Distributed Average Tracking for Lipschitz-Type Nonlinear Dynamical Systems

Yu Zhao a,*, Yongfang Liu a,

^aSchool of Automation, Northwestern Polytechnical University, Xi'an Shaanxi, 710129, China

Abstract

In this paper, a distributed average tracking problem is studied for Lipschitz-type nonlinear dynamical systems. The objective is to design distributed average tracking algorithms for locally interactive agents to track the average of multiple reference signals. Here, in both the agents' and the reference signals' dynamics, there is a nonlinear term satisfying the Lipschitz-type condition. Three types of distributed average tracking algorithms are designed. First, based on state-dependent-gain designing approaches, a robust distributed average tracking algorithm is developed to solve distributed average tracking problems without requiring the same initial condition. Second, by using a gain adaption scheme, an adaptive distributed average tracking algorithm is proposed in this paper to remove the requirement that the Lipschitz constant is known for agents. Third, to reduce chattering and make the algorithms easier to implement, a continuous distributed average tracking algorithm based on a time-varying boundary layer is further designed as a continuous approximation of the previous discontinuous distributed average tracking algorithms.

Key words: Distributed average tracking, nonlinear dynamics, adaptive algorithm, continuous algorithm.

1 Introduction

In the past two decades, there have been lots of interests in the distributed cooperative control [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], and [13], for multiagent systems due to its potential applications in formation flying, path planning and so forth. Distributed average tracking, as a generalization of consensus and cooperative tracking problems, has received increasing attentions and been applied in many different perspectives, such as distributed sensor networks [14], [15] and distributed coordination [16], [17]. For practical applications, distributed average tracking should be investigated for signals modeled by more and more complex dynamical systems.

The objective of distributed average tracking problems is to design a distributed algorithm for multi-agent systems to track the average of multiple reference signals. The motivation of this problem comes from the coordinated tracking for multiple camera systems. Spurred by the pioneering works in [18], and [19] on the distributed average tracking via linear algorithms, real applications of related results can be found in distributed sensor fusion

[14], [15], and formation control [16]. In [20], distributed average tracking problems were investigated by considering the robustness to initial errors in algorithms. The above-mentioned results are important for scientific researchers to build up a general framework to investigate this topic. However, a common assumption in the above works is that the multiple reference signals are constants [19] or achieving to values [18]. In practical applications, reference signals may be produced by more general dynamics. For this reason, a class of nonlinear algorithms were designed in [21] to track multiple reference signals with bounded deviations. Then, based on non-smooth control approaches, a couple of distributed algorithms were proposed in [22] and [23] for agents to track arbitrary time-varying reference signals with bounded deviations and bounded second deviations, respectively. Using discontinuous algorithms, further, [24] studied the distributed average tracking problems for multiple signals generated by linear dynamics.

Motivated by the above mentioned observations, this paper is devoted to solving the distributed average tracking problem for Lipschitz-type nonlinear dynamical systems. Three DAT algorithms are proposed in this paper. First of all, based on relative states of neighboring agents, a class of distributed discontinuous DAT algorithms are proposed with robustness to initial condi-

^{*} Corresponding author

Email address: yuzhao5977@gmail.com (Yu Zhao).

tions. Then, a novel class of distributed algorithms with adaptive coupling strengths are designed by utilizing an adaptive control technique. Different from [22], [23] and [24], the proposed algorithms are based on node adaptive lows. Further, a class of continuous algorithms are given to reduce chattering. Compared with the above existing results, the contributions of this paper are threefold. First, main results of this paper extend the dynamics of the reference signals and agents from linear systems [22] and [23] to nonlinear systems, which can describe more complex dynamics. Second, by using adaptive control approaches, the requirements of all global information are removed, which greatly reduce the computational complexity for large-scale networks. Third, compared with existing results in [24], new continuous algorithms are redesigned via the boundary layer concept to reduce the chattering phenomenon. Continuous algorithms in this paper is more appropriate for real engineering applications.

Notations: Let R^n and $R^{n\times n}$ be sets of real numbers and real matrices, respectively. I_n represents the identity matrix of dimension n. Denote by $\mathbf{1}$ a column vector with all entries equal to one. The matrix inequality $A > (\geq) B$ means that A-B is positive (semi-) definite. Denote by $A \otimes B$ the Kronecker product of matrices A and B. For a vector $x=(x_1,x_2,\cdots,x_n)^T \in R^n$, let $\|x\|$ denote the 2-norm of x, $h(x)=\frac{x}{\|x\|}$, $h_{\varepsilon}(x)=\frac{x}{\|x\|+\varepsilon e^{-ct}}$. For a set V, |V| represents the number of elements in V.

2 Preliminaries

2.1 Graph Theory

An undirected (simple) graph \mathcal{G} is specified by a vertex set $\mathcal V$ and an edge set $\mathcal E$ whose elements characterize the incidence relations between distinct pairs of \mathcal{V} . The notation $i \sim j$ is used to denote that node i is connected to node j, or equivalently, $(i,j) \in \mathcal{E}$. We make use of the $|\mathcal{V}| \times |\mathcal{E}|$ incidence matrix, $D(\mathcal{G})$, for a graph with an arbitrary orientation, i.e., a graph whose edges have a head (a terminal node) and a tail (an initial node). The columns of $D(\mathcal{G})$ are then indexed by the edge set, and the ith row entry takes the value 1 if it is the initial node of the corresponding edge, -1 if it is the terminal node, and zero otherwise. The diagonal matrix $\Delta(\mathcal{G})$ of the graph contains the degree of each vertex on its diagonal. The adjacency matrix, $A(\mathcal{G})$, is the $|\mathcal{V}| \times |\mathcal{V}|$ symmetric matrix with zero in the diagonal and one in the (i, j)th position if node i is adjacent to node j. The graph Laplacian [25] of \mathcal{G} , $L := \frac{1}{2} \mathring{D}(\mathcal{G}) D(\mathcal{G})^T = \mathring{\Delta}(\mathcal{G}) A(\mathcal{G})$, is a rank deficient positive semi-definite matrix.

