State and Input Simultaneous Estimation for a Class of Time-Delay Systems With Uncertainties

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***Abstract*—This brief addresses the problem of estimation of both the states and the unknown inputs of a class of systems that are subject to a time-varying delay in their state variables, to an unknown input, and also to an additive uncertain, nonlinear disturbance. Conditions are derived for the solvability of the design matrices of a reduced-order observer for state and input estimation, and for the stability of its dynamics. To improve computational efficiency, a delay-dependent asymptotic stability condition is then developed using the linear matrix inequality formulation. A design procedure is proposed and illustrated by a numerical example.**

***Index Terms*—Reduced-order observer, simultaneous estimation, time delay, unknown input.**

I. Introduction

In RECENT years, there has been significant research effort devoted to the problem of estimating simultaneously both the states and unknown inputs for time-invariant systems and uncertain/nonlinear systems [1]–[9]. This problem is motivated in part by applications requiring fault detection and isolation, and of fault-tolerant control [10], or where measurement of the system inputs is either too expensive or perhaps physically not possible [4]. The concept of unknown input observability and its relationship to unknown input observers was provided in [5] for decomposed systems such as multi-port networks. In chaotic systems, one may need to estimate the state for chaos synchronization and also the information signal input for secure communication [6].

Estimation of unknown inputs normally requires the derivatives of the output measurements [1], [2]. Dealing with this problem, a combined state/input estimator is proposed in [4], but the system states and unknown inputs can only be estimated to any desired degree of accuracy. An extension of the work in [4] is found in [8], enabling the estimation of a linear function of the state vector under less conservative conditions. In another approach, the problem of estimating simultaneously the states and unknown inputs can be handled by means of a state-space observer for a descriptor system obtained from original systems with unknown inputs [7], [9]. However, this approach does not consider the state delay problem, which is frequently encountered in many practical engineering applications.

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Although uncertain time-delay systems have been a subject of extensive studies (see, e.g., [11]), the design of reduced-order observers for simultaneous estimation of both the states and unknown inputs seems to receive little attention in the literature. Addressing this problem, our intention is to develop a design technique for reduced-order observers that can ensure exact asymptotic convergence of the state and unknown input estimates for systems with a time-varying delay in the state variables.

II. Problem Statement

Consider the following time-delay system:

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + Bu(t) + Wf(t,x) \quad t>0 \tag{1a}$$

$$x(t) = \varphi(t), \qquad t \in [-\tau_u, 0] \tag{1b}$$

$$0 \le \tau(t) \le \tau_u, \qquad \dot{\tau}(t) \le \alpha < 1 \tag{1c}$$

$$y(t) = Cx(t) + Du(t) \tag{1d}$$

where $y(t) \in \Re^r$ is the measured output vector, $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are the state and unknown input vectors to be estimated, and $\varphi(t)$ is a continuous initial function. Matrices A, A_d, B, W, C and D of appropriate dimensions are real constant. Here, the time-varying time delay $\tau(t)$ is subject to (1c), which means that the delay may vary from time to time but the rate of changing is bounded, as in the filter design problem [12]. In this brief, unknown disturbance $f(t,x) \in \Re^d$, not required to be estimated, represents the uncertainties, nonlinearities and time-varying terms [4], [8], and matrix W, assumed to have full-column rank, represents the distribution of this disturbance in the system dynamics.

Let us define an augmented state vector, $z(t)$, where $z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \Re^{(n+m)}$. Accordingly, the time-delay system (1) can be expressed as

$$E\dot{z}(t) = \overline{A}z(t) + \overline{A}_d z(t-\tau(t)) + Wf(t), \qquad t>0 \tag{2a}$$

$$z(t) = \overline{\varphi}(t), \qquad t \in [-\tau_u, 0] \tag{2b}$$

$$y(t) = Hz(t) \tag{2c}$$

where $\overline{\varphi}(t) = \begin{bmatrix} \varphi(t) \\ 0_{m\times 1} \end{bmatrix}$, $E = [I_n \; 0_{n\times m}]$, $\overline{A} = [A \; B]$, $\overline{A}_d = [A_d \; 0_{n\times m}]$, and $H = [C \; D]$.

