Protocol-Based Control for Semi-Markov Jump Systems With Dynamic Quantization

Jiangming Xu, Jun Cheng, Huaicheng Yan, *Member, IEEE*, Ju H. Park, *Senior Member, IEEE*, and Wenhai Qi, *Senior Member, IEEE*

***Abstract*—This brief is devoted to protocol-based control method design for semi-Markov switching systems with dynamic quantization. Essential different from the semi-Markov process with arbitrary switchings, by proposing a novel semi-Markov process subjected to a deterministic switching signal, a generalized framework of semi-Markov switching systems is set up. With respect to the quantization range of the dynamic quantizer, an improved event-triggered protocol associated with the quantizer parameter is proposed. On account of mode-dependent Lyapunov theory, sufficient conditions are established to ensure the stochastic finite-time boundedness of the resulting system. In the end, a tunnel diode circuit model shows the practicability and effectiveness of the presented method.**

***Index Terms*—Semi-Markov process, event-triggered protocol, quantization effect, switching signal.**

I. Introduction

As typical hybrid systems, Markov jump systems (MJSs) have a wide range of applications in various areas, which are composed of a series of modes and subsystems. In reality, MJSs are capable of characterizing dynamics with unexpected variations, external perturbations, etc [1]. To be specific, in light of the assumption that sojourn-time (ST) obeys exponential distribution, the time-invariant transition rates have been well exploited in MJSs [2], [3]. As indicated in [4], it is more general to explore non-exponential distribution in depicting the ST, namely, semi-Markov process (SMP), and the corresponding system is called semi-MJS (SJSS) [5]. Owing to their distinguishing ability, the ST-independent transition rates/probabilities are time-varying, which are essentially different from the reporting MSSs subject to time-invariant transition rates/probabilities. On the other hand, SMP in the afore-discussed controllers/filters design is pre-supposed to be described by arbitrary switchings. As pointed out in [6], SMP with arbitrary switchings may increase the conservativeness of derived results, and such drawbacks can be well tackled by a deterministic switching signal (DSS). It is noteworthy that SMP subject to DSS has not attracted suitable attention due probably to its inherent difficulty.

For a resource-saving network with massive packets exchange, quantization is frequently occurred in the control community [7], [8]. In view of packets should be quantized before accessing the communication channels, the quantization errors should be taken into consideration since they affect the control performance. To date, many efficient quantization techniques have been studied in the principle of easy-to-implement, see, uniform case, logarithmic case, etc. These techniques are so-called static quantization strategies. Comparably, the dynamic quantization mechanism has distinguishing flexibility by introducing dynamic parameters [9]. In contrast to the fruitful achievements of static quantization strategies, the dynamic case has not gained enough research interest, which motivates the present study.

On another research front, massive packets transmitted via the shared channels may result in many undesired issues, such as data dropouts, time delays, and packet collisions. As such, event-triggered protocol (ETP) has been verified to be an efficient tool in dealing above shortcomings [10]. Apart from the existing time-triggered scheme, the packets will not be transferred to the actuator unless the preset conditions are met. Added by its merits, several types of ETPs have been presented, including static ETP [11], and dynamic ETP [12], [13], etc. Notice that all these protocols are based on the current and last transferred information in triggering criteria, which omit the particularity of the quantization effect in constructing ETP scheduling. Up to now, little research interest has been devoted to ETP involving the quantization effect. Thus, it would be an interesting issue to investigate the improved ETP by utilizing a dynamic quantization strategy.

On the basis of the above observation, the finite-time asynchronous control method for SMJS with dynamic quantization

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Jiangming Xu is with the School of Mathematics and Statistics, Guangxi Normal University, Guilin 541006, China (e-mail: jiangmingxumath@163.com; jcheng@gxnu.edu.cn).

Jun Cheng is with the School of Mathematics and Statistics, Guangxi Normal University, Guilin 541006, China, and also with the School of Information Science and Engineering, Chengdu University, Chengdu 610106, China (e-mail: jcheng@gxnu.edu.cn).

Huaicheng Yan is with the Key Laboratory of Advanced Control and Optimization for Chemical Process of Ministry of Education, East China University of Science and Technology, Shanghai 200237, China (e-mail: hcyan@ecust.edu.cn).

Ju H. Park is with the Department of Electrical Engineering, Yeungnam University, Kyongsan 38541, Republic of Korea (e-mail: jessie@ynu.ac.kr).

Wenhai Qi is with the School of Engineering, Qufu Normal University, Rizhao 276826, China (e-mail: qiwhtanedu@163.com).

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TABLE I
THE PARAMETER DEFINITIONS

$\mathcal{R}_k$	Markov chain
$\mathcal{T}_k$	ST of SMP, i.e., $\mathcal{T}_k = t_k - t_{k-1}$
$\mathcal{G}_p$	probability distribution function
χ_p	probability density function

and improved ETP (IETP) is explored. The key contributions are generalized as follows: (1) A generalized framework of SMJS is constructed, in which the variation of SMP is regulated by a higher-level DSS, and the average residence time strategy improves the stability analysis of SMJS. Clearly, the time-varying transition rates are jointly described by the switching signal and SMP. (2) Considering the dynamic quantization effect rather than traditional system states, an IETP is effectively scheduled, which is capable of speeding up the convergence rate since the introduction of adjustable quantization parameters. (3) Based on the hidden semi-Markov model, an asynchronous control method associated with the dynamic quantization effect is designed. By means of Lyapunov theory, sufficient criteria are achieved to ensure the stochastic finite-time boundedness (SFTB) of the resulting systems.

