

Title	A general method to predict the amplitude of oscillation in nearly-sinusoidal oscillators
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Publication date	2004-08
Original Citation	Maggio, G.M., de Feo O. and Kennedy, M.P. (2004) 'A general method to predict the amplitude of oscillaton in nearly-sinusoidal oscillators'. IEEE Transactions On Circuits and Systems I-Regular Papers, 51 (8):1586-1595.
Type of publication	Article (peer-reviewed)
Link to publisher's version	10.1109/TCSI.2004.832793
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# A General Method to Predict the Amplitude of Oscillation in Nearly Sinusoidal Oscillators

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**Abstract**—In this paper, a general methodology for predicting the amplitude of oscillation in nearly sinusoidal oscillators is presented. The method relies on the recently proposed projection technique for the computation of the center manifold and on the Hopf normal form theory to approximate the corresponding limit cycle in state space. The Colpitts oscillator is selected as a case study and, for this circuit, a closed-form expression for the amplitude of oscillation is derived as a function of the circuit parameters.

**Index Terms**—Center manifold theorem, Colpitts oscillator, Hopf normal form theorem, oscillation amplitude.

## I. INTRODUCTION

IN THE context of  $L$ - $C$  sinusoidal oscillators, the oscillation frequency is quite well determined by the selective network, while the amplitude of the oscillation is generally unknown *a priori* [1]. Linear analysis, including the Barkhausen criterion [1], typically fails to provide any useful information about the amplitude of oscillation. In fact, in contrast to other networks, nearly sinusoidal oscillators are nominally nearby to nonhyperbolic conditions at their equilibria [2]. Normally, a Hopf bifurcation [3] takes place at the oscillation condition, implying that a pair of complex conjugate eigenvalues have vanishing real parts. This means that, in general, the flow of the system near the equilibrium is *not* equivalent to its linearization as would be the case for hyperbolic systems, according to the Hartman–Grobman theorem [4]. In other words, oscillators do not usually possess the same qualitative behavior as their linearizations. Note that the fact that the Barkhausen criterion actually relies on the Hartman–Grobman theorem was already contained implicitly in [5] and then pointed out explicitly in [6].

In previous literature, the amplitude of oscillation in nearly sinusoidal oscillators has been approximated by means of Volterra series [7], averaging techniques [8], algebraic methods [9], the describing function [10], and the harmonic balance methods [11], [12]. Moreover, in [13], Buonomo *et al.* have derived simple asymptotic formulas for nearly sinusoidal oscil-

lations by using Poincaré perturbation theory. However, such methods rely on the assumption that the oscillator model is in Lur'e form [14], that is, it admits a single feedback loop with a scalar nonlinearity. On the other hand, the work by Mees *et al.* on the Hopf bifurcation theorem [15] is more general but it involves very complicated mathematical expressions. In addition, the method is not very practical since most oscillator models have a state-space dimension exceeding two. The rationale behind our approach is to exploit the projection technique proposed by Kuznetsov [16] to carry out a decomposition of the describing equations. Indeed, this approach applies in general to  $n$ -dimensional systems and leads to much simpler formula than [15].

In this study, we introduce a methodology for deriving an analytical approximation of the amplitude of oscillation in nearly sinusoidal oscillators [17]. We exploit the nonhyperbolicity of the oscillator to reduce the system to its center manifold representation [18], using the center manifold theorem [19] combined with the projection method [16]. The center manifold theorem permits to locate (in state space) the limit cycle associated with the oscillation in a small neighborhood of the oscillation condition. Its amplitude can then be determined by applying the Hopf bifurcation normal form theorem [15]. Our approach has the advantage of being analytical and therefore can provide, in closed form, the functional dependence of the amplitude of oscillation on the control parameters. Moreover, the approach is rather general, being applicable to any oscillator configuration exhibiting an Hopf bifurcation. The main limitation of the proposed method is that the predictions tend to be accurate only for parameters value relatively close to the oscillation condition. As a case study, we derive an approximate analytical expression for the amplitude of the output voltage waveform in a Colpitts oscillator [1] as a function of the circuit parameters. For this case, it is shown that predictions are reliable for all parameter values for which the distortion from nearly sinusoidal behavior is not significant (see also [20]).

The paper is organized as follows. Section II introduces the circuit model of the Colpitts oscillator. In Section III, the birth of the oscillation due to a Hopf bifurcation is described. Section IV deals with the approximation of the limit cycle in the state space by exploiting center manifold theory; the results are then transformed back in terms of the output signal amplitude. Finally, in Section V, the theoretical results are compared with the results from numerical and SPICE simulations.

## II. COLPITTS OSCILLATOR

We consider the Colpitts oscillator shown in Fig. 1(a), which contains a bipolar junction transistor (BJT) as the gain element

Manuscript received August 16, 2002; revised June 26, 2003. The work of G.M. Maggio was supported in part by the Army Research Office under Grant DAAG55-98-1-0269. The work of O. De Feo was supported by the Swiss National Science Foundation under Grant FN-2000-63789.00. This paper was recommended by Associate Editor C. P. Silva.

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Digital Object Identifier 10.1109/TCSI.2004.832793

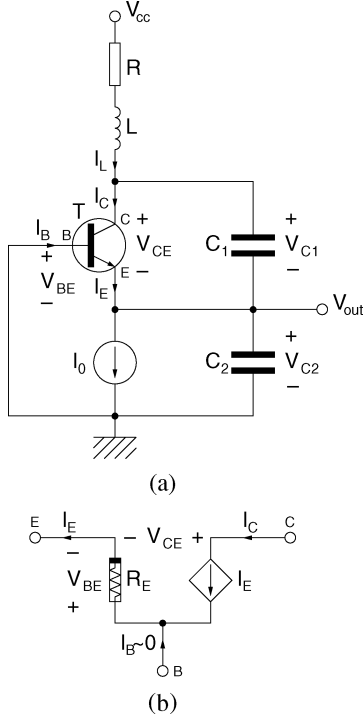


Fig. 1. Colpitts oscillator. (a) Circuit schematic. (b) Transistor model in CB configuration.

and a resonant network consisting of an inductor and a pair of capacitors. We model the transistor  $T$  by a (voltage-controlled) nonlinear resistor  $R_E$  and a linear current-controlled current source, neglecting the base current, as shown in Fig. 1(b). The latter assumption is equivalent to setting the common-base short-circuit forward current gain  $\alpha_F = 1$ . Note that parasitic capacitors  $C_{be}$  and  $C_{ce}$  add in parallel with  $C_2$  and  $C_1$ , respectively, so they are not included explicitly in our model.

