$\mathcal{H}_\infty$ Stabilization of Discrete-Time Nonlinear Semi-Markov Jump Singularly Perturbed Systems With Partially Known Semi-Markov Kernel Information

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***Abstract*—In this paper, the $\mathcal{H}_\infty$ stabilization problem is studied for discrete-time semi-Markov jump singularly perturbed systems (SMJSPSs) with repeated scalar nonlinearities. As the exact statistical information of the sojourn time or the mode transition is difficult to obtain, the case with only partial semi-Markov kernel information available is considered. Furthermore, introducing an external disturbance or nonlinearity into the analysis of discrete-time semi-Markov jump systems (DTSMJSs) meets critical obstacles, since the relation between the system state vectors at two nonadjacent instants is difficult to determine. To address this issue, the variation trend of the Lyapunov function for a semi-Markov jump sequence is analyzed in detail. Subsequently, criteria of mean-square exponential stability (MSES) for DTSMJSs are established for the first time based on the Lyapunov stability theory. By virtue of the criteria obtained and the cone complementary linearization algorithm, a controller ensuring MSES and $\mathcal{H}_\infty$ performance for discrete-time nonlinear SMJSPSs is constructed. Finally, the effectiveness and applicability of the proposed method are validated by simulation examples including an inverted pendulum model.**

***Index Terms*—Semi-Markov jump singularly perturbed systems, repeated scalar nonlinearities, mean-square exponential stability, $\mathcal{H}_\infty$ performance.**

I. Introduction

Due to the powerful capability of modeling hybrid systems encountering abrupt variations in structures or parameters, switched systems have achieved great progress in both theory and application during the past several decades [1]–[8]. To describe the switching behavior displaying stochastic features, the Markov chain has been extensively utilized and many excellent achievements on Markov jump systems (MJSs) have emerged [9]–[14]. However, as pointed out in [15], MJSs have tight restrictions on the type of sojourn time probability distribution (STPD), which may lead to inapplicability of the Markov jump model in many practical scenarios. To overcome this deficiency, the semi-Markov jump systems (SMJSs) were proposed subsequently. For SMJSs, the STPD is not confined to memoryless random distributions, and the system jump at a certain instant may depend on the time it remains in the current mode. Therefore, SMJSs can be seen as generalizations of MJSs [16]–[21]. In addition, the dynamic behavior of many practical systems usually exhibits the multiple-time-scales property due to the parasitism of some small parameters, such as electromagnetic transient processes in power systems or time constants of actuators in control systems [22]–[24]. The existence of these small parameters makes controlling of the system particularly intricate as it is difficult to conduct effective analysis of all dynamics on a single time scale. As a consequence, the singularly perturbed models are employed to deal with such case, where a singularly perturbed parameter (SPP) ϵ is utilized to describe the discrepancies between the "fast" and "slow" dynamics [25]. Significant achievements have been made on singularly perturbed systems with Markov jump parameters (e.g., [26]). However, when it comes to semi-Markov jump singularly perturbed systems (SMJSPSs), research on relevant issues is far away from maturity and many key problems still remain open, e.g., how to deal with the sophisticated systems with SPP and memory transition probabilities (TPs) exsit simultaneously. Subsequent analysis on obtaining numerically checkable conditions that independent of time-varying TPs and SPP also deserves further investigation.

On the other hand, with regard to the analysis and synthesis for SMJSPSs, a prevailingly adopted method is the semi-Markov kernel (SMK) approach [27]. Different from [28], [29], where the sojourn time of each subsystem is assumed to obey a specified probability distribution with fixed parameters, the SMK is described by an embedded Markov

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chain (EMC) and a sojourn time probability density function (STPDF) depending on both the current and next system modes [30]. In doing so, the STPDF can be considered separately for different system modes and, as a consequence, this method can be applied to a larger scope of systems. Focused on a class of linear SMJSs, the stabilization issue was addressed in [31] by virtue of multiple Lyapunov functions and the SMK approach. With the assumption that the slow state variables of the system are available, the mode-dependent controller design problem for SMJSPSs was discussed in [32]. Visibly, the existing results about SMJSs/SMJSPSs are generally restricted by an implicit assumption that the accurate information of SMK is completely available. However, in practical applications, acquiring the detailed information on SMK is a thorny problem and it is more common that only part of the statistical information about sojourn time or TPs can be obtained precisely [33]. Therefore, it is more reasonable to address the analysis and synthesis problems for SMJSPSs with partially known sojourn time or TPs. Although the problem with partially known TPs has been already studied in the last few decades [34], [35], the obtained results are mainly confined to MJSs and few papers are devoted to SMJSPSs with partially known SMK. This triggers our great interest to address this issue.

Furthermore, it can be noted that most of the currently developed approaches to discrete-time SMJSs (DSMJSs) focus on the stability and stabilization analysis for linear systems based on the σ-error mean-square stability (σ-EMSS) lemma given in [27], and the performance analysis is rarely involved. Despite the advantages of the proposed methods in dealing with discrete-time SMJSs in [36], [37], the form of the investigated systems is restricted. Especially, the developed methods are difficult to apply to the performance analysis of nonlinear discrete-time SMJSPSs. The main reason is that, the state equation of the system needs to be iterated from time k to time $k+\tau$ to express the relationship between Lyapunov function (LF) values at times k and $k+\tau$ [37]. In this situation, introducing nonlinear and external disturbance terms into the system brings additional difficulties to the problem solving. Although the $\mathcal{H}_\infty$ control problem has been investigated for SMJSs in [38], the LF is required to decrease between any instants k and $k+1$, which may result in more conservative results. In order to find an applicable and easy-to-use method to solve those problems, the variation trends of the LF at jumping and non-jumping instants for a semi-Markov jump sequence are discussed in detail in this paper based on the Lyapunov stability theory. Consequently, a set of stability criteria that can be applied to a broader class of DSMJSs, namely, mean-square exponential stability (MSES) criteria, are obtained.

Motivated by the above discussion, this paper focuses on developing a mode-dependent controller ensuring the MSES and prescribed $\mathcal{H}_\infty$ performance for a class of discrete-time nonlinear SMJSPSs with partially known SMK information. The main contribution can be summarized as follows.

(i) An eminent stability concept, i.e., MSES, is considered for DSMJSs for the first time, and the corresponding novel stability lemma is established. As a consequence, an external disturbance or nonlinearity can be introduced into the analysis of DSMJSs, which means the proposed method are capable of generalizing some outstanding works regarding DSMJSs, e.g., [30], [36], [39].

(ii) From a new perspective, that is, how the LF varies with the semi-Markov jump sequence, the stabilization analysis of the semi-Markov jump nonlinear systems is carried out. By considering that the Lyapunov function can decrease or increase at non-jumping instants while at the jumping instant the Lyapunov function value is less than that at the previous jumping instant, the utilization of $x(k+\kappa)=\bar{A}_a^{\kappa}x(k)$ [30] is avoided. Consequently, the complexity of the subsequent decoupling process is reduced greatly.

(iii) As the first attempt, the $\mathcal{H}_\infty$ control problem is studied for discrete-time SMJSPSs with repeated scalar nonlinearities. To make the problem under investigation more comprehensive, we consider that only partial SMK information is available. In addition, in contrast to [37], the cone complementary linearization (CCL) algorithm is adopted to deal with the coexistence of an unknown matrix and its inverse in the derived stabilization criteria. Besides, a dimension-adjusting matrix is introduced for matrix processing.

Notations: The notation employed in this work is standard. $\mathbb{R}$: the set of real numbers; $\mathbb{Z}$: the set of non-negative integers; $\mathbb{R}_{[a_1,a_2]}$: the set $\{a\in\mathbb{R}|a_1\le a\le a_2\}$; $\mathbb{Z}_{[a_1,a_2]}$: the set $\{a\in\mathbb{Z}|a_1\le a\le a_2\}$; $\mathcal{E}\{\cdot\}|_\zeta$: the conditional expectation operator conditioned on ζ; $sym\{A\}$: $A+A^T$; $tr(A)$: the trace of the square matrix A; $\lambda_{\max}(A)/\lambda_{\min}(A)$: the maximum or minimum eigenvalue of matrix A; $P>0$: matrix P is positive definite. Other conventional notations and corresponding interpretations can be found in [27].

II. Problem Statement

A. System Description and Controller Design

For a fixed complete probability space $(\hat{\Phi},\hat{\mathcal{F}},\Pr)$, we consider the following discrete-time SMJSPSs with repeated scalar nonlinearities (Σ):

$$\begin{cases} x(k+1)=A_{\vartheta(k)}E_\epsilon x(k)+B_{\vartheta(k)}u(k)\\ \qquad\qquad +C_{\vartheta(k)}g(E_\epsilon x(k))+D_{\vartheta(k)}\omega(k),\\ z(k)=U_{\vartheta(k)}E_\epsilon x(k)+F_{\vartheta(k)}u(k),\end{cases}\tag{1}$$

where $E_\epsilon\triangleq\operatorname{diag}\{I_{n_1},\epsilon I_{n_2}\}$ and ϵ is the SPP. $x(k)\triangleq\begin{bmatrix}x_s^T(k) & x_f^T(k)\end{bmatrix}^T\in\mathbb{R}^{n_x}$ with $x_s(k)\in\mathbb{R}^{n_1}$ and $x_f(k)\in\mathbb{R}^{n_2}$ being the slow and fast state vectors, respectively. $z(k)\in\mathbb{R}^{n_z}$, $u(k)\in\mathbb{R}^{n_u}$ and $\omega(k)\in\mathbb{R}^{n_\omega}$ are the system output, a control input and an external disturbance that belongs to $l_2[0,\infty)$, respectively. $g(\cdot)$ is a bounded nonlinear function satisfying

$$|g(\nu)+g(\xi)|\le|\nu+\xi|,\forall\nu,\xi\in\mathbb{R}.\tag{2}$$

$\{\vartheta(k)\}_{k\in\mathbb{Z}}$ is a right-continuous semi-Markov chain (SMC), and for $\forall k\in\mathbb{Z}$, $\vartheta(k)\in\mathcal{N}\triangleq\{1,2,\dots,N\}$, $A_{\vartheta(k)}$, $B_{\vartheta(k)}$, $C_{\vartheta(k)}$, $D_{\vartheta(k)}$, $U_{\vartheta(k)}$ and $F_{\vartheta(k)}$ are given matrices with suitable dimensions.