Organization: Section II formulates the problem description of SMJS and IETP. Section III provides the main results. Section IV presents the simulation results. Section V draws a conclusion.

Notations: $\Pr\{\cdot\}$ signifies the occurrence probability. $\mathscr{X}^\top$ symbolizes the transpose of the matrix $\mathscr{X}$. $\mathbb{R}^{n_x}$ stands for the n_x-dimensional Euclidean space. diag$\{\cdot\}$ means the diagonal matrix. $\lambda_{\min}(\cdot)/\lambda_{\max}(\cdot)$ symbols the minimal/maximal eigenvalue. $\|\cdot\|$ refers to Euclidean vector norm. $\mathcal{E}\{\cdot\}$ signifies the expectation. $\mathbb{N}$ describes the set of nonnegative integers.

II. Problem Formulations

A. Semi-Markov Jump System

Consider the continuous-time SMJS in the following form

$$\dot{x}(t) = A(r(t))x(t) + B(r(t))u(t) + C(r(t))w(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, and $w(t) \in \mathcal{L}_2[0,\infty)$ signify the state, control input and disturbance input, respectively. The stochastic process $\{r(t), t \geq 0\} \in \mathcal{M} = \{1, 2, \ldots, M\}$ represents a continuous time semi-Markov process (SMP). Let $N(t) = \sup\{k : t_k \leq t\}$, and $r(t) = \mathcal{R}_{N(t)}$, where the parameter definitions are provided in Table I, and $\{(\mathcal{R}_k, t_k)\}$ is called the renewal process. The parameter $\{\theta(t), t > 0\} \in \mathcal{N} = \{1, 2, \ldots, N\}$ describes a DSS. In this regard, the corresponding transition probability of renewal process subject to DSS $\theta(t)$ can be described as

$$\Pr\{\mathcal{R}_{k+1} = q \mid \mathcal{R}_k = p, \theta(t) = \eth\} = \begin{cases} \delta_{pq}^{\eth}, & p \neq q \\ 0, & p = q \end{cases}$$

Additionally, inspired by the work of [5], the TR matrix can be inferred as $\Psi^{\eth}(h) = [\pi_{pq}^{\eth}(h)]_{M\times M}$ with TR

$$\begin{aligned} &\Pr\{r(t+\sigma) = q \mid r(t) = p, \theta(t) = \eth\} \\ &= \begin{cases} \Pr\{\mathcal{T}_{k+1} \leq h+\sigma, \mathcal{R}_{k+1} = q \mid \mathcal{T}_{k+1} > h, \\ \mathcal{R}_k = p, \theta(t) = \eth\} = \pi_{pq}^{\eth}(h)\sigma + o(\sigma), p \neq q \\ \Pr\{\mathcal{T}_{k+1} > h+\sigma, \mathcal{R}_{k+1} = q \mid \mathcal{T}_{k+1} > h, \\ \mathcal{R}_k = p, \theta(t) = \eth\} = 1 + \pi_{pq}^{\eth}(h)\sigma + o(\sigma), p = q \end{cases} \end{aligned} \tag{2}$$

where $\lim_{\sigma\to 0} = o(\sigma)/\sigma = 0$, and the cumulative distribution function (CDF) $\mathcal{G}_p = \Pr\{\mathcal{T}_{k+1} < h | r(t_k) = p, \theta(t) = \eth\}$. For $p \neq q$, $\pi_{pq}^{\eth}(h) \geq 0$ signifies the mode transfer rate, it yields $\pi_{pq}^{\eth}(h) = \lim_{\sigma\to 0} \frac{\Pr\{r(t+\sigma)=q|r(t)=p,\theta(t)=\eth\}}{\sigma} = \delta_{pq}^{\eth}\frac{\chi_p(h)}{1-\mathcal{G}_p(h)}, p \neq q$ and $\pi_{pp}^{\eth}(h) = -\sum_{q=1,q\neq p}^{M} \pi_{pq}^{\eth}(h) < 0$.

Remark 1: Notably, the ST plays a crucial role in describing the dynamic behavior of MSSs, and conventional ST subjected to exponential distribution renders the restriction of widespread applications. Lately, SMPs have been introduced to deal with the afore-discussed limitation, in which ST obeys generalized distributions (i.e., Weibull, Bayes). Nevertheless, on the basis of SMP with arbitrary switching, the control laws may increase the conservativeness of the systems. As a result, SMP associated with a deterministic signal $\{\theta(t), t \geq 0\}$ rather than an arbitrary signal is proposed in this brief, the time-varying transition rates can be described in a more general form, which contains the existing Markov process and SMP as special cases.