The problem of designing a reduced-order observer to estimate both $x(t)$ and $u(t)$ of system (1) now becomes that of designing a reduced-order observer to estimate the state, $z(t)$, of system (2)

$$\dot{\omega}(t) = N\omega(t) + N_d\omega(t-\tau(t)) + Ly(t) + L_d y(t-\tau(t)), \quad t>0 \tag{3a}$$

$$\omega(t) = \rho(t), \qquad t \in [-\tau_u, 0] \tag{3b}$$

$$\hat{z}(t) = M\omega(t) + Fy(t) \tag{3c}$$

where $\omega(t) \in \Re^{(n+m-r)}$ is the observer state vector, $\rho(t)$ is a continuous vector-valued initial function, and $\hat{z}(t)$ denotes the estimate of $z(t)$. The design problem is to determine matrices N, N_d, L, L_d, M and F such that $\hat{z}(t)$ converges asymptotically to $z(t)$ (i.e., $\hat{z}(t) \rightarrow z(t)$ as $t \rightarrow \infty$).

III. Main Results

Let $T \in \Re^{(n+m-r)\times n}$ be a full-row rank matrix such that $\det\left[\begin{smallmatrix}TE\\ H\end{smallmatrix}\right] \neq 0$, and let us define the error vectors $\varepsilon(t) \in \Re^{(n+m-r)}$ and $e(t) \in \Re^{(n+m)}$ as

$$\varepsilon(t) = \omega(t) - TEz(t) \tag{4a}$$
$$e(t) = \hat{z}(t) - z(t). \tag{4b}$$

Theorem 1: For the observer (3), the estimate $\hat{z}(t)$ will converge asymptotically to $z(t)$ if there exists a matrix T such that the following conditions hold.

Condition 1: $\varepsilon(t)$ determined by the observer error system

$$\begin{aligned}\dot{\varepsilon}(t) &= N\varepsilon(t) + N_d\varepsilon\left(t-\tau(t)\right), \qquad t>0\\ \varepsilon(t) &= \rho(t) - TE\overline{\varphi}(t), \qquad t \in [-\tau_u, 0]\end{aligned}$$

converges asymptotically to zero for all $\varepsilon(\theta)$, $\forall\theta \in [-\tau_u, 0]$.

Condition 2:

$$\begin{cases}\text{(i)} & NTE + LH - T\overline{A} = 0\\ \text{(ii)} & N_dTE + L_dH - T\overline{A}_d = 0\\ \text{(iii)} & TW = 0.\end{cases}$$

Condition 3: $[M\ F]\left[\begin{smallmatrix}TE\\ H\end{smallmatrix}\right] = I_{(n+m)}$.

Proof: From (4a), (2) and (3), the following error dynamics equation is obtained:

$$\begin{aligned}\dot{\varepsilon}(t) &= \dot{\omega}(t) - TE\dot{z}(t)\\ &= N\varepsilon(t)N_d\varepsilon\left(t-\tau(t)\right)(NTE + LH - T\overline{A})z(t)\\ &\quad + (N_dTE + L_dH - T\overline{A}_d)z\left(t-\tau(t)\right) - TWf(t)\\ &\qquad t>0\end{aligned} \tag{5a}$$
$$\varepsilon(t) = \rho(t) - TE\overline{\varphi}(t), \qquad t \in [-\tau_u, 0]. \tag{5b}$$

From (4b), (3c), and (4a), $e(t)$ can be expressed as

$$e(t) = M\varepsilon(t) + (MTE + FH - I_{n+m})z(t). \tag{6}$$

From (5), $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$ if Conditions 1 and 2 of Theorem 1 are satisfied. Now, if Condition 3 of Theorem 1 is satisfied, then from (6), one has $e(t) = M\varepsilon(t)$. Thus, since $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$, it follows that $e(t) \rightarrow 0$ as $t \rightarrow \infty$ and hence $\hat{z}(t) \rightarrow z(t)$. This completes the proof of Theorem 1.

Remark 1: The observer design problem now rests with the determination of matrices T, N, N_d, L, L_d, M, and F to satisfy Conditions 1–3 of Theorem 1. The first two equations, (i) and (ii), of Condition 2 are the generalized Sylvester equations which must be satisfied under the constraint of (iii). Here, the key for solving the set of matrix equations in Conditions 2 and 3 renders to the determination of matrix T.

