Adaptive Fuzzy Output-Feedback Control Design for a Class of p-Norm Stochastic Nonlinear Systems With Output Constraints

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***Abstract*— This paper considers the control problem of *p*-norm stochastic nonlinear systems with output constraints, while the system nonlinearities are completely unknown and the system states are unavailable except the output. A nonlinear observer is constructed to estimate the unmeasurable states. Then, based on the constructed observer and a tan-type barrier Lyapunov function (BLF), an adaptive fuzzy output-feedback control strategy is developed by combining the technique of adding a power integrator with the fuzzy logic systems (FLSs). The proposed scheme enables that all the signals of the considered closed-loop systems are bounded in probability while the prespecified output constraint is not violated. Finally, a numerical example verifies the validation of the proposed scheme.**

***Index Terms*— Fuzzy output-feedback control, stochastic nonlinear systems, adding a power integrator, output constraints, state observer.**

I. Introduction

OVER past decades, various control problems of nonlinear systems have always been concerned in the control field, such as event-triggered control [1], sliding-mode control [2], [3], finite-time control [4], [5], neural/fuzzy control [6]–[8] and so on. In recent years, the constrained control problem of nonlinear systems has received growing attention since many real systems are often subject to output/state constraints for performance specifications and/or safety reasons [9]. Since the barrier Lyapunov function has been proposed and verified to be a useful tool for dealing with the constraints [10], lots of BLF-based strategies have been subsequently developed to address the control issues for different deterministic and stochastic nonlinear systems with output/state constraints. After the work of [10], references [11] and [12] have respectively solved the adaptive state-feedback control and event-triggered control by means of log-type BLFs for pure-feedback and interconnected nonlinear systems with state constraints, and the integral BLFs have been proposed to address the adaptive control problem in [13], [14]. Later, the BLF-based approaches have been extended to stochastic nonlinear systems. For examples, an adaptive controller and a finite-time controller have been successively designed by using tan-type BLFs in [15] and [16] for strict-feedback stochastic systems with output or full-state constraints. Ulteriorly, references [17], [18] have proposed two novel tan-type BLFs and considered the second-order sliding mode control of nonlinear constrained systems. Meanwhile, some adaptive neural or fuzzy control schemes have also been recently presented for some kinds of nonlinear systems subject to output/state constraints and unknown nonlinear functions. For instances, reference [19] has addressed the adaptive fuzzy constrained control for multi-input multi-output nonstrict-feedback nonlinear systems; the adaptive fuzzy and neural control problems have been considered for stochastic constrained nonlinear switched systems with unknown nonlinearities in [20] and [21], respectively.

It should be noted that above methods are strictly restricted on the state-feedback control field. Nevertheless, it is generally known that the system states are rarely completely measurable, or are observable with requiring high observation costs in the practical systems [22], [23]. For this reason, the output-feedback control has been paid much attention since it does not need the measurability of the states. For stochastic nonlinear systems without constraints, the results about output-feedback control are rich. References [24]–[29] have focused on studying the output-feedback control problem for stochastic nonlinear systems with satisfying some growth conditions or with unknown nonlinearities. In contrast with stochastic systems without constraints, few valuable results of output-feedback control have been obtained for stochastic constrained nonlinear systems. An observer-based control strategy has been proposed for strict-feedback stochastic nonlinear systems with an output constraint in [30], and reference [31] has investigated the neural output-feedback control for stochastic full-state-constrained systems.

However, the stochastic nonlinear systems considered in above-mentioned works are all strict-feedback systems rather than p-norm stochastic ones with the ratios of positive odd integers as the fractional powers. Compared with the

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strict-feedback systems, the research on p-norm stochastic nonlinear systems has started relatively later and is more challenging due to the lack of controllability/existence in the Jacobian linearization. Also for this reason, such a class of stochastic systems have attracted more interest. By taking advantage of the adding a power integrator technique [32], the state-feedback or output-feedback control have been well studied under some nonlinear growth conditions for such kinds of systems without any constraints [33]–[39]. Recently, the researching topics of p-norm stochastic nonlinear systems have been extended to the control problems in the cases of unknown nonlinearities or output constraints. The requirements of growth conditions in [33]–[39] have been removed by references [40], [41], and subsequently the decentralized neural state-feedback controller has been constructed for large-scale stochastic high-order nonlinear systems. Also, references [42], [43] have further addressed the neural control for switched stochastic high-order systems. The progress of constrained control of p-norm stochastic nonlinear systems has only been made in recent three years, which has been motivated by the achievements of p-norm deterministic constrained systems [44]–[46]. Based on some growth conditions, two state-feedback control algorithms have been successively developed for p-norm stochastic nonlinear systems with symmetric and asymmetric output constraints in [47] and [48]. Meanwhile, references [49]–[51] have further considered the neural or fuzzy control for this kind of systems subject to output/state constraints.

It is however worth noting that all above works [47]–[51] about p-norm stochastic constrained systems are strictly required to be available on the states, which implies that the proposed constrained control schemes would be invalid when only the output is available. What's more, different from the strict-feedback nonlinear system, the defined error dynamic systems of p-norm stochastic nonlinear systems definitely contain the items with exponential powers, which causes the proposed output-feedback constrained control schemes in [30], [31] can't be extended to solve the output-feedback control problem of p-norm stochastic constrained systems. To our best knowledge, up to now, no results are available for p-norm stochastic nonlinear systems with output constraints, unknown nonlinearities, and some unmeasurable states.

Motivated by this observation, in this paper, we investigate the output-feedback control issue for p-norm stochastic nonlinear systems with output constraints and unknown nonlinearities. This issue is challenging in at least two aspects. First, an adaptive fuzzy controller needs to be contrived to concurrently handle unknown nonlinearities, unmeasurable states and output constraints. Second, both the reconstructed observer-based system and the error dynamic system are p-norm nonlinear systems, which makes the stability analysis very difficult. To resolve these obstacles, a BLF-based adaptive fuzzy output-feedback control approach is developed, which shows all the signals of the considered system are bounded in probability while the output constraint requirement is achieved. Compared with the existing results, the contributions of this paper are mainly reflected in three aspects:

1) *It is first time to investigate p-norm stochastic nonlinear systems with output constraints, unavailable states and unknown nonlinearities.* Notably, the existing control strategies for p-norm stochastic constrained nonlinear systems are mainly based on the requirement that all the states are measurable (e.g., [49]–[51]), while this work considers the constrained control problem in the case of that only the system output is available for feedback design. Moreover, the system nonlinearities in this paper are totally unknown instead of satisfying some growth conditions which is required in [33]–[39]. In other words, three types of restrictions (i.e., output constraints, unknown nonlinearities and unavailable states) are simultaneously taken into account. Hence, our work can develop the control theory of p-norm systems.
2) *A nonlinear observer is constructed to estimate the unmeasurable states.* Different from the strict-feedback stochastic nonlinear systems [30], [31], the state-observer for p-norm stochastic nonlinear systems is a p-norm nonlinear system. Thus, the observer gains can't be directly gotten from the Hurwitz matrix, but need to be obtained by analyzing the stability of the defined error dynamic system.
3) *A BLF-based fuzzy output-feedback control strategy is developed.* A tan-type BLF is adopted to handle the output constraint. Based on this BLF and the constructed observer, an output-feedback control strategy is proposed by combining the FLSs into the adding a power integrator technique. It is proved that the proposed strategy can guarantee the boundness of all the signals in the considered systems without violating the output constraint.

This article is organized as follows. The problem statement and preliminaries are presented in Section II. Section III provides the fuzzy output-feedback controller design and theoretical analysis. In Section IV, a numerical simulation example is given. Finally, Section V concludes this paper.

II. Problem Statement and Preliminaries

A. Problem Statement

Consider the following class of p-norm stochastic nonlinear systems

$$\begin{aligned} dx_i &= x_{i+1}^{p_i}dt + f_i(\bar{x}_i)dt + g_i^T(\bar{x}_i)d\omega, \\ &\quad i = 1, \cdots, n-1, \\ dx_n &= u^{p_n}dt + f_n(x)dt + g_n^T(x)d\omega, \\ y &= x_1, \end{aligned} \tag{1}$$

where $x = (x_1, \cdots, x_n)^T \in R^n$, $u \in R$ and $y \in R$ are the system state vector, control input and output, respectively; ω denotes a m-dimensional independent standard Brownian motion; for $i = 1, \cdots, n$, $\bar{x}_i = (x_1, \cdots, x_i)^T \in R^i$, the fractional power $p_i \in R_{odd}^{\geq 1} := \{s/r | s \geq r,\ s$ and r are positive odd integers$\}$, $f_i : R^i \to R$ and $g_i : R^i \to R^m$ are unknown functions satisfying locally Lipschitz continuous condition and $f_i(0) = 0$, $g_i(0) = 0$. The system output $y = x_1$ is measurable and required to remain in the set $\Pi_1 = \{y(t) \in R, |y(t)| < \varepsilon\}$ with a given constant $\varepsilon > 0$, while other states $x_2, \cdots, x_n$ are all unmeasurable.

As an important kind of stochastic nonlinear systems, the p-norm stochastic nonlinear systems (1) can be applied to model many practical systems, such as, the under-actuated and weakly coupled mechanical system [33] and coupled inverted double pendulums [34]. It should be noted that, system (1) is an extension of the standard strict-feedback stochastic nonlinear systems (i.e., $p_1 = \cdots = p_n = 1$), which means the proposed method can also be applicable to strict-feedback ones.

The objective of the paper is to develop a fuzzy output-feedback control scheme for system (1), which can enable that all the variables of system (1) are bounded in probability with keeping the system output within the pre-given set.

B. Preliminaries

As preliminaries, a definition, some lemmas and an assumption are following listed.

Consider the stochastic system as below

$$dx = f(x)dt + g(x)d\omega, \tag{2}$$

where ω and x respectively represent a standard Brownian motion and the system state; $f(x)$ and $g(x)$ are locally Lipschitz continuous functions with $f(0) = 0,\ g(0) = 0$.

Definition 1: [5] For any C^2 function $V(x)$ related to system (2), the differential operator $\mathcal{L}$ is defined as:

$$\mathcal{L}V = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} tr\left\{ g^T(x) \frac{\partial^2 V}{\partial x^2} g(x) \right\}. \tag{3}$$

Lemma 1: [19] For system (2), if there exists a C^2 Lyapunov function $V(x)$ such that

$$\begin{cases} \varpi_1(|x|) \le V(x) \le \varpi_2(|x|) \\ \mathcal{L}V(x) \le -\pi_0 V(x) + \upsilon_0 \end{cases}, \forall x \in R^n, t \ge 0,$$

then, the system has a unique solution satisfying

$$E\{V(x(t))\} \le V(x_0)e^{-\pi_0 t} + \upsilon_0/\pi_0, \forall t \ge 0,$$

and all variables in system (2) are bounded in probability, where ϖ_1, ϖ_2 are $\mathcal{K}_\infty$ functions, and $\pi_0, \upsilon_0 > 0$ are constants.

Lemma 2: [3] For any real numbers $q \ge 1$ and any variables $\zeta_1, \zeta_2 \in R$, one has

(i) $|\zeta_1^q - \zeta_2^q| \le q(2^{q-2}+2)|\zeta_1 - \zeta_2|(|\zeta_1 - \zeta_2|^{q-1} + \zeta_2^{q-1})$,

(ii) $(|\zeta_1| + |\zeta_2|)^{\frac{1}{q}} \le |\zeta_1|^{\frac{1}{q}} + |\zeta_2|^{\frac{1}{q}} \le 2^{1-\frac{1}{q}}(|\zeta_1| + |\zeta_2|)^{\frac{1}{q}}$.