B. Dynamic Quantizer

Concerning the limited capacity of network bandwidth, it is essential to consider the quantized signals over the digital processing. As such, a generalized dynamic quantizer is forwarded in this brief. It is assumed that there exist range and sensitivity scalars $M > 0$ and $\Delta > 0$ such that

$$q(x(t), \mu(t)) = 0, \text{ if } |x(t)| \leq \Delta\mu(t), \tag{3}$$

$$|q(x(t), \mu(t)) - x(t)| \leq \Delta\mu(t), \text{if } \Delta\mu(t) \leq |x(t)| \leq M\mu(t), \tag{4}$$

$$|q(x(t), \mu(t))| \geq (M - \Delta)\mu(t), \text{if } |x(t)| \geq M\mu(t). \tag{5}$$

From one-parameter family of quantizer, it holds that

$$q(x(t), \mu(t)) = \mu(t)q\left(\frac{x(t)}{\mu(t)}\right). \tag{6}$$

For better analysis the quantification purpose, we define

$$\mu(t) = \frac{\sqrt{\theta}}{M}|x(t)|, \tag{7}$$

where $\mu(t) > 0$ signifies a dynamic quantizer parameter.

C. Improved Event-Triggered Protocol

To save network burden and maintain the control performance, the ETP has been established to arrange the data transmission. By means of ETP, one can effectively judge whether to release the data to networked channels. Before presenting the IETP, let's recall the ETP given in [14], [15]:

$$\begin{aligned} t_{k+1}h = t_k h + \min_{l\in\mathbb{N}} \Big\{ lh \mid\, & e^\top(t_k h + lh)\Omega e(t_k h + lh) \\ & \geq \rho x^\top(t_k h + lh)\Omega x(t_k h + lh) \Big\}, \end{aligned} \tag{8}$$

where $e(t_k h + lh) = x(t_k h + lh) - x(t_k h)$, and matrix $\Omega > 0$. $\rho \in (0, 1)$ is a preset threshold scalar, h, and $t_k h$ are the sampling interval, and the latest released instant, respectively.

In this brief, apart from the ETP (8), the quantization information is involved in the IETP, and the next released time instant is determined by

$$t_{k+1}h = t_k h + \min_{l\in\mathbb{N}} \Big\{ lh \mid q^\top(e(t_k h + lh), \mu(t_k h + lh))\Omega_p$$

$$\begin{aligned} &\times q(e(t_kh+lh), \mu(t_kh+lh)) \\ &\geq \rho\mu^\top(t_kh+lh)\Omega_p\mu(t_kh+lh)\Big\}, \end{aligned} \tag{9}$$

where matrix $\Omega_p > 0$ is mode-dependent, and $q(e(t_kh+lh), \mu(t_kh+lh)) = q(x(t_kh+lh), \mu(t_kh+lh)) - q(x(t_kh), \mu(t_kh))$. Besides, the time interval of two adjacent instant is described as $[t_kh+\eta_k, t_{k+1}h+\eta_{k+1}) = \bigcup_{\mathcal{I}}^{d}\mathcal{I}_l$, where $\mathcal{I}_l = [t_kh+lh+\eta_{k+l}, t_kh+lh+h+\eta_{k+l+1}]$, $d = t_{k+1}-t_k-1$. In addition, let $\eta(t) = t - t_kh - lh$ for all $t \in \mathcal{I}_l$, it yields $\eta(t) \in [\eta_1, \eta_2]$, where η_1 and η_2 mean the lower and upper bounds of $t - t_kh - lh$. Summarized the above-observation, a controller is devised as

$$u(t) = K_{\varphi(t)}q(x(t_kh), \mu(t_kh)), \tag{10}$$

where $K_{\varphi(t)}$ stands for the control gain to be solved. A homogeneous Markov process, $\varphi(t) \in \mathcal{S} = \{1, 2, \ldots, S\}$, is built via a conditional probability matrix $\Gamma = [\Phi_{ps}]_{M\times S}$:

$$\Pr\{\varphi(t) = s \mid r(t) = p\} = \Phi_{ps}, \tag{11}$$

where $\Phi_{ps} \in [0, 1]$, and $\sum_{s=1}^{S}\Phi_{ps} = 1$.

Remark 2: Essential different from the existing ETPs, by resorting to the dynamically quantified parameter $\mu(t)$, a novel IETP is scheduled to arrange the transmission of the packets efficiently. As such, more information can be applied based on IETP (9), which renders higher freedom in analysis control performance. Compared with ETPs (8), fewer packets will be released while maintaining the desired control performance. With this property, one can achieve fewer triggering times and further alleviate the communication burden to fulfill the practical industrial requirements. Such a point will be verified in the simulation section.

Remark 3: The dynamic quantization scheme in this brief is capable of reducing the frequency of packet exchange. In contrast to the static case (i.e., logarithmic quantization, uniform quantization), the dynamic quantization scheme has impressive flexibility since the presence of several adjustable parameters.

For brevity, let $\varphi(t) = s$, and $r(t) = p$. From (1) and (10), the closed-loop SMJS is established as

$$\begin{aligned} \dot{x}(t) = A_px(t) + B_pK_s(&q(x(t-\eta(t)), \mu(t)) \\ &- q(e(t_kh+lh), \mu(t_kh+lh))) + C_pw(t). \end{aligned} \tag{12}$$

Definition 1 [16]: For switching signals $\theta(t)$, if there exist real numbers $N_0 > 0$ and $\tau_a > 0$, such that the number of switches on interval $(\tau_1, \tau_2]$ meets $N_\theta(\tau_2, \tau_1) \leq (\tau_2-\tau_1)/\tau_a + N_0$, τ_a is average residence time, and N_0 is the chatter bound.