An undirected path between node i_1 and node i_s on undirected graph means a sequence of ordered undirected edges with the form $(i_k; i_{k+1}), k = 1, \dots, s-1$. A graph \mathcal{G} is said to be connected if there exists a path between each pair of distinct nodes.

Assumption 1 Graph G is undirected and connected.

Lemma 1 [25] Under Assumption 1, zero is a simple eigenvalue of L with **1** as an eigenvector and all the other eigenvalues are positive. Moreover, the smallest nonzero eigenvalue λ_2 of L satisfies $\lambda_2 = \min_{x \neq 0, 1^T x = 0} \frac{x^T L x}{x^T x}$.

3 Main results

3.1 Robust distributed average tracking algorithms design

Consider a multi-agent system consisting of N physical agents described by the following nonlinear dynamics

$$\dot{x}_i(t) = Ax_i(t) + Bf(x_i, t) + Bu_i, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$ both are constant matrices with compatible dimensions, $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^p$ is the state and control input of the *i*th agent, respectively, and $f: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^p$ is a nonlinear function. Suppose that there is a time-varying reference signal, $r_i(t) \in \mathbb{R}^n, i = 1, 2, \dots, N$, which generated by the following Lipschitz-type nonlinear dynamical systems:

$$\dot{r}_i(t) = Ar_i(t) + Bf(r_i, t), \tag{2}$$

where $r_i(t) \in \mathbb{R}^n$ is the state of the *i*th reference signal.

It is assumed that agent i has access to $r_i(t)$, and agent i can obtain the relative information from its neighbors denoted by \mathcal{N}_i .

Assumption 2 (A, B) is stabilizable.

The main objective of this paper is to design a class of distributed controller $u_i(t)$ for physical agent i in (1) to track the average of multiple reference signals $r_i(t)$ generated by the general nonlinear dynamics (2), i.e.,

$$\lim_{t \to \infty} \left(x_i(t) - \frac{1}{N} \sum_{i=1}^{N} r_i(t) \right) = 0,$$

where each agent has only local interaction with its neighbors.

Assumption 3 For $\forall \theta_i(t) \in R^n$, i = 1, 2 and $\forall t > 0$, the nonlinear function $f: R^n \times R^+ \to R^p$ satisfies a Lipschitz-type condition: $||f(\theta_1, t) - f(\theta_2, t)|| \leq \gamma ||\theta_1 - \theta_2||$, where $\gamma \in R^+$ and f(0, t) = 0.

As it was mentioned, there are many applications that the physical agents should track a time varying trajectory, where each agent has an incomplete copy of this

trajectory. While, the physical agents and reference trajectory might be described by more complicated dynamics rather than the linear dynamics in real applications. Therefore, we consider a more general group of physical agents, where the nonlinear function $f(\cdot,t)$ in their dynamics satisfies the Lipschitz-type condition.

Therefore, a distributed average tracking controller algorithm is designed as

$$u_i(t) = K_1(p_i(t) - r_i(t)) + K_2\tilde{x}_i(t) + \mu\phi_i h[K_2\tilde{x}_i(t)]$$
$$+\alpha\vartheta_i Bh\left(\sum_{j\in\mathcal{N}_i} K_1(p_i(t) - p_j(t))\right), \tag{3}$$

with a distributed average tracking filter algorithm is proposed as follows:

$$p_{i}(t) = s_{i}(t) + r_{i}(t),$$

$$\dot{s}_{i}(t) = As_{i}(t) + BK_{1}(p_{i}(t) - r_{i}(t))$$

$$+\alpha \vartheta_{i}Bh\left(\sum_{j \in \mathcal{N}_{i}} K_{1}(p_{i}(t) - p_{j}(t))\right),$$

$$(4)$$

where $\tilde{x}_{i}(t) = x_{i}(t) - p_{i}(t), s_{i}(t), i = 1, 2, \dots, N$, are the states of the DAT algorithm, $\phi_i = ||x_i(t) - r_i(t)|| +$ ν , and $\vartheta_i = ||r_i(t)|| + \beta$ state-dependent time-varying parameters, μ , ν , α and β constant parameters, K_1 and K_2 control gain matrices, respectively, to be determined.

Then, using the controller (3) for (1), one gets the tracking error system

$$\dot{\tilde{x}}_i(t) = (A + BK_2)\tilde{x}_i(t) + B(f(x_i, t) - f(r_i, t)) + \mu \phi_i Bh[K_2\tilde{x}_i(t)].$$
 (5)

Following from (2) and (4), one gets

$$\dot{p}_i(t) = (A + BK_1)p_i(t) - BK_1r_i(t) + Bf(r_i, t) + \alpha \vartheta_i Bh\left(\sum_{j \in \mathcal{N}_i} K_1(p_i(t) - p_j(t))\right). \tag{6}$$

Let $\tilde{x}(t) = (\tilde{x}_1^T(t), \tilde{x}_2^T(t), \dots, \tilde{x}_N^T(t))^T$, $p(t) = (p_1^T(t), p_2^T(t), \dots, p_N^T(t))^T$, $p(t) = (r_1^T(t), r_2^T(t), \dots, r_N^T(t))^T$, and $p(t) = (r_1^T(t), r_2^T(t), \dots, r_N^T(t))^T$. In matrix form, one obtains the algorithm of the property of follows: trix form, one obtains the closed-loop system as follows:

$$\dot{\tilde{x}}(t) = (I \otimes (A + BK_2))\tilde{x}(t) + (I \otimes B)(F(x, t) - F(r, t)) + \mu(\Phi \otimes B)H[(I \otimes K_2)\tilde{x}(t)],$$
(7)

with

$$\dot{p}(t) = (I \otimes (A + BK_1))p(t) - (I \otimes BK_1)r(t) + (I \otimes B)F(r,t) + \alpha(\Theta \otimes B)H((L \otimes K_1)p(t)), \quad (8)$$

where

$$H((I \otimes K_2)\tilde{x}(t)) = \begin{pmatrix} h(K_2\tilde{x}_1(t)) \\ \vdots \\ h(K_2\tilde{x}_N(t)) \end{pmatrix},$$

and

$$H((L \otimes K_1)p(t)) = \begin{pmatrix} h\left(\sum_{j \in \mathcal{N}_1} K_2(p_1(t) - p_j(t))\right) \\ \vdots \\ h\left(\sum_{j \in \mathcal{N}_N} K_2(p_N(t) - p_j(t))\right) \end{pmatrix}.$$