Lemma 3: [44] For any positive real numbers $k_1, k_2, \vartheta, \varsigma$ and any variables $\zeta_1, \zeta_2 \in R$, the following inequality holds

$$\vartheta|\zeta_1|^{k_1}|\zeta_2|^{k_2} \le \varsigma \frac{k_1}{k_1+k_2}|\zeta_1|^{k_1+k_2} + \frac{k_1}{k_1+k_2}\vartheta^{\frac{k_1+k_2}{k_2}}\varsigma^{-\frac{k_1}{k_2}}|\zeta_2|^{k_1+k_2}.$$

Lemma 4: [3] For $q \in (0, \infty)$, $c = \max\{n^{q-1}, 1\}$ and $\forall \zeta_i \in R, i = 1, \cdots, n,$ we have

$$(|\zeta_1| + \cdots + |\zeta_n|)^q \le c(|\zeta_1|^q + \cdots + |\zeta_n|^q).$$

Lemma 5: [36] If $p > 1$ is an odd number and ζ_1, ζ_2 are any real numbers, then

$$-(\zeta_1 - \zeta_2)(\zeta_1^p - \zeta_2^p) \le -\frac{1}{2^{p-1}}(\zeta_1 - \zeta_2)^{p+1}.$$

Lemma 6: [52] Suppose $F(Z)$ is a continuous function defined on a compact set Π_0. Then, for any positive constant δ, there is a fuzzy logic system $\Upsilon^T\Psi(X)$ such that

$$\sup_{Z \in \Pi_0} |F(Z) - \Upsilon^T\Psi(Z)| \le \delta,$$

where $\Upsilon = (\upsilon_1, \cdots, \upsilon_N)^T$ is the ideal constant weight vector, $N > 1$ is the number of the fuzzy rules, $\Psi(Z) = \frac{(\psi_1(Z), \cdots, \psi_N(Z))^T}{\sum_{j=1}^N \psi_j(Z)}$ is the basis function vector with $\psi_j(Z)(j = 1, \cdots, N)$ being usually chosen as Gaussian functions.

Assumption 1: Suppose the fractional powers satisfy

$$p_1 \ge p_2 \ge \cdots \ge p_n \ge 1.$$

Remark 1: 1) In view of Lemma 6, any continuous function $F(Z)$ defined on a compact set Π_0 could be approximated as $F(Z) = \Upsilon^T\Psi(Z) + \epsilon(Z)$, where $\epsilon(Z)$ is the approximation error with $|\epsilon(Z)| < \delta$ ($\delta > 0$ is a given constant).

2) Assumption 1 shows that the fractional powers in system (1) have a known constant upper bound, and there is a certain relation among these powers. Evidently, Assumption 1 is standard and common, which can be found in the existing related works (e.g. [35], [38]).

III. Main Results

First of all, introduce a coordinate transformation as

$$\chi_i = \frac{x_i}{H^{r_i}}, i = 1, \cdots, n, \tilde{u} = \frac{u}{H^{r_{n+1}}}, \tag{4}$$

where $r_1 = 0$, $r_{j+1} = \frac{r_j+1}{p_j}(j = 1, \cdots, n)$ and the scaling gain $H > 1$ is a constant to be determined later. By adjusting the value of H, the drift and diffusion terms (i.e., $f_i(\cdot)$'s and $g_i(\cdot)$'s) can be dominated. Thus, H plays an important role in the later controller design and stability analysis.

According to (4), system (1) turns into an equivalent system

$$\begin{aligned} d\chi_i &= H\chi_{i+1}^{p_i}dt + \phi_i(\overline{\chi}_i)dt + h_i^T(\overline{\chi}_i)d\omega, \\ &\quad i = 1, \cdots, n-1, \\ d\chi_n &= H\tilde{u}^{p_n}dt + \phi_n(\chi)dt + h_n^T(\chi)d\omega, \\ y &= \chi_1, \end{aligned} \tag{5}$$

where $\chi = (\chi_1, \cdots, \chi_n)^T$, $\overline{\chi}_i = (\chi_1, \cdots, \chi_i)^T$, $\phi_i(\cdot) = \frac{f_i(\cdot)}{H^{r_i}}$, $h_i(\cdot) = \frac{g_i(\cdot)}{H^{r_i}}$.

A. State-Observer Design

In this section, we will construct an observer to estimate the state vector χ. The variable vectors $\widehat{\chi} = (\widehat{\chi}_1, \cdots, \widehat{\chi}_n)^T$ and $\overline{\widehat{\chi}}_i = (\widehat{\chi}_1, \cdots, \widehat{\chi}_i)^T$ respectively represent the estimators of the state vectors χ and $\overline{\chi}_i$. For the sake of brevity, the functions $\phi_i(\cdot)$'s and $h_i(\cdot)$'s are replaced with ϕ_i's and h_i's.

Firstly, motivated by [36], [37], a full-order nonlinear state-observer is constructed as

$$\begin{aligned} \dot{\widehat{\chi}}_i &= H\widehat{\chi}_{i+1}^{p_i} + H\gamma_i \ldots \gamma_1(\chi_1^{p_1} - \widehat{\chi}_1^{p_1}), \\ &\quad i = 1, \cdots, n-1, \\ \dot{\widehat{\chi}}_n &= H\tilde{u}^{p_n} + H\gamma_n \ldots \gamma_1(\chi_1^{p_1} - \widehat{\chi}_1^{p_1}), \end{aligned} \tag{6}$$

where $\gamma_j(j = 1, \cdots, n)$ are the gains to be determined later. Then, the error dynamics of above observer can be defined as

$$e_i = \chi_i - \widehat{\chi}_i, i = 1, \cdots, n.$$

Further, we introduce the following coordinate transformation

$$\tilde{e}_1 = e_1,\ \tilde{e}_i = e_i - \gamma_i e_{i-1},\ i = 2, \cdots, n. \tag{7}$$

From the equivalent system (5) and the state-observer (6), one can infer an error dynamics system

$$\begin{aligned} d\tilde{e}_1 &= H(\chi_2^{p_1} - \widehat{\chi}_2^{p_1})dt - H\gamma_1(\chi_1^{p_1} - \widehat{\chi}_1^{p_1})dt \\ &\quad + \phi_1 dt + h_1^T d\omega, \\ d\tilde{e}_i &= H(\chi_{i+1}^{p_i} - \widehat{\chi}_{i+1}^{p_i})dt - H\gamma_i(\chi_i^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}})dt \\ &\quad + (\phi_i - \gamma_i\phi_{i-1})dt + (h_i - \gamma_i h_{i-1})^T d\omega, \\ &\quad\ i = 2, \cdots, n-1, \\ d\tilde{e}_n &= -H\gamma_n(\chi_n^{p_{n-1}} - \widehat{\chi}_n^{p_{n-1}})dt \\ &\quad + [\phi_n - \gamma_n\phi_{n-1}]dt + [h_n - \gamma_n h_{n-1}]^T d\omega. \end{aligned} \tag{8}$$

Besides, we reconstruct a new system from (5) and (6)

$$\begin{aligned} d\chi_1 &= H\chi_2^{p_1}dt + \phi_1 dt + h_1^T d\omega, \\ d\widehat{\chi}_i &= H\widehat{\chi}_{i+1}^{p_i}dt + H\gamma_i \ldots \gamma_1(\chi_1^{p_1} - \widehat{\chi}_1^{p_1})dt, \\ &\quad\ i = 1, \cdots, n-1, \\ d\widehat{\chi}_n &= H\tilde{u}^{p_n}dt + H\gamma_n \ldots \gamma_1(\chi_1^{p_1} - \widehat{\chi}_1^{p_1})dt. \end{aligned} \tag{9}$$

Remark 2: Observer (6) is constructed by borrowing from the work of [36], [37]. However, observer (6) is a full-order nonlinear observer different from the reduced-order ones used in [36], [37]. It should be mentioned that, the systems investigated in [36], [37] are p-norm systems without constraints and with some given growth conditions, while this paper considers the output-feedback control problem for the systems with the output constraint and removing the growth conditions.

In addition, the output-feedback controller design method proposed in [36], [37] is first constructing a state-feedback controller and then applying the equivalent principle. In a word, this method is however invalid for p-norm systems with output constraints since the coefficient gains of the constrained state-feedback controller are functions of the origin state vector rather than constants. Thus, motivated by the output-feedback control design method for the strict-feedback constrained nonlinear systems (e.g., [30], [31]), this paper designs the state-observer at first, and then reconstructs a new system (9) from the origin system (5) and the state-observer (6). Then, based on the new system (9), the output-feedback controller is directly designed by the adding a power integrator technique, which will be explicitly shown in next subsection.

B. Output-Feedback Controller Design

Set up the following transformation

$$\eta_1 = \chi_1,\ \ \eta_i = \widehat{\chi}_i - \xi_{i-1}, i = 2, \ldots, n, \tag{10}$$

where ξ_{i-1}'s are the virtual controllers to be designed later.

Next, the controller design procedure of system (9) will be explicitly shown.

Step 1. From (10), we can get

$$d\eta_1 = d\chi_1 = (H\chi_2^{p_1} + \phi_1)dt + h_1^T d\omega. \tag{11}$$

Define the first Lyapunov function

$$V_1 = \frac{\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) + \frac{\tilde{\theta}_1^2}{2b_1} \triangleq V_B(\eta_1) + \frac{\tilde{\theta}_1^2}{2b_1}, \tag{12}$$

where $b_1 > 0$ is an adjustment parameter, $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$ is the error with $\hat{\theta}_1$ being the estimation of θ_1 to be given later.

Clearly, the tan-type BLF $V_B(\eta_1) = \frac{\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)$ is used to handle the system output constraint. It is clear to see that V_1 is positive definite. In light of (3), (11) and (12), it is easy to obtain

$$\begin{aligned} \mathcal{L}V_1 &= \frac{\partial V_B}{\partial\eta_1}(H\chi_2^{p_1} + \phi_1) + \frac{1}{2}\frac{\partial^2 V_B}{\partial\eta_1^2}h_1^T h_1 - \frac{1}{b_1}\tilde{\theta}_1\dot{\hat{\theta}}_1 \\ &= K(\eta_1)\eta_1^3(H\chi_2^{p_1} + \phi_1) + \frac{3}{2}K(\eta_1)\|h_1\|^2\eta_1^2 \\ &\quad + \frac{2\pi}{\varepsilon^4}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)K(\eta_1)\|h_1\|^2\eta_1^6 - \frac{1}{b_1}\tilde{\theta}_1\dot{\hat{\theta}}_1, \end{aligned}$$

where $K(\eta_1) = \sec^2\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)$.

Applying Lemma 3, one gets

$$\frac{3}{2}K(\eta_1)\|h_1\|^2\eta_1^2 \le \frac{3}{4}(K(\eta_1))^2\|h_1\|^4\eta_1^4 + \frac{3}{4}.$$

Then, we have

$$\begin{aligned} \mathcal{L}V_1 &\le HK(\eta_1)\eta_1^3(\widehat{\chi}_2^{p_1} - \xi_1^{p_1}) + HK(\eta_1)\eta_1^3\xi_1^{p_1} \\ &\quad + HK(\eta_1)\eta_1^3 F_1(Z_1) - \frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) \\ &\quad - \frac{\tilde{\theta}_1}{b_1}\dot{\hat{\theta}}_1 + \frac{3}{4}, \end{aligned} \tag{13}$$

where $Z_1 = (\overline{\chi}_2^T, \overline{\widehat{\chi}}_2^T)^T$, $\tau_1 > 0$ is an adjustment parameter,

$$\begin{aligned} F_1(Z_1) &= \chi_2^{p_1} - \widehat{\chi}_2^{p_1} + \frac{\tau_1\varepsilon^4}{2\pi H\eta_1^3}\sin\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)\cos\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) \\ &\quad + \frac{\phi_1}{H} + \frac{3\eta_1}{4H}K(\eta_1)\|h_1\|^4 + \frac{2\pi\eta_1^3}{H\varepsilon^4}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)\|h_1\|^2. \end{aligned}$$

Obviously, $F_1(Z_1)$ is an unknown nonlinear function needing to be approximated. Analysing the expression of this function, it is easy to deduce that $F_1(Z_1)$ is continuous, and a reduced fuzzy system proposed in reference [49] is an appropriate approximator for it. Based on Lemma 6 and [49], $F_1(Z_1)$ can be approximated as

$$F_1(Z_1) = \Upsilon_1^T\Psi_1(\widetilde{Z}_1) + \epsilon_1(\widetilde{Z}_1), \tag{14}$$

where $\widetilde{Z}_1 = (\chi_1, \widehat{\chi}_1)^T$, $|\epsilon_1(\widetilde{Z}_1)| \le \delta_1$ and δ_1 is a given positive constant. In view of Lemma 3, it is easy to obtain

$$\begin{aligned} &K(\eta_1)\eta_1^3 F_1(Z_1) \\ &\quad \le K(\eta_1)|\eta_1|^3(\|\Upsilon_1\|\|\Psi_1\| + \delta_1) \\ &\quad \le \frac{3\sigma_{11}\theta_1}{p_1+3}(K(\eta_1)\|\Psi_1\|)^{\frac{p_1+3}{3}}\eta_1^{p_1+3} + \frac{p_1}{p_1+3}\sigma_{11}^{-\frac{3}{p_1}} \\ &\quad\quad + \frac{3}{p_1+3}K(\eta_1)^{\frac{p_1+3}{3}}\eta_1^{p_1+3} + \frac{p_1}{p_1+3}\delta_1^{\frac{p_1+3}{p_1}}, \end{aligned} \tag{15}$$

where $\theta_1 = \|\Upsilon_1\|^{\frac{p_1+3}{3}}$ and $\sigma_{11} > 0$ is an adjustment parameter.