Definition 2 [17]: For the bounded disturbance $\mathbb{W}_{[t_1,t_2]} \triangleq \{w^\top(t)w(t) \leq d^2, \forall t \in [t_1, t_2]\}$, SMJS (12) is called SFTB with respect to (w.r.t.) $(c_1, c_2, T_f, \mathcal{R}, d)$ such that $\sup_{-\eta_2<\theta<0}\mathcal{E}\{x^\top(\theta)\mathcal{R}x(\theta), \dot{x}^\top(\theta)\mathcal{R}\dot{x}(\theta)\} \leq c_1 \implies \mathcal{E}\{x^\top(t)\mathcal{R}x(t)\} \leq c_2$, where scalars $T_f > 0$, $c_1 > 0$, $c_2 > 0$, and matrix $\mathcal{R} > 0$.

Lemma 1: [18] For any $t_0 \leq t \leq t_1$, and continuous and summable functions $\nu(t)$, $\zeta(t)$, and $\eta(t)$, if $\dot{\nu}(t) \leq \zeta(t)\nu(t) + \eta(t)$ holds, one can derive that $\nu(t) \leq \nu(t_0)\exp(\int_{t_0}^{t}\zeta(r)dr) + \int_{t_0}^{t}\eta(s)\exp(\int_{s}^{t}\zeta(r)dr)ds$.

III. Main Results

Theorem 1: For given scalars η_1, $\eta_2 > 0$, $d > 0$, $a > 0$, $b > 0$, $\rho > 0$, $\theta > 0$, the closed-loop SMJS (12) is SFTB w.r.t. $(c_1, c_2, T_f, \mathcal{R}, d)$, if there exist matrices $P_p^{\eth} > 0$, $M > 0$, $R_\varsigma > 0$, $Q_\varsigma > 0$, $(\varsigma = 1, 2)$, $\Omega_p > 0$, and any matrix Z, for any $\eth \in \mathcal{N}$, $p, q \in \mathcal{M}$, $s \in \mathcal{S}$, such that

$$\Xi_{pm\eth} = \begin{bmatrix} \Xi_{pm\eth}^{\imath} & \Xi_{pm\eth}^{J} \\ * & \Xi_{pm}^{\ell} \end{bmatrix} < 0, \tag{13}$$

$$P_p^{\eth} \leq \mu P_p^{\beta}, (\eth \neq \beta) \tag{14}$$

$$c_2\kappa_2 > c_1e^{-bt}\kappa^{\frac{T_f}{\tau_a}}\kappa_1 + d^2\kappa^{\frac{T_f}{\tau_a}}, \tag{15}$$

$$\tau_a > \tau_a^* = \frac{T_f\ln\kappa}{\ln(c_2\kappa_2) - \ln(e^{-bT_f}\kappa_1c_1 + d^2)}, \tag{16}$$

where $\Xi_{pm\eth}^{\imath} = \begin{bmatrix} \Xi_{pm}^1 & \Xi_{pm}^7 \\ * & \Xi_{pm}^8 \end{bmatrix}$, $\Xi_{pm\eth}^{J} = \begin{bmatrix} e^{-b\eta_1}R_1 & 0 & \Xi_{pm}^2 & \Xi_{pm}^3 & ZC_P \\ e^{-b\eta_2}R_2 & e^{-b\eta_2}R_2 & a\Xi_{pm}^7 & 0 & 0 \end{bmatrix}$, $\Xi_{pm}^{\ell} = \mathrm{diag}\{\Xi_{pm}^4, \Xi_{pm}^5, \Xi_{pm}^6, -\Omega_p, -I\}$, $\Xi_{pm}^1 = \sum_{q=1}^{M}\bar{\pi}_{pq}^{\eth}P_q^{\eth} + Q_1 + Q_2 + P_p^{\eth} - e^{-b\eta_1}R_1 - \frac{\theta}{M^2}\Delta^\top\Omega_p\Delta + He\{ZA_p\}$, $\Xi_{pm}^2 = P_p^{\eth} - Z + aA_p^\top Z^\top$, $\Xi_{pm}^3 = \frac{\sqrt{\theta}}{M}\Delta^\top\Omega_p - \sum_{s=1}^{S}\Phi_{ps}ZB_pK_s$, $\Xi_{pm}^4 = -e^{-b\eta_1}Q_1 - e^{-b\eta_1}R_1$, $\Xi_{pm}^5 = -e^{-b\eta_2}Q_2 - e^{-b\eta_2}R_2$, $\Xi_{pm}^6 = -Z - Z^\top + \eta_1^2R_1 + \eta_{12}^2R_2$, $\Xi_{pm}^7 = \sum_{s=1}^{S}\Phi_{ps}ZB_pK_s$, $\Xi_{pm}^8 = -2e^{-b\eta_2}R_2 + \frac{\theta}{M^2}\rho\Omega_p$, $\bar{\pi}_{pq}^{\eth} = \mathcal{E}\{\pi_{pq}^{\eth}(h)\} = \int_0^\infty\pi_{pq}^{\eth}(h)\chi_p(h)dh$, $\eta_{12} = \eta_2 - \eta_1$. $\kappa = \kappa_1/\kappa_2$, $\kappa_1 = \kappa_{11} + \kappa_{12} + \kappa_{13}$, $\kappa_2 = \kappa_{21} + \kappa_{22}$, $\kappa_{11} = \lambda_{\max}\{\mathcal{R}^{-\frac{1}{2}}P_p^{\eth}\mathcal{R}^{-\frac{1}{2}}\}$, $\kappa_{12} = \sum_{\varsigma=1}^{2}\frac{1-e^{-b\eta_\varsigma}}{b}\lambda_{\max}\{\mathcal{R}^{-\frac{1}{2}}Q_\varsigma\mathcal{R}^{-\frac{1}{2}}\}$, $\kappa_{13} = \lambda_{\max}\{\mathcal{R}^{-\frac{1}{2}}R_1\mathcal{R}^{-\frac{1}{2}}\}\frac{\eta_1(b\eta_1-1+e^{-b\eta_1})}{b^2} + \lambda_{\max}\{\mathcal{R}^{-\frac{1}{2}}R_2\mathcal{R}^{-\frac{1}{2}}\}\frac{\eta_{12}(b\eta_{12}-\eta_{12}-e^{-b\eta_1}+e^{-b\eta_2})}{b^2}$, $\kappa_{21} = \lambda_{\min}\{\mathcal{R}^{-\frac{1}{2}}P_p^{\eth}\mathcal{R}^{-\frac{1}{2}}\}$, $\kappa_{22} = \sum_{\varsigma=1}^{2}\lambda_{\min}\{\mathcal{R}^{-\frac{1}{2}}Q_\varsigma\mathcal{R}^{-\frac{1}{2}}\}\frac{1-e^{-b\eta_\varsigma}}{b}$.