Define $M = I_N - \frac{1}{N} \mathbf{1} \mathbf{1}^T$. Then M satisfies following properties: Firstly, it is easy to see that 0 is a simple eigenvalue of M with 1 as the corresponding right eigenvector and 1 is the other eigenvalue with multieigenvector and 1 is the other eigenvalue with indular plicity N-1, i.e., $M\mathbf{1} = \mathbf{1}^T M = 0$. Secondly, since $L^T = L$, one has $LM = L(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T) = L - \frac{1}{N}L\mathbf{1}\mathbf{1}^T = L = L - \frac{1}{N}\mathbf{1}\mathbf{1}^T L = (I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T)L = ML$. Finally, $M^2 = M(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T) = M - \frac{1}{N}M\mathbf{1}\mathbf{1}^T = M$.

Define $\xi(t) = (M \otimes I)p(t)$, where $\xi(t) = (\xi_1^T(t), \xi_2^T(t), \cdots, \xi_N^T(t))^T$. Then, it follows that $\xi(t) = 0$ if and only if $p_1(t) = p_2(t) = \cdots = p_N(t)$. Therefore, the consensus problem of (6) is solved if and only if $\xi(t)$ asymptotically converges to zero. Hereafter, we refer to $\xi(t)$ as the consensus error. By noting that LM = L = ML, it is not difficult to obtain from (8) that the consensus error $\xi(t)$ satisfies

$$\dot{\xi}(t) = (M \otimes (A + BK_1))\xi(t) - (M \otimes BK_1)r(t) + \alpha(M\Theta \otimes B)H(L \otimes K_1)\xi(t) + (M \otimes B)F(r, t).(9)$$

Algorithm 1: Under Assumptions 1 and 2, for multiple reference signals in (2), the distributed average tracking algorithms (4) and (3) can be constructed as follows

(1) Solve the following algebraic Ricatti equations (AREs):

$$P_{i}A + A^{T}P_{i} - P_{i}BB^{T}P_{i} + Q_{i} = 0, (10)$$

with $Q_i > 0$ to obtain matrices $P_i > 0$, where i =

1, 2. Then, choose $K_i = -B^T P_i$, i = 1, 2. (2) Choose the parameters $\alpha \geq \gamma + \|B^T P_1\|, \beta > 0$ $\mu \geq \gamma$ and $\nu > 0$.

Theorem 1 Under Assumptions 1-3, by using the distributed average tracking controller algorithm (3) with the distributed average tracking filter algorithm (4), the state $x_i(t)$ in (1) will track the average of multiple reference signals $r_i(t)$, $i=1,2,\cdots,N$, generated by the Lipschitz-type nonlinear dynamical systems (2) if the parameters α , β , μ , ν and the feedback gains K_i , i=1,2, are designed by Algorithm 1.

Proof: The proof contains three steps. First, it is proved that for the *i*th agent, $\lim_{t\to\infty} \left(p_i(t) - \frac{1}{t} \right)$

 $\frac{1}{N}\sum_{k=1}^{N}p_k(t)$ = 0. Consider the Lyapunov function candidate

$$V_1(t) = \xi^T(L \otimes P_1)\xi. \tag{11}$$

By the definition of $\xi(t)$, it is easy to see that $(\mathbf{1}^T \otimes I)\xi = 0$. For the connected graph \mathcal{G} , it then follows from Lemma 1 that

$$V_1(t) \ge \lambda_2 \lambda_{\min}(P_1) \|\xi\|^2, \tag{12}$$

where $\lambda_{\min}(P_1)$ is the smallest eigenvalue of the positive matrix P_1 . The time derivative of V_1 along (9) can be obtained as follows

$$\dot{V}_{1} = \dot{\xi}^{T} (L \otimes P_{1}) \xi + \xi^{T} (L \otimes P_{1}) \dot{\xi}
= \xi^{T} (M \otimes (A + BK_{1})^{T}) (L \otimes P_{1}) \xi
+ \xi^{T} (L \otimes P_{1}) (M \otimes (A + BK_{1})) \xi
- 2\xi^{T} (L \otimes P_{1}) (M \otimes BK_{1}) r(t)
+ 2\alpha \xi^{T} (L \otimes P_{1}) (M \Theta \otimes B) H(L \otimes K_{1}) \xi(t)
+ 2\xi^{T} (L \otimes P_{1}) (M \otimes B) F(r, t).$$
(13)

Substituting $K_1 = -B^T P_1$ into (13), it follows from the fact LM = ML = L and Assumption 3 that

$$\dot{V}_{1} = \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi
+ 2\xi^{T} (L \otimes P_{1} B B^{T} P_{1}) r(t)
- 2\alpha \xi^{T} (L \otimes P B) H [(L \otimes B^{T} P_{1}) \xi]
+ 2\xi^{T} (L \otimes P_{1} B) F(r, t)
= \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi
+ 2\sum_{i=1}^{N} \left(\sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1} (\xi_{i}(t) - \xi_{j}(t))] \right)^{T} B^{T} P_{1} r_{i}
- 2\alpha \sum_{i=1}^{N} \vartheta_{i} \left(\sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1} (\xi_{i}(t) - \xi_{j}(t))] \right)^{T}
h \left(\sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1} (\xi_{i}(t) - \xi_{j}(t))] \right)^{T} [f(r_{i}, t) - f(0, t)]
+ 2\sum_{i=1}^{N} \left(\sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1} (\xi_{i}(t) - \xi_{j}(t))] \right)^{T} [f(r_{i}, t) - f(0, t)]
\leq \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi$$

$$+2 \left\| \sum_{i=1}^{N} \left(\sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right)^{T} \right\| \|B^{T} P_{1} r_{i}\|$$

$$-2\alpha \sum_{i=1}^{N} \vartheta_{i} \left\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right\|$$

$$+2 \sum_{i=1}^{N} \left\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right\| \|f(r_{i}, t) - f(0, t)\|$$

$$\leq \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi$$

$$-2\alpha \sum_{i=1}^{N} \vartheta_{i} \left\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right\|$$

$$+2 \sum_{i=1}^{N} \left\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right\| (\gamma + \|B^{T} P\|_{1}) \|r_{i}\|$$

$$= \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi$$

$$-2 \sum_{i=1}^{N} [(\alpha - \gamma - \|B^{T} P_{1}\|) \|r_{i}\| + \alpha\beta]$$

$$\left\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right\|. \tag{14}$$

Since $\alpha > \gamma + ||B^T P_1||, \beta > 0$, one has

$$\dot{V}_1 \le \xi^T (L \otimes (P_1 A + A^T P_1 - 2P_1 B B^T P_1)) \xi
\le \lambda_2 \xi^T (I \otimes (P_1 A + A^T P_1 - 2P_1 B B^T P_1)) \xi.$$
(15)