Combining (15) with (13), gets

$$\begin{aligned}\mathcal{L}V_1 \le\ & HK(\eta_1)\eta_1^3(\widehat{\chi}_2^{p_1}-\xi_1^{p_1})+HK(\eta_1)\eta_1^3\xi_1^{p_1}\\&-\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)-\frac{1}{b_1}\tilde{\theta}_1\dot{\hat{\theta}}_1\\&+\frac{3H}{p_1+3}(K(\eta_1))^{\frac{p_1+3}{3}}\left[\|\Psi_1\|^{\frac{p_1+3}{3}}\sigma_{11}\hat{\theta}_1+1\right]\eta_1^{p_1+3}\\&+\tilde{\theta}_1\left[\frac{3H\sigma_{11}}{p_1+3}(K(\eta_1)\|\Psi_1\|)^{\frac{p_1+3}{3}}\eta_1^{p_1+3}-\frac{\dot{\hat{\theta}}_1}{b_1}\right]\\&+\frac{p_1H}{p_1+3}\sigma_{11}^{-\frac{3}{p_1}}+\frac{3}{4}+\frac{p_1H}{p_1+3}\delta_1^{\frac{p_1+3}{p_1}}.\end{aligned}\tag{16}$$

Thus, we can design

$$\begin{aligned}\xi_1(\chi_1,\widehat{\chi}_1,\hat{\theta}_1)&=-(M_1(\chi_1,\widehat{\chi}_1,\hat{\theta}_1))^{\frac{1}{p_1}}\eta_1\\&\triangleq-\varphi_1(\chi_1,\widehat{\chi}_1,\hat{\theta}_1)\eta_1,\end{aligned}\tag{17}$$

$$\begin{aligned}\dot{\hat{\theta}}_1=&\ \frac{3Hb_1\sigma_{11}}{p_1+3}(K(\eta_1)\|\Psi_1\|)^{\frac{p_1+3}{3}}\eta_1^{p_1+3}\\&-d_1\hat{\theta}_1,\end{aligned}\tag{18}$$

where $M_1^{\frac{1}{p_1}}(\cdot)$ is a positive smooth function satisfying $M_1(\cdot)\ge\frac{3}{p_1+3}(K(\eta_1))^{\frac{p_1}{3}}\left[\|\Psi_1\|^{\frac{p_1+3}{3}}\sigma_{11}\hat{\theta}_1+1\right]+\frac{\rho_1(\eta_1)}{K(\eta_1)}+\tau_1$; $\rho_1(\eta_1)$ is a positive smooth function to be given in the next step and $d_1>0$ is an adjustment parameter.

Apparently, the property of $M_1(\cdot)$ infers that the virtual controller $\xi_1(\cdot)$ is smooth. Meanwhile, if $\hat{\theta}_1(0)\ge 0$, it is easily gotten from Lemma 2 in reference [29] and (18) that $\hat{\theta}_1(t)\ge 0$ for $\forall t\ge 0$. This implies the positivity of $M_1(\cdot)$ can be guaranteed. Similar characteristics will be always fulfilled in next steps of the design procedure.

Then, substituting (17) and (18) into (16), gets

$$\begin{aligned}\mathcal{L}V_1\le&-\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)-H\tau_1\eta_1^{p_1+3}\\&-H\rho_1(\eta_1)\eta_1^{p_1+3}+HK(\eta_1)\eta_1^3(\widehat{\chi}_2^{p_1}-\xi_1^{p_1})\\&+\frac{d_1}{b_1}\tilde{\theta}_1\hat{\theta}_1+\frac{3}{4}+\frac{Hp_1}{p_1+3}\sigma_{11}^{-\frac{3}{p_1}}+\frac{Hp_1}{p_1+3}\delta_1^{\frac{p_1+3}{p_1}}.\end{aligned}$$

Additionally, it is easy to get

$$\frac{d_1}{b_1}\tilde{\theta}_1\hat{\theta}_1=\frac{d_1}{b_1}(\theta_1-\tilde{\theta}_1)\tilde{\theta}_1\le-\frac{d_1\tilde{\theta}_1^2}{2b_1}+\frac{d_1\theta_1^2}{2b_1}.$$

Hence, we can obtain

$$\begin{aligned}\mathcal{L}V_1\le&-\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)-\frac{d_1\tilde{\theta}_1^2}{2b_1}-H\rho_1(\eta_1)\eta_1^{p_1+3}\\&-H\tau_1\eta_1^{p_1+3}+Q_1+HK(\eta_1)\eta_1^3(\widehat{\chi}_2^{p_1}-\xi_1^{p_1}),\end{aligned}\tag{19}$$

where $Q_1=\frac{3}{4}+\frac{p_1H}{p_1+3}\sigma_{11}^{-\frac{3}{p_1}}+\frac{Hp_1}{p_1+3}\delta_1^{\frac{p_1+3}{p_1}}+\frac{d_1\theta_1^2}{2b_1}$.

Remark 3: The tan-type BLF $V_B(\eta_1)$ is adopted to handle the output constraint issue, since it possesses the property $\lim_{\varepsilon\to\infty}V_B(\eta_1)=\frac{\eta_1^4}{4}$ which implies that the proposed method is also applicable to the systems without output constraints. What's more, it is worth mentioning that the function $K(\eta_1)$ is the key to keep the output variable to remain in the constrained region, since $K(\eta_1)$ increases along with the state x_1 approaching the boundaries $x_1=|\varepsilon|$.

Remark 4: (i) Obviously, one can verify

$$\lim_{\eta_1\to 0}\frac{\sin\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)\cos\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)}{2\pi H\eta_1^3}=\lim_{\eta_1\to 0}\frac{\frac{\pi\eta_1^4}{2\varepsilon^4}\cos\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)}{2\pi H\eta_1^3}=0,$$

which further ensures that $F_1(Z_1)$ is continuous.

(ii) Note that $F_1(Z_1)$ in (14) includes variables χ_1, χ_2, $\widehat{\chi}_1$, $\widehat{\chi}_2$, hence the FLS $\Upsilon_1^T\Psi_1(\widetilde{Z}_1)$ is a reduced fuzzy system utilized to perform the approximation task. In fact, if a traditional fuzzy system is used to approximate it, the first virtual controller $\xi_1(\cdot)$ will include all the variables $\chi_1,\chi_2,\widehat{\chi}_1,\widehat{\chi}_2$. That means the backstepping approach can't be continually applied in the later design procedure. As stated in [49], the reduced FLS is more proper to estimate the unknown functions.

Step 2. Based on (10) and It $\hat{o}$'s formula, one has

$$\begin{aligned}d\eta_2=&[H\widehat{\chi}_3^{p_2}+H\gamma_2\gamma_1(\chi_1^{p_1}-\widehat{\chi}_1^{p_1})-\mathcal{L}\xi_1]dt\\&-\left(\frac{\partial\xi_1}{\partial\chi_1}h_1\right)^T d\omega,\end{aligned}\tag{20}$$

where $\mathcal{L}\xi_1=\frac{\partial\xi_1}{\partial\chi_1}(H\chi_2^{p_1}+\phi_1)+\frac{\partial\xi_1}{\partial\widehat{\chi}_1}\dot{\widehat{\chi}}_1+\frac{\partial\xi_1}{\partial\hat{\theta}_1}\dot{\hat{\theta}}_1+\frac{1}{2}\frac{\partial^2\xi_1}{\partial\chi_1^2}h_1^Th_1$ is a smooth function due to the smoothness of ξ_1.

Choose the Lyapunov function as

$$V_2=V_1+\Lambda_2\tag{21}$$

with

$$\Lambda_2=\frac{\eta_2^{p_1-p_2+4}}{p_1-p_2+4}+\frac{\tilde{\theta}_2^2}{2b_2},\tag{22}$$

where $b_2>0$ is an adjustment parameter, $\tilde{\theta}_2=\theta_2-\hat{\theta}_2$ is the error with $\hat{\theta}_2$ being the estimation of θ_2 to be given later.

By the definition of V_2, it is easily seen that V_2 is positive definite. According to (3), (20) and (22), it can be gotten that

$$\begin{aligned}\mathcal{L}\Lambda_2=&\ \eta_2^{p_1-p_2+3}[H\widehat{\chi}_3^{p_2}+H\gamma_2\gamma_1(\chi_1^{p_1}-\widehat{\chi}_1^{p_1})-\mathcal{L}\xi_1]\\&-\frac{1}{b_2}\tilde{\theta}_2\dot{\hat{\theta}}_2+\frac{p_1-p_2+3}{2}\left\|\frac{\partial\xi_1}{\partial\chi_1}h_1\right\|^2\eta_2^{p_1-p_2+2}.\end{aligned}\tag{23}$$

Using Lemma 3, one can directly deduce

$$\begin{aligned}&\frac{p_1-p_2+3}{2}\left\|\frac{\partial\xi_1}{\partial\chi_1}h_1\right\|^2\eta_2^{p_1-p_2+2}\\&\le q_2\left\|\frac{\partial\xi_1}{\partial\chi_1}h_1\right\|^{\frac{2(p_1-p_2+4)}{p_1-p_2+2}}\eta_2^{p_1-p_2+4}+\frac{p_1-p_2+3}{p_1-p_2+4},\end{aligned}\tag{24}$$

where $q_2=(p_1-p_2+3)(p_1-p_2+2)/2(p_1-p_2+4)$.

Besides, it can be obtained from Lemmas 2 and 3 that

$$\begin{aligned}&K(\eta_1)\eta_1^3(\widehat{\chi}_2^{p_1}-\xi_1^{p_1})\\&\le K(\eta_1)|\eta_1|^3D_1[|\eta_2|^{p_1}+\varphi_1^{p_1-1}|\eta_2||\eta_1|^{p_1-1}]\\&\le(K(\eta_1))^{\frac{p_1+3}{3}}\frac{3D_1^{\frac{p_1+3}{3}}}{p_1+3}\sigma_{12}^{-\frac{p_1}{3}}\eta_1^{p_1+3}+\frac{p_1\sigma_{12}}{p_1+3}\eta_2^{p_1+3}\\&\quad+\frac{p_1+2}{p_1+3}K(\eta_1)\eta_1^{p_1+3}+\frac{(\varphi_1(\cdot))^{(p_1-1)(p_1+3)}}{p_1+3}K(\eta_1)\eta_2^{p_1+3}\\&\le\rho_1(\eta_1)\eta_1^{p_1+3}+\frac{p_1\sigma_{12}}{p_1+3}\eta_2^{p_1+3}+\varrho_1(\cdot)\eta_2^{p_1+3},\end{aligned}\tag{25}$$

where $D_1 = (2^{p_1-2}+2)p_1$ is a positive constant; $\varrho_1(\cdot) = (\varphi_1(\cdot))^{(p_1-1)(p_1+3)}K(\eta_1)/(p_1+3)$ and $\rho_1(\eta_1) = \frac{3}{p_1+3}D_1^{\frac{p_1+3}{3}}\sigma_{12}^{-\frac{p_1}{3}}(K(\eta_1))^{\frac{3}{p_1+3}} + \frac{p+2}{p_1+3}K(\eta_1)$ are positive smooth functions; and $\sigma_{12}>0$ is an adjustment parameter.