Proof: The Lyapunov function is chosen as

$$V(x(t), r(t), \theta(t)) = \sum_{s=1}^{3}V_s(x(t), r(t), \theta(t)), \tag{17}$$

where $V_1(x(t), r(t), \theta(t)) = x^\top(t)P_{r(t)}^{\theta(t)}x(t)$, $V_2(x(t), r(t), \theta(t)) = \int_{t-\eta_1}^{t}e^{b(s-t)}x^\top(s)Q_1x(t)ds + \int_{t-\eta_2}^{t}e^{b(s-t)}x^\top(s)Q_2x(t)ds$, and $V_3(x(t), r(t), \theta(t)) = \eta_1\int_{-\eta_1}^{0}\int_{t+s}^{t}e^{b(\theta-t)}\dot{x}^\top(\theta)R_1\dot{x}(\theta)d\theta ds + \eta_{12}\int_{-\eta_2}^{-\eta_1}\int_{t+s}^{t}e^{b(\theta-t)}\dot{x}^\top(\theta)R_2\dot{x}(\theta)d\theta ds$.

On the basis of the probability density function and the infinitesimal operator, the expectations of derivation about $V(x(t), r(t), \theta(t))$ carries out

$$\begin{aligned} &\mathcal{E}\{\mathcal{L}V_1(x(t), r(t), \theta(t))\} \\ &= \lim_{\sigma\to 0}\frac{1}{\sigma}\Bigg[\mathcal{E}\Bigg\{\sum_{q\neq p}\Pr\{\mathcal{T}_{k+1} \leq h+\sigma, \mathcal{R}_{k+1} = q \mid \mathcal{T}_{k+1} > h, \\ &\mathcal{R}_k = p, \theta(t) = \eth\}x^\top(t+\sigma)P_q^{\eth}x(t+\sigma) + \Pr\{\mathcal{T}_{k+1} > h \\ &+\sigma \mid \mathcal{T}_{k+1} > h, \mathcal{R}_k = p, \theta(t) = \eth\}x^\top(t+\sigma)P_p^{\eth}x(t+\sigma)\Bigg\} \\ &- x^\top(t)P_p^{\eth}x(t)\Bigg] \\ &= x^\top(t)\left(\sum_{q=1}^{S}\bar{\pi}_{pq}^{\eth}P_q^{\eth}\right)x(t) + \mathbf{He}\Big\{\dot{x}^\top(t)P_p^{\eth}x^\top(t)\Big\}, \end{aligned} \tag{18}$$

$$\begin{aligned}&\mathcal{E}\{\mathcal{L}V_2(x(t), r(t), \theta(t))\}\\&\quad= -bV_2(t) + x^\top(t)(Q_1 + Q_2)x(t) - e^{-b\eta_1}x^\top(t-\eta_1)\\&\quad\times Q_1x(t-\eta_1) - e^{-b\eta_2}x^\top(t-\eta_2)Q_1x(t-\eta_2),\end{aligned} \tag{19}$$

$$\begin{aligned}&\mathcal{E}\{\mathcal{L}V_3(x(t), r(t), \theta(t))\}\\&\quad\le -bV_3(t) + \dot{x}^\top(t)\Big(\eta_1^2R_1 + \eta_{12}^2R_2\Big)\dot{x}(t)\\&\quad\quad - e^{-b\eta_1}x_{t-\eta_1}^{t\top}R_1x_{t-\eta_1}^t - e^{-b\eta_2}x_{t-\eta_2}^{t-\eta(t)\top}R_2x_{t-\eta_2}^{t-\eta(t)}\\&\quad\quad - e^{-b\eta_2}x_{t-\eta(t)}^{t-\eta_1\;\top}R_2x_{t-\eta(t)}^{t-\eta_1},\end{aligned} \tag{20}$$

where $x_{t-\eta_1}^t = x(t) - x(t-\eta_1)$, $x_{t-\eta_2}^{t-\eta(t)} = x(t-\eta(t)) - x(t-\eta_2)$, and $x_{t-\eta(t)}^{t-\eta_1} = x(t-\eta_1) - x(t-\eta(t))$.