Theorem 2: The system of equations constituted by Conditions 2 and 3 of Theorem 1 is solvable if and only if $\operatorname{rank}[D\ CW] = m + d$.

Proof: Define the following nonsingular matrix:

$$\begin{bmatrix}R\\ H\end{bmatrix} = \begin{bmatrix}I_{(n+m-r)} & V\\ 0 & I_r\end{bmatrix}\begin{bmatrix}TE\\ H\end{bmatrix} \tag{7}$$

where $R \in \Re^{(n+m-r)\times(n+m)}$ is a full-row rank matrix, and $V \in \Re^{(n+m-r)\times r}$ is an arbitrary matrix.

From (7), $TE + VH = R$. By coupling this with $TW = 0$ of Condition 2 one can combine into

$$[T \quad V]S = R_a \tag{8a}$$
$$S = \begin{bmatrix}E & W\\ H & 0_{r\times d}\end{bmatrix} = \begin{bmatrix}E_a\\ H_a\end{bmatrix}$$
$$R_a = \begin{bmatrix}R & 0_{(n+m-r)\times d}\end{bmatrix}. \tag{8b}$$

A solution for $[T\ V]$ in (8) exists if and only if

$$\operatorname{rank}\begin{bmatrix}S\\ R_a\end{bmatrix} = \operatorname{rank}[S]. \tag{9}$$

According to the full-column rank assumption for matrix W the left-hand side of (9) can be determined as

$$\begin{aligned}\operatorname{rank}\begin{bmatrix}S\\ R_a\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}E & W\\ H & 0\\ R & 0\end{bmatrix}\\ &= \operatorname{rank}\begin{bmatrix}H\\ R\end{bmatrix} + \operatorname{rank}[W] = (n+m+d)\end{aligned} \tag{10}$$

and its right-hand side as

$$\begin{aligned}\operatorname{rank}[S] &= \operatorname{rank}\begin{bmatrix}I_n & 0 & W\\ C & D & 0\end{bmatrix}\\ &= \operatorname{rank}\begin{bmatrix}I_n & 0\\ -C & I_r\end{bmatrix}\begin{bmatrix}I_n & 0 & W\\ C & D & 0\end{bmatrix}\\ &= n + \operatorname{rank}[D \quad CW].\end{aligned} \tag{11}$$

From (10) and (11), condition (9) is satisfied if and only if $\operatorname{rank}[D\ CW] = (m+d)$. Upon the satisfaction of this condition, the determination of the observer matrices can be obtained as follows. First, a solution to (8) can then be expressed as [13]

$$[T \quad V] = R_aS^+ + Z(I_{n+r} - SS^+) \tag{12}$$

where $S^+ = \Delta[E_a^T\ H_a^T]$, $\Delta = (S^TS)^{-1}$ and $Z \in \Re^{(n+m-r)\times(n+r)}$ is an arbitrary matrix.

From (12), matrix T is given as

$$T = R_a\Delta E_a^T + Z\begin{bmatrix}I_n - E_a\Delta E_a^T\\ -H_a\Delta E_a^T\end{bmatrix}. \tag{13}$$

Since $\left[\begin{smallmatrix}TE\\ H\end{smallmatrix}\right]$ is a square matrix and $\det\left[\begin{smallmatrix}TE\\ H\end{smallmatrix}\right] \neq 0$, Condition 3 of Theorem 1 yields $[M\ F] = \left[\begin{smallmatrix}TE\\ H\end{smallmatrix}\right]^{-1}$. Now, the first two equations of Condition 2 can be compactly expressed as $\begin{bmatrix}N & L\\ N_d & L_d\end{bmatrix}\begin{bmatrix}TE\\ H\end{bmatrix} = \begin{bmatrix}T\overline{A}\\ T\overline{A}_d\end{bmatrix}$. Thus, by post-multiplying both sides by $[M\ F]$ yields expressions for the four matrices N, N_d, L, and L_d

$$\begin{bmatrix}N & L\\ N_d & L_d\end{bmatrix} = \begin{bmatrix}T\overline{A}M & T\overline{A}F\\ T\overline{A}_dM & T\overline{A}_dF\end{bmatrix} \tag{14}$$

where T is as defined in (13). Finally, note that from (7) and by using $[M\ F] = \left[\begin{smallmatrix}TE\\ H\end{smallmatrix}\right]^{-1}$, matrices M and F can be obtained as $M = \left[\begin{smallmatrix}R\\ H\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}I_{n+m-r}\\ 0\end{smallmatrix}\right]$ and $F = \left[\begin{smallmatrix}TE\\ H\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}0\\ I_r\end{smallmatrix}\right]$.