Then, in light of Definition 1 and Eqs. (19), (21) and (23)-(25), one gets

$$\begin{aligned}\mathcal{L}V_2 \le& -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - H\tau_1\eta_1^{p_1+3} - \frac{d_1\tilde{\theta}_1^2}{2b_1}\\&+H\frac{p_1\sigma_{12}}{p_1+3}\eta_2^{p_1+3} + H\eta_2^{p_1-p_2+3}F_2(Z_2)\\&+H\eta_2^{p_1-p_2+3}(\widehat{\chi}_3^{p_2}-\xi_2^{p_2}) + H\eta_2^{p_1-p_2+3}\xi_2^{p_2}\\&+Q_1+\frac{p_1-p_2+3}{p_1-p_2+4} - \frac{1}{b_2}\tilde{\theta}_2\dot{\hat{\theta}}_2,\end{aligned}\tag{26}$$

where $Z_2 = (\overline{\chi}_2^T, \overline{\widehat{\chi}}_2^T, \hat{\theta}_1)^T$ and

$$\begin{aligned}F_2(Z_2) =& \frac{q_2\eta_2}{H}\left\|\frac{\partial\xi_1}{\partial\chi_1}h_1\right\|^{\frac{2(p_1-p_2+4)}{p_1-p_2+2}}\\&+\varrho_1(\cdot)\eta_2^{p_2} + \gamma_2\gamma_1(\chi_1^{p_1}-\widehat{\chi}_1^{p_1}) - \frac{\mathcal{L}\xi_1}{H}.\end{aligned}$$

Obviously, $F_2(Z_2)$ is an unknown and continuous nonlinear function. By following the same lines to obtain $F_1(Z_1)$ in Step 1, $F_2(Z_2)$ can be approximated as

$$F_2(Z_2) = \Upsilon_2^T\Psi_2(\widetilde{Z}_2) + \epsilon_2(\widetilde{Z}_2),$$

where $\widetilde{Z}_2 = (\chi_1, \overline{\widehat{\chi}}_2^T, \hat{\theta}_1)^T$, $|\epsilon_2(\widetilde{Z}_2)| \le \delta_2$ and δ_2 is a given positive constant.

By Lemma 3, one obtains

$$\begin{aligned}\eta_2^{p_1-p_2+3}F_2(Z_2) \le& |\eta_2|^{p_1-p_2+3}(\|\Upsilon_2\|\|\Psi_2\|+\delta_2)\\\le& \frac{(p_1-p_2+3)\sigma_{21}\theta_2}{p_1+3}\|\Psi_2\|^{\frac{p_1+3}{p_1-p_2+3}}\eta_2^{p_1+3}\\&+\frac{p_2\sigma_{21}^{-\frac{p_1-p_2+3}{p_1}}}{p_1+3} + \frac{p_1-p_2+3}{p_1+3}\eta_2^{p_1+3}\\&+\frac{p_2\delta_2^{\frac{p_1+3}{p_2}}}{p_1+3},\end{aligned}\tag{27}$$

where $\theta_2 = \|\Upsilon_2\|^{\frac{p_1+3}{p_1-p_2+3}}$ and $\sigma_{21}>0$ is an adjustment parameter.

Then, combining (26) with (27), it is easy to infer

$$\begin{aligned}\mathcal{L}V_2 \le& -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - H\tau_1\eta_1^{p_1+3} + \frac{Hp_1\sigma_{12}}{p_1+3}\eta_2^{p_1+3}\\&+\frac{(p_1-p_2+3)H}{p_1+3}\left[\|\Psi_2\|^{\frac{p_1+3}{p_1-p_2+3}}\sigma_{21}\hat{\theta}_2+1\right]\eta_2^{p_1+3}\\&+\tilde{\theta}_2\left[\frac{(p_1-p_2+3)H}{p_1+3}\|\Psi_2\|^{\frac{p_1+3}{p_1-p_2+3}}\sigma_{21}\eta_2^{p_1+3} - \frac{\dot{\hat{\theta}}_2}{b_2}\right]\\&+H\eta_2^{p_1-p_2+3}(\widehat{\chi}_3^{p_2}-\xi_2^{p_2}) + H\eta_2^{p_1-p_2+3}\xi_2^{p_2} - \frac{d_1\tilde{\theta}_1^2}{2b_1}\\&+\frac{p_1-p_2+3}{p_1-p_2+4} + \frac{Hp_2\sigma_{21}^{-\frac{p_1-p_2+3}{p_1}}}{p_1+3} + \frac{Hp_2\delta_2^{\frac{p_1+3}{p_2}}}{p_1+3} + Q_1.\end{aligned}\tag{28}$$

Thus, one can design

$$\begin{aligned}\xi_2(\chi_1,\overline{\widehat{\chi}}_2^T,\bar{\hat{\theta}}_2^T) =& -(M_2(\chi_1,\overline{\widehat{\chi}}_2^T,\bar{\hat{\theta}}_2^T))^{\frac{1}{p_2}}\eta_2\\\triangleq& -\varphi_2(\chi_1,\overline{\widehat{\chi}}_2^T,\bar{\hat{\theta}}_2^T)\eta_2,\end{aligned}\tag{29}$$

$$\begin{aligned}\dot{\hat{\theta}}_2 =& \frac{(p_1-p_2+3)Hb_2\sigma_{21}}{p_1+3}\\&\times\|\Psi_2\|^{\frac{p_1+3}{p_1-p_2+3}}\eta_2^{p_1+3} - d_2\hat{\theta}_2,\end{aligned}\tag{30}$$

where $M_2^{\frac{1}{p_2}}(\cdot)$ is a positive smooth function satisfying $M_2(\cdot) \ge \frac{p_1-p_2+3}{p_1+3}[\|\Psi_2\|^{\frac{p_1+3}{p_1-p_2+3}}\sigma_{21}\hat{\theta}_2+1] + \frac{p_1\sigma_{12}}{p_1+3}+\rho_2+\tau_2$; $\rho_2>0$ is a constant to be given in the next step; and $\tau_2,\ d_2>0$ are adjustment parameters. Combining the property of $M_2(\cdot)$ with the definition of $\xi_2(\cdot)$, infers that $\xi_2(\cdot)$ is also smooth.

What's more, it is not difficult to deduce

$$\frac{d_2}{b_2}\tilde{\theta}_2\hat{\theta}_2 = \frac{d_2}{b_2}(\theta_2-\tilde{\theta}_2)\tilde{\theta}_2 \le -\frac{d_2\tilde{\theta}_2^2}{2b_2} + \frac{d_2\theta_2^2}{2b_2}.\tag{31}$$

Substituting (29)-(31) into (28), yields

$$\begin{aligned}\mathcal{L}V_2 \le& -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - H\tau_1\eta_1^{p_1+3} - \frac{d_1\tilde{\theta}_1^2}{2b_1}\\&-H\rho_2\eta_2^{p_1+3} - H\tau_2\eta_2^{p_1+3} - \frac{d_2\tilde{\theta}_2^2}{2b_2}\\&+H\eta_2^{p_1-p_2+3}(\widehat{\chi}_3^{p_2}-\xi_2^{p_2}) + Q_1 + Q_2,\end{aligned}\tag{32}$$

where $Q_2 = \frac{p_1-p_2+3}{p_1-p_2+4} + \frac{Hp_2}{p_1+3}\sigma_{21}^{-\frac{p_1-p_2+3}{p_1}} + \frac{Hp_2\delta_2^{\frac{p_1+3}{p_2}}}{p_1+3} + \frac{d_2\theta_2^2}{2b_2}$.

Step $k(3\le k\le n)$**.** Suppose at Step $k-1$, there exist a Lyapunov function V_{k-1} and a series of smooth virtual controllers $\xi_j(\chi_1,\overline{\widehat{\chi}}_j^T,\bar{\hat{\theta}}_j^T)(j=1,\cdots,k-1)$, such that

$$\begin{aligned}\mathcal{L}V_{k-1} \le& -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k-1}\frac{d_j\tilde{\theta}_j^2}{2b_j}\\&-\sum_{j=1}^{k-1}H\tau_j\eta_j^{p_1+3} - H\rho_{k-1}\eta_{k-1}^{p_1+3}\\&+H\eta_{k-1}^{p_1-p_{k-1}+3}(\widehat{\chi}_k^{p_{k-1}}-\xi_{k-1}^{p_{k-1}})\\&+H\eta_{k-1}^{p_1-p_{k-1}+3}\xi_{k-1}^{p_{k-1}} + \sum_{j=1}^{k-1}Q_j,\end{aligned}\tag{33}$$

where $\eta_j, d_j(j=1,\cdots,k-1)$ are positive adjustment parameters; $\rho_{k-1}, Q_j(j=1,\cdots,k-1)$ are positive constants. Meanwhile, one can infer from (10) and It $\hat{o}$'s formula that

$$\begin{aligned}d\eta_k =& [H\widehat{\chi}_{k+1}^{p_k} + H\gamma_k\cdots\gamma_1(\chi_1^{p_1}-\widehat{\chi}_1^{p_1}) - \mathcal{L}\xi_{k-1}]dt\\&-\left(\frac{\partial\xi_{k-1}}{\partial\chi_1}h_1\right)^T d\omega,\end{aligned}\tag{34}$$

where $\mathcal{L}\xi_{k-1}$ is a smooth function in the following form

$$\begin{aligned}\mathcal{L}\xi_{k-1} =& \frac{\partial\xi_{k-1}}{\partial\chi_1}(H\chi_2^{p_1}+\phi_1) + \sum_{j=1}^{k-1}\frac{\partial\xi_{k-1}}{\partial\widehat{\chi}_j}\dot{\widehat{\chi}}_j\\&+\sum_{j=1}^{k-1}\frac{\partial\xi_{k-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j + \frac{1}{2}\frac{\partial^2\xi_{k-1}}{\partial\chi_1^2}h_1^Th_1.\end{aligned}$$

We choose the Lyapunov function as

$$V_k = V_{k-1} + \Lambda_k \tag{35}$$

with

$$\Lambda_k = \frac{\eta_k^{p_1-p_k+4}}{p_1-p_k+4} + \frac{\tilde{\theta}_k^2}{2b_k}, \tag{36}$$

where $b_k > 0$ is an adjustment parameter, $\tilde{\theta}_k = \theta_k - \hat{\theta}_k$ is the error with $\hat{\theta}_k$ being the estimation of θ_k to be given later.