Recalling (12), for any matrix Z, it holds that

$$\begin{aligned}0 &= 2(x^\top(t) + a\dot{x}^\top(t))Z\cdot[-\dot{x}(t) + A_px(t) + C_pw(t) + B_pK_s\\&\quad\times[q(x(t-\eta(t)), \mu(t)) - q(e(t_kh+lh), \mu(t_kh+lh))]].\end{aligned} \tag{21}$$

From the quantizer $(\cdot)$ given in (4), one can get that

$$\begin{aligned}q(x(t), \mu(t)) &= \mu(t)\left|q(\frac{x(t)}{\mu(t)}) - \frac{x(t)}{\mu(t)}\right| + x(t)\\&\le \mu(t)\Delta + x(t).\end{aligned} \tag{22}$$

Combining (9) and (17)-(22), it yields

$$\begin{aligned}&\mathcal{E}\{\mathcal{L}V(x(t), r(t), \theta(t))\}\\&\quad\le -bV(x(t), r(t), \theta(t)) + \xi^\top(t)\Xi_{ps}\xi(t) + w^\top(t)w(t),\end{aligned} \tag{23}$$

where $\xi(t) = [x(t)\ x(t-\eta(t))\ x(t-\eta_1)\ x(t-\eta_2)\ \dot{x}(t)\ e(t-\eta(t))\ w(t)]^\top$. It follows from (13) that $\Xi_{ps} < 0$, which equivalents $\mathcal{L}V(x(t), r(t), \theta(t)) < -bV(x(t), r(t), \theta(t)) + w^\top(t)w(t)$. On account of Lemma 1, for any $t \in [t_k, t_k+1]$, it follows from (23) that

$$\begin{aligned}\mathcal{E}\{V(x(t), r(t), \theta(t))\} &< -e^{b(t-t_k)}\mathcal{E}\{V(x(t_k), r(t_k), \theta(t_k))\}\\&\quad+\int_{t_k}^t e^{-b(t-s)}w^\top(s)w(s)ds.\end{aligned} \tag{24}$$

According to the condition (14), for the switching interval t_{k-1}, one derives the following formula

$$\mathcal{E}\{V(x(t_k), r(t_k), \theta(t_k))\} < \kappa\mathcal{E}\{V(x(t_k^-), r(t_k^-), \theta(t_k^-))\}. \tag{25}$$

Then, combining (24) and (25) gives

$$\mathcal{E}\{V(x(t), r(t), \theta(t))\} \le c_1e^{-bt}\kappa^{\frac{T_f}{\tau_a}}\kappa_1 + \kappa^{\frac{T_f}{\tau_a}}d^2. \tag{26}$$

According to (17) and (26), the following result holds

$$\mathcal{E}\{x^\top(t)\mathcal{R}x(t)\} \le c_1e^{-bT_f}\kappa^{\frac{T_f}{\tau_a}}\kappa_1 + \kappa^{\frac{T_f}{\tau_a}}d^2/\kappa_2. \tag{27}$$

On the basis of (15), it yields $\mathcal{E}\{x^\top(t)\mathcal{R}x(t)\} < c_2$. Thus, applying Definition 2, it can be concluded that SMJS (12) is SFTB, which completes the proof. ■

On the basis of Theorem 1, the control gains will be designed in Theorem 2.

Theorem 2: For given scalars η_1, $\eta_2 > 0$, $d > 0$, $a > 0$, $b > 0$, $\rho > 0$, and $\theta > 0$, the closed-loop SMJS (12) is SFTB w.r.t. $(c_1, c_2, T_f, \mathcal{R}, d)$, if there exist matrices $\bar{P}_p^{\eth} > 0$, $M > 0$, $\bar{R}_\varsigma > 0$, $\bar{Q}_\varsigma > 0$, $(\varsigma = 1, 2)$, $\bar{\Omega}_p > 0$, and compatible-dimensional matrix $\bar{Z}$, for any $\eth \in \mathcal{N}$, $p, q \in \mathcal{M}$, $s \in \mathcal{S}$, such that (14)-(16) hold, and

$$\hat{\Xi}_{pm\eth} = \begin{bmatrix}\hat{\Xi}_{pm\eth}^{\iota} & \hat{\Xi}_{pm\eth}^{J}\\ * & \hat{\Xi}_{pm}^{\ell}\end{bmatrix} < 0, \tag{28}$$