One of the main objectives of this paper is to design a state-feedback controller that guarantees stability and prescribed performance of the resulting closed-loop system. For this purpose, a mode-dependent controller is constructed as follows:

$$u(k) = \tilde{K}_{\vartheta(k)} x_s(k) = K_{\vartheta(k)} E_\epsilon x(k), \tag{3}$$

where $K_{\vartheta(k)} \triangleq \left[\tilde{K}_{\vartheta(k)} \; 0_{n_u \times n_2} \right]$ is the controller gain to be determined. To simplify the notation, we set $A_{\vartheta(k)} \triangleq A_a$ for $\forall \vartheta(k) \triangleq a \in \mathcal{N}$, and other symbols are similarly defined. Then, by denoting $\eta(k) \triangleq \left[x_s^T(k) \; \epsilon x_f^T(k) \right]^T$, the closed-loop system $(\hat{\Sigma})$ can be expressed based on (1) and (3) as follows:

$$\begin{cases} \eta(k+1) = E_\epsilon[\bar{A}_a \eta(k) + C_a g(\eta(k)) + D_a \omega(k)], \\ z(k) = \bar{F}_a \eta(k), \end{cases} \tag{4}$$

where

$$\bar{A}_a \triangleq A_a + B_a K_a, \bar{F}_a \triangleq U_a + F_a K_a.$$

Remark 1: In the circumstance that the mode information of the considered system is readily available, designing a mode dependent controller may be conducive to achieving better control effect. Furthermore, the fast variables are difficult to be measured directly in most practical SPSs. Thus, a mode dependent slow state feedback controller is designed in this paper. However, the negligence of fast state information may affect the ultimate control of the system. Besides, the implementation of actuator may fail in the operation of real systems [40], [41]. Therefore, designing a comprehensive controller which can not only make full use of system information but also tolerant sudden failure deserves further exploration.

B. Semi-Markov Jump Mechanism

The SMC $\{\vartheta(k)\}_{k\in\mathbb{Z}}$ is employed to describe the stochastic jump of systems among different modes. Thus, before proceeding further, two relevant concepts need to be introduced, i.e., the Markov renewal chain (MRC) and the SMK.

To facilitate the description of a semi-Markov jump sequence, for $\forall m \in \mathbb{Z}$ we define

$$\begin{cases} k_m\text{: time instant corresponding to the } m\text{th jump } (k_0 = 0); \\ G_m\text{: mode index corresponding to the } m\text{th jump;} \\ T_m\text{: sojourn time of } G_m \text{ between two jump instant } k_m \\ \qquad \text{and } k_{m+1} \; (T_m \triangleq k_{m+1} - k_m). \end{cases}$$

Then, the following two definitions are used to introduce the concept of the SMC.

Definition 1 [42]: For the stochastic process $\{(G_m, k_m)\}_{m\in\mathbb{Z}}$

(I) $\{(G_m, k_m)\}_{m\in\mathbb{Z}}$ is a discrete-time homogeneous MRC, if $\forall a, b \in \mathcal{N}$, $a \neq b$, and $\forall d \in \mathbb{Z}_{[1,\infty)}$

$$\begin{aligned} \Pr(G_{m+1} &= b, T_m = d | k_0, G_0; k_1, G_1; \ldots; k_m, G_m = a) \\ &= \Pr(G_{m+1} = b, T_m = d | G_m = a) \\ &= \Pr(G_1 = b, T_0 = d | G_0 = a). \end{aligned}$$

(II) The stochastic process $\{G_m\}_{m\in\mathbb{Z}}$ is called the embedded Markov chain (EMC) of the MRC. The TPs of the EMC are defined as

$$\pi_{ab} \triangleq \Pr(G_{m+1} = b | G_m = a), \quad \forall a, b \in \mathcal{N},$$

with $0 \leq \pi_{ab} \leq 1$, $\forall a \neq b$, $\pi_{aa} \triangleq 0$, and $\sum_{b=1}^{N} \pi_{ab} = 1$. $\Pi \triangleq [\pi_{ab}]_{a,b\in\mathcal{N}}$ is employed to denote the corresponding transition probability matrix (TPM).

Definition 2 [42], [43]: Given an MRC $\{(G_m, k_m)\}_{m\in\mathbb{Z}}$, $\{\vartheta(k)\}_{k\in\mathbb{Z}}$ represents a SMC related to the MRC, if for $\forall k \in \mathbb{Z}$, $\vartheta(k) = G_{M(k)}$ with $M(k) \triangleq \max\{m \in \mathbb{Z} | k \geq k_m\}$. The SMK of the SMC can be defined as $\Theta(d) \triangleq [\theta_{ab}(d)]_{a,b\in\mathcal{N}}$, where $\theta_{ab}(d) \triangleq \Pr(G_{m+1} = b, T_m = d | G_m = a)$ with $\sum_{d=1}^{\infty} \sum_{b\in\mathcal{N}} \theta_{ab}(d) = 1$, $0 \leq \theta_{ab}(d) \leq 1$, $\forall a, b \in \mathcal{N}$, $a \neq b$, $\forall d \in \mathbb{Z}_{[1,\infty)}$ and $\theta_{aa}(d) \triangleq 0$, $\forall a \in \mathcal{N}$, $\forall d \in \mathbb{Z}_{[1,\infty)}$. Then, by defining the STPDF as

$$\varsigma_{ab}(d) \triangleq \Pr(T_m = d | G_{m+1} = b, G_m = a),$$

the following equation can be obtained

$$\theta_{ab}(d) = \pi_{ab} \varsigma_{ab}(d), \forall a, b \in \mathcal{N}, \forall d \in \mathbb{Z}_{[1,\infty)}. \tag{5}$$

Remark 2: Note that in many practical scenarios, the completely accurate information of SMK, i.e., $\theta_{ab}(d)$ is difficult to obtain. Therefore, in this paper, the partially known SMK issue is considered. To facilitate the subsequent analysis, we make the following assumptions. For $\forall a, b \in \mathcal{N}$, $d \in \mathbb{Z}_{[1,\infty)}$,

$$\begin{aligned} &\bar{\mathcal{N}}_a \triangleq \{b | \theta_{ab}(d) \text{ is available}\}, \quad \mu_a \triangleq \sum\nolimits_{d=1}^{\bar{d}_a} \sum\nolimits_{b\in\mathcal{N}} \theta_{ab}(d) \\ &\tilde{\mathcal{N}}_a \triangleq \{b | \theta_{ab}(d) \text{ is unavailable }\}, \quad \mathcal{N} \triangleq \bar{\mathcal{N}}_a \cup \tilde{\mathcal{N}}_a. \end{aligned}$$

If for a scalar ϕ close enough to 1 such that $\mu_a > \phi$ holds, then it can be regarded that $\bar{d}_a \in \mathbb{Z}_{[1,\infty)}$ is closely enough to the sojourn time upper bound (STUB) for the ath mode [37].

Next, to study the stabilization and performance of SMJSPSs, the following lemmas and definitions are recalled.

Definition 3 [39]: Given the STUB $\bar{d}_a \in \mathbb{Z}_{[1,\infty)}$ for mode a $(\forall a \in \mathcal{N})$, the closed-loop system (4) with $\omega(k) \equiv 0$ is mean-square exponentially stable (MSESB), if there exist scales $\beta > 0$, $0 < \alpha < 1$ such that, for any initial conditions $\eta(0) \in \mathbb{R}^{n_x}$, $\vartheta(0) \in \mathcal{N}$, the following inequality holds:

$$\begin{aligned} &\mathcal{E}\{\|\eta(k)\|^2\}|_{x(0),\vartheta(0)(T_m\in\mathbb{Z}_{[1,\bar{d}_a]}|G_m=a)} \\ &\qquad \leq \beta\alpha^{k-k_0} \mathcal{E}\{\|\eta(k_0)\|^2\}, \forall k \geq k_0. \end{aligned} \tag{6}$$

Remark 3: It is generally known that most of the current research on stability and stabilization of DSMJSs are based on the σ-EMSS lemma proposed in [27]. Obviously, using Lemma 1 in [27] encounters restrictions on the form of the systems as the relationship between $\eta(k_m + t)$ and $\eta(k_m)$ needs to be determined. Therefore, to eliminate this restriction, the discussion of σ-EMSS is generalized to MSES and, subsequently, a new stability lemma applicable to a broader class of systems is proposed in this paper. In addition, the convergence rate can be tuned by the scalar α.

Lemma 1 [44]: A matrix Q is positive diagonally dominant, if and only if $Q \triangleq [q_{uv}]_{u,v\in[1,n_x]} > 0$ and there exists a symmetric matrix $\mathcal{R} \triangleq [r_{uv}]_{u,v\in[1,n_x]}$, such that

$$r_{uv} \geq 0, q_{uv} + r_{uv} \geq 0, \forall u \neq v, \tag{7}$$

$$q_{uu} \geq \sum\nolimits_{v\neq u} (q_{uv} + 2r_{uv}), \quad \forall u. \tag{8}$$

Lemma 2 [44]: If Q is a positive definite diagonally dominant matrix, then for all nonlinear functions $g(\cdot)$ satisfying (2), the following inequality holds:

$$\eta^T(k)Q\eta(k) - g^T(\eta(k))Qg(\eta(k)) \geq 0, \forall \eta(k). \tag{9}$$

Definition 4 [4]: If the closed-loop system (4) is MSES and under zero initial conditions there exists a scalar $\bar{\gamma} > 0$ such that the following inequality holds for any nonzero $\omega(k) \in l_2[0, \infty)$:

$$\sum_{k=0}^{\infty} \mathcal{E}\{z^T(k)z(k)\} \leq \bar{\gamma}^2 \sum_{k=0}^{\infty} \mathcal{E}\{\omega^T(k)\omega(k)\}, \tag{10}$$

then the closed-loop system (4) is MSESB with a prescribed $\mathcal{H}_\infty$ performance level $\bar{\gamma}$.

Lemma 3 [32]: For a positive scalar $\bar{\epsilon}$, if (i) $S_1 \geq 0$; (ii) $S_3 < 0$; (iii) $\bar{\epsilon}^2 S_1 + \bar{\epsilon} S_2 + S_3 < 0$ hold simultaneously, then $\epsilon^2 S_1 + \epsilon S_2 + S_3 < 0$ holds for $\forall \epsilon \in (0, \bar{\epsilon}]$.

III. Main Results

In this section, a set of stability and stabilization criteria for DSMJSs are presented. Based on these criteria, sufficient conditions guaranteeing the MSES and $\mathcal{H}_\infty$ performance of the closed-loop system (4) are derived. Then, by virtue of some proper matrix decoupling methods and the CCL algorithm, the specific form of the desired mode-dependent controller gain is given.

A. Mean-Square Exponential Stability Criteria

First of all, a result on the MSES of DSMJSs is given for subsequent analysis.