Remark 2: It is clear from (7) that one can always choose a matrix $R \in \Re^{(n+m-r)\times(n+m)}$ such that $\left[\begin{smallmatrix} R \\ H \end{smallmatrix}\right]$ is nonsingular since H is a full-row rank matrix, and therefore, $\det \left[\begin{smallmatrix} TE \\ H \end{smallmatrix}\right] \neq 0$. The condition provided in Theorem 2 gives the necessary and sufficient condition for the existence of matrices T and V.

Remark 3: When the system is subject only to $f(t,x)$, but not to unknown input $u(t)$, the condition of Theorem 2 implies $\mathrm{rank}(CW) = d$. When the system is subject only to the unknown input, $u(t)$, without uncertainties/nonlinearities $(W = 0)$, the condition of Theorem 2 becomes $\mathrm{rank}(D) = m$, i.e., matrix D has full-column rank. This is also the assumption required in [7] (or in [6] for the case of single output systems, $D \neq 0$) for the design of state and input observers. Here, to be able to cope with both $u(t)$ and $f(t,x)$, the number of measured outputs is chosen at least equal to the number of unknown inputs plus the size of the nonlinear term, i.e., $r \geq (m+d)$.

Now by substituting (13) and (14) into Condition 1 of Theorem 1, the following error dynamics equation is obtained:

$$\dot{\varepsilon}(t) = N\varepsilon(t) + N_d\varepsilon\left(t - \tau(t)\right), \qquad t > 0 \tag{15a}$$

$$\varepsilon(t) = \rho(t) - TE\overline{\varphi}(t), \qquad t \in [-\tau_u, 0] \tag{15b}$$

$$N = \Phi + Z\Omega; \quad N_d = \Phi_d + Z\Omega_d \tag{15c}$$

$$\Phi = R_a \Delta E_a^T \overline{A} M; \quad \Phi_d = R_a \Delta E_a^T \overline{A}_d M \tag{15d}$$

$$\Omega = \begin{bmatrix} \left(I_n - E_a\Delta E_a^T\right)\overline{A}M \\ -H_a \Delta E_a^T \overline{A} M \end{bmatrix}$$

$$\Omega_d = \begin{bmatrix} \left(I_n - E_a\Delta E_a^T\right)\overline{A}_d M \\ -H_a \Delta E_a^T \overline{A}_d M \end{bmatrix}. \tag{15e}$$

The design of observer (3) becomes thus the determination of a matrix Z such that the time-delay system (15) is asymptotically stable.

Lemma 1: Let $X \in \Re^{n\times p}$, $\mathrm{rank}(X) = q$ and $q < p < n$. Let $X^{\perp} \in \Re^{p\times(p-q)}$ be an orthogonal basis for the null space of X. Let $Y \in \Re^{p\times w}$ be a full-column rank matrix with $w \leq q$. Then $\mathrm{rank}(XY) = \mathrm{rank}(Y)$ if and only if $\mathrm{rank}[Y \ X^{\perp}] = (w + p - q)$.