where $\hat{\Xi}_{pm\eth}^{\iota} = \begin{bmatrix}\hat{\Xi}_{pm}^1 & \hat{\Xi}_{pm}^7\\ * & \hat{\Xi}_{pm}^8\end{bmatrix}$, $\hat{\Xi}_{pm\eth}^{J} = \begin{bmatrix}e^{-b\eta_1}\bar{R}_1 & 0 & \hat{\Xi}_{pm}^2 & \hat{\Xi}_{pm}^3 & \bar{Z}C_P & \frac{\sqrt{\theta}}{M}\Delta^\top & \Lambda_m^1\\ e^{-b\eta_2}\bar{R}_2 & e^{-b\eta_2}\bar{R}_2 & a\hat{\Xi}_{pm}^7 & 0 & 0 & 0 & 0\end{bmatrix}$, $\hat{\Xi}_{pm}^{\ell} = \mathrm{diag}\ \{\hat{\Xi}_{pm}^4, \hat{\Xi}_{pm}^5, \hat{\Xi}_{pm}^6, -\bar{\Omega}_p, -I, -\bar{\Omega}_p, \Lambda_m\}$, $\hat{\Xi}_{pm}^1 = \bar{\pi}_{pp}^{\eth}\bar{P}_p^{\eth} + \bar{Q}_1 + \bar{Q}_2 + \bar{P}_p^{\eth} - e^{-b\eta_1}\bar{R}_1 + He\{\bar{Z}A_p\}$, $\hat{\Xi}_{pm}^2 = \bar{P}_p^{\eth} - \bar{Z} + aA_p^\top\bar{Z}^\top$, $\hat{\Xi}_{pm}^3 = \frac{\sqrt{\theta}}{M}\Delta^\top\bar{\Omega}_p - \sum_{s=1}^S\Phi_{ps}B_p\bar{K}_s$, $\hat{\Xi}_{pm}^4 = -e^{-b\eta_1}\bar{Q}_1 - e^{-b\eta_1}\bar{R}_1$, $\hat{\Xi}_{pm}^5 = -e^{-b\eta_2}\bar{Q}_2 - e^{-b\eta_2}\bar{R}_2$, $\hat{\Xi}_{pm}^6 = -\bar{Z} - \bar{Z}^\top + \eta_1^2\bar{R}_1 + \eta_{12}^2\bar{R}_2$, $\hat{\Xi}_{pm}^7 = \sum_{s=1}^S\Phi_{ps}B_p\bar{K}_s$, $\hat{\Xi}_{pm}^8 = -2e^{-b\eta_2}\hat{R}_2 + \frac{\theta}{M^2}\rho\hat{\Omega}_p$, $\Lambda_m = -diag\{P_1^{\eth}, P_2^{\eth}, \ldots, P_{p-1}^{\eth}, P_{p+1}^{\eth}, \ldots, P_M^{\eth}\}$, $\Lambda_m^1 = \bar{Z}[\sqrt{\bar{\pi}_{p1}^{\eth}}P_1^{\eth} \cdots \sqrt{\bar{\pi}_{p(p-1)}^{\eth}}P_{p-1}^{\eth}\ \sqrt{\bar{\pi}_{p(p+1)}^{\eth}}P_{p+1}^{\eth} \cdots \sqrt{\bar{\pi}_{pM}^{\eth}}P_M^{\eth}]$.

Furthermore, the controller gains are devised as

$$\bar{K}_s = K_sZ^{-1}. \tag{29}$$

Proof: Let $\bar{Z} = Z^{-1}$, $\bar{Q}_\varsigma = Z^{-1}Q_\varsigma Z^{-1}$, $\bar{R}_\varsigma = Z^{-1}R_\varsigma Z^{-1}$, $(\varsigma = 1, 2)$, $\bar{\Omega}_p = Z^{-1}\Omega_pZ^{-1}$, $\bar{P}_p^{\eth} = Z^{-1}P_p^{\eth}Z^{-1}$. Pre- and post-multiplying (13) with $\mathrm{diag}\{\bar{Z}, \bar{Z}, \bar{Z}, \bar{Z}, \bar{Z}, \bar{Z}, I\}$, which yields $\hat{\Xi}_{pm\eth} < 0$. This ends the proof. ■

IV. Illustrative Example

In the section, a tunnel diode circuit model (TDCM) is employed to verify the feasibility of the proposed methodology. As signified in [19], the dynamic equation of TDCM is described as

$$\begin{cases}C\dot{V}_c(t) = -\frac{1}{R_l}V_c(t) - \frac{1}{R_d}V_c(t) + i_l(t),\\ L\dot{i}_l(t) = -V_c(t) - R_ei_l(t) + V_s(t) + u(t) + w(t),\end{cases} \tag{30}$$

where $C = 0.1F$ signifies the capacitance, $L = 2mH$ represents the inductance, $R_d = \frac{1}{0.002+0.01V_d^2(t)}$, and $V_s(t) = 0.5V_c(t)$. In this example, the values of resistances R_e and R_l vary with the complex environment, and three different sets of values are considered: (1) $R_{l1} = 220\Omega$, $R_{e1} = 1\Omega$; (2) $R_{l2} = 180\Omega$, $R_{e2} = 2\Omega$; (3) $R_{l3} = 150\Omega$, $R_{e3} = 3\Omega$. Let $x_1(t) = V_c(t)$, $x_2(t) = i_l(t)$, and set $\mathcal{M} = 3$, $\mathcal{N} = 2$, $\mathcal{S} = 3$.