#### A. State Equations

The state equations for the schematic in Fig. 1(a) are as follows:

$$\begin{cases} C_1 \frac{dV_{C1}}{dt} = -f(-V_{C2}) + I_L \\ C_2 \frac{dV_{C2}}{dt} = I_L - I_0 - G_0 V_{C2} \\ L \frac{dI_L}{dt} = -V_{C1} - V_{C2} - R I_L + V_{cc} \end{cases} \quad (1)$$

where  $f(\cdot)$  is the driving-point characteristic of the nonlinear resistor. This characteristic can be expressed in the form  $I_E = f(V_{BE}) = f(-V_{C2})$  which we approximate with an exponential function [1], namely

$$I_E \simeq I_S \exp\left(\frac{V_{BE}}{V_T}\right)$$

where  $I_S$  is the B-E junction saturation current and  $V_T \simeq 26$  mV (room temperature).

#### B. Normalization and Parameters

We introduce a set of dimensionless state variables  $(x_1, x_2, x_3)$  by choosing the equilibrium point of system

(1) to be the origin of the new coordinate system and by normalizing voltages, currents, and time w.r.t.  $V_{ref} = V_T$ ,  $I_{ref} = I_0$ , and  $t_{ref} = 1/\omega_0$ , respectively, where  $\omega_0 = 1/\sqrt{L(C_1 C_2 / (C_1 + C_2))}$ , i.e., the resonant frequency of the unloaded tank circuit. With this transformation, the state equation(1) of the Colpitts oscillator can be rewritten as [20]

$$\begin{cases} \dot{x}_1 = \frac{g^*}{Q(1-k)} [-n(x_2) + x_3] \\ \dot{x}_2 = \frac{g^*}{Qk} x_3 \\ \dot{x}_3 = -\frac{Qk(1-k)}{g^*} [x_1 + x_2] - \frac{1}{Q} x_3 \end{cases} \quad (2)$$

where

$$n(x_2) = e^{-x_2} - 1 \quad \text{and} \quad k = C_2 / (C_1 + C_2).$$

System (2) depends only on the two parameters:

- $g^*$ , the “loop gain” of the oscillator;
- $Q = \omega_0 L / R$ , the quality factor of the (unloaded) tank circuit.

The parameter  $k$  has only a scaling effect on the variables; it has no influence on the dynamics. Note that  $g^*$  represents the value of the loop gain for which the phase condition of the Barkhausen criterion [1] is satisfied (and  $\alpha_F = 1$ ). In particular, the circuit will oscillate only if the start-up condition  $g^* > 1$  is fulfilled.

### III. BIRTH OF OSCILLATION

The birth of oscillation in a sinusoidal oscillator is typically associated with a supercritical Hopf bifurcation. At this bifurcation, a pair of complex and conjugate eigenvalues associated with an equilibrium point crosses the imaginary axis, the equilibrium loses its stability, and a stable limit cycle appears around it [21].

#### A. Hopf Bifurcation

Here, we recall briefly the Andronov–Hopf bifurcation theorem [21]. Let us consider a one-parameter family of ordinary differential equations in the form

$$\dot{x} = f(x; \mu), \quad x \in \mathbb{R}^n; \quad \mu \in \mathbb{R}^1 \quad (3)$$

where  $f$  is a smooth function representing the vector field,  $x$  is the state vector, and  $\mu$  is the parameter. At the parameter value  $\mu = \mu^H$ , an equilibrium point  $x^H$  of system (3)

$$f(x^H; \mu^H) = 0$$

undergoes a (nondegenerate) supercritical Hopf bifurcation when the following conditions are satisfied [21]

- (H.1) At  $\mu = \mu^H$ , a simple pair of complex conjugate eigenvalues  $\lambda(\mu) = \sigma(\mu) + j\omega(\mu)$  of the linearized vector field at the equilibrium  $x^H$  crosses the imaginary axis with respect to the parameter  $\mu$ .
- (H.2) Nonzero “speed” crossing:  $(\partial\sigma(\mu)/\partial\mu)|_{\mu=\mu^H} \neq 0$ .
- (H.3) The first Lyapunov coefficient is negative:  $l_1 < 0$ .

As will be shown later, the first Lyapunov coefficient  $l_1$  is a smooth function of the coefficients, up to the third order, of the multivariable Taylor expansion of the vector field in the neighborhood of the equilibrium  $x^H$ .

In the case of the Colpitts oscillator, the vector field is given by the right-hand side of system (2), thus  $n = 3$ . The equilibrium  $x^H$  undergoing the Hopf bifurcation coincides with the origin  $O$ , and the bifurcation parameter corresponds to  $g^*$ . The Jacobian of the vector field (2) evaluated at the equilibrium point  $O$  turns out to be

$$A = \frac{\partial F}{\partial x} \Big|_{x=O} = \begin{bmatrix} 0 & \frac{g^*}{Q(1-k)} & \frac{g^*}{Q(1-k)} \\ 0 & 0 & \frac{g^*}{Qk} \\ -\frac{Qk(1-k)}{g^*} & -\frac{Qk(1-k)}{g^*} & -\frac{1}{Q} \end{bmatrix}$$

whose characteristic equation  $\det(A - \lambda I) = 0$  is

$$\lambda^3 + \frac{1}{Q}\lambda^2 + \lambda + \frac{g^*}{Q} = 0. \quad (4)$$

By imposing that  $\lambda_{1,2} = \pm j\omega_H$ , we obtain  $\{g^* = 1, \omega_H = 1\}$ , confirming that for  $g^* = 1$  the equilibrium point  $O$  is characterized by a pair of purely imaginary eigenvalues with unitary angular frequency (due to the normalization). Substituting  $\lambda_{1,2} = \pm j$  into (4), we determine the real eigenvalue to be  $\lambda_R = -1/Q$ .

The calculations above show that, for our model of the Colpitts oscillator, condition (H.1) is satisfied at  $g^* = 1$ . Also, it can easily be shown that the nondegeneracy condition (H.2) is always satisfied [cf. (11)]. In order to verify condition (H.3), i.e., that the Hopf bifurcation is supercritical, we compute the first Lyapunov coefficient  $l_1$  according to the projection method described in [21], i.e.,

$$l_1 = \frac{1}{2\omega_H} \text{Re} \left[ \langle p, C(q, q, \bar{q}) \rangle - 2\langle p, B(q, A^{-1}B(q, \bar{q})) \rangle + \langle p, B(\bar{q}, (2j\omega_H I - A)^{-1}B(q, q)) \rangle \right]$$

where  $\langle \cdot, \cdot \rangle$  is the standard scalar product in  $\mathbb{C}^2$ .  $q$  and  $p$  are the right and left eigenvectors associated to the purely imaginary eigenvalue at the Hopf bifurcation, namely,

$$Aq = j\omega_H q \quad A^T p = -j\omega_H p$$

satisfying the normalization  $\langle p, q \rangle = 1$ . The bilinear and trilinear vector functions  $B(x, y)$  and  $C(x, y, z)$  are the second- and third-order coefficients of the multivariable Taylor expansion of the vector field, evaluated in a neighborhood of the equilibrium  $x^H$  undergoing the Hopf bifurcation at the bifurcation parameter value  $\mu = \mu^H$ , namely,

$$B_i(x, y) = \sum_{j,k=1}^n \frac{\partial^2 f_i(\xi, \mu^H)}{\partial \xi_j \partial \xi_k} \Big|_{\xi=x^H} x_j y_k$$

$$C_i(x, y, z) = \sum_{j,k,l=1}^n \frac{\partial^3 f_i(\xi, \mu^H)}{\partial \xi_j \partial \xi_k \partial \xi_l} \Big|_{\xi=x^H} x_j y_k z_l$$

for  $i = 1, 2, \dots, n$ .