Proof: Define $J^T = X^{\perp}$, then it is clear that $\left[\begin{smallmatrix} X \\ J \end{smallmatrix}\right]$ is a full-column rank matrix and the following equation holds:

$$\mathrm{rank}\left(\begin{bmatrix} X \\ J \end{bmatrix}[Y \quad X^{\perp}]\right) = \mathrm{rank}[Y \quad X^{\perp}]$$

where the left-hand side can be determined as

$$\begin{aligned} \mathrm{rank}\left(\begin{bmatrix} X \\ J \end{bmatrix}[Y \quad X^{\perp}]\right) &= \mathrm{rank}\begin{bmatrix} XY & 0 \\ JY & JJ^T \end{bmatrix} \\ &= \mathrm{rank}(XY) + \mathrm{rank}(JJ^T) \\ &= \mathrm{rank}(XY) + (p-q). \end{aligned}$$

Therefore, it follows that

$$\mathrm{rank}(XY) - \mathrm{rank}[Y \quad X^{\perp}] - (p-q). \tag{16}$$

Now, when $\mathrm{rank}[Y \ X^{\perp}] = (w + p - q)$, from (16) one has $\mathrm{rank}(XY) = w$. When $\mathrm{rank}[Y \ X^{\perp}] \neq (w + p - q)$, i.e., $\mathrm{rank}[Y \ X^{\perp}] < (w + p - q)$, it is also clear from (16) that $\mathrm{rank}(XY) \neq w$, i.e., $\mathrm{rank}(XY) < w$.

Lemma 2: Let X be a $(2n + 2r + m) \times (n + 2r)$-matrix defined by (17), shown at the bottom of the page. Then

$$\text{(i) } \mathrm{rank}(X) = (n + 2r - d)$$

$$\text{(ii) } \mathrm{X}^{\perp} = \begin{bmatrix} W \\ 0 \\ 0 \end{bmatrix}.$$

Proof: Let us first prove part [*i*]. From (17), it follows that

$$\mathrm{rank}(X) = r + \mathrm{rank}\begin{bmatrix} R_a\Delta E_a^T & R_a\Delta H_a^T \\ H_a\Delta E_a^T & H_a\Delta H_a^T \\ \left(I_n - E_a\Delta E_a^T\right) & -E_a\Delta H_a^T \\ -H_a\Delta E_a^T & \left(I_r - H_a\Delta H_a^T\right) \end{bmatrix}. \tag{18}$$

Using S and S^+ as defined in (8b) and (12), respectively, the term in the right-hand side of the above can be evaluated as

$$\begin{aligned} &\mathrm{rank}\begin{bmatrix} \left[\begin{smallmatrix} R_a \\ H_a \end{smallmatrix}\right] S^+ \\ \left(I_{(n+r)} - SS^+\right) \end{bmatrix} \\ &= \mathrm{rank}\begin{bmatrix} \left[\begin{smallmatrix} R_a \\ H_a \end{smallmatrix}\right] S^+ \\ \left(I_{(n+r)} - SS^+\right) \end{bmatrix} \left[S \quad \left(I_{(n+r)} - SS^+\right)\right] \\ &= \mathrm{rank}\begin{bmatrix} R_a \\ H_a \end{bmatrix} + \mathrm{rank}\left(I_{(n+r)} - SS^+\right). \end{aligned}$$

As $\mathrm{rank}\left[\begin{smallmatrix} R_a \\ H_a \end{smallmatrix}\right] = n + m$ and $\mathrm{rank}(I_{(n+r)} - SS^+) = r - m - d$, it follows from (18) that $\mathrm{rank}(X) = (n + 2r - d)$.

To prove part (*ii*), first note that $\mathrm{rank}(X^{\perp}) = d$. By recalling the assumption $\mathrm{rank}(W) = d$ and using the fact that $\Delta E_a^T W = \left[\begin{smallmatrix} 0_{(n+m)\times d} \\ I_d \end{smallmatrix}\right]$, it is easy to verify that $XX^{\perp} = 0$.

Theorem 3: Matrix $N = (\Phi + Z\Omega)$ is Hurwitz if and only if

$$\mathrm{rank}\begin{bmatrix} sI_n - A & -B & W \\ C & D & 0 \end{bmatrix} = (n + m + d) \qquad \forall s \in \mathbf{C} \quad \mathrm{Re}(s) \geq 0. \tag{19}$$

Proof: Matrix $N = (\Phi + Z\Omega)$ is Hurwitz if and only if the pair (Ω, Φ) is detectable, i.e.