Evidently, V_k is positive definite. Based on Definition 1, it can be obtained from Eqs. (34) and (36) that

$$\mathcal{L}\Lambda_k = \eta_k^{p_1-p_k+3}[H\widehat{\chi}_{k+1}^{p_k} + H\gamma_k\cdots\gamma_1(\chi_1^{p_1} - \widehat{\chi}_1^{p_1}) - \mathcal{L}\xi_{k-1}] - \frac{1}{b_k}\tilde{\theta}_k\dot{\hat{\theta}}_k + \frac{p_1-p_k+3}{2}\eta_k^{p_1-p_k+2}\left\|\frac{\partial\xi_{k-1}}{\partial\chi_1}h_1\right\|^2. \tag{37}$$

Apparently, it is not hardly gotten from Lemma 3 that

$$\begin{aligned}\frac{p_1-p_k+3}{2}&\eta_k^{p_1-p_k+2}\left\|\frac{\partial\xi_{k-1}}{\partial\chi_1}h_1\right\|^2\\ &\le q_k\left\|\frac{\partial\xi_{k-1}}{\partial\chi_1}h_1\right\|^{\frac{2(p_1-p_k+4)}{p_1-p_k+2}}\eta_k^{p_1-p_k+4} + \frac{p_1-p_k+3}{p_1-p_k+4},\end{aligned} \tag{38}$$

where $q_k = (p_1-p_k+3)(p_1-p_k+2)/2(p_1-p_k+4)$.

In addition, it can be verified from Lemmas 2 and 3 that

$$\begin{aligned}\eta_{k-1}^{p_1-p_{k-1}+3}&\left(\widehat{\chi}_k^{p_{k-1}} - \xi_{k-1}^{p_{k-1}}\right)\\ &\le |\eta_{k-1}|^{p_1-p_{k-1}+3}D_{k-1}\left[|\eta_k|^{p_{k-1}} + |\eta_k||\xi_{k-1}|^{p_{k-1}-1}\right]\\ &\le \rho_{k-1}\eta_{k-1}^{p_1+3} + \frac{p_{k-1}\sigma_{k-1\ 2}}{p_1+3}\eta_k^{p_1+3} + \varrho_{k-1}(\cdot)\eta_k^{p_1+3},\end{aligned} \tag{39}$$

where $D_{k-1} = (2^{p_{k-1}-2}+2)p_{k-1}$ and $\rho_{k-1} = (D_{k-1})^{\frac{p_1+3}{p_1-p_{k-1}+3}}\sigma_{k-1\ 2}^{-\frac{p_{k-1}}{p_1-p_{k-1}+3}}/(p_1+3) + \frac{p_1+2}{p_1+3}$ are positive constants; $\varrho_{k-1}(\cdot) = [D_{k-1}(\varphi_{k-1}(\cdot))^{p_{k-1}-1}]^{p_1+3}/(p_1+3)$ is a smooth positive function; and $\sigma_{k-1\ 2} > 0$ is an adjustment parameter.

In light of (28), (37)-(39), one has

$$\begin{aligned}\mathcal{L}V_k \le& -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k-1}H\tau_j\eta_j^{p_1+3} - \frac{\tilde{\theta}_k}{b_k}\dot{\hat{\theta}}_k\\ &+H\frac{p_{k-1}\sigma_{k-1\ 2}}{p_1+3}\eta_k^{p_1+3} + H\eta_k^{p_1-p_k+3}F(Z_k)\\ &-\sum_{j=1}^{k-1}\frac{d_j\tilde{\theta}_j^2}{2b_j} + \sum_{j=1}^{k-1}Q_j + \frac{p_1-p_k+3}{p_1-p_k+4}\\ &+H\eta_k^{p_1-p_k+3}(\widehat{\chi}_{k+1}^{p_k} - \xi_k^{p_k}) + H\eta_k^{p_1-p_k+3}\xi_k^{p_k},\end{aligned} \tag{40}$$

where $Z_k = (\overline{\chi}_2^T, \overline{\widehat{\chi}}_k^T, \bar{\hat{\theta}}_{k-1}^T)^T$, $\bar{\hat{\theta}}_{k-1} = (\hat{\theta}_1, \cdots, \hat{\theta}_{k-1})^T$ and

$$F_k(Z_k) = \gamma_k\cdots\gamma_1(\chi_1^{p_1} - \widehat{\chi}_1^{p_1}) - \frac{\mathcal{L}\xi_{k-1}}{H} + \varrho_{k-1}(\cdot)\eta_k^{p_k} + \frac{q_k\eta_k}{H}\left\|\frac{\partial\xi_{k-1}}{\partial\chi_1}h_1\right\|^{\frac{2(p_1-p_k+4)}{p_1-p_k+2}}.$$

Clearly, $F_k(Z_k)$ is an unknown and continuous nonlinear function. Similarly to the first two steps, we rewrite $F_k(Z_k)$ by applying a reduced FLS $\Upsilon_k^T\Psi_k(\widetilde{Z}_k)$ as

$$F_k(Z_k) = \Upsilon_k^T\Psi_k(\widetilde{Z}_k) + \epsilon_k(\widetilde{Z}_k),$$

where $\widetilde{Z}_k = (\chi_1, \overline{\widehat{\chi}}_k^T, \bar{\hat{\theta}}_{k-1}^T)^T$, $|\epsilon_k(\widetilde{Z}_k)| \le \delta_k$ and δ_k is a given positive constant.

Hence, from Lemma 3, it is easy to obtain

$$\begin{aligned}\eta_k^{p_1-p_k+3}&F_k(Z_k)\\ \le& \frac{(p_1-p_k+3)\sigma_{k1}\theta_k}{p_1+3}\|\Psi_k\|^{\frac{p_1+3}{p_1-p_k+3}}\eta_k^{p_1+3}\\ &+\frac{p_k}{p_1+3}\sigma_{k1}^{-\frac{p_1-p_k+3}{p_k}} + \frac{p_1-p_k+3}{p_1+3}\eta_k^{p_1+3} + \frac{p_k\delta_k^{\frac{p_1+3}{p_k}}}{p_1+3},\end{aligned} \tag{41}$$

where $\theta_k = \|\Upsilon_k\|^{\frac{p_1+3}{p_1-p_k+3}}$ and $\sigma_{k1} > 0$ is an adjustment parameter.

In light of (41) and (40), it is easy to get

$$\begin{aligned}\mathcal{L}V_k \le& -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k-1}H\tau_j\eta_j^{p_1+3} - \sum_{j=1}^{k-1}\frac{d_j\tilde{\theta}_j^2}{2b_j}\\ &+\sum_{j=1}^{k-1}Q_j + H\frac{p_{k-1}\sigma_{k-1\ 2}}{p_1+3}\eta_k^{p_1+3} + \frac{Hp_k}{p_1+3}\delta_k^{\frac{p_1+3}{p_k}}\\ &+H\frac{p_1-p_k+3}{p_1+3}[\|\Psi_k\|^{\frac{p_1+3}{p_1-p_k+3}}\sigma_{k1}\hat{\theta}_k + 1]\eta_k^{p_1+3}\\ &+\tilde{\theta}_k[\frac{(p_1-p_k+3)H}{p_1+3}\|\Psi_k\|^{\frac{p_1+3}{p_1-p_k+3}}\sigma_{k1}\eta_k^{p_1+3} - \frac{\dot{\hat{\theta}}_k}{b_k}]\\ &+H\eta_k^{p_1-p_k+3}(\widehat{\chi}_{k+1}^{p_k} - \xi_k^{p_k}) + H\eta_k^{p_1-p_k+3}\xi_k^{p_k}\\ &+\frac{Hp_k}{p_1+3}\sigma_{k1}^{-\frac{p_1-p_k+3}{p_k}} + \frac{p_1-p_k+3}{p_1-p_k+4}.\end{aligned} \tag{42}$$

Hence, the smooth virtual controller ξ_k with the adaptive law of $\hat{\theta}_k$ can be designed as

$$\begin{aligned}\xi_k(\chi_1, \overline{\widehat{\chi}}_k^T, \bar{\hat{\theta}}_k^T) &= -(M_k(\chi_1, \overline{\widehat{\chi}}_k^T, \bar{\hat{\theta}}_k^T))^{\frac{1}{p_k}}\eta_k\\ &\triangleq -\varphi_k(\chi_1, \overline{\widehat{\chi}}_k^T, \bar{\hat{\theta}}_k^T)\eta_k,\end{aligned} \tag{43}$$

$$\begin{aligned}\dot{\hat{\theta}}_k =& \frac{(p_1-p_k+3)H\sigma_{k1}b_k}{p_1+3}\\ &\times\|\Psi_k\|^{\frac{p_1+3}{p_1-p_k+3}}\eta_k^{p_1+3} - d_k\hat{\theta}_k,\end{aligned} \tag{44}$$

where $\bar{\hat{\theta}}_k = (\hat{\theta}_1, \cdots, \hat{\theta}_k)^T$; $M_k^{\frac{1}{p_k}}(\cdot)$ is a positive smooth function satisfying

$$M_k(\cdot) \ge \frac{p_1-p_k+3}{p_1+3}[\sigma_{k1}\hat{\theta}_k\|\Psi_k\|^{\frac{p_1+3}{p_1-p_k+3}} + 1] + \frac{p_{k-1}\sigma_{k-1\ 2}}{p_1+3} + \rho_k + \tau_k;$$

$\rho_k > 0$ is a constant to be given in the next step; and $\tau_k, d_k > 0$ are adjustment parameters.

Furthermore, we have

$$\frac{d_k}{b_k}\hat{\theta}_k\tilde{\theta}_k \le -\frac{d_k}{2b_k}\tilde{\theta}_k^2 + \frac{d_k}{2b_k}\theta_k^2. \tag{45}$$

From (42)-(45), it can be deduced that

$$\begin{aligned}\mathcal{L}V_k \le& -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k}\frac{d_j\tilde{\theta}_j^2}{2b_j} - \sum_{j=1}^{k}H\tau_j\eta_j^{p_1+3}\\ &-H\rho_k\eta_k^{p_1+3} + H\eta_k^{p_1-p_k+3}(\widehat{\chi}_{k+1}^{p_k} - \xi_k^{p_k}) + \sum_{j=1}^{k}Q_j,\end{aligned} \tag{46}$$

where $Q_k = \frac{Hp_k}{p_1+3}\sigma_{k1}^{-\frac{p_1-p_k+3}{p_k}} + \frac{Hp_k}{p_1+3}\delta_k^{\frac{p_1+3}{p_k}} + \frac{p_1-p_k+3}{p_1-p_k+4} + \frac{d_k\theta_k^2}{2b_k}$.

That is to say, for $\forall k = 3, \cdots, n$, Eqs. (43), (44) and (46) all hold.

Step n. It easily renders from above steps that, there exist a sequence of smooth virtual signals with corresponding adaptive parameter laws $(\xi_i, \hat{\theta}_i)(i = 1, \cdots, n)$ such that Eq. (46) holds for $k = n$ with $\chi_{n+1} = \tilde{u}$. That is, the Lyapunov function can be chosen as

$$\begin{aligned} V_n &= V_{n-1} + \Lambda_n \\ &= \frac{\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) + \sum_{j=2}^{n}\frac{\eta_j^{p_1-p_j+4}}{p_1-p_j+4} + \sum_{j=1}^{n}\frac{\tilde{\theta}_j^2}{2b_j}, \quad (47) \end{aligned}$$

and the controller $\tilde{u}$ can be designed as

$$\begin{aligned} \tilde{u} &= \xi_n(\chi_1, \widehat{\chi}^T, \hat{\theta}^T) = -(M_n(\chi_1, \widehat{\chi}^T, \hat{\theta}^T))^{\frac{1}{p_n}}\eta_n \\ &\triangleq -\varphi_n(\chi_1, \widehat{\chi}^T, \hat{\theta}^T)\eta_n \end{aligned} \quad (48)$$

with

$$\dot{\hat{\theta}}_n = \frac{(p_1-p_n+3)H\sigma_{n1}b_n}{p_1+3}\|\Psi_n\|^{\frac{p_1+3}{p_1-p_n+3}}\eta_n^{p_1+3} - d_n\hat{\theta}_n, \quad (49)$$

where $\hat{\theta} = (\hat{\theta}_1, \cdots, \hat{\theta}_n)^T$, $M_n^{\frac{1}{p_n}}(\cdot)$ is a positive smooth function satisfying

$$M_n(\cdot) \geq \frac{p_1-p_n+3}{p_1+3}[\sigma_{n1}\hat{\theta}_n\|\Psi_n\|^{\frac{p_1+3}{p_1-p_n+3}} + 1] + \frac{p_{n-1}\sigma_{n-1\ 2}}{p_1+3} + \tau_n.$$

Clearly, one can further get

$$\begin{aligned} \mathcal{L}V_n \leq & -\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{n}\frac{d_j\tilde{\theta}_j^2}{2b_j} \\ & -\sum_{j=1}^{n}H\tau_j\eta_j^{p_1+3} + \sum_{j=1}^{n}Q_j. \end{aligned} \quad (50)$$

Then, by the definitions of $\eta_k(k = 1, \cdots, n)$, we obtain the implementable output-feedback controller of system (9)

$$\tilde{u} = -(\bar{\varphi}_1\chi_1 + \bar{\varphi}_2\widehat{\chi}_2 + \cdots + \bar{\varphi}_n\widehat{\chi}_n), \quad (51)$$

where $\bar{\varphi}_k = \prod_{j=k}^{n}\varphi_j$ for $k = 1, \ldots, n$.