For the Colpitts oscillator, it follows that

$$B(x, y) = \left[ -\frac{x_2 y_2}{(1-k)Q}, 0, 0 \right]$$

$$C(x, y, z) = \left[ \frac{x_2 y_2 z_2}{(1-k)Q}, 0, 0 \right]$$

$$p = \left[ \frac{Q(1-k)(Q-j)}{2(1+Q^2)}, \frac{(Q(1-k)+j)(Q-j)}{2(1+Q^2)}, \frac{1+jQ}{2k(1+Q^2)} \right]$$

$$q = \left[ \frac{kQ-j}{(1-k)Q}, 1, jkQ \right].$$

Thus,

$$l_1(Q) = -\frac{Q^3}{(1+Q^2)(1+4Q^2)}. \quad (5)$$

Therefore, it follows that for our model of the Colpitts oscillator  $l_1(Q) < 0$ ,  $\forall Q > 0$ , confirming that the condition (H.3) is always satisfied. Hence, at  $g^* = 1$ , in agreement with the Barkhausen criterion, a stable limit cycle appears through a Hopf bifurcation.

#### IV. LIMIT CYCLE APPROXIMATION

This section deals with the approximation in the state space of the limit cycle which is born from the Hopf bifurcation. We exploit center manifold theory [21] and Hopf normal form theory [15].

##### A. Center Manifold Approximation

Bifurcation analysis of equilibria in multidimensional systems reduces to that for the equations restricted to the so-called *center manifold*  $W_\mu^c$  [21]. Note also that, for all codim-1 bifurcations of equilibria, it is sufficient, in general, to obtain the restricted equations up to third-order terms only.

1) *Projection Method for Center Manifold Approximation*: The projection method for the center manifold computation avoids the transformation of the system into its eigenbasis [16]. Only eigenvectors corresponding to the critical eigenvalues of the Jacobian matrix  $A$ , evaluated at the equilibrium point  $x^*$ , and its transpose  $A^T$  are used to “project” the system into the critical eigenspace and its complement. In the case of a Hopf bifurcation, the critical real eigenspace  $T^c$  corresponding to the imaginary eigenvalues is two-dimensional and is spanned by  $\{\text{Re}[q], \text{Im}[q]\}$  where, as in the previous section,  $q$  is the eigenvector of the critical eigenvalue  $\lambda_1 = j\omega_H$ . In general, the eigenspace  $T^{\text{su}}$  corresponding to the remaining eigenvalues<sup>1</sup> of  $A$  has dimension  $(n-2)$ . For our particular case,  $T^{\text{su}}$  is a one-dimensional real eigenspace. The situation is illustrated schematically in Fig. 2. It follows that one can decompose any  $x \in W_\mu^c$  as

$$x = zq + \bar{z}\bar{q} + y \quad (6)$$

where  $z \in \mathbb{C}$  is a coordinate on  $T^c$ , indeed  $zq + \bar{z}\bar{q} \in T^c$ , while  $y \in T^{\text{su}}$  describes how the center manifold  $W_\mu^c$  develops off

<sup>1</sup>Note that the superscript su means stable-unstable while the superscript c stands for center.

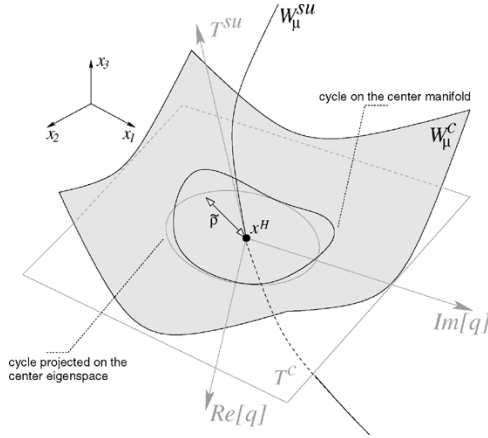


Fig. 2. Sketch of the center manifold  $W^c$  at a Hopf bifurcation.

the critical eigenspace  $T^c$ , i.e., is the component of  $W_\mu^c$  in the  $T^{su}$  direction.

In summary, the behavior of system (3) restricted to the center manifold in the neighborhood of the Hopf bifurcation (up to the quadratic terms) is given, for sufficiently small  $|\mu - \mu^H|$ , by [21]

$$\dot{w} = \lambda(\mu)w + c_1(\mu)w^2\bar{w} + O(|w|^4) \quad (7)$$

where  $\lambda(\mu)$  is the bifurcating eigenvalue. Equation (7) contains only one cubic term  $w^2\bar{w}$ , called the *resonant* term. The coefficient  $c_1(\mu)$  of the resonant term is given by

$$c_1(\mu) = \frac{g_{20}g_{11}(2\lambda + \bar{\lambda})}{2|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{g_{21}}{2} \quad (8)$$

with the  $g_{ij}$  coefficients

$$\begin{aligned} g_{20} &= \langle p, B(q, q) \rangle \\ g_{11} &= \langle p, B(q, \bar{q}) \rangle \\ g_{02} &= \langle p, B(\bar{q}, \bar{q}) \rangle \\ g_{21} &= \langle p, C(q, q, \bar{q}) \rangle - 2\langle p, B(q, A^{-1}B(q, \bar{q})) \rangle \\ &\quad + \langle p, B(\bar{q}, (2j\omega_H I - A)^{-1}B(q, q)) \rangle \\ &\quad + \frac{1}{j\omega_H} \langle p, B(q, q) \rangle \langle p, B(q, \bar{q}) \rangle \\ &\quad - \frac{2}{j\omega_H} |\langle p, B(q, \bar{q}) \rangle|^2 - \frac{1}{3j\omega_H} |\langle p, B(\bar{q}, \bar{q}) \rangle|^2. \end{aligned}$$

Note that, exactly at the Hopf bifurcation ( $\mu = \mu^H$ ), the following relationship holds:

$$l_1 = \frac{\text{Re}[c_1(\mu_H)]}{\omega_H} \quad (9)$$

### B. Linear Approximation

It is proposed here to approximate the limit cycle by a linearization of the center manifold. To this end, one should expand up to the linear term (first order approximation) the dependence of the eigenvalues  $\lambda(\mu)$  and eigenvectors  $p(\mu)$  and  $q(\mu)$  upon the sensitive parameters, in this case  $g^*$ . However, to simplify the computation, we assume that, in a neighborhood

of the Hopf bifurcation, only the bifurcating eigenvalues depend upon the parameters while their left and right eigenvectors remain approximately fixed. This hypothesis greatly simplifies the calculations and yet gives, as will be shown later, a good approximation. This is the simplest possible parameter dependent restriction of the system to the center manifold.