In this example, the ST of the SMP is supposed to obey the Weibull distribution. As such, the transition rate function is expressed as $\pi_{pq}^{\eth}(h) = \delta_{pq}^{\eth}\frac{k}{c^k}h^{k-1}$, whose distribution density function is inferred as $\chi_p(h) = \frac{k}{c^k}h^{k-1}e^{-(\frac{h}{c})^k}$. By considering the switching signal $\theta(t)$, one has $\delta_{pq}^1 = \begin{bmatrix}0 & 0.3 & 0.7\\ 0.4 & 0 & 0.6\\ 0.5 & 0.5 & 0\end{bmatrix}$ and $\delta_{pq}^2 = \begin{bmatrix}0 & 0.4 & 0.6\\ 0.5 & 0 & 0.5\\ 0.2 & 0.8 & 0\end{bmatrix}$.

Without loss of generality, when $p = 1$, one selects $c = 1$ and $k = 2$. When $p = 2$, one chooses $c = 1$, $k = 3$. When $p = 3$, one sets $c = 1$ and $k = 4$. According to the above description, the transition rate matrices can be obtained as

$$\Psi^1(h) = \begin{bmatrix}-2h & 0.6h & 1.4h\\ 1.2h^2 & -3h^2 & 1.8h^2\\ 2h^3 & 2h^3 & -4h^3\end{bmatrix}, \Psi^2(h) = \begin{bmatrix}-2h & 0.8h & 1.2h\\ 1.5h^2 & -3h^2 & 1.5h^2\\ 0.8h^3 & 3.2h^3 & -4h^3\end{bmatrix}$$

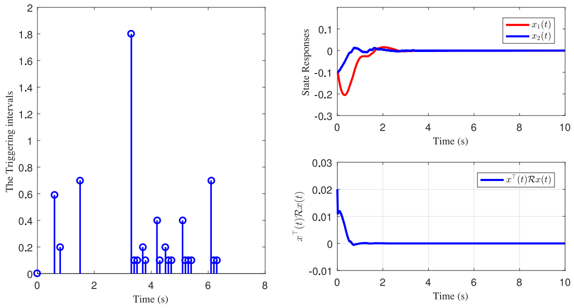

Fig. 1. Simulation results of (9).

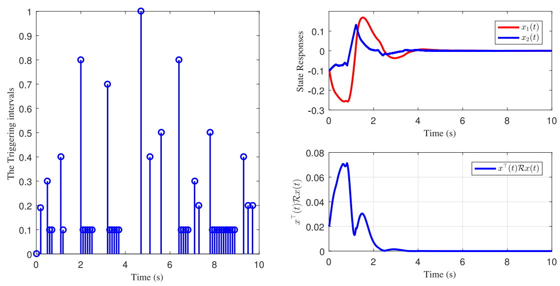

Fig. 2. Simulation results of (8).

Besides, the conditional probability matrix Γ is set to be $\Gamma = [0.3\ 0.3\ 0.4; 0.4\ 0.2\ 0.4; 0.2\ 0.5\ 0.3]$. Other parameters are set as $b = 0.07$, $\eta_1 = 1$, $\eta_2 = 5$, $d = 0.7$, $\rho = 0.4$, $c_1 = 0.05$, $c_2 = 1$, $\mathcal{R} = I$, $\kappa = 3$, $\theta = 3.3456$, $\Delta = 0.7$, and $M = 50$. By solving the conditions of Theorem 2, the controller gains are devised as $K_1 = [2.5341 \ \ -0.3790]^\top$, $K_2 = [0.8042 \ \ -0.0151]^\top$, and $K_3 = [-3.3755\ 0.7261]^\top$.

For given external disturbance $w(t) = \frac{\sqrt{2}}{5}\cos(3.14t)$ and select the initial states $x_0(t) = [-0.1 -0.1]^\top$, the simulation results performed are plotted in Figs. 1–2. Under the framework of IETP, the triggering instants and release intervals, trajectories of dynamic state $x(t)$, and evolution of $x^\top(t)\mathcal{R}x(t)$ are depicted in Fig. 1. The averaged response of $x^\top(t)\mathcal{R}x(t)$ not exceed c_2 within a preset finite-time.

Aim to examine the merits of the IETP on the dynamic performance, Fig. 2 is provided for the simulation comparison, which reveals the triggering instants and release intervals, trajectories of dynamic state $x(t)$, and evolution of $x^\top(t)\mathcal{R}x(t)$. As observed from Figs. 1–2, by means of IETP scheduling, faster convergence is achieved, and much better control performance can be expected. Meanwhile, for analysis clarity, we define the triggering rate $\mathbb{TR} = \frac{\text{triggering times}}{\text{the number of sampled-data}}$. In the proposed IETP (9), the triggering rate is calculated as 24.98%. However, based on the ETP (8), the triggering rate is computed as 47.11%. The benefit of the proposed IETP, fewer packets are transmitted along with a small triggering rate. These simulation results confirm the effectiveness and superiority of the IETP and controller design strategy.

V. Conclusion

In this brief, the SFTB of SMJS has studied by an quantization-dependent ETP. A novel SMP subjected to a DSS has been constructed to describe the generalized framework of SMJS, and an IETP associated with the quantizer parameter has been proposed. Finally, a TDCM has been applied to illustrate the practicability of the proposed method. Besides, the extension of the attained results to multi-agent will be our future research topic [20], [21].

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