Thus, the fuzzy output-feedback controller of system (1) is

$$u = H^{r_{n+1}}\tilde{u} = -H^{r_{n+1}}(\bar{\varphi}_1\chi_1 + \bar{\varphi}_2\widehat{\chi}_2 + \cdots + \bar{\varphi}_n\widehat{\chi}_n), \quad (52)$$

where $\widehat{\chi}_i(i = 2, \cdots, n)$ are provided by the observer (6). In light of the design procedure, the actual controller (52) is evident smooth.

C. Selection of the Observer Gains

In this part, the values of the unknown gains $\gamma_i(i = 1, \cdots, n)$ will be obtained by the Lyapunov analysis. At the same time, one will get the values of some other parameters appearing in the output-feedback controller.

Based on the error dynamic system (8) and Assumption 1, we define the Lyapunov function

$$U_n = \frac{\gamma\tilde{e}_1^4}{4} + \sum_{i=2}^{n}\frac{\gamma\tilde{e}_i^{p_1-p_{i-1}+4}}{p_1-p_{i-1}+4}, \quad (53)$$

where $\gamma > 0$ is an adjustment constant.

By the definition of U_n, we can easily know U_n is positive definite and obtain that

$$\begin{aligned} \mathcal{L}U_n = & H\gamma\tilde{e}_1^3(\chi_2^{p_1} - \widehat{\chi}_2^{p_1}) - H\gamma\gamma_1\tilde{e}_1^3(\chi_1^{p_1} - \widehat{\chi}_1^{p_1}) + \gamma\tilde{e}_1^3\phi_1 \\ & + \frac{3}{2}\gamma\tilde{e}_1^2\|h_1\|^2 + \sum_{i=3}^{n}H\gamma\tilde{e}_{i-1}^{p-p_{i-2}+3}(\chi_i^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}}) \\ & - \sum_{i=2}^{n}H\gamma\gamma_i\tilde{e}_i^{p_1-p_{i-1}+3}(\chi_i^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}}) \\ & + \sum_{i=2}^{n}H\left[\frac{\gamma}{H}\tilde{e}_i^{p_1-p_{i-1}+3}(\phi_i - \gamma_i\phi_{i-1})\right] \\ & + \sum_{i=2}^{n}H\left[\frac{p_1-p_{i-1}+3}{2H}\gamma\tilde{e}_i^{p_1-p_{i-1}+2}\|h_i - \gamma_i h_{i-1}\|^2\right]. \end{aligned} \quad (54)$$

To find the appropriate values of γ_i's, the upper bound of each item in the right side of Eq. (54) will be given in the next steps. Firstly, we can directly infer from Lemma 5 that

$$-\gamma\gamma_1\tilde{e}_1^3(\chi_1^{p_1} - \widehat{\chi}_1^{p_1}) \leq -\frac{\gamma\gamma_1}{2^{p_1-1}}\tilde{e}_1^{p_1+3}. \quad (55)$$

Furthermore, from Lemmas 2 and 3, one can get

$$\begin{aligned} \gamma\tilde{e}_1^3(\chi_2^{p_1} - \widehat{\chi}_2^{p_1}) & \leq \gamma|\tilde{e}_1|^3 \cdot 2^{p_1}[|e_2|^{p_1} + |\chi_2|^{p_1}] \quad (56) \\ & \leq a_{21}\tilde{e}_1^{p_1+3} + a_{22}\tilde{e}_2^{p_1+3} + \lambda_{21}|\chi_2|^{p_1+3}, \\ & \times \frac{\gamma}{H}\tilde{e}_1^3\phi_1 + \frac{3\gamma}{2H}\tilde{e}_1^2\|h_1\|^2 \\ & \leq \frac{9}{p_1+3}\cdot\left(\frac{\gamma}{H}\right)^{\frac{p_1+3}{3}}\tilde{e}_1^{p_1+3} + \widetilde{F}_1(\chi_1), \quad (57) \end{aligned}$$

where $a_{21} = a_{21}(\gamma_2) > 0$ is a constant independent of H and γ_1; $a_{22} > 0$ is a constant independent of H and $\gamma_k(k = 1, 2)$; and $\widetilde{F}_1(\chi_1) = \frac{p_1}{p_1+3}\phi_1^{\frac{p_1+3}{p_1}} + \frac{3}{2}(\frac{\gamma}{H})^{\frac{p_1+3}{3(p_1+1)}}\|h_1\|^{\frac{2(p_1+3)}{p_1+1}}$.

On the other hand, it can be given from Assumption 1 that $4 \leq p_1 + p_{i-1} - p_{i-2} + 3 \leq p_1 + 3$ for all $i = 3, \cdots, n$.

Then, for $\forall 3 \leq i \leq n$, it is verified from Lemmas 2-5 that

$$\begin{aligned} & \gamma\tilde{e}_{i-1}^{p_1-p_{i-2}+3}(\chi_i^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}}) \\ & \quad \leq \gamma|\tilde{e}_{i-1}|^{p_1-p_{i-2}+3}\cdot 2^{p_{i-1}}[|e_i|^{p_{i-1}} + |\chi_i|^{p_{i-1}}] \\ & \quad \leq \sum_{j=1}^{i}a_{ij}\tilde{e}_j^{p_1+3} + \lambda_{i1}\chi_i^{p_1+p_{i-1}-p_{i-2}+3} + \widetilde{Q}_i, \end{aligned} \quad (58)$$

where $a_{ij} = a_{ij}(\gamma_i, \cdots, \gamma_{j+1})(j = 1, \cdots, i-1)$ are positive constants independent of H and $\gamma_k(k = 1, \cdots, j)$; a_{ii}, $\lambda_{i1} > 0$ are constants independent of H and $\gamma_j(j = 1, \cdots, i)$, $\widetilde{Q}_i > 0$ is also a constant.

For $\forall 2 \leq i \leq n$, it is obvious that

$$\begin{aligned} & -\gamma\gamma_i\tilde{e}_i^{p_1-p_{i-1}+3}(\chi_i^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}}) \\ & \quad = -\gamma\gamma_i\tilde{e}_i^{p_1-p_{i-1}+3}[\chi_i^{p_{i-1}} - (\widehat{\chi}_i + \tilde{e}_i)^{p_{i-1}}] \\ & \quad\quad -\gamma\gamma_i\tilde{e}_i^{p_1-p_{i-1}+3}[(\widehat{\chi}_i + \tilde{e}_i)^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}}]. \end{aligned} \quad (59)$$

Then, for $\forall 2 \leq i \leq n$, it can be inferred from $\tilde{e}_i = (\widehat{\chi}_i + \tilde{e}_i) - \widehat{\chi}_i$ and Lemma 5 that

$$\begin{aligned} & -\gamma\gamma_i\tilde{e}_i^{p_1-p_{i-1}+3}[(\widehat{\chi}_i + \tilde{e}_i)^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}}] \\ & \quad -\gamma\gamma_i\tilde{e}_i^{p_1-p_{i-1}+2}\cdot\tilde{e}_i[(\widehat{\chi}_i + \tilde{e}_i)^{p_{i-1}} - \widehat{\chi}_i^{p_{i-1}}] \\ & \leq -\frac{\gamma\gamma_i}{2^{p_{i-1}-1}}\tilde{e}_i^{p_1+3}. \end{aligned} \quad (60)$$

Meanwhile, for $\forall 2 \le i \le n$, we can deduce from Lemmas 2 and 3 that the following inequalities hold

$$\begin{aligned}
&|-\gamma\gamma_i\tilde{e}_i^{p_1-p_{i-1}+3}[\chi_i^{p_{i-1}}-(\widehat{\chi}_i+\tilde{e}_i)^{p_{i-1}}]| \\
&\quad \le \gamma\gamma_i D_{i-1}|\tilde{e}_i|^{p_1-p_{i-1}+3} \\
&\qquad \times\left[|\gamma_i e_{i-1}|^{p_{i-1}}+|\gamma_i e_{i-1}||\chi_i|^{p_{i-1}-1}\right] \\
&\quad \le \sum_{j=1}^{i} c_{ij}\tilde{e}_j^{p_1+3}+\lambda_{i2}\chi_i^{p_1+3}, \qquad (61)\\
&\quad \frac{\gamma}{H}\tilde{e}_i^{p_1-p_{i-1}+3}(\phi_i-\gamma_i\phi_{i-1}) \\
&\quad +\frac{p_1-p_{i-1}+3}{2H}\gamma\,\tilde{e}_i^{p_1-p_{i-1}+2}\|h_i-\gamma_i h_{i-1}\|^2 \\
&\quad \le \frac{(p_1-p_{i-1}+3)^2}{p_1+3}\cdot\left(\frac{\gamma}{H}\right)^{\frac{p_1+3}{p_1-p_{i-1}+3}}\tilde{e}_i^{p_1+3}+\widetilde{F}_i(\overline{\chi}_i),
\end{aligned} \tag{62}$$

where $c_{ij}=c_{ij}(\gamma_i,\cdots,\gamma_{j+1})(j=1,\cdots,i-1)$ are positive constants independent of H and $\gamma_k(k=1,\cdots,j)$; c_{ii}, $\lambda_{i2}>0$ are constants independent of H and $\gamma_j(j=1,\cdots,i)$; and

$$\begin{aligned}
\widetilde{F}_i(\overline{\chi}_i)&=\frac{p_{i-1}}{p_1+3}(\phi_i-\gamma_i\phi_{i-1})^{\frac{p_1+3}{p_{i-1}}} \\
&+\frac{p_1-p_{i-1}+3}{2}\left(\frac{\gamma}{H}\right)^{\frac{p_1+3}{(p_1-p_{i-1}+3)(p_{i-1}+1)}}\|h_i-\gamma_i h_{i-1}\|^{\frac{2(p_1+3)}{p_{i-1}+1}}.
\end{aligned}$$

Substituting (55)-(62) into (54) yields

$$\begin{aligned}
\mathcal{L}U_n \le& -H\left(\frac{\gamma\gamma_1}{2^{p_1-1}}-\sum_{j=2}^{n}(a_{j1}+c_{j1})\right)\tilde{e}_1^{p_1+3} \\
&-H\sum_{i=2}^{n-1}\left(\frac{\gamma\gamma_i}{2^{p_{i-1}-1}}-\sum_{j=2}^{n}(a_{ji}+c_{ji})\right)\tilde{e}_i^{p_1+3} \\
&-H\left(\frac{\gamma\gamma_n}{2^{p_1-1}}-(a_{nn}+c_{nn})\right)\tilde{e}_n^{p_1+3} \\
&+H\sum_{i=2}^{n}\frac{(p_1-p_{i-1}+3)^2}{p_1+3}\left(\frac{\gamma}{H}\right)^{\frac{p_1+3}{p_1-p_{i-1}+3}}\tilde{e}_i^{p_1+3} \\
&+\frac{9H}{p_1+3}\left(\frac{\gamma}{H}\right)^{\frac{p_1+3}{3}}\tilde{e}_1^{p_1+3}+\widetilde{F}(\chi)+\sum_{i=2}^{n}H\widetilde{Q}_i,
\end{aligned} \tag{63}$$

where $\widetilde{F}(\chi)=H\lambda_{21}\chi_2^{p_1+3}+\sum_{i=3}^{n}H\lambda_{i1}\chi_i^{p_1+p_{i-1}-p_{i-2}+3}+\sum_{i=2}^{n}H\lambda_{i2}\chi_i^{p_1+3}+\sum_{i=1}^{n}H\widetilde{F}_i(\overline{\chi}_i)$ is an unknown continuous function.