The linear approximation consists of dropping all the nonlinear terms in the expression of  $y$  and  $z$  (see also [21]). Namely, we make the following approximations.

- (A.1) In a neighborhood of the Hopf bifurcation, the limit cycle born from it lies on the center (critical) eigenspace  $T^c$ , i.e.,

$$y = 0.$$

- (A.2) In a neighborhood of the Hopf bifurcation, the amplitude  $|z|$  of the emerging limit cycle is sufficiently small that the nonlinear terms may be neglected, i.e.,

$$z = w.$$

Then, from (6), it follows that

$$x = zq + \bar{z}\bar{q}. \quad (10)$$

In order to combine (10) with (7) to uniquely determine the dependence of the limit cycle upon the sensitive parameter, it is necessary to compute the parameter dependent functions  $\lambda(\mu)$  and  $c_1(\mu)$ , at least up to their linear approximation.

We now proceed, assuming that:

- (A.3)  $\lambda(\mu)$  depends upon the parameter  $\mu$ , while we keep all the other quantities  $q$ ,  $p$ , and  $g_{ij}$  evaluated at the Hopf bifurcation ( $\mu = \mu^H$ ).

*1) Eigenvalues in the Neighborhood of the Hopf:* In order to compute the simplest parameter-dependent center manifold normal form, as given above, we need to express the dependence of the eigenvalues upon the bifurcation parameter,  $g^*$ , in a neighborhood of the Hopf bifurcation. Since in general the eigenvalues do not admit a simple closed form, we need to express these dependencies in an approximate way. For small deviations from the value at which the Hopf bifurcation occurs, we can express the dependencies up to first order and so obtain a simpler closed form. In particular, we make the following approximation:

- (A.4) The eigenvalues  $\lambda(\mu)$  in a neighborhood of the Hopf bifurcation are calculated as a linear approximation with respect to  $\mu$ .

To this aim, it is necessary to express the Jacobian and its characteristic polynomial in a neighborhood of the Hopf bifurcation, i.e., for small deviations from the Hopf:  $g^* = 1 + dg^*$ . The Jacobian and its characteristic polynomial in the neighborhood of the Hopf are given, respectively, by

$$\begin{aligned} A(g^*) &= A(1 + dg^*) = A(dg^*) \\ &= \begin{bmatrix} 0 & \frac{1+dg^*}{Q(1-k)} & \frac{1+dg^*}{Q(1-k)} \\ 0 & 0 & \frac{1+dg^*}{Q^k} \\ -\frac{Qk(1-k)}{1+dg^*} & -\frac{Qk(1-k)}{1+dg^*} & -\frac{1}{Q} \end{bmatrix} \\ \lambda^3 + \frac{1}{Q}\lambda^2 + \lambda + \frac{1+dg^*}{Q} &= 0. \end{aligned}$$

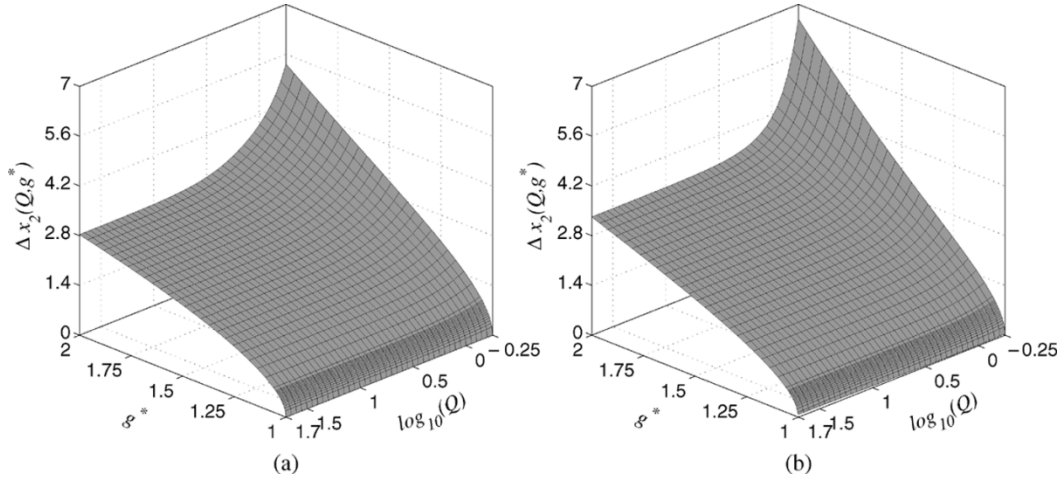


Fig. 3. Dependence of the output amplitude  $\Delta x_2 = (\max(x_2) - \min(x_2))/2$  upon the parameters  $g^*$  and  $Q$ : (a) theoretical, using the linear approximation of the center manifold projection given by (14), and (b) obtained by simulation of the mathematical model (2).

Now, let's compute the variation of the characteristic polynomial in such a way to determine the local dependencies of the eigenvalues upon the variation  $dg^*$ . The characteristic polynomial must satisfy the total differentiation rule

$$3\lambda(g^*)^2 \frac{\partial \lambda(g^*)}{\partial g^*} + \frac{2\lambda(g^*)}{Q} \frac{\partial \lambda(g^*)}{\partial g^*} + \frac{\partial \lambda(g^*)}{\partial g^*} + \frac{1}{Q} = 0.$$

Thus, the derivative of the eigenvalues with respect to the parameter  $g^*$  can be obtained as

$$\frac{\partial \lambda(g^*)}{\partial g^*} = -\frac{1}{3Q\lambda(g^*)^2 + 2\lambda(g^*) + Q}.$$

Therefore, the eigenvalues in a neighborhood of the Hopf bifurcation are given by

$$\begin{cases} \lambda_{1,2} = \frac{Q(g^*-1)}{2(1+Q^2)} \pm j \left[ 1 + \frac{(g^*-1)}{2(1+Q^2)} \right] \\ \lambda_R = -\frac{1+g^*Q}{Q(1+Q^2)}. \end{cases} \quad (11)$$

Consequently, one can compute the coefficient of the resonant term  $c_1 = c_1(g^*)$  according to (8), whose cumbersome expression is reported in the Appendix as (16).