It follows from (10) that $P_1A + A^TP_1 - P_1BB^TP_1 \le -Q_1$. Therefore, we have

$$\dot{V}_1 < -\eta_1 V_1,\tag{16}$$

where $\eta_1 = \frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)}$. Thus, one has

$$\lim_{t \to \infty} \xi_i(t) = \lim_{t \to \infty} \left(p_i(t) - \frac{1}{N} \sum_{k=1}^N p_k(t) \right) = 0.$$

Second, it is proved that $\lim_{t\to\infty} \left(p_i(t) - \frac{1}{N} \sum_{k=1}^N r_k(t) \right) = 0$. Let $r^*(t) = \frac{1}{N} \sum_{i=1}^N r_i(t)$. It follows from (2) that

$$\dot{r}^*(t) = Ar^*(t) + \frac{1}{N}B\sum_{i=1}^N f(r_i(t), t).$$
(17)

Let $p^*(t) = \frac{1}{N} \sum_{i=1}^{N} p_i(t)$. It follows from (2) that

$$\dot{p}^{*}(t) = (A + BK_{1})p^{*}(t) - BK_{1}r^{*}(t) + \frac{1}{N}B\sum_{i=1}^{N} f(r_{i}(t), t) + \alpha \sum_{i=1}^{N} \vartheta_{i}h\left(\sum_{j \in \mathcal{N}_{i}} K_{1}(p_{i}(t) - p_{j}(t))\right).$$
(18)

Denote $\zeta(t) = p^*(t) - r^*(t)$, one has

$$\dot{\zeta}(t) = \dot{p}^{*}(t) - \dot{r}^{*}(t)
= (A + BK_{1})p^{*}(t) - BK_{1}r^{*}(t) - Ar^{*}(t)
+ \alpha \sum_{i=1}^{N} \vartheta_{i}h \left(\sum_{j \in \mathcal{N}_{i}} K_{1}(p_{i}(t) - p_{j}(t)) \right)
= (A + BK_{1})\zeta(t) + \omega(t),$$
(19)

where
$$\omega(t) = \alpha \sum_{i=1}^{N} \vartheta_i h \left(\sum_{j \in \mathcal{N}_i} K_1(p_i(t) - p_j(t)) \right)$$

We then use input-to-state stability to analyze the system (19) by treating the term $\omega(t)$ as the input and $\zeta(t)$ as the states. Since (10) with $K_1 = -B^T P_1$, one has $A + BK_1$ is Hurwitz. Thus, the system (19) with zero input is exponentially stable and hence input-to-state stable. Since $\lim_{t\to\infty} \left(p_i(t) - \frac{1}{2} p_i(t) - \frac{1}{2} p_i$

$$\frac{1}{N}\sum_{k=1}^{N}p_k(t) = 0. \text{ One has } \lim_{t\to\infty}\omega(t) = 0. \text{ Thus,}$$
 it follows that $\lim_{t\to\infty}\zeta(t) = 0$, which implies that $\lim_{t\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}p_i(t)-\frac{1}{N}\sum_{i=1}^{N}r_i(t)\right)=0.$ Therefore, one obtains $\lim_{t\to\infty}\left(p_i(t)-\frac{1}{N}\sum_{k=1}^{N}r_k(t)\right)=\lim_{t\to\infty}\left(p_i(t)-\frac{1}{N}\sum_{i=1}^{N}p_i(t)\right)+\lim_{t\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}p_i(t)-\frac{1}{N}\sum_{i=1}^{N}r_i(t)\right)=0.$

Third, it is proofed that $\lim_{t\to\infty} \left(x_i(t) - \frac{1}{N} \sum_{i=1}^N r_i(t) \right) = 0$. Consider the candidate Lyapunov function

$$V_2 = \tilde{x}^T (I \otimes P_2) \tilde{x}, \tag{20}$$

with $P_2 > 0$. By taking the derivative of V_2 along (7), one gets

$$\dot{V}_{2} = \tilde{x}^{T} (I \otimes ((A + BK_{2})^{T} P_{2} + P_{2}(A + BK_{2}))) \tilde{x}
+ 2\tilde{x}^{T} (I \otimes P_{2}B) (F(x,t) - F(r,t))
+ 2\mu(\Phi \otimes P_{2}B) H[(I \otimes K_{2})\tilde{x}(t)].$$
(21)

Using $K_2 = -B^T P_2$, one has

$$\dot{V}_{2} = \tilde{x}^{T} (I \otimes (A^{T}P_{2} + P_{2}A - 2P_{2}BB^{T}P_{2}))\tilde{x}
+2\tilde{x}^{T} (I \otimes P_{2}B)(F(x,t) - F(r,t))
-2\mu\tilde{x}^{T} (\Phi \otimes P_{2}B)H[(I \otimes B^{T}P_{2})\tilde{x}(t)]
= \tilde{x}^{T} (I \otimes (A^{T}P_{2} + P_{2}A - 2P_{2}BB^{T}P_{2}))\tilde{x}
+2\sum_{i=1}^{N} (B^{T}P_{2}\tilde{x}_{i}(t))^{T} (f(x_{i},t) - f(r_{i},t))
-2\mu\sum_{i=1}^{N} \phi_{i}(B^{T}P_{2}\tilde{x}_{i}(t))^{T} h(B^{T}P_{2}\tilde{x}_{i})$$