Apparently, a FLS can be applied to handle the unknown function $\widetilde{F}(\chi)$. Consequently, using Lemmas 4 and 6, one easily deduces that

$$\widetilde{F}(\chi)=\Upsilon_0^T\Psi_0+\epsilon_0(\chi)\le\frac{3\theta_0}{p_1+3}+\frac{p_1}{p_1+3}+\delta_0,$$

where $\theta_0=\|\Upsilon_0\|^{\frac{p_1+3}{3}}$, and $\delta_0>0$ is a given constant.

Therefore, the observer gains $\gamma_1,\cdots,\gamma_n$ and constant H can be selected as bellow:

$$\begin{aligned}
\gamma_n &\ge \max\left\{\frac{2^{p_{n-1}-1}}{\gamma}(\bar{c}_n+1+\kappa_n),1\right\}, \\
\gamma_{n-1} &\ge \max\left\{\frac{2^{p_{n-2}-1}}{\gamma}(\bar{c}_{n-1}(\gamma_n)+1+\kappa_{n-1}),1\right\}, \\
&\vdots \\
\gamma_2 &\ge \max\left\{\frac{2^{p_1-1}}{\gamma}(\bar{c}_2(\gamma_n,\cdots,\gamma_3)+1+\kappa_2),1\right\}, \\
\gamma_1 &\ge \max\left\{\frac{2^{p_1-1}}{\gamma}(\bar{c}_1(\gamma_n,\cdots,\gamma_2)+1+\kappa_1),1\right\}, \\
H &\ge \max\Bigg\{1,\left(\frac{9}{p_1+3}\right)^{\frac{3}{p_1+3}},\cdots, \\
&\qquad \times\left(\frac{(p_1-p_{n-1}+3)^2}{p_1+3}\right)^{\frac{p_1-p_{n-1}+3}{p_1+3}}\Bigg\},
\end{aligned} \tag{64}$$

where $\bar{c}_n=a_{nn}+c_{nn}$ is a positive constant independent of H and γ_i's, $\bar{c}_i=\bar{c}_i(\gamma_n,\cdots,\gamma_{i+1})=\sum_{j=2}^{n}(a_{ji}+c_{ji})$ $(i=1,\cdots,n-1)$ are positive constants independent of H and $\gamma_j(j=1,\cdots,i)$, and $\kappa_i(i=1,\cdots,n)$ are adjustment parameters.

Then, substituting (64) into (63) yields

$$\mathcal{L}U_n\le-\sum_{j=1}^{n}H\kappa_j\tilde{e}_j^{p_1+3}+\overline{Q}, \tag{65}$$

where $\overline{Q}=\sum_{i=2}^{n}H\widetilde{Q}_i+\frac{3\theta_0}{p_1+3}+\frac{p_1}{p_1+3}+\delta_0$ is a constant.

So far, we have completed the design of the output-feedback controller with appropriate observer gains.

Remark 5: It should be pointed out that, since the error dynamic system is also a p-norm stochastic nonlinear system, the observer gains can't be directly assumed to satisfy the Hurwitz matrix widely used in strict-feedback constrained nonlinear constrained systems. In this paper, the gains are devised by the rigorous analysis of the stability of the error dynamic system. The exponential terms with p_i's as the fractional powers are handled by fully and subtly taking advantage of the properties of the FLSs for the error dynamic system.

D. Verification of Keeping the Output Constraint and Stability

A theorem is first provided in the following to summarize the main result of this paper.

Theorem 1: For system (1), there is a fuzzy output-feedback controller (52) with the parameter adaptive laws (18), (30), (44) and (49), such that

1) the output constraint is not violated in the sense of probability, i.e., $P\{|y(t)|<\varepsilon\}=1$;
2) all the signals in system (1) are bounded in probability.

Proof. Firstly, we define the overall Lyapunov function for system (9) as

$$V=V_n+U_n,$$

where V_n and U_n have been respectively defined in (47) and (53).

Then, based on Eqs. (3), (50) and (65), it is easy to infer

$$\begin{aligned}
\mathcal{L}V\le&-\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right)-\sum_{j=1}^{n}\frac{d_j\tilde{\theta}_j^2}{2b_j} \\
&-\sum_{j=1}^{n}H\tau_j\eta_j^{p_1+3}-\sum_{j=1}^{n}H\kappa_j\tilde{e}_j^{p_1+3}+\sum_{j=1}^{n}Q_j+\overline{Q}.
\end{aligned} \tag{66}$$

In addition, it is easily obtained from Lemma 3 that

$$\begin{aligned}\tilde{e}_1^4 &\leq \frac{4}{p_1+3}\tilde{e}_1^{p_1+3} + \frac{p_1-1}{p_1+3},\\ \tilde{e}_j^{p_1-p_{j-1}+4} &\leq \frac{p_1-p_{j-1}+4}{p_1+3}\tilde{e}_j^{p_1+3} + \frac{p_{j-1}-1}{p_1+3},\\ \eta_j^{p-p_j+4} &\leq \frac{p_1-p_j+4}{p_1+3}\eta_j^{p_1+3} + \frac{p_j-1}{p_1+3}, j=2,\cdots,n,\end{aligned} \tag{67}$$

which renders

$$\begin{aligned}-\sum_{j=1}^{n} H\kappa_j \tilde{e}_j^{p_1+3} \leq &-\frac{\bar{\kappa}_1\tilde{e}_1^4}{4} - \sum_{j=2}^{n}\frac{\bar{\kappa}_j\tilde{e}_j^{p_1-p_{j-1}+4}}{p_1-p_{j-1}+4}\\ &+\frac{(p_1-1)H\kappa_1}{4} + \sum_{j=2}^{n}\frac{(p_{j-1}-1)H\kappa_j}{p_1-p_{j-1}+4},\\ -\sum_{j=1}^{n} H\tau_j\eta_j^{p_1+3} \leq &-\sum_{j=2}^{n}\frac{\bar{\tau}_j\eta_j^{p_1-p_j+4}}{p_1-p_j+4} + \sum_{j=2}^{n}\frac{(p_j-1)H\tau_j}{p_1-p_j+4},\end{aligned} \tag{68}$$

where $\bar{\kappa}_j = (p_1+3)H\kappa_j$, $\bar{\tau}_j = (p_1+3)H\tau_j (j=2,\cdots,n)$.

Further, we get

$$\begin{aligned}\mathcal{L}V \leq &-\frac{\tau_1\varepsilon^4}{2\pi}\tan\left(\frac{\pi\eta_1^4}{2\varepsilon^4}\right) - \sum_{j=1}^{n}\frac{d_j\tilde{\theta}_j^2}{2b_j} - \frac{\bar{\kappa}_1\tilde{e}_1^4}{4}\\ &-\sum_{j=2}^{n}\frac{\bar{\tau}_j\eta_j^{p_1-p_j+4}}{p_1-p_j+4} - \sum_{j=2}^{n}\frac{\bar{\kappa}_j\tilde{e}_j^{p_1-p_{j-1}+4}}{p_1-p_{j-1}+4} + Q^*,\end{aligned} \tag{69}$$

where $Q^* = \sum_{j=1}^{n} Q_j + \overline{Q} + \frac{(p_1-1)H\kappa_1}{4} + \sum_{j=2}^{n}[\frac{(p_{j-1}-1)H\kappa_j}{p_1-p_{j-1}+4} + \frac{(p_j-1)H\tau_j}{p_1-p_j+4}]$.

1) Since $V = V_n + U_n$ is continuous and positive definite, we can obtain from Lemma 4.3 in [53] that there exist $\mathcal{K}_\infty$ class functions $\varpi_1(\|Z\|)$ and $\varpi_2(\|Z\|)$ such that

$$\varpi_1(\|Z\|) \leq V \leq \varpi_2(\|Z\|), \tag{70}$$

where $Z = (\eta_1,\cdots,\eta_n,\theta_1,\cdots,\theta_n,\tilde{e}_1,\cdots,\tilde{e}_n)^T$.

Let $\pi_0 = \min\{\tau_1,\bar{\tau}_2,\cdots,\bar{\tau}_n,\bar{\kappa}_1,\cdots,\bar{\kappa}_n,d_1,\cdots,d_n\}$ and $\upsilon_0 = Q^*$. Then, Eq. (69) can be rewritten as

$$\mathcal{L}V \leq -\pi_0 V + \upsilon_0. \tag{71}$$

According to Lemma 1, one can directly verify from Eqs. (70) and (71) that

$$EV(t) \leq V(0)e^{-\pi_0 t} + \frac{\upsilon_0}{\pi_0}.$$

Due to $0 < e^{-\pi_0 t} < 1$, it is easy to get

$$EV(t) \leq V(0) + \frac{\upsilon_0}{\pi_0}. \tag{72}$$

For any initial state $x(0) = (x_1(0),\cdots,x_n(0))^T$, if $|x_1(0)| < \varepsilon$, then $|\eta_1(0)| = |\chi_1(0)| < \varepsilon$. Hence, by the definition of $V(t)$, it is easy to gain $V(0) < \infty$ for $\forall x(0)$ with $|x_1(0)| < \varepsilon$, which gives

$$EV(t) \leq V(0) + \frac{\upsilon_0}{\pi_0} < \infty. \tag{73}$$

Eq. (73) renders the boundness of the mean of $V(t)$, which also indicates $V(t)$ is bounded with probability one. Keeping this in mind and noting that $V = V_B + \frac{\tilde{\theta}_1^2}{2b_1} + \sum_{j=2}^{n}\Lambda_j + U_n$, one directly gains

$$P\{V_B(\eta_1) < \infty\} = 1.$$

Thus, it can be deduced that $P\{|\eta_1(t)| < \varepsilon\} = 1$, that is

$$P\{|y(t)| < \varepsilon\} = P\{|x_1(t)| < \varepsilon\} = 1. \tag{74}$$

Consequently, the output of system (1) remains in the set Π_1 with probability one.

2) On the other hand, combining Eq. (71) with Lemma 1 directly infers that all the variables in system (9) (i.e., χ_1, $\widehat{\chi}_i$'s, $\tilde{\theta}_i$'s, ξ_i's and $\tilde{u}$) are bounded in probability, which indicates $\hat{\theta}_i$'s are also bounded owing to $\hat{\theta}_i = \theta_i - \tilde{\theta}_i$. Meanwhile, noting that the controller $u = H^{r_{n+1}}\tilde{u}$, one can easily deduce that the controller u is also bounded. Equally, one directly gets from Eq. (71) and Lemma 1 that all the error variables (i.e., e_i's and $\tilde{e}_i$'s) are bounded in probability. Since $\chi_i = e_i + \widehat{\chi}_i$, all the χ_i's are bounded. Moreover, by (4), one has $x_i = \chi_i H^{r_i}$, which verifies that x_i is bounded, $\forall i = 1,\cdots,n$. Till now, we drive the conclusion that all the signals in system (1) are bounded in probability. ■

Remark 6: 1) This paper designs a fuzzy output-feedback controller rather than the state-feedback controllers in existing literatures about output constraints. Compared with the existing results, the main feature is that three restrictions (i.e., output constraints, unknown nonlinearities and unavailable states) are simultaneously taken into account, which implies the topic is quite difficult and challenging.

2) Although the paper has made some contributions on p-norm stochastic systems with output constraints. However, the proposed method still has some shortcomings. On one hand, the proposed control strategy has been developed under Assumption 1, which is still somewhat restrictive from the practical point of view. On the other hand, the considered output constraint is constant and symmetric, which causes the proposed scheme cannot be directly applied or further extended to the cases of state constraints or asymmetric output constraints. We will focus on these issues in the future.