Lemma 4: For a discrete-time nonlinear stochastic jump system $\eta(k+1) = f_{\vartheta(k)}(\eta(k))$ with bounded sojourn time, $\eta(k)$ and $\vartheta(k)$ denote the state vector and the jumping signal, respectively. $k_0, k_1, \ldots, k_m, \ldots$ are the jumping instants. If there exist scalars $\beta_1 > 0$, $\beta_2 > 0$, and a set of $\mathcal{C}^1$ functions $V(\eta(k), \vartheta(k), k - k_m) : \mathbb{R}^{n_x} \to \mathbb{R}$ with $k \in \mathbb{Z}_{[k_m, k_{m+1})}$, such that for given positive constants $h_{\vartheta(k_m)}$, $l_{\vartheta(k_m)}$, $\bar{d}_{\vartheta(k_m)}$, $\vartheta(k_m) \in \mathcal{N}$, the following inequalities hold under any initial conditions $\eta(0) \in \mathbb{R}^{n_x}$, $\vartheta(0) \in \mathcal{N}$

$$\begin{aligned} &\bar{\delta}_{\vartheta(k_m)} \\ &\quad \triangleq \begin{cases} l_{\vartheta(k_m)} h_{\vartheta(k_m)}^{\bar{d}_{\vartheta(k_m)}} < 1, & \text{if } h_{\vartheta(k_m)} > 1, \\ l_{\vartheta(k_m)} h_{\vartheta(k_m)} < 1, & \text{if } 0 < h_{\vartheta(k_m)} \leq 1, \end{cases} \end{aligned} \tag{11}$$

$$\begin{aligned} &\beta_1 \|\eta(k)\|^2 \\ &\quad \leq V(\eta(k), \vartheta(k), k - k_m) \leq \beta_2 \|\eta(k)\|^2, \end{aligned} \tag{12}$$

$$\begin{aligned} &\quad h_{\vartheta(k)} V(\eta(k-1), \vartheta(k-1), k - k_m - 1) \\ &\quad \geq V(\eta(k), \vartheta(k), k - k_m), \forall k \in \mathbb{Z}_{[k_m+1, k_{m+1})}, \end{aligned} \tag{13}$$

$$\begin{aligned} &\quad h_{\vartheta(k_m)} V(\eta(k_{m+1} - 1), \vartheta(k_{m+1} - 1), k_{m+1} - k_m - 1) \\ &\quad \geq V(\eta(k_{m+1}), \vartheta(k_m), k_{m+1} - k_m), \end{aligned} \tag{14}$$

$$\begin{aligned} &\quad \mathcal{E}\{l_{\vartheta(k_m)} V(\eta(k_{m+1}), \vartheta(k_m), k_{m+1} - k_m)\}|_{\eta(k_m), \vartheta(k_m)} \\ &\quad \geq \mathcal{E}\{V(\eta(k_{m+1}), \vartheta(k_{m+1}), 0)\}|_{\eta(k_m), \vartheta(k_m)}, \end{aligned} \tag{15}$$

then the system is MSESB.

Proof: Consider that the STUB for mode a ($\forall a \in \mathcal{N}$) is $\bar{d}_a$. Define a σ-algebra generated by $\{\rho_\iota \triangleq (\eta(k_\iota), \vartheta(k_\iota)), \iota \in \mathbb{Z}_{[0,m]}\}$ as $\Upsilon_m \triangleq \sigma\{\rho_0, \rho_1, \ldots, \rho_m\}$. First, it can be derived from (11) and (13)-(15) that

$$\begin{aligned} &\mathcal{E}\{V(\eta(k_{m+1}), \vartheta(k_{m+1}), 0)\}|_{\rho_m} \\ &\quad \leq \mathcal{E}\{l_{\vartheta(k_m)} h_{\vartheta(k)}^{T_m} V(\eta(k_m), \vartheta(k_m), 0)\}|_{\eta(k_m), \vartheta(k_m)} \\ &\quad \leq \bar{\delta}_{\vartheta(k_m)} V(\eta(k_m), \vartheta(k_m), 0). \end{aligned} \tag{16}$$

Then, recalling the property of the conditional expectation [37], one can infer from (16) that

$$\begin{aligned} 0 &\leq \mathcal{E}\{\bar{\delta}_{\vartheta(k_m)} V(\eta(k_m), \vartheta(k_m), 0) \\ &\quad - \mathcal{E}\{V(\eta(k_{m+1}), \vartheta(k_{m+1}), 0)\}|_{\rho_m}\}|_{\rho_0} \\ &\leq \mathcal{E}\{\bar{\delta}_{\vartheta(k_m)} V(\eta(k_m), \vartheta(k_m), 0)\}|_{\rho_0} \\ &\quad - \mathcal{E}\{V(\eta(k_{m+1}), \vartheta(k_{m+1}), 0)\}|_{\rho_0}, \end{aligned}$$

which means that

$$\begin{aligned} &\mathcal{E}\{V(\eta(k_{m+1}), \vartheta(k_{m+1}), 0)\}|_{\rho_0} \\ &\qquad\qquad \leq \bar{\delta}\mathcal{E}\{V(\eta(k_m), \vartheta(k_m), 0)\}|_{\rho_0}, \end{aligned} \tag{17}$$

with $\bar{\delta} \triangleq \max_{\forall \vartheta(k_m) \in \mathcal{N}}\{\bar{\delta}_{\vartheta(k_m)}\}$.

Considering $k \in [k_m, k_{m+1})$, it follows from (13) and (17) that

$$\begin{aligned} &\mathcal{E}\{V(\eta(k), \vartheta(k), k - k_m)\}|_{\rho_0} \\ &\quad \leq \bar{\delta}^m \mathcal{E}\{h_{\vartheta(k)}^{k-k_m} V(\eta(k_0), \vartheta(k_0), 0)\}|_{\rho_0} \\ &\quad \leq \bar{\beta}\alpha^{k-k_0+1}\mathcal{E}\{V(\eta(k_0), \vartheta(k_0), 0)\}|_{\rho_0}, \end{aligned}$$

with $\bar{\beta} \triangleq \max_{\forall \vartheta(k_m) \in \mathcal{N}}\{h_{\vartheta(k)}^{\bar{d}_{\vartheta(k_m)}}, 1\}$, $\alpha \triangleq \max_{\forall k \in \mathbb{Z}_{[k_0, \infty)}}\{\bar{\delta}^{m/(k-k_0+1)}\}$. Obviously, $\alpha \in (0, 1)$. Then, combining with (12), one can get

$$\mathcal{E}\{\|\eta(k)\|^2\}|_{\rho_0} \leq \alpha \frac{\beta_2}{\beta_1} \bar{\beta} \alpha^{k-k_0} \mathcal{E}\{\|\eta(k_0)\|^2\}. \tag{18}$$

As (18) is derived with the bounded sojourn time $\bar{d}_a$ ($\forall a \in \mathcal{N}$), one finally obtains (6).

Remark 4: From the perspective of the LF variation trend, an illustration of Lemma 4 is presented in Fig. 1, where a possible evolution of LF under the restrictions (11)-(15) is given. It can be observed that the LF can increase or decrease at non-jumping instants, while at the adjacent two jumping instants it is required to decrease. Notice that $\vartheta(k)$ is right-continuous, then by introducing a virtual point $(k_{m+1}, V(\eta(k_{m+1}), a, k_{m+1} - k_m))$, the comparison between the expectation values of LF at jumping instants k_{m+1} and k_m can be made. Meanwhile, by virtue of condition (11),

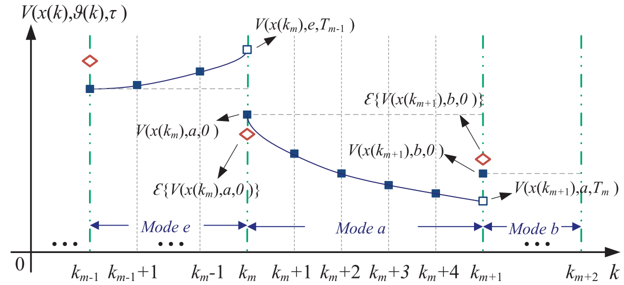

Fig. 1. A possible evolution of Lyapunov function.

$\bar{\delta}_{\vartheta(k_m)} < 1$ can be ensured, which combining with (17) implies that the expectation of $V(\eta(k_{m+1}), b, 0)$ is smaller than $V(\eta(k_m), a, 0)$, i.e., the expectation of the LF value at the current jump instant is smaller than that at the previous jump instant. As a result, the LF will tend to zero in the mean-square sense in spite of the increase of its values at some non-jumping instants.

Remark 5: In [37], the authors considered that the LF value at any non-jumping instant $k_m + t$ can be increase or decrease relative to the LF value at the jumping instant k_m, and at the jumping instant k_{m+1} the LF value is required to be less than that at the instant k_m. Although this consideration can yield results with less conservatism, the relationship between $\eta(k_m + t)$ and $\eta(k_m)$ needs to be determined. Specifically, even for a simple system $\eta(k+1) = \bar{A}_a\eta(k)$, a coupling term $\bar{A}_a^t$ should be introduced, which would make the decoupling process more difficult. If a nonlinearity or disturbance is presented, relevant analysis would be even more complicated. The method proposed in this paper enables one to avoid this problem and, therefore, would be applicable to a broader class of systems.

Remark 6: The main reason for considering the LF discussed above is to reach a favourable compromise between the easily-checked conditions and the less conservative results. Different from conventional construction of LF, the approach utilized in this paper focuses on coordinating the variations of the LF at the jumping and non-jumping instants of semi-Markov jump signal, aiming at providing more possibilities and easily implemented approach to demonstrate the stability.