$$\leq \tilde{x}^{T} (I \otimes (A^{T} P_{2} + P_{2} A - 2P_{2} B B^{T} P_{2})) \tilde{x}
+ 2 \sum_{i=1}^{N} \|B^{T} P_{2} \tilde{x}_{i}(t)\| \|(f(x_{i}, t) - f(r_{i}, t))\|
- 2\mu \sum_{i=1}^{N} \phi_{i} \|B^{T} P_{2} \tilde{x}_{i}(t)\|
\leq \tilde{x}^{T} (I \otimes (A^{T} P_{2} + P_{2} A - 2P_{2} B B^{T} P_{2})) \tilde{x}
+ 2 \sum_{i=1}^{N} \|B^{T} P_{2} \tilde{x}_{i}(t)\| \gamma \|x_{i} - r_{i}\|
- 2\mu \sum_{i=1}^{N} (\|x_{i} - r_{i}\| + \nu) \|B^{T} P_{2} \tilde{x}_{i}(t)\|
\leq \tilde{x}^{T} (I \otimes (A^{T} P_{2} + P_{2} A - 2P_{2} B B^{T} P_{2})) \tilde{x}
- 2 \sum_{i=1}^{N} ((\mu - \gamma) \|x_{i} - r_{i}\| + \mu \nu) \|B^{T} P_{2} \tilde{x}_{i}(t)\|. \quad (22)$$

Since $\mu \geq \gamma$ and $\nu > 0$, one has

$$\dot{V}_2 \le \tilde{x}^T (I \otimes (A^T P_2 + P_2 A - 2P_2 B B^T P_2)) \tilde{x}. \tag{23}$$

Using $A^{T}P_{2} + P_{2}A - 2P_{2}BB^{T}P_{2} \leq -Q_{2}$, one has

$$\dot{V}_2 \le -\eta_2 V_2. \tag{24}$$

where $\eta_2 = \frac{\lambda_{\min}(Q_2)}{\lambda_{\max}(P_2)}$. Thus, one has $\lim_{t\to\infty} \left(x_i(t) - \frac{1}{N}\sum_{i=1}^N r_i(t)\right) = \lim_{t\to\infty} (x_i(t) - p_i(t)) + \left(p_i(t) - \frac{1}{N}\sum_{i=1}^N r_i(t)\right) = 0$. Therefore, the distributed average tracking problem is solved. This completes the proof.

3.2 Adaptive distributed average tracking algorithms design

Note that, in above subsection, the proposed distributed average tracking algorithms (3) and (4) require that the parameters α and μ satisfy the conditions $\alpha \geq \gamma + \|B^T P_1\|$ and $\mu \geq \gamma$, which depend the Lipschitz constant γ . Since the γ is a global information, for a local agent, it becomes difficult to obtain γ . Therefore, to overcome the global information restriction, we design an adaptive distributed average tracking controller algorithm

$$u_{i}(t) = K_{1}(p_{i}(t) - r_{i}(t)) + K_{2}\tilde{x}_{i}(t) + \mu_{i}(t)\phi_{i}h[K_{2}\tilde{x}_{i}(t)] + \alpha_{i}(t)\vartheta_{i}Bh\left(\sum_{j \in \mathcal{N}_{i}} K_{1}(p_{i}(t) - p_{j}(t))\right), \tag{25}$$

and an adaptive distributed average tracking filter algorithm $\,$

$$p_i(t) = s_i(t) + r_i(t),$$

$$\dot{s}_i(t) = As_i(t) + BK_1(p_i(t) - r_i(t)) + \alpha_i(t)\vartheta_i Bh\left(\sum_{j \in \mathcal{N}_i} K_1(p_i(t) - p_j(t))\right), \tag{26}$$

with two time-varying parameters $\mu_i(t)$ and $\alpha_i(t)$ satisfying the following adaptive update strategies:

$$\dot{\mu}_i(t) = \kappa_i \phi_i \| K_2 \widetilde{x}_i(t) \|, \tag{27}$$

and

$$\dot{\alpha}_i(t) = \chi_i \vartheta_i \left\| \sum_{j \in \mathcal{N}_i} K_1(\xi_i(t) - \xi_j(t)) \right\|, \tag{28}$$

respectively, where κ_i, χ_i are adaptive parameters to be determined.

By substituting adaptive controller (25) into (1), one obtains

$$\dot{\tilde{x}}_i(t) = (A + BK_2)\tilde{x}_i(t) + B(f(x_i, t) - f(r_i, t)) + \mu_i(t)\phi_i Bh[K_2\tilde{x}_i(t)],$$
(29)

where $\mu_i(t)$ is given by (27). According to (2) and (26), one has

$$\dot{p}_i(t) = (A + BK_1)p_i(t) - BK_1r_i(t) + Bf(r_i, t) + \alpha_i(t)\vartheta_i Bh\left(\sum_{j \in \mathcal{N}_i} K_1(p_i(t) - p_j(t))\right), \tag{30}$$

where $\alpha_i(t)$ is given by (28).

Then, the closed-loop systems in matrix form are obtained,

$$\dot{\tilde{x}}(t) = (I \otimes (A + BK_2))\tilde{x}(t) + (I \otimes B)(F(x, t) - F(r, t)) + (\mu(t)\Phi \otimes B)H[(I \otimes K_2)\tilde{x}(t)], \tag{31}$$

with

$$\dot{\xi}(t) = (I \otimes (A + BK_1))\xi(t) - (M \otimes BK_1)r(t) + (M \otimes B)F(r,t) + (M\alpha(t)\Theta \otimes B)H((L \otimes K_1)\xi(B))$$

where $\mu(t) = \operatorname{diag}(\mu_1(t), \mu_2(t), \dots, \mu_N(t))$, and $\alpha(t) = \operatorname{diag}(\alpha_1(t), \alpha_2(t), \dots, \alpha_N(t))$, respectively.

Assumption 4 It is assumed that r_i is bounded.

Algorithm 2: Under Assumptions 1-4, for multiple reference signals in (2), the adaptive distributed average tracking algorithms (25)-(28) is designed by the following two steps:

(1) Solve the AREs (10) to obtain K_i , i = 1, 2.

(2) Choose the parameters $\kappa > 0, \chi > 0, \beta > 0$, and $\nu > 0$.