$$\mathrm{rank}\begin{bmatrix} sI_{(n+m-r)} - \Phi \\ \Omega \end{bmatrix} = (n+m-r) \qquad \forall s \in \mathbf{C},\ \mathrm{Re}(s) \geq 0. \tag{20}$$

$$X = \begin{bmatrix} R_a\Delta E_a^T & R_a\Delta H_a^T & -R_a\Delta E_a^T(sE - \overline{A})F \\ H_a\Delta E_a^T & H_a\Delta H_a^T & -H_a\Delta E_a^T(sE - \overline{A})F \\ \left(I_n - E_a\Delta E_a^T\right) & -E_a\Delta H_a^T & -\left(I_n - E_a\Delta E_a^T\right)(sE - \overline{A})F \\ -H_a\Delta E_a^T & \left(I_r - H_a\Delta H_a^T\right) & H_a\Delta E_a^T(sE - \overline{A})F \\ 0 & 0 & I_r \end{bmatrix} \tag{17}$$

Sufficiency: Let us first define the following matrix:

$$Y = \begin{bmatrix} (sE-\overline{A})M & (sE-\overline{A})F \\ 0 & 0 \\ 0 & I_r \end{bmatrix} \in \Re^{(n+2r)\times(n+m)}. \tag{21}$$

Then condition (19) can be expressed equivalently as

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} sI_n - A & -B & W \\ C & D & 0 \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} sE-\overline{A} & W \\ H & 0 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} sE-\overline{A} & W \\ 0 & 0 \\ H & 0 \end{bmatrix}\begin{bmatrix} M & F & 0 \\ 0 & 0 & I_d \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} (sE-\overline{A})M & (sE-\overline{A})F & W \\ 0 & 0 & 0 \\ 0 & I_r & 0 \end{bmatrix} \\ &= \operatorname{rank}[Y \quad X^{\perp}] \\ &= (n+m+d) \qquad \forall s \in \mathbf{C},\ \mathrm{Re}(s) \geq 0. \end{aligned} \tag{22}$$

Now, if

$$\operatorname{rank}\begin{bmatrix} sI_n - A & -B & W \\ C & D & 0 \end{bmatrix} = \operatorname{rank}[Y \quad X^{\perp}] = (n+m+d)$$

$\forall s \in \mathbf{C}, \mathrm{Re}(s) \geq 0$ then according to Lemma 1, it follows that $\operatorname{rank}(XY) = (n+m), \forall s \in \mathbf{C}, \mathrm{Re}(s) \geq 0$. Using matrix X as defined in Lemma 2, it is easy to show the following:

$$\begin{aligned} \operatorname{rank}(XY) &= \operatorname{rank}\begin{bmatrix} sI_{(n+m-r)} - R_a\Delta E_a^T\overline{A}M & 0 \\ -H_a\Delta E_a^T\overline{A}M & 0 \\ -\left(I_n - E_a\Delta E_a^T\right)\overline{A}M & 0 \\ H_a\Delta E_a^T\overline{A}M & 0 \\ 0 & I_r \end{bmatrix} \\ &= (n+m) \qquad \forall s \in \mathbf{C},\ \mathrm{Re}(s) \geq 0 \end{aligned}$$

which gives

$$\operatorname{rank}\begin{bmatrix} sI_{(n+m-r)-\Phi} \\ \Omega \end{bmatrix} = (n+m-r) \qquad \forall s \in \mathbf{C},\ \mathrm{Re}(s) \geq 0.$$

Necessity: From the proof of Lemma 1, when $\operatorname{rank}[Y\ X^{\perp}] \neq (n+m+d), \forall s \in \mathbf{C}, \mathrm{Re}(s) \geq 0$ (i.e., $\operatorname{rank}[Y\ X^{\perp}] < (n+m+d)$), it is implied that $\operatorname{rank}(XY) < (n+m), \forall s \in \mathbf{C}, \mathrm{Re}(s) \geq 0$ and therefore, $\operatorname{rank}\left[\begin{smallmatrix} sI_{n+m-r}-\Phi \\ \Omega \end{smallmatrix}\right] < (n+m-r)$, $\forall s \in \mathbf{C}, \mathrm{Re}(s) \geq 0$. This completes the proof of Theorem 3.