B. Stabilization and Performance Analysis

In the following, the stabilization and performance analysis for the closed-loop system (4) is performed. Before proceeding further, we introduce the following notation

$$\tilde{\delta} \triangleq \min_{\forall a\in\mathcal{N}}\{\tilde{\delta}_a\}, \tilde{\delta}_a \triangleq \begin{cases} l_a h_a, & \text{when } h_a > 1, \\ l_a h_a^{\bar{d}_a}, & \text{when } 0 < h_a \le 1, \end{cases}$$
$$\bar{h} \triangleq \max_{\forall a\in\mathcal{N}}\{h_a\}, \bar{d}_{\max} \triangleq \max_{\forall a\in\mathcal{N}}\{\bar{d}_a\}, \bar{\delta} \triangleq \max_{\forall a\in\mathcal{N}}\{\bar{\delta}_a\},$$
$$\tilde{h} \triangleq \min_{\forall a\in\mathcal{N}}\{h_a\}, h_{\max} \triangleq \max\{\bar{h}, 1\}, h_{\min} \triangleq \min\{\tilde{h}, 1\}.$$

Theorem 1: For given positive scalars h_a, l_a, $\bar{d}_a$, $a \in \mathcal{N}$, and γ, if there exist symmetric positive definite matrices $P_a(\tau)$, $\tau \in \mathbb{Z}_{[0,\bar{d}_a]}$, positive diagonally dominant matrices $Q_{a(\tau)} \triangleq [q_{uv}^{a\tau}]_{u,v\in[1,n_x]}$, and symmetric matrices $R_a(\tau) \triangleq [r_{uv}^{a\tau}]_{u,v\in[1,n_x]}$, $\tau \in \mathbb{Z}_{[0,\bar{d}_a-1]}$, such that for $\forall a$, $b \in \mathcal{N}$, $a \neq b$, and $\forall\tau \in \mathbb{Z}_{[1,\bar{d}_a]}$, the following relations hold:

$$\bar{\delta}_a \triangleq \begin{cases} l_a h_a^{\bar{d}_a} < 1, & \text{when } h_a > 1, \\ l_a h_a < 1, & \text{when } 0 < h_a \le 1, \end{cases} \tag{19}$$

$$\begin{bmatrix} \psi_{a(\tau-1)}^1 & 0 & 0 & \bar{A}_a^T & \bar{F}_a^T \\ * & -Q_{a(\tau-1)} & 0 & C_a^T & 0 \\ * & * & -\gamma^2 I & D_a^T & 0 \\ * & * & * & -P_a(\tau) & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \tag{20}$$

$$\begin{cases} q_{uu}^{a(\tau-1)} - \sum_{v,v\neq u}(q_{uv}^{a(\tau-1)} + 2r_{uv}^{a(\tau-1)}) \ge 0, & \forall u, \\ r_{uv}^{a(\tau-1)} \ge 0, q_{uv}^{a(\tau-1)} + r_{uv}^{a(\tau-1)} \ge 0, & \forall u \neq v, \end{cases} \tag{21}$$

$$P_b^{-1}(0) - l_a P_a^{-1}(\tau) < 0, \tag{22}$$

where $\psi_{a(\tau-1)}^1 \triangleq Q_{a(\tau-1)} - h_a E_\epsilon^{-1} P_a^{-1}(\tau-1) E_\epsilon^{-1}$, then the closed-loop system (4) is MSESB with the prescribed $\mathcal{H}_\infty$ performance index

$$\bar{\gamma} \triangleq \gamma\sqrt{\frac{h_{\max}^{\bar{d}_{\max}}\bar{h}(1-\bar{\delta})}{h_{\min}^{\bar{d}_{\max}}\tilde{h}\tilde{\delta}(1-\bar{\delta}^{1/\bar{d}_{\max}})}}.$$

Proof: Construct a LF as

$$V(\eta(k), \vartheta(k), \tau) \triangleq \eta^T(k) E_\epsilon^{-1} P_{\vartheta(k)}^{-1}(\tau) E_\epsilon^{-1} \eta(k) \tag{23}$$

where $\tau \triangleq k - k_m$, with $k_m \triangleq \max\{k_i \in \mathbb{Z} | k \ge k_i, i \in \mathbb{Z}\}$. Denote $J(k) \triangleq z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k)$ and $\vartheta(k_m) = a$, $\vartheta(k_{m+1}) = b$, $\forall a, b \in \mathcal{N}$, $a \neq b$. Since the bounded nonlinear function $g(\cdot)$ satisfies the constraint (2), then based on condition (21), it can be derived from Lemma 1 and Lemma 2 that

$$0 \le \eta^T(k_m+\tau-1) Q_{a(\tau-1)} \eta(k_m+\tau-1) - g^T(\eta(k_m+\tau-1)) Q_{a(\tau-1)} g(\eta(k_m+\tau-1)).$$

Subsequently, by applying Schur complement to (20), and combining (4), (23) with the above analysis, we can immediately get

$$\begin{aligned} &V(\eta(k_m+\tau), a, \tau) + J(k_m+\tau-1) \\ &\quad - h_a V(\eta(k_m+\tau-1), a, \tau-1) \\ &\le 0, \forall a \in \mathcal{N}, \forall\tau \in \mathbb{Z}_{[1,T_m]}. \end{aligned} \tag{24}$$

Step 1: In the first place, we prove that the closed-loop system (4) is MSES for $\omega(l) \equiv 0$.

Obviously, (19) is equivalent to (11).

From (23), one can get

$$\upsilon_1 \|\eta(k)\|^2 \le V(\eta(k), \vartheta(k), \tau) \le \upsilon_2 \|\eta(k)\|^2$$

with $\upsilon_1 \triangleq \min_{\forall a\in\mathcal{N},\tau\in\mathbb{Z}_{[1,\bar{d}_a]}} \lambda_{\min}(E_\epsilon^{-1} P_a^{-1}(\tau) E_\epsilon^{-1})$, $\upsilon_2 \triangleq \max_{\forall a\in\mathcal{N},\tau\in\mathbb{Z}_{[1,\bar{d}_a]}} \lambda_{\max}(E_\epsilon^{-1} P_a^{-1}(\tau) E_\epsilon^{-1})$. Therefore, (12) holds.

Combining (23), (24) with $\omega(l) \equiv 0$, the following inequality is obtained:

$$\begin{aligned} &h_{\vartheta(k_m)} V(\eta(k_m+\tau-1), \vartheta(k_m), \tau-1) \\ &\quad \ge V(\eta(k_m+\tau), \vartheta(k_m), \tau), \quad \forall\tau \in \mathbb{Z}_{[1,T_m]}, \end{aligned} \tag{25}$$

which means that (13) and (14) are satisfied.

Furthermore, one can deduce from (22) that

$$\begin{aligned} 0 &\ge \mathcal{E}\{V(\eta(k_{m+1}), b, 0) - l_a V(\eta(k_{m+1}), a, T_m)\}|_{\rho_m} \\ &= \sum_{T_m=1}^{\bar{d}_a} \sum_{b\in\mathcal{N}} \frac{\theta_{ab}(T_m)}{\mu_a} x^T(k_m+T_m) \\ &\quad \times \left[P_b^{-1}(0) - l_a P_a^{-1}(T_m)\right] x(k_m+T_m) \end{aligned} \tag{26}$$

which means that (15) is satisfied.

Thus, by Lemma 4, the MSES of the closed-loop system (4) is guaranteed.

Step 2: In the following, the $\mathcal{H}_\infty$ performance is analyzed. Denote $\hat{l}(k_m,t) \triangleq l_{\vartheta(k_{m-1})} h_{\vartheta(k_{m-1})}^{k_m-t-1}$. One can get from (15), (19) and (24) that for $k \in (k_m+1, k_{m+1})$

$$\mathcal{E}\{V(\eta(k),a,\tau)\} \le \mathcal{E}\left\{h_{\max}^{\bar{d}_{\max}} V(\eta(k_m),a,0) - \sum_{t=k_m}^{k-1} h_a^{k-t-1} J(t)\right\}. \quad (27)$$

Moreover, for $k = k_m$

$$\mathcal{E}\{V(\eta(k_m),a,0)\} \le \mathcal{E}\left\{\bar{\delta} V(\eta(k_{m-1}),\vartheta(k_{m-1}),0) - \sum_{t=k_{m-1}}^{k_m-1} \hat{l}(k_m,t)\, J(t)\right\}. \quad (28)$$

Obviously, (19) ensures $\tilde{\delta} \le \bar{\delta} < 1$ and $\tilde{\delta} \le l_{\vartheta(k_j)} h_{\vartheta(k_j)}^{\tau_j} \le \bar{\delta}$, $\forall j \in \mathcal{N}, \tau_j \in \mathbb{Z}_{[1,T_j]}$. Then, under zero initial conditions, by iterating inequalities (27) and (28), one can derive

$$\begin{aligned} 0 > \mathcal{E}\{&h_{\max}^{\bar{d}_{\max}} \bar{\delta}^{m-1} \sum_{t=k_0}^{k_1-1} l_{\vartheta(k_0)} h_{\vartheta(k_0)}^{k_1-t-1} J(t) \\ &+ \ldots + h_{\max}^{\bar{d}_{\max}} \bar{\delta}^0 \sum_{t=k_{m-1}}^{k_m-1} l_{\vartheta(k_{m-1})} h_{\vartheta(k_{m-1})}^{k_m-t-1} J(t) \\ &+ \sum_{t=k_m}^{k-1} h_a^{k-t-1} J(t)\}, \end{aligned}$$

which yields

$$\begin{aligned} &\frac{h_{\min}^{\bar{d}_{\max}} \tilde{\delta}}{\bar{h}} \mathcal{E}\{\bar{\delta}^{m-1} \sum_{t=k_0}^{k_1-1} z^T(t)z(t) + \ldots \\ &\quad + \bar{\delta}^0 \sum_{t=k_{m-1}}^{k_m-1} z^T(t)z(t) + \bar{\delta}^{-1} \sum_{t=k_m}^{k-1} z^T(t)z(t)\} \\ &\le \frac{h_{\max}^{\bar{d}_{\max}} \gamma^2}{\tilde{h}} \mathcal{E}\{\bar{\delta}^m \sum_{t=k_0}^{k_1-1} \omega^T(t)\omega(t) + \ldots \\ &\quad + \bar{\delta} \sum_{t=k_{m-1}}^{k_m-1} \omega^T(t)\omega(t) + \sum_{t=k_m}^{k-1} \omega^T(t)\omega(t)\}. \quad (29) \end{aligned}$$

Denoting $\bar{M}(t,k)$ as the number of jumping instants in the interval $(t,k]$, one concludes that $(k-t)/\bar{d}_{\max} - 1 \le \bar{M}(t,k) \le k-t$. Then, the following inequality is deduced from (29):

$$\mathcal{E}\{\sum_{t=k_0}^{k-1} \bar{\delta}^{k-t-2} z^T(t)z(t)\} \le \frac{h_{\max}^{\bar{d}_{\max}} \bar{h}\gamma^2}{h_{\min}^{\bar{d}_{\max}} \tilde{h}\tilde{\delta}} \mathcal{E}\{\sum_{t=k_0}^{k-1} \bar{\delta}^{\frac{k-t-1}{\bar{d}_{\max}}-1} \omega^T(t)\omega(t)\},$$

which yields

$$\mathcal{E}\{\sum_{k=k_0+1}^{\infty} \sum_{t=k_0}^{k-1} \bar{\delta}^{k-t-1} z^T(t)z(t)\} \le \frac{h_{\max}^{\bar{d}_{\max}} \bar{h}\gamma^2}{h_{\min}^{\bar{d}_{\max}} \tilde{h}\tilde{\delta}} \mathcal{E}\{\sum_{k=k_0+1}^{\infty} \sum_{t=k_0}^{k-1} \bar{\delta}^{\frac{k-t-1}{\bar{d}_{\max}}} \omega^T(t)\omega(t)\}.$$

Subsequently, by exchanging the summation order and utilizing the equal ratio summation formula, one obtains

$$\mathcal{E}\{\sum_{t=k_0}^{\infty} z^T(t)z(t)\} \le \bar{\gamma}^2 \mathcal{E}\{\sum_{t=k_0}^{\infty} \omega^T(t)\omega(t)\}. \quad (30)$$

Thus, (10) is satisfied.