Remark 7: It could be observed from (71) that the control performance depends on the parameters π_0 and υ_0. As stated in [28]–[31], a better control performance could be obtained by carefully adjusting the values of all the adjustment parameters, the observer gains $\gamma_1,\cdots,\gamma_n$ and scaling gain H under the given rules (64). Undesirably, to arrive the better performance may make the altitude of the controller u larger. Consequently, in practical applications, the careful adjustment of all the parameters and gains also should take into consideration the tradeoff between the better performance and control action.

IV. Simulation Example

A p-norm stochastic nonlinear system is presented in the following to demonstrate the validation of the proposed scheme.

$$\begin{cases} dx_1 = x_2^5 dt + 2x_1^3 dt + 4x_1 d\omega,\\ dx_2 = u^3 dt + x_1 x_2^2 dt + 3x_2^2 d\omega,\\ y = x_1.\end{cases} \tag{75}$$

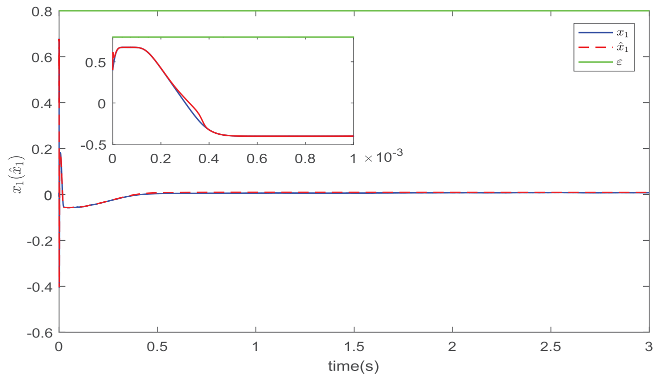

Fig. 1. Trajectories of $x_1(t)$ and $\hat{x}_1(t)$ with $\varepsilon = 0.8$.

Suppose $\varepsilon = 0.8$, that means the output $y = x_1$ is required to be kept in $\Pi_1 = \{y(t) \in R, |y(t)| < 0.8\}$. Obviously, $r_1 = 0$, $r_2 = \frac{1}{5}$, $r_3 = \frac{2}{5}$. Letting $\chi_1 = \frac{x_1}{H^{r_1}}$, $\chi_2 = \frac{x_2}{H^{r_2}}$ and $\tilde{u} = \frac{u}{H^{r_3}}$, the following equivalent system is gotten

$$\begin{cases} d\chi_1 = H\chi_2^5 dt + 2\chi_1^3 dt + 4\chi_1 d\omega, \\ d\chi_2 = H\tilde{u}^3 dt + H^{\frac{4}{5}}\chi_1\chi_2^2 dt + 3H^{\frac{4}{5}}\chi_2^2 d\omega, \\ y = \chi_1. \end{cases} \tag{76}$$

Based on system (76), a two-order observer is first constructed as

$$\begin{cases} \dot{\widehat{\chi}}_1 = H\widehat{\chi}_2^5 + H\gamma_1(\chi_1^5 - \widehat{\chi}_1^5), \\ \dot{\widehat{\chi}}_2 = H\tilde{u}^3 + H\gamma_2\gamma_1(\chi_1^5 - \widehat{\chi}_1^5). \end{cases} \tag{77}$$

Thus, the fuzzy output-feedback controller with adaptive laws is designed from Theorem 1 as

$$\begin{aligned} u &= H^{\frac{2}{5}}\tilde{u} = -H^{\frac{2}{5}}(M_2^{\frac{1}{3}}\widehat{\chi}_2 + M_2^{\frac{1}{3}}M_1^{\frac{1}{5}}\chi_1), \\ M_1 &= \frac{3}{8}(K(\eta_1))^{\frac{5}{3}}\left[\|\Psi_1(\widetilde{Z}_1)\|^{\frac{8}{3}}\sigma_{11}\hat{\theta}_1 + 1\right] + \frac{\rho_1(\eta_1)}{K(\eta_1)} + \tau_1, \\ M_2 &= \frac{5}{8}[\|\Psi_2(\widetilde{Z}_2)\|^{\frac{8}{5}}\sigma_{21}\hat{\theta}_2 + 1] + \frac{5\sigma_{12}}{8} + \tau_2, \\ \dot{\hat{\theta}}_1 &= \frac{3Hb_1\sigma_{11}}{8}(K(\eta_1)\|\Psi_1(\widetilde{Z}_1)\|)^{\frac{8}{3}}\eta_1^8 - d_1\hat{\theta}_1, \\ \dot{\hat{\theta}}_2 &= \frac{5Hb_2\sigma_{21}}{8}\|\Psi_2(\widetilde{Z}_2)\|^{\frac{8}{5}}\eta_2^8 - d_2\hat{\theta}_2, \end{aligned} \tag{78}$$

where $\eta_1 = \chi_1 = x_1$, $\eta_2 = \widehat{\chi}_2 - \xi_1$, $\xi_1 = -M_1^{\frac{1}{5}}\eta_1$, $\widetilde{Z}_1 = (\chi_1, \widehat{\chi}_1)^T$, $\widetilde{Z}_2 = (\chi_1, \widehat{\chi}_2^T, \hat{\theta}_1)^T$; $\rho_1(\eta_1) = \frac{3}{8} \cdot 50^{\frac{8}{3}}\sigma_{12}^{-\frac{5}{3}}(K(\eta_1))^{\frac{8}{3}} + \frac{7}{8}K(\eta_1)$; $\Psi_i(\widetilde{Z}_i)(i = 1, 2)$ are the fuzzy basis functions; σ_{11}, σ_{12}, $\sigma_{21}, \tau_i(i = 1, 2)$, $b_i(i = 1, 2)$, and $d_i(i = 1, 2)$ are adjustment parameters; H and $\gamma_i(i = 1, 2)$ are constant parameters to be designed. Choose

$$\Psi_1(\widetilde{Z}_1) = \frac{(\psi_{11}(\widetilde{Z}_1), \cdots, \psi_{17}(\widetilde{Z}_1))^T}{\sum_{j=1}^N \psi_{1j}(\widetilde{Z}_1)}$$

$$\Psi_2(\widetilde{Z}_2) = \frac{(\psi_{21}(\widetilde{Z}_2), \cdots, \psi_{27}(\widetilde{Z}_1))^T}{\sum_{j=1}^N \psi_{2j}(\widetilde{Z}_2)}$$

with

$$\psi_{1j}(\widetilde{Z}_1) = \exp\left[\frac{-(\chi_1 - 8 + 2j)^2}{2}\right]\exp\left[\frac{-(\widehat{\chi}_1 - 8 + 2j)^2}{2}\right],$$

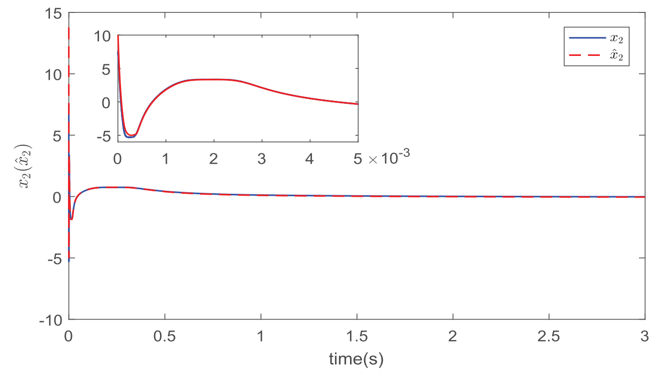

Fig. 2. Trajectories of $x_2(t)$ and $\hat{x}_2(t)$.

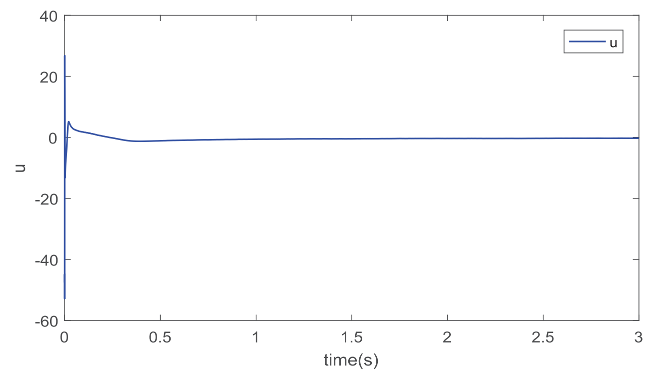

Fig. 3. Trajectory of u.

$$\begin{aligned} \psi_{2j}(\widetilde{Z}_2) = \exp&\left[\frac{-(\chi_1 - 8 + 2j)^2}{2}\right]\exp\left[\frac{-(\widehat{\chi}_1 - 8 + 2j)^2}{2}\right] \\ &\times \exp\left[\frac{-(\widehat{\chi}_2 - 8 + 2j)^2}{2}\right] \\ &\times \exp\left[\frac{-(\hat{\theta}_1 - 8 + 2j)^2}{2}\right], \quad j = 1, \ldots, 7. \end{aligned}$$

Further, we choose the adjustment parameters as $\sigma_{11} = 0.2$, $\sigma_{12} = 0.1$, $\sigma_{21} = 5$, $b_1 = b_2 = 1$, $d_1 = d_2 = 0.05$, $\tau_1 = \tau_2 = 0.5$. At the same time, one chooses and designs other parameters appearing in the controller as $\gamma = 20$, $\kappa_1 = \kappa_2 = 0.05$, $H = 21$, $\gamma_1 = 300$, $\gamma_2 = 2$. Then, the initial values of the states and adaptive parameters are selected as $[x_1(0), x_2(0), \widehat{\chi}_1(0), \widehat{\chi}_2(0), \hat{\theta}_1, \hat{\theta}_2]^T = [0.4, 7.5, 0.4, 7.5, 10, 10]^T$. The simulation results are displayed in Figs. 1-3.

Fig.1 shows the trajectories of $x_1(t)$ and $\hat{x}_1(t)$ under controller (78), which clearly demonstrates that the system output $x_1(t)$ and the estimate $\hat{x}_1(t)$ are kept within Π_1. In addition, since $x_1(t)$ is available, the error between it and the estimate $\hat{x}_1(t)$ constructed by the aid of the known $x_1(t)$ can reach to be very small. Further, the trajectories of $x_2(t)$ and $\hat{x}_2(t)$ are provided by Fig. 2, which illustrates that $x_2(t)$ can be well estimated by $\hat{x}_2(t)$. In addition, the trajectory of controller u is given by Fig. 3. Also, it is easy to observe from Figs. 1-3 that all the variables in system (75) remain bounded under controller (78) all the time.

V. Conclusion

This paper has investigated the controller design problem for a class of p-norm stochastic nonlinear systems which suffer from output constraints, unmeasurable states, and unknown nonlinearities. A tan-type BLF has been used to handle the output constraint issue, and a full-order observer has been designed to estimate the unmeasurable states. On these basic, the fuzzy control algorithm has been developed by applying the FLSs and the adding a power integrator technique, thereby an output-feedback controller being constructed. The designed controller ensures the stability of the considered system in the sense of boundness without violating the given constraint. However, there are still some challenging and unsolved problems, such as the strong restriction of Assumption 1, how applying the type-2 FLSs to enhance the computational issue, how to solve the output-feedback control for systems with state constraints or asymmetric output constraints and so on. In our future work, we will try to investigate these issues.

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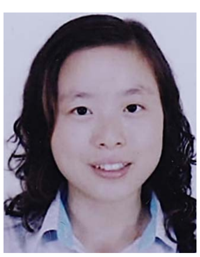

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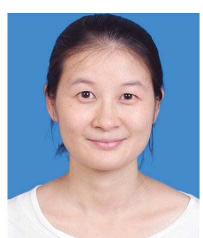

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