### C. Restriction to the Center Manifold

Substituting  $w = \rho e^{j\varphi}$  into the Hopf normal form equation restricted to the center manifold (7), and separating the real and imaginary parts, we obtain

$$\begin{cases} \dot{\rho} = \rho (\operatorname{Re}[\lambda(\mu)] + \operatorname{Re}[c_1(\mu)]\rho^2) \\ \dot{\varphi} = \operatorname{Im}[\lambda(\mu)] + \operatorname{Im}[c_1(\mu)]\rho^2. \end{cases}$$

Hence, one can impose  $\dot{\rho} = 0$  and compute the corresponding radius  $\tilde{\rho}$  of the limit cycle in the variable  $w$ , namely

$$\tilde{\rho} = \sqrt{-\frac{\operatorname{Re}[\lambda]}{\operatorname{Re}[c_1]}}.$$

On the other hand, from the imaginary part, we obtain the frequency as

$$\tilde{\omega} = \dot{\varphi} = \operatorname{Im}[\lambda_C] + \tilde{\rho}^2 \operatorname{Im}[c_1].$$

The explicit expressions of the above formulas, of both the amplitude and frequency, are quite involved and, for the sake of clarity, are reported in the Appendix. Nonetheless, it should be noted that, as far as the (normalized) frequency is concerned, as shown in the Appendix, it deviates significantly from 1 only for very small values of  $Q$ , where the approximation of the model is not valid. In fact, it is shown in [22] that, if the common-base current gain of the transistor  $\alpha_F \neq 1$ , the system admits a degenerate Hopf, where  $l_1 = 0$ , for small  $Q$ .

### D. Projection of the Restriction Onto the Real Coordinates

Using (10), it is now possible to use the approximation above, namely,  $\tilde{\omega} \simeq 1$ ,  $\forall g^*, Q$ , to project the solution

$$w(t) = \tilde{\rho} e^{j\tilde{\omega}t} \simeq \tilde{\rho} e^{jt}$$

onto the original coordinates  $x$ . Hence, the solution corresponding to the limit cycle born from the Hopf bifurcation can be written as

$$x_{cyc}(t) = \left[ 2\frac{\tilde{\rho}(kQ \cos(t) + \sin(t))}{(1-k)Q}, 2\tilde{\rho} \cos(t), -2kQ\tilde{\rho} \sin(t) \right]. \quad (12)$$

The amplitude of  $x_2$  (proportional to  $V_{C_2}$ , which is the usual output of the Colpitts oscillator) is then  $A_{x_2} = 2\tilde{\rho}$ .

### E. A Simplified Formula

It is shown in the Appendix that, for reasonably high values of  $Q (> 3)$ ,  $c_1(g^*)$  is a very slowly varying function of  $g^*$ . Hence, we can make a further approximation, namely:

$$(A.5) \quad \operatorname{Re}[c_1(\mu)] \simeq \operatorname{Re}[c_1(\mu_H)].$$

Thus, from (9), taking into account that  $\omega_H = 1$ , we obtain the simplified formula

$$\tilde{\rho} \simeq \sqrt{-\frac{\operatorname{Re}[\lambda(\mu)]}{l_1}}.$$

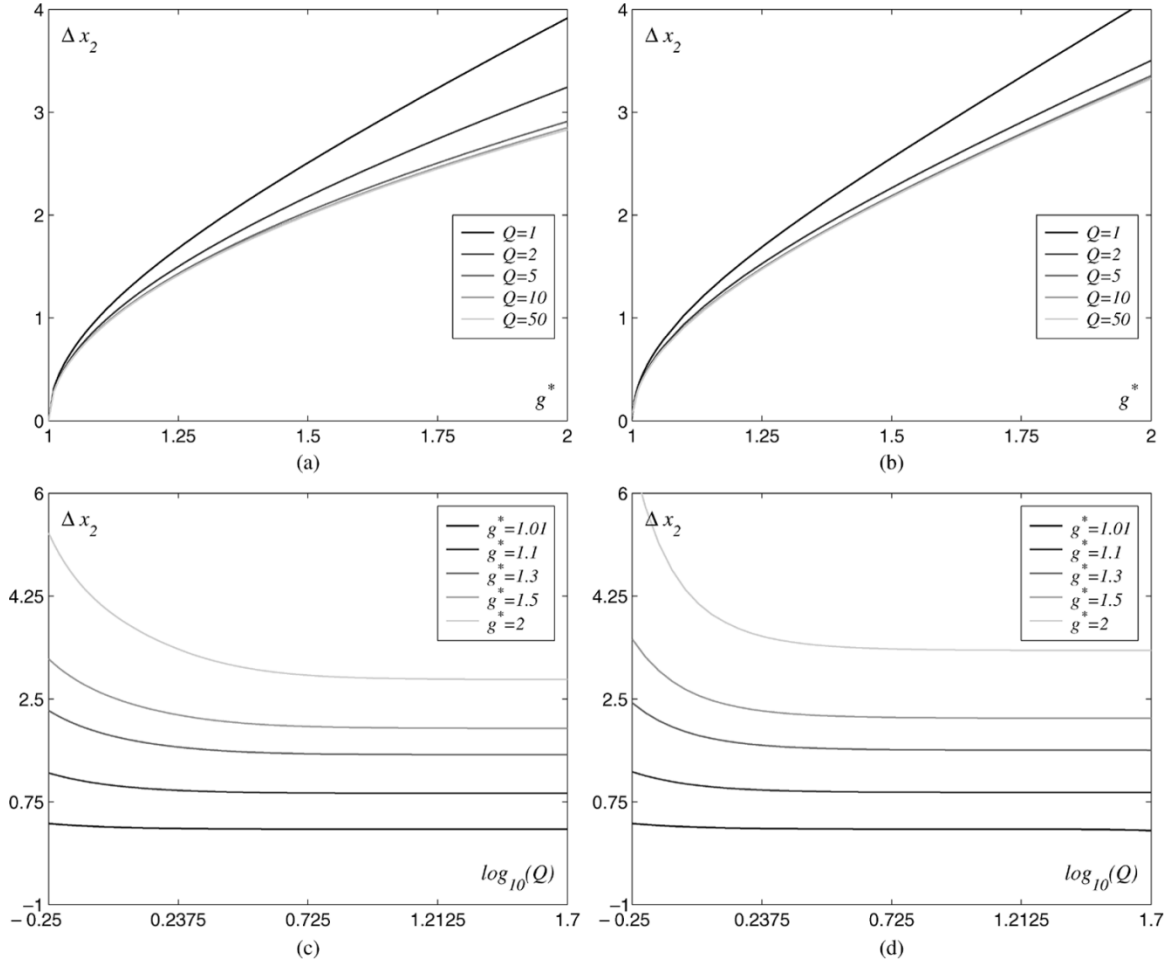


Fig. 4. Dependence of the output amplitude  $\Delta x_2 = (\max(x_2) - \min(x_2))/2$  upon the parameter  $g^*$  for five significant values of  $Q$ : (a) theoretical, using the linear approximation of the center manifold projection given by (14), (b) obtained by simulation of the mathematical model (2). Dependence of  $\Delta x_2$  on the parameter  $Q$  for five significant values of  $g^*$ , (c) theoretical, using the linear approximation of the center manifold projection given by (14), and (d) obtained by simulation of the mathematical model (2).