C. Controller Design

In this subsection, based on Theorem 1, a mode-dependent controller is constructed by utilizing the CCL algorithm.

Theorem 2: For given positive scalars $\bar{\epsilon}$, h_a, l_a, $\bar{d}_a$, $a \in \mathcal{N}$, and γ, if there exist matrices $\bar{Y}_a$, symmetric positive definite matrices $P_a(\tau)$, $\tau \in \mathbb{Z}_{[0,\bar{d}_a]}$, positive diagonally dominant matrices $Q_{a(\tau)} \triangleq [q_{uv}^{a\tau}]_{u,v\in[1,n_x]}$, and symmetric matrices $R_a(\tau) \triangleq [r_{uv}^{a\tau}]_{u,v\in[1,n_x]}$, $\tau \in \mathbb{Z}_{[0,\bar{d}_a-1]}$, such that for $\forall\tau \in \mathbb{Z}_{[1,\bar{d}_a]}$, $\forall a$, $b \in \mathcal{N}(a \ne b)$, and $\varkappa = 1, 2$, the relations (19), (21), (22) and the following inequality hold:

$$\begin{bmatrix} \phi_{a(\tau-1)}^{\varkappa} & 0 & 0 & \tilde{A}_a^T & \tilde{\psi}_a^{15} \\ * & -Q_{a(\tau-1)} & 0 & C_a^T & 0 \\ * & * & -\gamma^2 I & D_a^T & 0 \\ * & * & * & -P_a(\tau) & 0 \\ * & * & * & * & \psi_{a(\tau-1)}^3 \end{bmatrix} < 0, \quad (31)$$

where

$$\begin{aligned} \phi_{a(\tau-1)}^1 &\triangleq h_a[H_1 P_a(\tau-1)H_1 - sym\{\bar{Y}_a^T\}], \\ \phi_{a(\tau-1)}^2 &\triangleq h_a[E_{\bar{\epsilon}} P_a(\tau-1)E_{\bar{\epsilon}} - sym\{\bar{Y}_a^T\}], \\ \psi_{a(\tau-1)}^3 &\triangleq \mathrm{diag}\{-I, -Q_{a(\tau-1)}^{-1}\},\ E_{\bar{\epsilon}} \triangleq \mathrm{diag}\{I_{n_1}, \bar{\epsilon} I_{n_2}\}, \\ \tilde{A}_a &\triangleq \bar{A}_a\bar{Y}_a,\ \tilde{\psi}_a^{15} \triangleq \begin{bmatrix} \bar{Y}_a^T \bar{F}_a^T & \bar{Y}_a^T \end{bmatrix},\ H_1 \triangleq \mathrm{diag}\{I_{n_1}, 0_{n_2}\}, \end{aligned}$$

then the system (4) is MSESB with the prescribed $\mathcal{H}_\infty$ performance index $\bar{\gamma}$ for $\forall\epsilon \in (0, \bar{\epsilon}]$.

Proof: For inequality (31), one can infer from Lemma 3 that

$$\begin{bmatrix} \tilde{\psi}_{a(\tau-1)}^{11} & 0 & 0 & \tilde{A}_a^T & \tilde{\psi}_a^{15} \\ * & -Q_{a(\tau-1)} & 0 & C_a^T & 0 \\ * & * & -\gamma^2 I & D_a^T & 0 \\ * & * & * & -P_a(\tau) & 0 \\ * & * & * & * & \psi_{a(\tau-1)}^3 \end{bmatrix} < 0, \quad (32)$$

where $\tilde{\psi}_{a(\tau-1)}^{11} \triangleq h_a[E_\epsilon P_a(\tau-1)E_\epsilon - sym\{\bar{Y}_a^T\}]$.

For (32), by virtue of the inequality

$$-\bar{Y}_a^T E_\epsilon^{-1} P_a^{-1}(\tau-1) E_\epsilon^{-1} \bar{Y} a \le E_\epsilon P_a(\tau-1) E_\epsilon - sym\{\bar{Y}_a^T\}$$

and Schur complement and pre- and post-multiplying the last inequality by $\mathrm{diag}\{(\bar{Y}_a^{-1})^T, I, I, I, I, I\}$ and $\mathrm{diag}\{\bar{Y}_a^{-1}, I, I, I, I, I\}$, respectively, one obtains that (32) implies (20). This completes the proof.

Theorem 3: For given scalar ε, positive scalars $\bar{\epsilon}$, h_a, l_a, $\bar{d}_a$ $(a \in \mathcal{N})$, γ and a matrix $\bar{I} \triangleq \begin{bmatrix} \bar{I}_1 & \bar{I}_4 \\ \bar{I}_2 & \bar{I}_3 \end{bmatrix}$ satisfying $\det(\bar{I}_1 + \varepsilon\bar{I}_2) \ne 0$, if there exist matrices $\bar{Y}_a \triangleq Y_a\bar{I}$ with $Y_a \triangleq \begin{bmatrix} Y_{11a} & \varepsilon Y_{11a} \\ Y_{21a} & Y_{22a} \end{bmatrix}$ and $\tilde{\mathcal{K}}_a$, symmetric positive definite matrices $P_a(\tau)$, $\tau \in \mathbb{Z}_{[0,\bar{d}_a]}$, positive diagonally dominant matrices $Q_{a(\tau)} \triangleq [q_{uv}^{a\tau}]_{u,v\in[1,n_x]}$, and symmetric matrices $R_a(\tau) \triangleq [r_{uv}^{a\tau}]_{u,v\in[1,n_x]}$, $\tau \in \mathbb{Z}_{[0,\bar{d}_a-1]}$, such that for $\forall\tau \in \mathbb{Z}_{[1,\bar{d}_a]}$, $\hat{\tau} \in \mathbb{Z}_{[0,\bar{d}_a]}$, $\forall a$, $b \in \mathcal{N}(a \ne b)$, and $\varkappa = 1, 2$, the relations (19), (21) and the following conditions hold:

$$\begin{bmatrix} \phi^{\varkappa}_{a(\tau-1)} & 0 & 0 & \hat{A}^T_a & \hat{\psi}^{15}_a \\ * & -Q_{a(\tau-1)} & 0 & C^T_a & 0 \\ * & * & -\gamma^2 I & D^T_a & 0 \\ * & * & * & -P_a(\tau) & 0 \\ * & * & * & * & \hat{\psi}^3_{a(\tau-1)} \end{bmatrix} < 0, \tag{33}$$

$$\bar{P}_b(0) - l_a \bar{P}_a(\tau) < 0, \tag{34}$$

$$Q_{a(\tau-1)} \bar{Q}_{a(\tau-1)} = I,\, P_a(\hat{\tau})\, \bar{P}_a(\hat{\tau}) = I, \tag{35}$$

where

$$\mathcal{K}_a \triangleq \begin{bmatrix} \tilde{\mathcal{K}}_a & \varepsilon\tilde{\mathcal{K}}_a \end{bmatrix},$$
$$\hat{\psi}^{15}_a \triangleq \begin{bmatrix} \bar{I}^T Y^T_a U^T_a + \bar{I}^T \mathcal{K}^T_a F^T_a & \bar{I}^T Y^T_a \end{bmatrix},$$
$$\hat{A}_a \triangleq A_a Y_a \bar{I} + B_a \mathcal{K}_a \bar{I},\, \hat{\psi}^3_{a(\tau-1)} \triangleq -\mathrm{diag}\{I, \bar{Q}_{a(\tau-1)}\},$$

then the closed-loop system (4) is MSESB with the prescribed $\mathcal{H}_\infty$ performance index $\bar{\gamma}$ and the desired controller gain in (3) is given by

$$\tilde{K}_a = \tilde{\mathcal{K}}_a Y^{-1}_{11a}. \tag{36}$$

Proof: By setting $\bar{Y}_a \triangleq Y_a \bar{I}$ and $\tilde{\mathcal{K}}_a \triangleq \tilde{K}_a Y_{11a}$, it can be concluded from (33)-(35) that (22) and (31) hold. Moreover, condition (33) implies $\phi^{\varkappa}_{a(\tau-1)} < 0$. Since $P_a(\tau-1)$ is a symmetric positive definite matrix, one can obtain $\bar{Y}_a > 0$. Then, combining with the form of Y_a and $\bar{I}$ presented above, one can obtain that Y_{11a} is invertible. Thus, the desired controller gain can be calculated by (36). This completes the proof.

Remark 7: Note that the conditions presented in Theorem 3 are not convex due to the constraint (35). To address this issue, the CCL algorithm is employed, by which the original nonconvex feasibility problem is transformed into the minimization problem constrained by a set of linear matrix inequalities [45]. The details are given as follows:

$$\text{Min } tr\left\{\sum\nolimits_{a=1}^{N}\left(\sum\nolimits_{\tau=0}^{\bar{d}_a-1} Q_{a\tau}\bar{Q}_{a\tau} + \sum\nolimits_{\hat{\tau}=0}^{\bar{d}_a} P_a(\hat{\tau})\bar{P}_a(\hat{\tau})\right)\right\}, \tag{37}$$

subject to (19), (21), (33), (34), and

$$\begin{bmatrix} Q_a(\tau) & I \\ I & \bar{Q}_a(\tau) \end{bmatrix} > 0, \quad \forall a \in \mathcal{N}, \tau \in \mathbb{Z}_{[0,\bar{d}_a-1]}, \tag{38}$$

$$\begin{bmatrix} P_a(\hat{\tau}) & I \\ I & \bar{P}_a(\hat{\tau}) \end{bmatrix} > 0, \quad \forall a \in \mathcal{N}, \hat{\tau} \in \mathbb{Z}_{[0,\bar{d}_a]}. \tag{39}$$

Detailedly, the critical process of the CCL algorithm is to transfer the solving of condition (19), (21) and (33)-(35) into verifying the feasibility of (37) subject to (19), (21), (33), (34), (38) and (39). If they are feasible, the matrices $Q_{a\tau}$, $P_a(\hat{\tau})$, $\tilde{\mathcal{K}}_a$ should be calculated. Moreover, we have to substitute these solved matrices into condition (19), (21), (22) and (31) to determine whether they are feasibility. Only when these condition are feasible, the expected controller gain can be calculated by (36). Although the adopted method are powerful in dealing with the circumstance where there are complex coupled nonlinear terms in the conditions to be solved, the implementation of this process is intricate and the presented conditions should be checked for $\forall\tau \in \mathbb{Z}_{[1,\bar{d}_a]}$, $\hat{\tau} \in \mathbb{Z}_{[0,\bar{d}_a]}$, $\forall a, b \in \mathcal{N}(a \neq b)$, which may bring in huge computation burden. Thus, exploring effective method with simplicity deserves further investigation.