Theorem 2 Under Assumptions 1-4, the adaptive distributed average tracking algorithms (25)-(28) solve the distributed average tracking problem of the multi-agent system (1) with the reference dynamical system (2) if the parameters are given by Algorithm 2.

Proof: First, consider the following Lyapunov candidate.

$$V_3 = \xi^T (L \otimes P_1) \xi + \sum_{i=1}^N \frac{\widetilde{\alpha}_i(t)^2}{\chi_i}, \tag{33}$$

where $\tilde{\alpha}_i(t) = \alpha_i(t) - \alpha$. As proved in Theorem 1, the derivation of (33) along (32) and (28) is given by

$$\dot{V}_{3} \leq \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi
-2 \sum_{i=1}^{N} [(\alpha_{i}(t) - \gamma - \|B^{T} P_{1}\|) \|r_{i}\| + \alpha_{i}(t)\beta]
\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \|
+2 \sum_{i=1}^{N} \widetilde{\alpha}_{i}(t) \vartheta_{i} \| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \|
= \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi
-2 \sum_{i=1}^{N} [(\alpha - \gamma - \|B^{T} P_{1}\|) \|r_{i}\| + \alpha\beta]
\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \|.$$
(34)

Adaptively updating $\alpha > \gamma + ||B^T P_1|| > 0$, and choosing $\beta > 0$, one has

$$\dot{V}_3 < -\xi^T (L \otimes Q_1) \xi \triangleq -U(t) < 0, \tag{35}$$

which implies that $V_3(t)$ is non-increasing. Then, according to (33), it follows that $\xi, \alpha_i(t)$ are bounded. It is following from Assumption 4 that r is bounded. One has $\|F(r,t)\| = \|F(r,t) - F(0,t)\| \leq \gamma \|r\|$, which implies that F(r,t) is bounded. Therefore, $\dot{\xi}$ is bounded, which implies that $\lim_{t\to\infty} V_3(t)$ exists and is finite. Since (35), one has one has $\int_0^\infty U(t)dt$ exists and is finite. By noting that $\dot{U}(t)$ is also bounded. Therefore, U(t) is uniform continuity. By utilizing Barbalat's Lemma, it guarantees $\lim_{t\to\infty} U(t) = 0$. Thus, one has $\lim_{t\to\infty} \xi(t) = 0$. Noting that $\chi > 0, \beta > 0$, one has $\alpha_i(t)$ is monotonically increasing and bounded. Thus, $\alpha_i(t)$ converges to some finite constants. Thus, it follows that $\lim_{t\to\infty} \xi_i(t) = 0$.

$$\lim_{t \to \infty} \left(p_i(t) - \frac{1}{N} \sum_{k=1}^{N} p_k(t) \right) = 0.$$

Second, similar to the proof in Theorem 1, one has

$$\dot{\zeta}(t) = (A + BK_1)\zeta(t) + \varpi(t), \tag{36}$$

where
$$\varpi(t) = \sum_{i=1}^{N} \alpha_i(t) \vartheta_i h \left(\sum_{j \in \mathcal{N}_i} K_1(p_i(t)) - \frac{1}{N} \right)$$

 $p_j(t)$). Note that $\alpha_i(t)$ converges to some finite constants. By leveraging input-to-state stability to analyze the system (36), one has $\lim_{t\to\infty}\zeta(t)=0$. Then, one has $\lim_{t\to\infty}\left(p_i(t)-\frac{1}{N}\sum_{k=1}^N r_k(t)\right)=0$.

Third, consider the following Lyapunov candidate

$$V_4 = \tilde{x}^T (I \otimes P_2) \tilde{x} + \sum_{i=1}^N \frac{\tilde{\mu}_i(t)^2}{\kappa_i}, \tag{37}$$

where $\widetilde{\mu}_i(t) = \mu_i(t) - \mu$. As the proof given by Theorem 1, one has the derivation of (37) along (31) and (27),

$$\dot{V}_{4} \leq \tilde{x}^{T} (I \otimes (A^{T}P_{2} + P_{2}A - 2P_{2}BB^{T}P_{2}))\tilde{x}
-2 \sum_{i=1}^{N} ((\mu_{i}(t) - \gamma) \|x_{i} - r_{i}\| + \mu_{i}(t)\nu) \|B^{T}P_{2}\tilde{x}_{i}(t)\|
+2 \sum_{i=1}^{N} \tilde{\mu}_{i}(t)\phi_{i} \|B^{T}P_{2}\tilde{x}_{i}(t)\|
\leq \tilde{x}^{T} (I \otimes (A^{T}P_{2} + P_{2}A - 2P_{2}BB^{T}P_{2}))\tilde{x}
-2 \sum_{i=1}^{N} ((\mu - \gamma) \|x_{i} - r_{i}\| + \mu\nu) \|B^{T}P_{2}\tilde{x}_{i}(t)\|. \quad (38)$$

Adaptively updating $\mu \geq \gamma$ and choosing $\nu > 0$, one has

$$\dot{V}_4 \le -\tilde{x}^T (I \otimes Q_2) \tilde{x} \triangleq -W(t) \le 0, \tag{39}$$

which implies that $V_4(t)$ is non-increasing. Then, according to (37), it follows that $\widetilde{x}, \mu_i(t)$ are bounded. It is following from Assumption 4 and (30) that r and p are bounded. One has $\|F(x,t)-F(r,t)\| \leq \gamma \|x-r\| \leq \gamma(\|\widetilde{x}\|+\|p\|+\|r\|)$, which implies that F(x,t)-F(r,t) is bounded. Therefore, from (31), one has \widetilde{x} is bounded, which implies that $\lim_{t\to\infty} V_4(t)$ exists and is finite. Thus, $\int_0^\infty W(t)dt$ exists and is finite. By noting that $\dot{W}(t)$ is also bounded. Therefore, W(t) is uniform continuity. By utilizing Barbalat's Lemma, it guarantees $\lim_{t\to\infty} W(t)=0$. Thus, one has $\lim_{t\to\infty} \widetilde{x}(t)=0$. Noting that $\kappa_i>0, \nu>0$, one has $\mu_i(t)$ is monotonically increasing and bounded. Thus, $\mu_i(t)$ converges to some finite constants. It follows that $\lim_{t\to\infty} \widetilde{x}_i(t)=0$, which implies $\lim_{t\to\infty} \left(x_i(t)-\frac{1}{N}\sum_{k=1}^N r_k(t)\right)=0$. The proof is completed.