Then, the combination of (5) and (11) results in

$$\tilde{\rho} = \frac{\sqrt{2}\sqrt{(g^* - 1)(1 + 4 * Q^2)}}{2Q}. \quad (13)$$

Finally, combining (12) with (13), we obtain

$$x_2(t) = \frac{\sqrt{2}\sqrt{(g^* - 1)(1 + 4 * Q^2)}}{Q} \cos(\omega_0 t). \quad (14)$$

Denormalizing, a very compact closed-form formula of the amplitude of the oscillation can be obtained as

$$V_{out}(t) = V_T \frac{\sqrt{2}\sqrt{(g^* - 1)(1 + 4 * Q^2)}}{Q} \cos(\omega_0 t) - V_T \ln\left(\frac{I_0}{I_S}\right) \quad (15)$$

which, as will be shown in Section V, turns out to be a good quantitative approximation, despite our simplifying approximations.

## V. SIMULATIONS VERSUS THEORETICAL PREDICTIONS

Fig. 3 shows the numerical results for the dependence of the output amplitude  $\Delta x_2 = (\max(x_2) - \min(x_2))/2$  upon the

parameters  $g^*$  and  $Q$ , versus the corresponding theoretical prediction given by (14). The comparison is carried out in more detail in Fig. 4, showing the dependence of the output amplitude when one of the parameters is fixed and the other is varied, while Fig. 5 reports a direct comparison of the amplitude predicted by the linear approximation of the center manifold projection with the amplitude obtained by simulation of the mathematical model (2). In all cases, the theoretical predictions are in good agreement with the simulation results.

Fig. 6 shows the results from SPICE simulations for the dependence of the output amplitude  $\Delta V_{out} = (\max(V_{out}) - \min(V_{out}))/2$  upon the parameters  $g^*$  and  $Q$ , versus the corresponding theoretical prediction given by (15). Fig. 7 reports a direct comparison of the amplitude predicted by the linear approximation of the center manifold projection [see (15)] with the amplitude obtained by SPICE simulations. Overall, the theoretical predictions are in good agreement with the SPICE simulation results. The little discrepancies visible in Figs. 6 and 7, close to  $g^* = 1$ , are due to the fact that the mathematical model (2) does not correspond exactly to the circuit since the transistor model used in the SPICE simulations has a nominal value of the common-base

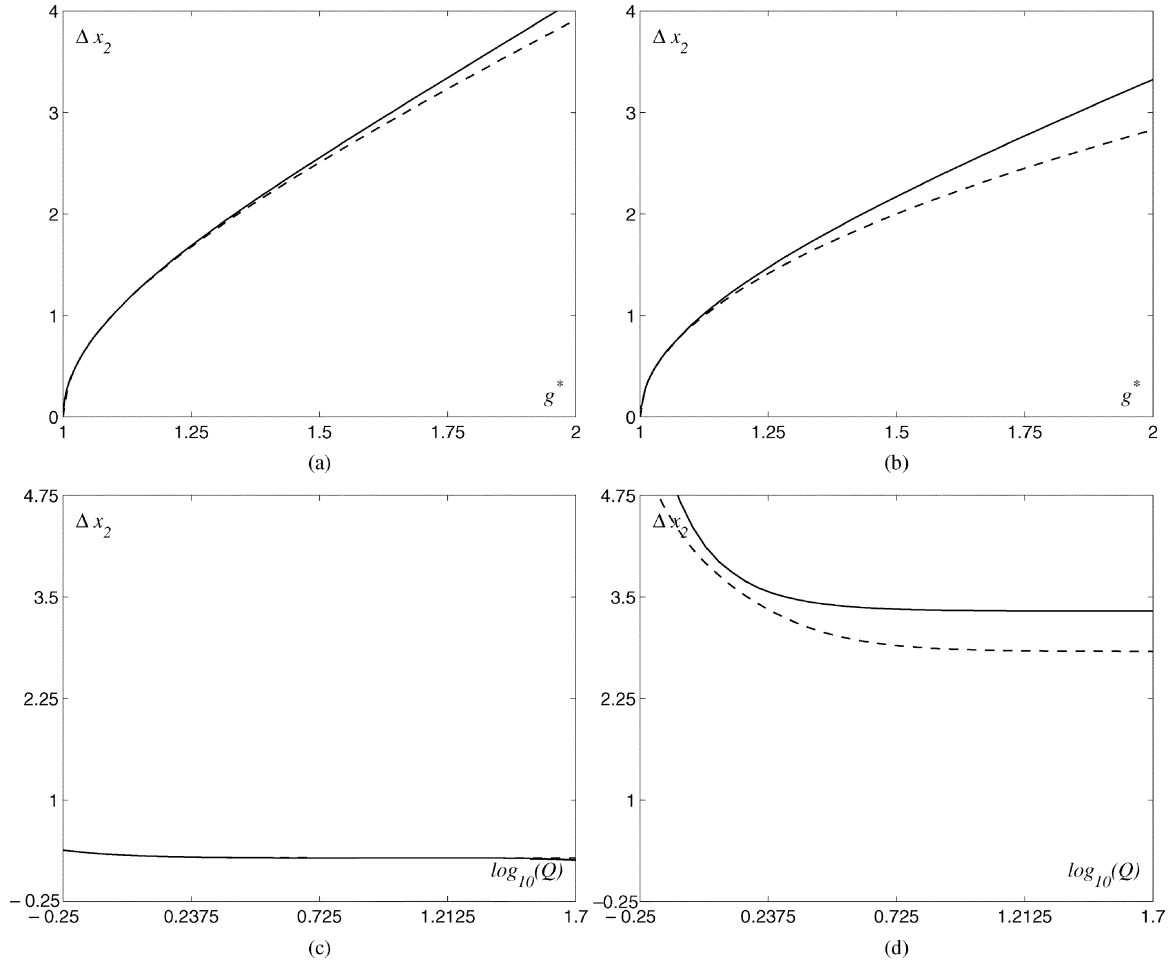


Fig. 5. Direct comparison of the amplitude predicted by the linear approximation of the center manifold projection given by (14) (dashed curve) with the amplitude obtained by simulation (solid curve) of the mathematical model (2): (a) dependence upon  $g^*$  for a small value of  $Q$  ( $Q = 1$ ), (b) dependence upon  $g^*$  for a large value of  $Q$  ( $Q = 50$ ), (c) dependence upon  $Q$  for a small value of  $g^*$  ( $g^* = 1.01$ ), and (d) dependence upon  $Q$  for a large value of  $g^*$  ( $g^* = 2$ ).