Remark 8: In the decoupling process of Theorem 3, a dimension-adjusting matrix $\bar{I}$ is introduced. The main reason is that in the construction of matrix Y_a, the dimensions of Y_{11a} and εY_{11a} are the same. If the dimension-adjusting matrix $\bar{I}$ is not applied, one will have $\bar{Y}_a = Y_a$. As $\bar{Y}_a$ are square matrices, it means that the dimension of $\bar{Y}_a$ is required to be even, i.e., n_x is an even number. This would impose restrictions on applications of the proposed methods. Therefore, the matrix $\bar{I}$ is utilized for adjusting dimensions, as illustrated in Example 2. Furthermore, it can be noticed that condition (33) requires $Y_{11a}(\bar{I}_1 + \varepsilon\bar{I}_2) > 0$. Thus, the selected parameters in $\bar{I}$ satisfy the constraint that $\bar{I}_1 + \varepsilon\bar{I}_2$ is nonsingular.

IV. Examples

Example 1: Consider the discrete-time SMJSPS with three modes and the following parameters

$$A_1 = \varrho\begin{bmatrix} -1.36 & 0.69 \\ -1.81 & 0.57 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.31 & 0.24 \\ 0.68 & 0.43 \end{bmatrix},$$
$$A_2 = \varrho\begin{bmatrix} 1.34 & 0.62 \\ -0.37 & 0.36 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.42 & -0.61 \\ 0.21 & 0.34 \end{bmatrix},$$
$$A_3 = \varrho\begin{bmatrix} 1.31 & 1.14 \\ 0.21 & -0.61 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.36 & -0.13 \\ 0.24 & 0.62 \end{bmatrix},$$
$$B_1 = B_2 = B_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \quad F_1 = F_2 = F_3 = 1,$$
$$D_1 = D_2 = D_3 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T, \; U_1 = \begin{bmatrix} 0.6 & 0.3 \end{bmatrix},$$
$$U_2 = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \; \bar{I} = \mathrm{diag}\{1, 1\},$$

and $\varrho = 1$; $E_\epsilon = \mathrm{diag}\{1, \epsilon\}$ with $\bar{\epsilon} = 0.05$. The jumping among these modes is described by the SMC with the STUB being $\bar{d}_1 = 10$, $\bar{d}_2 = 6$ and $\bar{d}_3 = 8$. The TPM $\Pi \triangleq [\pi_{ab}]_{a,b\in\mathcal{N}}$ and STPDF $\Lambda(d) \triangleq [\varsigma_{ab}(d)]_{a,b\in\mathcal{N}}$ with partially unavailable information are given as

$$\Pi = \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0.15 & 0 & 0.85 \\ \tilde{\pi}_{31} & \tilde{\pi}_{32} & 0 \end{bmatrix},$$

$$\Lambda(d) = \begin{bmatrix} 0 & \dfrac{e^{-7}\cdot 7^d}{d!} & \tilde{\varsigma}_{13}(d) \\ \dfrac{e^{-5}\cdot 5^d}{d!} & 0 & \dfrac{0.6^d\cdot 0.4^{6-d}\cdot 6!}{(6-d)!d!} \\ 0.6^{(d-1)^{0.8}} - 0.6^{d^{0.8}} & \dfrac{0.5^8\cdot 8!}{(8-d)!d!} & 0 \end{bmatrix},$$

respectively, where $\tilde{\pi}_{31}$, $\tilde{\pi}_{32}$ and $\tilde{\varsigma}_{13}(d)$ are unavailable elements. Without loss of generality, for the unavailable elements, we consider that $\tilde{\pi}_{31} = s_1(k)$, $\tilde{\pi}_{32} = 1 - s_1(k)$, and $\tilde{\varsigma}_{13}(d) = s_2(k)(1 - s_2(k))^{d-1}$ with $s_1(k) \in [0, 1]$, $s_2(k) \in [0.65, 0.8]$. Besides, for the variation rate of LF, we set $h_a = 0.8$ and $l_a = 1.1$ with $a \in \{1, 2, 3\}$. Other parameters are assigned as $\gamma = 2$, $\varepsilon = 0.5$.

First of all, we show the effectiveness of the designed controller by comparing the state and output responses of the open-loop and closed-loop systems. Based on the conditions

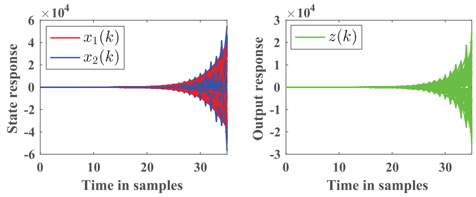

Fig. 2. The state and output responses of the open-loop system over 100 realizations.

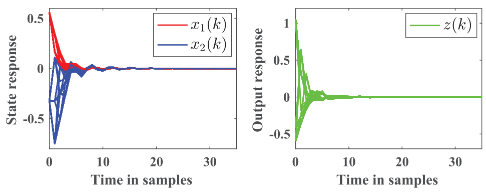

Fig. 3. The state and output responses of the closed-loop system over 100 realizations.

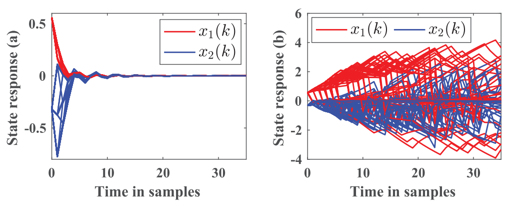

Fig. 4. The state responses of the closed-loop SMJSPS with (a) $l_1 = 0.75$, $l_2 = 0.82$, $l_3 = 0.88$, $h_1 = 0.92$, $h_2 = 0.95$, $h_3 = 0.98$ and (b) $l_1 = 16$, $l_2 = 15$, $l_3 = 17$, $h_1 = 14$, $h_2 = 17$, $h_3 = 16$ over 100 realizations.

presented in Theorem 3, the following controller gains are derived by using the CCL algorithm.

$$\left[\ \tilde{K}_1 \ \middle|\ \tilde{K}_2 \ \middle|\ \tilde{K}_3\ \right] = \left[\ 1.2791 \ \middle|\ -1.2556 \ \middle|\ -1.1270\ \right].$$

The external disturbance $\omega(k)$ is assumed as

$$\omega(k) = 0.8 \cdot \exp(-0.2k) \cdot \sin(8k).$$

The nonlinear function satisfying (2) is chosen as $g(\eta(k)) = \tanh(\eta(k))$. The initial condition is assigned as $x(0) = \left[0.56\ -0.32\right]^T$. Then, Fig. 2 shows the state and output responses of the open-loop system over 100 realizations. With the calculated controller gains and employing 100 randomly generated jump sequences, the state and output responses of the closed-loop system are depicted in Fig. 3. Obviously, an unstable SMJSPS turns to a stable one with the using of the designed controller, which implies the validity of the proposed control scheme.

In what follows, the influence of the LF variation rate at non-jumping and jumping instants on the SMJSPS stability is investigated. The state responses of the closed-loop SMJSPS with different LF variation rates, i.e., different h_a and l_a, $a \in \{1, 2, 3\}$, are depicted in Fig. 4 (over 100 random generated jump sequences), where h_a and l_a are taken as: (a) $l_1 = 0.75$, $l_2 = 0.82$, $l_3 = 0.88$, $h_1 = 0.92$, $h_2 = 0.95$, $h_3 = 0.98$ (the LF is required to decay at both non-jumping and jumping instants); (b) $l_1 = 16$, $l_2 = 15$, $l_3 = 17$, $h_1 = 14$, $h_2 = 17$, $h_3 = 16$ (the LF is allowed to increase at both non-jumping and jumping instants). It can be noted from Fig. 4 that to ensure the MSES and $\mathcal{H}_\infty$ performance of the SMJSPS, the LF should decrease at some non-jumping or jumping instants to make the LF attenuating as a whole.

TABLE I

THE MAXIMUM $\bar{\epsilon}_{\max}$ CORRESPONDING TO DIFFERENT γ

γ	0.5	1.0	1.5	2.0	2.5	3.0
$\bar{\epsilon}_{\max}$	0.0675	0.1288	0.1407	0.1452	0.1472	0.1488

TABLE II

THE OPTIMAL $\mathcal{H}_\infty$ PERFORMANCE INDEX $\bar{\gamma}_{\min}$ FOR DIFFERENT ε, $\bar{d}_a$ AND l_a, $a \in \{1, 2, 3\}$ WITH THE MAXIMUM ITERATION NUMBER $T_{max} = 100$

$\bar{\gamma}_{\min}$	$\bar{d}_1$=10, $\bar{d}_2$=6, $\bar{d}_3$=8			$\bar{d}_1$=8, $\bar{d}_2$=5, $\bar{d}_3$=6		
	ε=0.45	ε=0.50	ε=0.55	ε=0.45	ε=0.50	ε=0.55
$l_a = 0.6$	4.6368	38.8477	infeasible	4.0468	13.1851	infeasible
$l_a = 0.7$	2.3467	3.0829	8.8198	1.9713	2.5444	4.8006
$l_a = 0.8$	1.6146	1.9418	2.3198	1.4579	1.6454	2.0232

Algorithm 1 Calculate the Minimum γ for Different $\bar{d}_a$

Input: System parameters; Scalar: $\bar{d}_a$; Accuracy coefficient: $o(\gamma)$; Search interval: $[\gamma_{low}, \gamma_{up}]$; Maximum number of iterations: T_{max}; Number of iterations: T.

Output: The minimum $\gamma_{\min}$.

1: Set $Flag = 0$;

2: Let $\gamma_{use} = (\gamma_{low} + \gamma_{up})/2$. Check the conditions (19), (21), (33), (34), (38) and (39). If they are feasible, $Flag = 1$ and go to 3; else, go to 6;

3: Minimize (37) and calculate $\tilde{K}_a$, $P_a(\hat{\tau})$, $Q_a(\tau)$. Then, check the conditions (20), (22) with the calculated $\tilde{K}_a$, $P_a(\hat{\tau})$, $Q_a(\tau)$; If they are feasible, $\gamma_{up} = \gamma_{use}$ and go to 4; else, and go to 5;

4: If $\left|\gamma_{up} - \gamma_{low}\right| < o(\gamma)$, then $\gamma_{\min} = \gamma_{up}$ and go to 7; else, go to 2;

5: If $T < T_{max}$, go to 3; else, $\gamma_{low} = \gamma_{use}$ and go to 4;

6: If $Flag = 1$, then $\gamma_{low} = \gamma_{use}$ and go to 4; else, $\gamma_{\min} = Null$, go to 7;

7: **return** $\gamma_{\min}$.