Corollary 1: Matrix $\overline{N} = (N+N_d)$ is Hurwitz if and only if

$$\operatorname{rank}\begin{bmatrix} sI_n-(A+A_d) & -B & W \\ C & D & 0 \end{bmatrix} = (n+m+d) \qquad \forall s \in \mathbf{C}\ \ \mathrm{Re}(s) \geq 0. \tag{23}$$

Remark 4: For the case, the system is subject only to the unknown input, $u(t)$, without uncertainties/nonlinearities ($W = 0$), since $d = 0$ matrix N is Hurwitz if and only if $\operatorname{rank}\begin{bmatrix} sI_n - A & -B \\ C & D \end{bmatrix} = (n+m)$ $\forall s \in \mathbf{C}, \mathrm{Re}(s) \geq 0$. Similarly, matrix $\overline{N} = (N+N_d)$ is Hurwitz if and only if $\operatorname{rank}\begin{bmatrix} sI_n - (A+A_d) & -B \\ C & D \end{bmatrix} = (n+m)$ $\forall s \in \mathbf{C}, \mathrm{Re}(s) \geq 0$.

Now for the determination of a matrix Z in (15), a delay-dependent condition is provided in the following theorem.

Theorem 4: Upon the satisfaction of the conditions given, respectively, in Theorem 2 and Corollary 1, for given scalars $\tau_u > 0$, $\alpha < 1$, there exists a matrix Z such that the error $\varepsilon(t)$ of system (15) converges asymptotically to zero provided there exist matrices $P = P^T > 0$ and G; and positive scalars δ_1 and δ_2 such that the linear matrix inequality (LMI) shown in (24), at the bottom of the page, holds, where

$$\Gamma = P(\Phi+\Phi_d)+(\Phi+\Phi_d)^TP+G(\Omega+\Omega_d)+(\Omega+\Omega_d)^TG^T.$$

Moreover, parameter matrix Z is then given by $Z = P^{-1}G$.

Proof: Choose a Lyapunov function candidate $V(\varepsilon, t)$ for (15) as

$$\begin{aligned} V(\varepsilon,t) &= \varepsilon^T(t)P\varepsilon(t) + \frac{1}{\delta_1}\int_0^{\tau_u}\int_{t-\theta}^{t} \varepsilon^T(s)N^TPN\varepsilon(s)ds d\theta \\ &\quad + \frac{1}{\delta_2(1-\alpha)^2}\int_{\tau(t)}^{\tau_u+\tau(t)}\int_{t-\theta}^{t} \varepsilon^T(s)N_d^TPN_d\varepsilon(s)dsd\theta \end{aligned} \tag{25}$$

where $P > 0$, and δ_1 and δ_2 are positive scalars.

When taking the time derivative of the Lyapunov function, we first make use of the Leibniz–Newton formula to rewrite (15a) in the form of

$$\begin{aligned} \dot{\varepsilon}(t) = (N+N_d)\varepsilon(t) - N_dN\int_{t-\tau(t)}^{t}\varepsilon(s)ds \\ -N_dN_d\int_{t-\tau(t)}^{t}\varepsilon\left(s-\tau(s)\right)ds. \end{aligned} \tag{26}$$

Now the integral terms in the right-hand side of $V(\varepsilon, t)$ in (25) can be handled by applying directly Lemma 1 given in [14]. Finally, by using the inequality $-2u^Tv \leq \delta u^TP^{-1}u + (1/\delta)v^TPv$, where u and v are vectors of appropriate dimension and δ is a positive scalar,

$$\dot{V}(\varepsilon,t) \leq \varepsilon^T(t)\Psi\varepsilon(t) \tag{27}$$

$$\begin{bmatrix} \Gamma & \tau_uP\Phi_d+\tau_uG\Omega_d & \tau_u\Phi^TP+\tau_u\Omega^TG^T & \tau_u\Phi_d^TP+\tau_u\Omega_d^TG^T \\ \tau_u\Phi_d^TP+\tau_u\Omega_d^TG^T & -\tau_u(\delta_1+\delta_2)^{-1}P & 0 & 0 \\ \tau_uP\Phi+\tau_uG\Omega & 0 & -\tau_u\delta_1P & 0 \\ \tau_uP\Phi_d+\tau_uG\Omega_d & 0 & 0 & -\tau_u\delta_2(1-\alpha)^2P \end{bmatrix} < 0 \tag{24}$$

where

$$\Psi = P(N+N_d)+(N+N_d)^T P+(\delta_1+\delta_2)\tau_u PN_d P^{-1}N_d^T P + \frac{\tau_u}{\delta_1}N^T PN + \frac{\tau_u}{\delta_2(1-\alpha)^2}N_d^T PN_d.$$

In the last step, condition (24) is obtained using the Schur decomposition by substituting (15c)–(15e) into (27) and letting $G = PZ$. This completes the proof of Theorem 4.