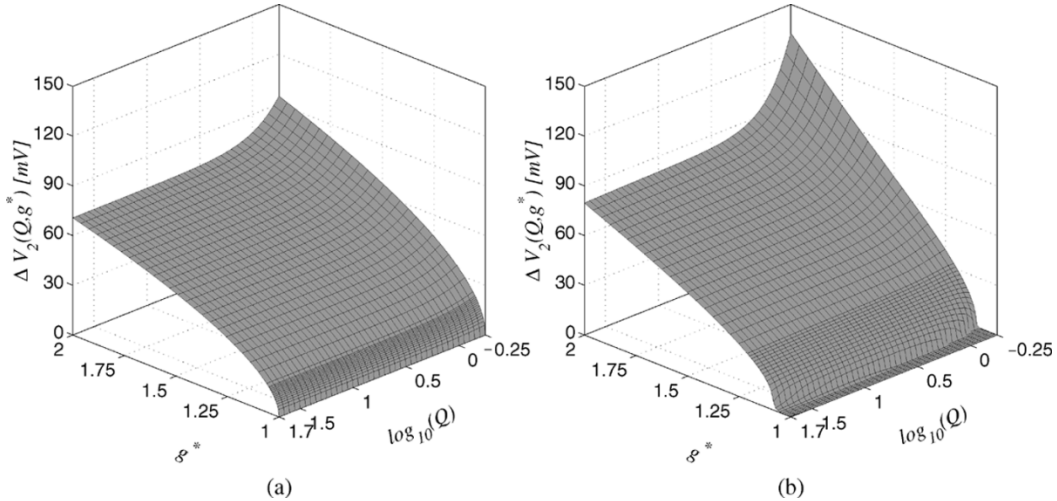


Fig. 6. Dependence of the output amplitude  $\Delta V_{\text{out}} = (\max(V_{\text{out}}) - \min(V_{\text{out}}))/2$  upon the parameters  $g^*$  and  $Q$ : (a) theoretical, using the linear approximation of the center manifold projection given by (15), and (b) obtained by SPICE simulation. Note that in (b) the oscillation starts for values of  $g^*$  slightly larger than unity, as for the SPICE model  $\alpha_F < 1$ .

short-circuit forward current gain  $\alpha_F = 0.996 < 1^2$ . Consequently, the required value of  $g^*$  to start up the oscillation is slightly larger than unity.

<sup>2</sup>The nominal values of the components used in the SPICE simulations are reported in the Appendix.

## VI. CONCLUSION

In this study, we have presented a general methodology for predicting the amplitude of oscillation in nearly sinusoidal oscillators. The method used is based upon a linear approximation of the center manifold close to the Hopf bifurcation re-



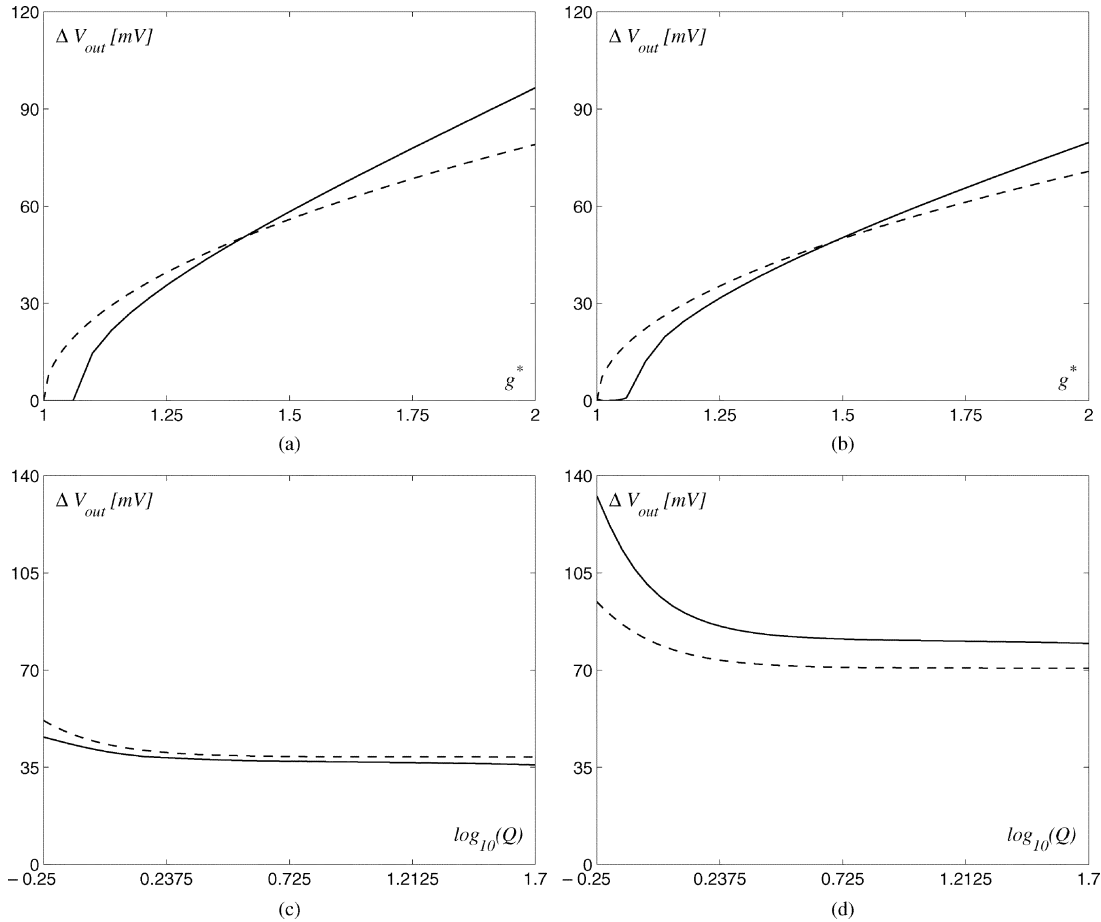


Fig. 7. Direct comparison of the output amplitude predicted by the linear approximation of the center manifold projection given by (15) (dashed curve) with the amplitude obtained by SPICE simulations (solid curve): (a) dependence upon  $g^*$  for a small value of  $Q$  ( $Q = 1$ ), (b) dependence upon  $g^*$  for a large value of  $Q$  ( $Q = 50$ ), (c) dependence upon  $Q$  for a small value of  $g^*$  ( $g^* = 1.3$ ), and (d) dependence upon  $Q$  for a large value of  $g^*$  ( $g^* = 2$ ). Again, note that in (a) and (b) the oscillation starts for values of  $g^*$  slightly larger than unity, as for the SPICE model  $\alpha_F < 1$ .

sponsible for the birth of the oscillation. The Colpitts oscillator has been selected as a case study. For this circuit, we have derived a simple closed-form expression for the amplitude of oscillation, as a function of the parameters. Although several approximations have been made, our theoretical predictions are in good qualitative and quantitative agreement with both the simulations of the mathematical model of the Colpitts oscillator and the SPICE simulations. The simplified model of the oscillation amplitude may find application for design purposes.

#### APPENDIX I

##### EXPRESSION OF THE RESONANT TERM

Substituting (11) into (8) leads to the following expression of the resonant term,  $c_1$ :

$$\begin{aligned}
 c_1(g^*, Q) &= \frac{(2Q(1-Q^2) + j(2Q^2 - 1))(2Q + j)(Q + j)}{4(1 + 4Q^2)(1 + Q^2)^2} \\
 &\quad - \frac{((11 - 5g^*)Q^2 + 3g^* + 3 + j2Q(Q^2 + 1))(Q + j)}{4(1 + Q^2)^2(4Q^2 + g^{*2} + 2g^* + 1)} \\
 &\quad + \frac{Qg^* - Q - j3(2Q^2 + g^* + 1)}{144Q^4 + 4Q^2g^{*2} + 136Q^2g^* + 148Q^2 + 36g^{*2} + 72g^* + 36}.
 \end{aligned} \quad (16)$$

#### APPENDIX II

##### AMPLITUDE

Substituting the above expression of  $c_1(g^*, Q)$ , together with the expression (11) of the critical eigenvalue, in the expression of  $\tilde{\rho}$  leads to (17), shown at the bottom of the next page.