Remark 9: To achieve the optimal performance index under different parameters or investigate the effect of the scalar γ on the upper bound of the SPP, the minimum γ is calculated for different $\bar{d}_a$. For this purpose, Algorithm 1 is provided based on Theorem 3 and Remark 7.

Next, the relationship between γ and the SPP upper bound $\bar{\epsilon}$ is discussed. Using our calculation method, the maximum $\bar{\epsilon}$ corresponding to a certain γ is denoted as $\bar{\epsilon}_{\max}$ and its values are listed in Table I. It can be observed that the maximum SPP upper bound grows larger as γ increases, which means that weak system performance may result in a larger upper bound of the SPP.

Furthermore, the influence of ε, STUB $\bar{d}_a$, and variation rate l_a on system performance $\bar{\gamma}$ is explored. For simplicity, we set $h_a = 1$. Other parameters are the same as before. The minimum values of $\bar{\gamma}$, i.e., $\bar{\gamma}_{\min}$, for different ε, $\bar{d}_a$ and l_a, are presented in Table II, from which one can observe that within a certain range a decrease of l_a or an increase of ε may lead to worse system performance.

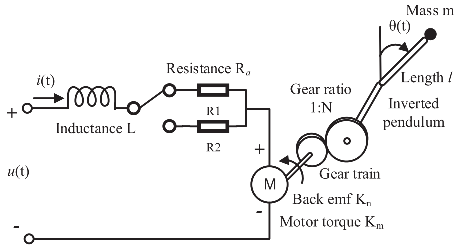

Fig. 5. Structure of inverted pendulum controlled by DC motor.

Example 2: To further verify the applicability of the proposed methods, we consider an inverted pendulum controlled by a DC motor [46], which is presented in Fig. 5 and can be described as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = g/l \sin(x_1(t)) + NK_m/(ml^2)x_3(t), \\ L\dot{x}_3(t) = u(t) - K_n N x_2(t) - R_a x_3(t) + \omega(k), \quad a \in \{1,2\}, \end{cases}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ represent the angle $\theta(t)$, the angular velocity $\dot{\theta}(t)$ and the current $i(t)$, respectively. $u(t)$ is an input voltage and $\omega(k)$ is a disturbance. Denote $\epsilon = L = 0.05H$. Then, by employing discretizing methods similar to those in [46] with the same system parameters, one can obtain the following system model:

$$\begin{cases} x(k+1) = A_a E_\epsilon x(k) + B_a u(k) + D_a \omega(k), \\ z(k) = U_a E_\epsilon x(k) + F_a u(k), \end{cases}$$

where

$$E_\epsilon = \mathrm{diag}\{1, 1, \epsilon\}, \quad U_a = \left[\begin{array}{cc|c} 1 & 1 & 1 \end{array}\right], F_a = 0.4,$$

$$A_1 = \left[\begin{array}{cc|c} 1.0313 & 0.0796 & 0.0020 \\ 0.7805 & 0.9913 & 0.0397 \\ \hline -0.0196 & -0.0397 & 0.0089 \end{array}\right],$$

$$A_2 = \left[\begin{array}{cc|c} 1.0313 & 0.0797 & 0.0018 \\ 0.7812 & 0.9944 & 0.0354 \\ \hline -0.0181 & -0.0354 & 0.0063 \end{array}\right],$$

$$B_1 = D_1 = \left[\begin{array}{c} 0.0239 \\ 0.7993 \\ \hline 0.7818 \end{array}\right], \quad B_2 = D_2 = \left[\begin{array}{c} 0.0180 \\ 0.5596 \\ \hline 0.4708 \end{array}\right].$$

In addition, the STUBs are given as $\bar{d}_1 = 8$, $\bar{d}_2 = 5$. The TPM and STPDF are assigned as

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Lambda(d) = \begin{bmatrix} 0 & \tilde{\varsigma}_{12}(d) \\ \dfrac{0.6^d \cdot 0.4^{5-d} \cdot 5!}{(5-d)!d!} & 0 \end{bmatrix},$$

where $\tilde{\varsigma}_{12}(d)$ is the partially unavailable element of SMK. Considering that $\tilde{\varsigma}_{12}(d) = s_3(k)(1 - s_3(k))^{d-1}$ with $s_3(k) \in [0.5, 0.8]$, then, it can be calculated from (5) and Remark 2 that $\mu_1 = \sum_{d=1}^{\bar{d}_1} \tilde{\varsigma}_{12}(d) \geq 0.9961$ and $\mu_2 = \sum_{d=1}^{\bar{d}_2} \frac{0.6^d \cdot 0.4^{5-d} \cdot 5!}{(5-d)!d!} = 0.9898$. Thus, the selection of $\bar{d}_1$ and $\bar{d}_2$ is reasonable when the partially known STPDF $\Lambda(d)$ is taken as above. The dimension-adjusting matrix $\bar{I}$ is chosen as $\bar{I} = \begin{bmatrix} \bar{I}_1 & 0 \\ \bar{I}_2 & \bar{I}_3 \end{bmatrix}$ with $\bar{I}_2 = \begin{bmatrix} 0 & 0 \\ 0.8 & 0.6 \end{bmatrix}$, $\bar{I}_3 = \begin{bmatrix} 0.9 & -0.3 \end{bmatrix}$, $\bar{I}_1 = \mathrm{diag}\{0.8, 0.5\}$.

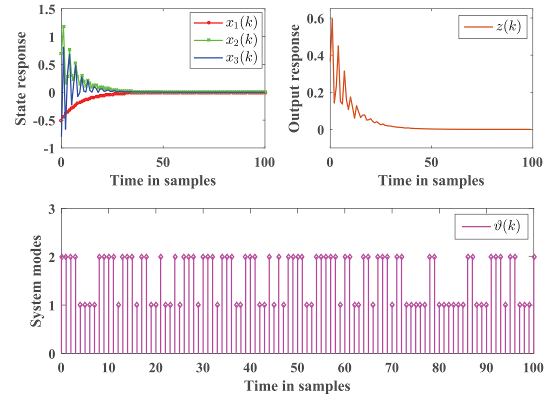

Fig. 6. The state and output responses of the closed-loop system and the evolution of system mode $\vartheta(k)$.

In addition, we set $\gamma = 5$, $\varepsilon = 0.4$, $h_a = 0.88$, $l_a = 1.1$ with $a \in \{1, 2, 3\}$. Based on Theorem 3 and Lemma 7, one obtains

$$\left[\begin{array}{c|c} \tilde{K}_1 & \tilde{K}_2 \end{array}\right] = \left[\begin{array}{cc|cc} -2.5110 & -1.1421 & -3.3018 & -1.6249 \end{array}\right].$$

With the disturbance $\omega(k) = \exp(-0.2k) \cdot \sin(2k)$ and the initial condition $x(0) = \begin{bmatrix} -0.5 & 0.7 & -0.8 \end{bmatrix}^T$, the state and output responses, as well as the evolution of the system modes, are presented in Fig. 6. It can be observed from Fig. 6 that the state responses converge to zero with the designed controller, which validates effectiveness of the proposed methods.

V. Conclusion

The $\mathcal{H}_\infty$ controller design problem for discrete-time semi-Markov jump singularly perturbed systems with partially available semi-Markov kernel information has been addressed in this paper, where the repeated scalar nonlinearities and external disturbance have also been considered. A novel mean-square exponential stability criterion for discrete-time semi-Markov jump systems has been established via the analysis of the variation trend of the Lyapunov function. Then, based on the derived criteria, a set of sufficient conditions, which guarantee the mean-square exponential stability and $\mathcal{H}_\infty$ performance of the resulting closed system, have been constructed. Furthermore, the cone complementary linearization algorithm has been employed to deal with a nonconvex condition to obtain specific controller gain. Finally, simulation results have been provided to prove validity of the proposed method. It should be noted that the derived control scheme is centered on systems without time delays. As time delays are non-negligible in practical engineering [47], [48], extending our results to the study of the control synthesis issue for time-delayed hidden semi-Markov jump systems deserving further exploration. Besides, since the conditions obtained in this paper are sufficient, probing superior approaches to further reduce the conservatism of the results is also a significant issue worthy of in-depth investigation.

References

[1] K. Shi, Y. Tang, S. Zhong, C. Yin, X. Huang, and W. Wang, "Nonfragile asynchronous control for uncertain chaotic Lurie network systems with Bernoulli stochastic process," *Int. J. Robust Nonlinear Control*, vol. 28, no. 5, pp. 1693–1714, Mar. 2018.

[2] M. Zhang, P. Shi, C. Shen, and Z.-G. Wu, "Static output feedback control of switched nonlinear systems with actuator faults," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 8, pp. 1600–1609, Aug. 2020.

[3] M. Sathishkumar, R. Sakthivel, F. Alzahrani, B. Kaviarasan, and Y. Ren, "Mixed H_∞ and passivity-based resilient controller for nonhomogeneous Markov jump systems," *Nonlinear Anal., Hybrid Syst.*, vol. 31, pp. 86–99, Feb. 2019.

[4] J. Wang, Z. Huang, Z. Wu, J. Cao, and H. Shen, "Extended dissipative control for singularly perturbed PDT switched systems and its application," *IEEE Trans. Circuits Syst. I, Reg. Papers*, early access, Sep. 17, 2020, doi: 10.1109/TCSI.2020.3022729.

[5] P. Shi, M. Liu, and L. Zhang, "Fault-tolerant sliding-mode-observer synthesis of Markovian jump systems using quantized measurements," *IEEE Trans. Ind. Electron.*, vol. 62, no. 9, pp. 5910–5918, Sep. 2015.

[6] K. Shi, J. Wang, S. Zhong, Y. Tang, and J. Cheng, "Non-fragile memory filtering of T-S fuzzy delayed neural networks based on switched fuzzy sampled-data control," *Fuzzy Sets Syst.*, vol. 394, pp. 40–64, Sep. 2020.

[7] P. Bolzern, P. Colaneri, and G. De Nicolao, "Almost sure stability of Markov jump linear systems with deterministic switching," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 209–214, Jan. 2013.

[8] J. Man, X. Song, and J. Lu, "Nonfragile memory-based output feedback control for fuzzy Markov jump generalized neural networks with reaction-diffusion terms," *Int. J. Innov. Comput. Inf. Control*, vol. 15, no. 5, pp. 1609–1628, 2019.