Remark 5: (Constant time-delay): For the case where the delay is constant, i.e., $\dot{\tau}(t) = \alpha = 0$, all the results presented in this note are still valid with α in LMI condition (24), taking the value of 0 ($\alpha = 0$).

Design Procedure:

Step 1) Choose matrix R such that $\begin{bmatrix} R \\ H \end{bmatrix} \in \Re^{(n+m)\times(n+m)}$ is nonsingular. Obtain $M = \begin{bmatrix} R \\ H \end{bmatrix}^{-1}\begin{bmatrix} I_{(n+m-r)} \\ 0 \end{bmatrix}$.

Step 2) Use (15d), (15e) to derive matrices Φ, Φ_d, Ω and Ω_d.

Step 3) Solve the LMI (24) and obtain matrix $Z = P^{-1}G$.

Step 4) Use (13) to obtain matrix T. Obtain $F = \begin{bmatrix} TE \\ H \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I_r \end{bmatrix}$. Finally, from (14), obtain observer matrices N, N_d, L and L_d.

IV. Numerical Example

To illustrate the design procedure, let us consider a time-delay system described by (1), where

$$\dot{x}(t) = \begin{bmatrix} -10 & 1 & 2 \\ -48 & -2 & 0 \\ 1 & -1 & -20 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 & -1 \\ -0.5 & 1 & 0.5 \\ 0.25 & 0 & 0.5 \end{bmatrix} \cdot x(t-0.4) + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ tx_2(t)x_3(t) \\ 0 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(t).$$

$u(t)$ is an unknown input signal, the delay is constant with $\tau_u = 0.4$ sec and $\alpha = 0$. Here, $W = [0\ 1\ 0]^T$ and $f(t,x) = tx_2(t)x_3(t)$.

Step 1: Choose $R = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0.5 & -1 & 0 & 1 \end{bmatrix}$. Hence, matrix M is obtained as $M = \begin{bmatrix} 0.1379 & 0.0690 \\ -0.2759 & -0.1379 \\ 0.3103 & -0.3448 \\ -0.3448 & 0.8276 \end{bmatrix}$.

Step 2: Matrices Φ, Φ_d, Ω and Ω_d are obtained as

$$\Phi = \begin{bmatrix} -13.3793 & 5.3103 \\ 6.7586 & -20.6207 \end{bmatrix}$$

$$\Phi_d = \begin{bmatrix} -1.0172 & 1.7414 \\ -1.4655 & 2.0172 \end{bmatrix}$$

and $\Omega = 0$ and $\Omega_d = 0$. Thus, for this example, $N = \Phi$, $N_d = \Phi_d$ and hence $Z = 0$.

Steps 3 & 4: The LMI problem (22) is solved to obtain

$$T = \begin{bmatrix} 5 & 0 & 1 \\ 4.5 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} -6 & 10 \\ 12 & -5.5 \end{bmatrix}$$

$$L_d = 0_{2\times 2}$$

$$F = \begin{bmatrix} 0 & 2 & 0 & -2 \\ 0 & -1 & 0 & 2 \end{bmatrix}^T.$$

V. Conclusion

This brief has presented the design of a reduced-order observer for simultaneous state and input estimation for a class of dynamical systems that are subject to a time-varying delay in the state variables, an unknown input and an uncertain, nonlinear disturbance. The observer is developed for estimation of both the system states and unknown input without requiring derivatives of the measured outputs. Conditions are derived for the solvability of the design matrices, and for the existence of the proposed observer. To facilitate the computational effectiveness of the design method, a delay-dependent asymptotic stability condition is obtained using the LMI formulation. A design procedure is given and then illustrated through a numerical example.

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