Remark 1 Differing from the robust distributed average tracking algorithms (3) and (4) in above subsection, the

adaptive algorithms (25)-(28) are local fashion without knowing the global information γ .

3.3 Continuous distributed average tracking algorithms design

In the above subsections, the distributed average tracking algorithms are designed based on the discontinuous function h(z), which may generate chattering phenomenon. In order to reduce the chattering in real applications and make the controller easier to implement, based on the boundary layer concept, we replace the discontinuous function h(z) by a continuous approximation $h_{\varepsilon}(z)$, and propose a continuous distributed average tracking controller algorithm:

$$u_i(t) = K_1(p_i(t) - r_i(t)) + K_2\tilde{x}_i(t) + \mu\phi_i h_{\varepsilon}[K_2\tilde{x}_i(t)] + \alpha\vartheta_i Bh_{\varepsilon} \left(\sum_{j \in \mathcal{N}_i} K_1(p_i(t) - p_j(t))\right), \tag{40}$$

and an continuous distributed average tracking filter algorithm $\,$

$$p_{i}(t) = s_{i}(t) + r_{i}(t),$$

$$\dot{s}_{i}(t) = As_{i}(t) + BK_{1}(p_{i}(t) - r_{i}(t))$$

$$+\alpha \vartheta_{i}Bh_{\varepsilon} \left(\sum_{j \in \mathcal{N}_{i}} K_{1}(p_{i}(t) - p_{j}(t))\right). \tag{41}$$

Submitting (40) into (1), one obtains the closed-loop systems in matrix form like:

$$\dot{\tilde{x}}(t) = (I \otimes (A + BK_2))\tilde{x}(t) + (I \otimes B)(F(x, t) - F(r, t)) + (\mu \Phi \otimes B)H_{\varepsilon}[(I \otimes K_2)\tilde{x}(t)]. \tag{42}$$

It follows from (2) and (41) that

$$\dot{\xi}(t) = (I \otimes (A + BK_1))\xi(t) - (M \otimes BK_1)r(t) \\
+ (M \otimes B)F(r,t) + (\alpha M\Theta \otimes B)H_{\varepsilon}((L \otimes K_1)\xi(t))$$

Theorem 3 Under Assumptions 1-4, the adaptive DAT algorithms (40) and (41) solve the DAT problem of the multi-agent system (1) with the reference dynamical system (2) if the parameters are given by Algorithm 1.

Proof: First, consider the Lyapunov candidate (33). As proved in Theorem 1, the derivation of (33) along (43) is given by

$$\dot{V}_{1} \leq \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi
+ 2 \sum_{i=1}^{N} [(\gamma + ||B^{T} P_{1}||) ||r_{i}||] \left\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right\|
- 2 \sum_{i=1}^{N} \alpha \vartheta_{i} \left(\sum_{j \in \mathcal{N}_{i}} K_{1}(\xi_{i}(t) - \xi_{j}(t)) \right)$$

$$h_{\varepsilon} \bigg(\sum_{j \in \mathcal{N}_i} K_1(\xi_i(t) - \xi_j(t)) \bigg).$$

Since $\alpha > \gamma + ||B^T P_1||$ and $\beta > 0$, one has

$$\dot{V}_{1}(t) \leq \xi^{T} [L \otimes (A^{T} P_{1} + P_{1} A) - 2(L \otimes P_{1} B B^{T} P_{1})] \xi
+ 2 \sum_{i=1}^{N} \alpha \vartheta_{i} \left[\left\| \sum_{j \in \mathcal{N}_{i}} [B^{T} P_{1}(\xi_{i}(t) - \xi_{j}(t))] \right\| \right.
\left. - \left(\sum_{j \in \mathcal{N}_{i}} K_{1}(\xi_{i}(t) - \xi_{j}(t)) \right) \right.
\left. h_{\varepsilon} \left(\sum_{j \in \mathcal{N}_{i}} K_{1}(\xi_{i}(t) - \xi_{j}(t)) \right) \right]$$

$$\leq -\eta V_{1} + 2 \sum_{i=1}^{N} \alpha \vartheta_{i} \varepsilon e^{-ct}. \tag{45}$$

In light of the well-known Comparison Lemma, one gets that

$$V_1(t) \le e^{-\eta(t)} V_1(0) + 2 \sum_{i=1}^{N} \alpha \overline{\vartheta}_i \int_{0}^{t} \varepsilon e^{-\eta(t-\tau) - c\tau} d\tau, \quad (46)$$

where $\overline{\vartheta}_i$ is the supper bound of ϑ_i . According to $\lim_{t\to\infty}\int_0^t \varepsilon e^{-\eta(t-\tau)-c\tau}d\tau=0$, one has $V_1(t)$ exponentially converges to the origin as $t\to\infty$. Therefore, $\lim_{t\to\infty}\|p_i-\sum_{k=1}^N p_k\|=0$. Second, similar to Theorem 1, one has

$$\dot{\zeta}(t) = (A + BK_1)\zeta(t) + \varpi(t, \varepsilon), \tag{47}$$

where $\varpi(t,\varepsilon) = \sum_{i=1}^{N} \alpha \vartheta_i h_{\varepsilon} \left(\sum_{j \in \mathcal{N}_i} K_1(p_i(t) - p_j(t)) \right)$. Since $\lim_{t \to \infty} \sum_{j \in \mathcal{N}_i} K_1(p_i(t) - p_j(t)) = 0$. One has $\lim_{t \to \infty} \varpi(t,\varepsilon) = 0$. It follows that $\lim_{t \to \infty} \zeta(t) = 0$. Thus, $\lim_{t \to \infty} \|p_i - \sum_{k=1}^{N} r_k\| = 0$. Third, consider derivative of V_2 along (42), one gets

$$\dot{V}_{2} \leq \tilde{x}^{T} (I \otimes (A^{T} P_{2} + P_{2} A - 2 P_{2} B B^{T} P_{2})) \tilde{x}
+ 2 \sum_{i=1}^{N} \|B^{T} P_{2} \tilde{x}_{i}(t)\| \gamma \|x_{i} - r_{i}\|
- 2 \mu \sum_{i=1}^{N} \phi_{i} (B^{T} P_{2} \tilde{x}_{i}(t))^{T} h_{\varepsilon} (B^{T} P_{2} \tilde{x}_{i})
\leq - \eta_{2} V_{2} + 2 \sum_{i=1}^{N} \mu \phi_{i} \varepsilon e^{-ct}.$$
(48)

Thus, $\lim_{t\to\infty} V_2(t) = 0$, which implies $\lim_{t\to\infty} \|x_i(t) - p_i(t)\| = 0$. Thus, $\lim_{t\to\infty} \|x_i - \sum_{k=1}^N r_k\| = 0$. This completes the proof.

