)$$

$$= \sum_{s=0}^{N-1} \sum_{t=0}^{N-1-\tau} C_{s,t}^{(k,m_{1})} \cdot \left(C_{s,t+\tau}^{(k,m_{2})}\right)^{*}$$

$$= \sum_{s=0}^{N-1} \sum_{t=0}^{N-1-\tau} \omega_{N}^{f_{s}^{(k,m_{1})}(t)} \cdot \omega_{N}^{-f_{s}^{(k,m_{2})}(t+\tau)}$$

$$= \sum_{s=0}^{N-1} \sum_{t=0}^{N-1-\tau} \omega_{N}^{s(\pi_{k}(t)-\pi_{k}(t+\tau))+t(m_{1}-m_{2})-m_{2}\tau}.$$
(36)

consider the following cases.

Case 1: When  $\tau = 0, m_1 = m_2$ , then

$$\sum_{s=0}^{N-1} \tilde{R}_{C_s^{(k,m_1)}, C_s^{(k,m_2)}}(0) = N^2.$$
(37)

Case 2: When  $1 \le \tau \le N - 1, m_1 = m_2$ ,

$$\sum_{s=0}^{N-1} \tilde{R}_{C_s^{(k,m_1)}, C_s^{(k,m_2)}}(\tau)$$

$$= \sum_{s=0}^{N-1} \sum_{t=0}^{N-1-\tau} \omega_N^{-m_2\tau} \cdot \omega_N^{s(\pi_k(t) - \pi_k(t+\tau))}$$

$$= \omega_N^{-m_2\tau} \cdot \sum_{t=0}^{N-1-\tau} \sum_{s=0}^{N-1} \omega_N^{s(\pi_k(t) - \pi_k(t+\tau))} = 0.$$
(38)

When  $\tau \neq 0$ ,  $\pi_k(t) \neq \pi_k(t+\tau)$  because  $\pi_k(t)$  is a permutation on  $\mathbb{Z}_N$ . Also  $N \nmid (\pi_k(t) - \pi_k(t+\tau))$  because  $(\pi_k(t) - \pi_k(t+\tau) < N$ . Therefore (38) holds.

Case 3: When  $\tau = 0, m_1 \neq m_2$ ,  $\sum_{s=0}^{N-1} \tilde{R}_{C_s^{(k,m_1)}, C_s^{(k,m_2)}}(0) = \sum_{s=0}^{N-1} \sum_{t=0}^{N-1} \omega_N^{t(m_1-m_2)} = 0.$ (39)

Case 4: When  $1 \le \tau \le N - 1, m_1 \ne m_2$ ,

$$\sum_{s=0}^{N-1} \tilde{R}_{C_s^{(k,m_1)}, C_s^{(k,m_2)}}(\tau)$$

$$= \sum_{t=0}^{N-1-\tau} \omega_N^{t \cdot (m_1 - m_2) - m_2 \tau} \cdot \sum_{s=0}^{N-1} \omega_N^{s(\pi_k(t) - \pi_k(t+\tau))} = 0.$$
(40)

For  $\tau \neq 0$ ,  $\pi_k(t) \neq \pi_k(t+\tau)$ , beause  $\pi_k(t)$  is a permutation on  $\mathbb{Z}_N$ . Also  $N \nmid (\pi_k(t) - \pi_k(t+\tau))$ because  $(\pi_k(t) - \pi_k(t+\tau) < N$ . Therefore (40) holds.

From the above four cases, we conclude that  $\mathfrak{C}^k$ , for each  $0 \le k < F(N)$ , is an (N, N, N)-CCC.

Now let us prove the second part that  $\mathfrak{C}$  is a QCSS. Consider  $\mathcal{C}^{(k_1,m_1)} \in \mathfrak{C}^{k_1}$  and  $\mathcal{C}^{(k_2,m_2)} \in \mathfrak{C}^{k_2}$ . Then, the ACF of  $\mathcal{C}^{(k_1,m_1)}$  and  $\mathcal{C}^{(k_2,m_2)}$  is given by

$$\sum_{s=0}^{N-1} \tilde{R}_{C_{s}^{(k_{1},m_{1})},C_{s}^{(k_{2},m_{2})}}(\tau)$$

$$= \sum_{s=0}^{N-1} \sum_{t=0}^{N-1-\tau} \omega_{N}^{f_{s}^{(k_{1},m_{1})}(t)} \cdot \omega_{N}^{-f_{s}^{(k_{2},m_{2})}(t+\tau)}$$

$$= \sum_{s=0}^{N-1} \sum_{t=0}^{N-1-\tau} \omega_{N}^{t(m_{1}-m_{2})-m_{2}\tau+s\left(\pi_{k_{2}}(t+\tau)-\pi_{k_{1}}(t)\right)}$$

$$= \sum_{t=0}^{N-1-\tau} \omega_{N}^{t(m_{1}-m_{2})-m_{2}\tau} \cdot \sum_{s=0}^{N-1} \omega_{N}^{s\left(\pi_{k_{2}}(t+\tau)-\pi_{k_{1}}(t)\right)}.$$
(41)

Recall that permutations  $\pi_{k_1}$  and  $\pi_{k_2}$  satisfy Lemma 4. Therefore,  $\pi_{k_1}(t) - \pi_{k_2}(t+\tau) \equiv 0$ (mod N) for any  $0 \leq t \leq t + \tau \leq N - 1$ ,  $k_1 \neq k_2$  has at most one solution. Hence, if there is no solution, then  $\sum_{s=0}^{N-1} \tilde{R}_{C_s^{(k_1,m_1)},C_s^{(k_2,m_2)}}(\tau) = 0$  due to  $\sum_{s=0}^{N-1} \omega_N^{s(\pi_{k_2}(t+\tau)-\pi_{k_1}(t))} = 0$ . If there is one solution, say t', then for  $0 \leq t' \leq t' + \tau \leq N - 1$  or in other words for  $0 \leq t' \leq N - 1 - \tau$ , we have

$$\sum_{s=0}^{N-1} \tilde{R}_{C_{s}^{(k_{1},m_{1})},C_{s}^{(k_{2},m_{2})}}(\tau)$$

$$= \omega_{N}^{-m_{2}\tau} \cdot [\omega_{N}^{(m_{1}-m_{2})\cdot t'} \cdot N + \sum_{\substack{0 \le t \le N-1 \\ t \ne t'}} \omega_{N}^{(m_{1}-m_{2})t} \sum_{\substack{0 \le s \le N-1}} \omega_{N}^{(\pi_{k_{2}}(t+\tau)-\pi_{k_{1}}(t))\cdot s}]$$

$$= \omega_{N}^{-m_{2}\tau+(m_{1}-m_{2})t'} \cdot N.$$

Therefore,  $|\sum_{s=0}^{N-1} \tilde{R}_{C_s^{(k_1,m_1)}, C_s^{(k_2,m_2)}}(\tau)| \le N$  for all  $k_1 \ne k_2, 0 \le \tau \le N-1$  and  $0 \le m_1, m_2 \le N-1$ .

Therefore, the theorem is proved.

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