Fig. 8 reports the percentage error, as a function of the parameters  $g^*$  and  $Q$ , due to the approximation introduced by using the simplified formula (13) for  $\tilde{\rho}$ , rather than the correct formula (17). As can be noted, for  $Q > 3$ , there is basically no difference between the two predictions.

#### APPENDIX III

##### FREQUENCY

Substituting the expression for  $c_1(g^*, Q)$ , the expression (11) for the critical eigenvalue, and the above expression (17) for  $\tilde{\rho}$ , into the equation for  $\tilde{\omega}$  leads to (18), shown at the bottom of the next page.

Fig. 9 shows a plot of the predicted frequency  $\tilde{\omega}$  as a function of the parameters  $g^*$  and  $Q$ . As can be seen,  $\tilde{\omega}$  deviates significantly from 1 only for very small values of  $Q$ .

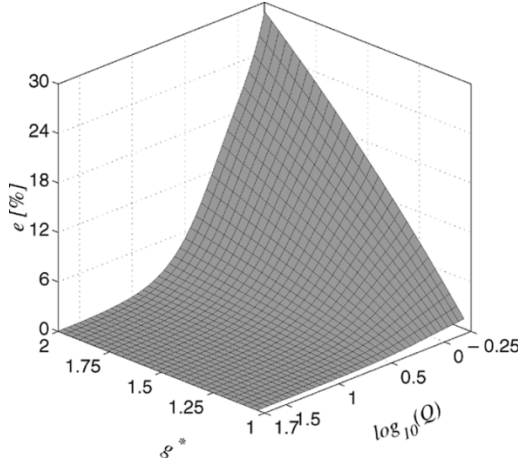


Fig. 8. Percentage error, as a function of the parameters  $g^*$  and  $Q$ , in using the amplitude  $\tilde{\rho}$  predicted by the simplified formula (13), rather than the correct formula (17).

#### APPENDIX IV

##### SPIKE SIMULATIONS—NOMINAL VALUES

The nominal values of the fixed components, for the schematic of the Colpitts oscillator shown in Fig. 1(a), used in the SPICE simulations are

$$\begin{aligned} L &= 18.2 \text{ mH} \\ C_1 &= 980 \text{ nF} \\ C_2 &= 980 \text{ nF} \end{aligned}$$

and the model used for the BJT is the standard 2N2222 transistor model, which has the nominal value of the common-base short-circuit forward current gain  $\alpha_F = 0.996$ . Finally, the values of

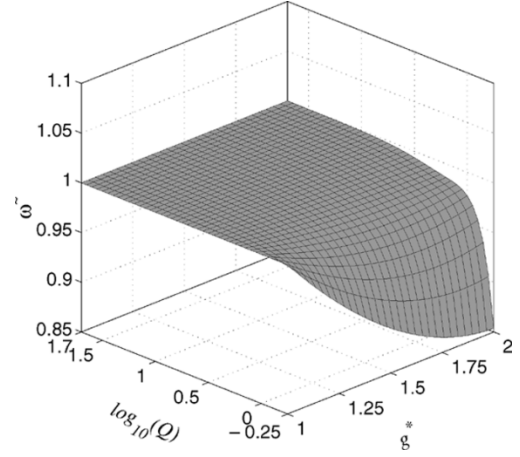


Fig. 9. Frequency  $\tilde{\omega}$  predicted by the formula (18) as a function of the parameters  $g^*$  and  $Q$ .

$R$  and  $I_0$  are varied with  $g^*$  and  $Q$  according to the following formula [20]:

$$\begin{cases} R = \frac{\omega_0 L}{Q} \\ I_0 = \frac{V_T g^* \omega_0 (C_1 + C_2)}{Q} \end{cases}$$

#### ACKNOWLEDGMENT

The authors would like to thank Prof. Y. A. Kuznetsov for the help in dealing with the projection technique and for stimulating discussions.

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$$\begin{aligned} \tilde{\rho}(g^*, Q) &= \sqrt{2} \sqrt{1 + 4Q^2} \sqrt{36Q^4 + Q^2 g^{*2} + 34Q^2 g^* + 37Q^2 + 9g^{*2} + 18g^* + 9} \\ &\cdot \frac{\sqrt{Q^2 g^{*2} + 2Q^2 g^* + 4Q^4 + 5Q^2 + g^{*2} + 2g^* + 1} \sqrt{g^* - 1}}{\sqrt{576Q^{10} + (160g^{*2} + 96g^* + 2048)Q^8 + (4g^{*4} + 120g^{*3} - 80g^{*2} + 1424g^* + 1988)Q^6}} \\ &\cdot \frac{+(36g^{*4} - 38g^{*3} + 382g^{*2} + 1198g^* + 726)Q^4 + (-g^{*4} + 24g^{*3} + 198g^{*2} + 264g^* + 91)Q^2}{-9g^{*4} - 10g^{*3} + 8g^{*2} + 10g^* + 1} \end{aligned} \quad (17)$$

$$\begin{aligned} \tilde{\omega}(g^*, Q) &= \frac{\left( (Q^4 + 9Q^2)g^{*5} + (4Q^6 + 63Q^4 + 16Q^2 - 11)g^{*4} + (144Q^6 - 22Q^2 - 22)g^{*3} \right. \\ &\quad \left. + (160Q^8 - 88Q^6 + 168Q^4 + 96Q^2)g^{*2} + (48Q^8 + 1176Q^6 + 1111Q^4 + 301Q^2 + 22)g^* \right. \\ &\quad \left. + 576Q^{10} + 2096Q^8 + 2220Q^6 + 961Q^4 + 176Q^2 + 11 \right)}{\left( (4Q^6 + 36Q^4 - Q^2 - 9)g^{*4} + (120Q^6 - 38Q^4 + 24Q^2 - 10)g^{*3} \right. \\ &\quad \left. + (160Q^8 - 80Q^6 + 382Q^4 + 198Q^2 + 8)g^{*2} (96Q^8 + 1424Q^6 + 1198Q^4 + 264Q^2 + 10)g^* \right. \\ &\quad \left. + 576Q^{10} + 2048Q^8 + 1988Q^6 + 726Q^4 + 91Q^2 + 1 \right)} \end{aligned} \quad (18)$$

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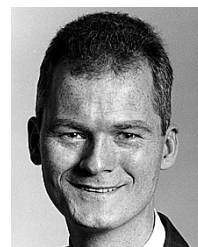
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