[9] S. Lakshmanan, F. A. Rihan, R. Rakkiyappan, and J. H. Park, "Stability analysis of the differential genetic regulatory networks model with time-varying delays and Markovian jumping parameters," *Nonlinear Anal., Hybrid Syst.*, vol. 14, pp. 1–15, Nov. 2014.

[10] T. Ru, J. Xia, X. Huang, X. Chen, and J. Wang, "Reachable set estimation of delayed fuzzy inertial neural networks with Markov jumping parameters," *J. Franklin Inst.*, vol. 357, no. 11, pp. 6882–6898, Jul. 2020.

[11] K. Shi, J. Wang, Y. Tang, and S. Zhong, "Reliable asynchronous sampled-data filtering of T–S fuzzy uncertain delayed neural networks with stochastic switched topologies," *Fuzzy Sets Syst.*, vol. 381, pp. 1–25, Feb. 2020.

[12] Y. Wang, X. Hu, K. Shi, X. Song, and H. Shen, "Network-based passive estimation for switched complex dynamical networks under persistent dwell-time with limited signals," *J. Franklin Inst.*, vol. 357, pp. 10921–10936, Oct. 2020.

[13] J. Wang, C. Yang, H. Shen, J. Cao, and L. Rutkowski, "Sliding-mode control for slow-sampling singularly perturbed systems subject to Markov jump parameters," *IEEE Trans. Syst., Man, Cybern. Syst.*, early access, Mar. 24, 2020, doi: 10.1109/TSMC.2020.2979860.

[14] J. Wang, J. Xia, H. Shen, M. Xing, and J. H. Park, "H_∞ synchronization for fuzzy Markov jump chaotic systems with piecewise-constant transition probabilities subject to PDT switching rule," *IEEE Trans. Fuzzy Syst.*, early access, Jul. 29, 2020, doi: 10.1109/TFUZZ.2020.3012761.

[15] H. Yan, Y. Tian, H. Li, H. Zhang, and Z. Li, "Input–output finite-time mean square stabilization of nonlinear semi-Markovian jump systems," *Automatica*, vol. 104, pp. 82–89, Jun. 2019.

[16] J. Wang, Y. Wang, H. Yan, J. Cao, and H. Shen, "Hybrid event-based leader-following consensus of nonlinear multiagent systems with semi-Markov jump parameters," *IEEE Syst. J.*, early access, Oct. 23, 2020, doi: 10.1109/JSYST.2020.3029156.

[17] F. Li, L. Wu, P. Shi, and C.-C. Lim, "State estimation and sliding mode control for semi-Markovian jump systems with mismatched uncertainties," *Automatica*, vol. 51, pp. 385–393, Jan. 2015.

[18] Y.-A. Liu, J. Xia, B. Meng, X. Song, and H. Shen, "Extended dissipative synchronization for semi-Markov jump complex dynamic networks via memory sampled-data control scheme," *J. Franklin Inst.*, vol. 357, pp. 10900–10920, Oct. 2020.

[19] J. Wang, T. Ru, H. Shen, J. Cao, and J. H. Park, "Finite-time $\mathcal{L}_2$-$\mathcal{L}_\infty$ synchronization for semi-Markov jump inertial neural networks using sampled data," *IEEE Trans. Netw. Sci. Eng.*, early access, Oct. 19, 2020, doi: 10.1109/TNSE.2020.3032025.

[20] T. Wu, X. Huang, X. Chen, and J. Wang, "Sampled-data H_∞ exponential synchronization for delayed semi-Markov jump CDNs: A looped-functional approach," *Appl. Math. Comput.*, vol. 377, Jul. 2020, Art. no. 125156.

[21] H. Shen, M. Dai, Y. Luo, J. Cao, and M. Chadli, "Fault-tolerant fuzzy control for semi-Markov jump nonlinear systems subject to incomplete SMK and actuator failures," *IEEE Trans. Fuzzy Syst.*, early access, Jul. 24, 2020, doi: 10.1109/TFUZZ.2020.3011760.

[22] W. Assawinchaichote and S. K. Nguang, "H_∞ filtering for fuzzy singularly perturbed systems with pole placement constraints: An LMI approach," *IEEE Trans. Signal Proc.*, vol. 52, no. 6, pp. 1659–1667, Jun. 2004.

[23] J. Dong and G.-H. Yang, "H_∞ control for fast sampling discrete-time singularly perturbed systems," *Automatica*, vol. 44, no. 5, pp. 1385–1393, 2008.

[24] J. Song, Y. Niu, and H.-K. Lam, "Reliable sliding mode control of fast sampling singularly perturbed systems: A redundant channel transmission protocol approach," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 66, no. 11, pp. 4490–4501, Nov. 2019.

[25] Y. Wang, P. Shi, and H. Yan, "Reliable control of fuzzy singularly perturbed systems and its application to electronic circuits," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 65, no. 10, pp. 3519–3528, Oct. 2018.

[26] H. Li, Y. Wang, D. Yao, and R. Lu, "A sliding mode approach to stabilization of nonlinear Markovian jump singularly perturbed systems," *Automatica*, vol. 97, pp. 404–413, Nov. 2018.

[27] L. Zhang, Y. Leng, and P. Colaneri, "Stability and stabilization of discrete-time semi-Markov jump linear systems via semi-Markov kernel approach," *IEEE Trans. Autom. Control*, vol. 61, no. 2, pp. 503–508, Feb. 2016.

[28] S. H. Kim, "Stochastic stability and stabilization conditions of semi-Markovian jump systems with mode transition-dependent sojourn-time distributions," *Inf. Sci.*, vols. 385–386, pp. 314–324, Apr. 2017.

[29] J. Huang and Y. Shi, "Stochastic stability and robust stabilization of semi-Markov jump linear systems," *Int. J. Robust Nonlinear Control*, vol. 23, no. 18, pp. 2028–2043, Dec. 2013.

[30] Z. Ning, L. Zhang, and W. X. Zheng, "Observer-based stabilization of nonhomogeneous semi-Markov jump linear systems with mode-switching delays," *IEEE Trans. Autom. Control*, vol. 64, no. 5, pp. 2029–2036, May 2019.

[31] B. Wang and Q. Zhu, "Stability analysis of discrete time semi-Markov jump linear systems," *IEEE Trans. Autom. Control*, early access, Mar. 3, 2020, doi: 10.1109/TAC.2020.2977939.

[32] H. Shen, F. Li, S. Xu, and V. Sreeram, "Slow state variables feedback stabilization for semi-Markov jump systems with singular perturbations," *IEEE Trans. Autom. Control*, vol. 63, no. 8, pp. 2709–2714, Aug. 2018.

[33] M. G. Todorov, M. D. Fragoso, and O. L. D. V. Costa, "Detector-based H_∞ results for discrete-time Markov jump linear systems with partial observations," *Automatica*, vol. 91, pp. 159–172, May 2018.

[34] L. Zhang and J. Lam, "Necessary and sufficient conditions for analysis and synthesis of Markov jump linear systems with incomplete transition descriptions," *IEEE Trans. Autom. Control*, vol. 55, no. 7, pp. 1695–1701, Jul. 2010.

[35] W.-J. Lin, Y. He, C.-K. Zhang, Q.-G. Wang, and M. Wu, "Reachable set estimation for discrete-time Markovian jump neural networks with generally incomplete transition probabilities," *IEEE Trans. Cybern.*, early access, Aug. 13, 2019, doi: 10.1109/TCYB.2019.2931008.

[36] L. Zhang, Z. Ning, and Y. Shi, "Analysis and synthesis for a class of stochastic switching systems against delayed mode switching: A framework of integrating mode weights," *Automatica*, vol. 99, pp. 99–111, Jan. 2019.

[37] Z. Ning, L. Zhang, and P. Colaneri, "Semi-Markov jump linear systems with incomplete sojourn and transition information: Analysis and synthesis," *IEEE Trans. Autom. Control*, vol. 65, no. 1, pp. 159–174, Jan. 2020.

[38] T. Yang, L. Zhang, and H.-K. Lam, "H_∞ fuzzy control of semi-Markov jump nonlinear systems under error mean square stability," *Int. J. Syst. Sci.*, vol. 48, no. 11, pp. 2291–2299, Aug. 2017.

[39] Y. Tian, H. Yan, H. Zhang, X. Zhan, and Y. Peng, "Dynamic output-feedback control of linear semi-Markov jump systems with incomplete semi-Markov kernel," *Automatica*, vol. 117, Jul. 2020.

[40] R. Sakthivel, K. Sundareswari, K. Mathiyalagan, A. Arunkumar, and S. M. Anthoni, "Robust reliable H_∞ control for discrete-time systems with actuator delays," *Asian J. Control*, vol. 17, no. 6, pp. 2133–2142, 2015.

[41] R. Sakthivel, K. Mathiyalagan, and S. M. Anthoni, "Robust $\mathcal{H}_\infty$ control for uncertain discrete-time stochastic neural networks with time-varying delays," *IET Control Theory Appl.*, vol. 6, no. 9, pp. 1220–1228, Jun. 2012.

[42] V. Barbu and N. Limnios, "Empirical estimation for discrete-time semi-Markov processes with applications in reliability," *J. Nonparametric Statist.*, vol. 18, nos. 7–8, pp. 483–498, Oct. 2006.

[43] E. Cinlar, *Introduction to Stochastic Processes*. Upper Saddle River, NJ, USA: Prentice-Hall, 1975.

[44] Y.-C. Chu and K. Glover, "Bounds of the induced norm and model reduction errors for systems with repeated scalar nonlinearities," *IEEE Trans. Autom. Control*, vol. 44, no. 3, pp. 471–483, Mar. 1999.

[45] L. El Ghaoui, F. Oustry, and M. AitRami, "A cone complementarity linearization algorithm for static output-feedback and related problems," *IEEE Trans. Autom. Control*, vol. 42, no. 8, pp. 1171–1176, Aug. 1997.

[46] F. Li, S. Xu, and B. Zhang, "Resilient asynchronous H_∞ control for discrete-time Markov jump singularly perturbed systems based on hidden Markov model," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 50, no. 8, pp. 2860–2869, Aug. 2019.

[47] H. Chen and P. Shi, "Impulsive control and exponential stability for nonlinear time-varying systems with time-varying delays," *ICIC Express Lett. B, Appl.*, vol. 8, no. 8, pp. 1181–1189, 2017.

[48] Z. Liu, J. Yu, L. Zhao, Y. Ma, B. Xue, and S. Cheng, "Adaptive H_∞ sliding mode control for a class of uncertain Markovian jump systems with time-delay," *ICIC Express Lett.*, vol. 14, no. 4, pp. 319–327, 2020.

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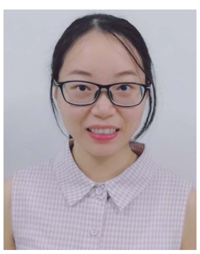

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