(AA) 4 Conclusions

In this paper, we have studied the distributed average tracking problem of multiple time-varying signals generated by nonlinear dynamical systems. In the distributed fashion, a pair of discontinuous algorithms with static and adaptive coupling strengths have been developed. Then, in light of the boundary layer concept, a continuous algorithm is designed. Besides, sufficient conditions for the existence of distributed algorithms are given. The future topic will be focused on the distributed average tracking problem for the case with only the relative output information of neighboring agents.

References

- R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," in Proc. IEEE, vol. 95, no. 1, pp. 215–233, 2007.
- [2] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, 2007.
- [3] Y. Hong, G. Chen, and L. Bushnell, "Distributed observers design for leader-following control of multi-agent networks," *Automatica*, vol. 44, no. 3, pp. 846–850, 2008.
- [4] Y. Cao and W. Ren, "Distributed coordinated tracking with reduced interaction via a variable structure approach," *IEEE Trans. Autom. Control*, vol. 56, no. 1, pp. 33–48, 2012.
- [5] Z. Li, X. Liu, W. Ren, and L. Xie, "Distributed tracking control for linear multiagent systems with a leader of bounded unknown input," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 518–523, 2013.
- [6] S. E. Tuna, "Synchronizing linear systems via partial-state coupling," Automatica, vol. 44, no. 8, pp. 2179-2184, 2008.
- [7] H. Zhang, F. Lewis, and A. Das, "Optimal design for synchronization of cooperative systems: State feedback, observer, and output feedback," *IEEE Trans. Autom.* Control, vol. 56, no. 8, pp. 1948–1952, 2011.
- [8] Y. F. Liu, and Z. Y. Geng, "Finite-time formation control for linear multi-agent systems: A motion planning approach," Systems and Control Letters, vol. 85, no. 11, pp. 54–60, 2015.
- [9] Y. F. Liu, Y. Zhao, and Z. Y. Geng, "Finite-time formation tracking control for multiple vehicles: A motion planning approach," *International Journal of Robust and Nonlinear* Control, DOI: 10.1002/rnc.3496, 2015.
- [10] Y. Zhao, Z. S, Duan, G. H. Wen, and G. R. Chen, "Distributed finite-time tracking of multiple non-identical second-order nonlinear systems with settling time estimation," *Automatica*, vol. 64, no. 2, pp. 86–93, 2016.
- [11] Y. Zhao, Z. S, Duan and G. H. Wen, "Distributed finite-time tracking of multiple Euler-Lagrange dynamics without velocity measurements," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 11, pp. 1688–1703, 2015.
- [12] Y. Zhao, Z. S, Duan, G. H. Wen, and G. R. Chen, "Distributed finite-time tracking for a multi-agent system under a leader with bounded unknown acceleration," Systems & Control Letters, vol. 81, no. 6, pp. 8–13, 2015.
- [13] M. Ji, G. Ferrari-Trecate, M. Egerstedt, and A. Buffa, "Containment control in mobile networks," *IEEE Trans. Autom. Control*, vol. 53, no. 8, pp. 1972–1975, 2008.

- [14] D. Spanos and R. Murray, "Distributed sensor fusion using dynamic consensus," in Proc. 16th IFAC World Congress, 2005.
- [15] H. Bai, R. Freeman, and K. Lynch, "Distributed kalman filtering using the internal model average consensus estimator," in Proc. Amer. Control Conf., pp. 1500–1505.
- [16] P. Yang, R. Freeman, and K. Lynch, "Multi-agent coordination by decentralized estimation and control," *IEEE Trans. Autom. Control*, vol. 53, no. 11, pp. 2480–2496, 2008.
- [17] Y. Sun and M. Lemmon, "Swarming under perfect consensus using integral action," in Proc. Amer. Control Conf., pp. 4594–4599, 2007.
- [18] D. Spanos, R. Olfati-Saber, and R. Murray, "Dynamic consensus on mobile networks," in Proc. 16th IFAC World Congress, 2005.
- [19] R. Freeman, P. Yang, and K. Lynch, "Stability and convergence properties of dynamic average consensus estimators," in Proc. 45th IEEE Conf. Decision Control, pp. 338–343, 2006.
- [20] H. Bai and R. F. nd K. Lynch, "Robust dynamic average consensus of time-varying inputs," in Proc. 49th IEEE Conf. Decision Control, pp. 3104–3109, 2010.
- [21] S. Nosrati, M. Shafiee, and M. Menhaj, "Dynamic average consensus via nonlinear protocols," *Automatica*, vol. 48, no. 9, pp. 2262–2270, 2012.
- [22] F. Chen, Y. Cao, and W. Ren, "Distributed average tracking of multiple time-varying reference signals with bounded derivatives," *IEEE Trans. Autom. Control*, vol. 57, no. 12, pp. 3169–3174, 2012.
- [23] F. Chen, W. Ren, W. Lan, and G. Chen, "Distributed average tracking for reference signals with bounded accelerations," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 863–869, 2015
- [24] Y. Zhao, Z. Duan and Z. Li, "Distributed average tracking for multiple signals with linear dynamics: an edge-based framework," in Proc. 11th IEEE Inter. Conf. Control Auto. 2014.
- [25] C. Godsil and G. Royle, Algebraic Graph Theory. New York: Springer, 2001.