# Exponential Consensus of Coupled Inertial Agents With the Fully Heterogeneous and Fully Variable Setting of the Control Gains

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***Abstract*—In this article, we consider the exponential consensus of coupled inertial (double-integrator) agents, particularly with the general setting of the damping and stiffness control gains. Each agent has one damping gain and one stiffness gain. Here, the damping and stiffness control gains of all agents can be both fully heterogeneous (FH) and fully variable (FV), which are called the FH-FV gains for convenience of reference. Specifically, the FH gains are defined as follows: 1) the damping gains of all agents are heterogeneous; 2) the stiffness gains of all agents are heterogeneous; and 3) the set of the damping gains and the set of the stiffness gains are distinct without dependence. Otherwise, the control gains are said partially heterogeneous (PH). The FV or partially variable (PV) aspect of control gains is defined similarly. The FH-FV gains setting is novel and generalizes the specially PH settings of constant gains in previous papers. We also consider the general FH-PV gains and the PH-PV gains. Then, we provide the series of conditions that ensure exponential convergence to consensus, for the agents with the FH-FV gains, the general FH-PV gains, and the PH-PV gains, respectively. The series of the conditions for each type of control gains has particular meaning for characterizing *heterogeneity* of the gains, especially, when the digraph of the agents is far-from-balanced.**

***Index Terms*—Consensus, constrained Rayleigh quotient, cooperative control, heterogeneous control gains, inertial agents, rendezvous, variable control gains.**

## I. Introduction

COOPERATIVE control of multiagent systems has attracted increasing interest in recent years, for example, consensus [1]–[14], [25], [26], [33], [34], [38]; flocking [11], [12], [17]–[19]; formation [30]; consensus filters [15]; distributed tracking [37], [39]; and the transient control of consensus [40], to mention a few.

For inertial (i.e., double-integrator) agents, since the state of a single agent includes both velocity and position, generally, there exist the damping and stiffness gains for the agent's velocity and position states, respectively. The damping and stiffness gains are generally distinct [11], [12], with the equal setting of the damping/stiffness gains being the special case.

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One type of heterogeneous settings of control gains (i.e., different gains for different agents) is considered in [25] for a network of inertial agents. A set of heterogeneous gains for steering agents is considered in [27]. There are other types of heterogeneous settings in the literature, for example, the heterogeneous inertias of agents; the heterogeneous setting of the velocity coupling and position coupling of agents [11]–[13]; the dynamic consensus of heterogeneous networked systems [16]; and the output consensus of heterogeneous linear or nonlinear systems [20]–[22].

Considering heterogeneous settings of control gains is meaningful [25], for either theoretical merits or applications. The physical meaning for heterogeneous gains arises for many reasons. For example, even theoretically homogeneous gains will possibly become heterogeneous in applications, as such gains in applications may have different values that are controlled or set by physical actuators. For another example, if a few agents have physical restrictions on their maximum values of the control gains, then the gains setting would become heterogeneous. Moreover, a heterogeneous setting of control gains is flexible to design control systems with more decentralized features [25].

In this article, we consider the exponential consensus of coupled double-integrator agents, particularly with a heterogeneous setting of the damping and stiffness gains while we will go further in this article to consider the fully heterogeneous (FH) setting of the gains described as follows, which is novel.

As each inertial agent has distinctly a damping gain and a stiffness gain, then the FH setting of the control gains (for abbreviation, the FH gains setting) is defined as follows: 1) the damping gains of all agents are heterogeneous; 2) the stiffness gains of all agents are heterogeneous; and 3) the set of the damping gains and the set of the stiffness gains are distinct without dependence.

As a comparison, a partially heterogeneous (PH) setting of the damping and stiffness gains (for abbreviation, the PH gains setting) includes one of the following cases: 1) only the damping gains of all agents are heterogeneous; 2) only the stiffness gains of all agents are heterogeneous; and 3) although the set of the damping gains and the set of the stiffness gains are, respectively, heterogeneous, the two sets have certain correlations.

For the correlations of the PH gains, a general case is that the ratios of the damping gains to the respective stiffness gains of the agents are *generally heterogeneous proportional coefficients*; as a special case, the ratios of the damping gains to the respective stiffness gains are a common proportional coefficient; a further special case is that, for each agent, its damping gain is just identical to its stiffness gain, as in many previous papers, for example, [25].

Similar to heterogeneous classification of the control gains, the variable aspect of the control gains can also be clarified as the fully variable (FV) control gains and partially variable (PV) control gains. The FV setting of the damping and stiffness gains is defined as follows: 1) the damping gains of all agents are all variables; 2) the stiffness gains of all agents are all variables; and 3) the ratios of the stiffness gains to the respective damping gains are also variables (instead of the constant coefficients). Otherwise, the control gains are said to be PV.

In this article, we consider the agents with only position coupling while with the FH and FV control gains, for abbreviation, the FH-FV gains. The main contributions of this article are as follows.

First, we propose the general FH-FV setting of the control gains for double-integrator agents, in which the setting is novel and generalizes the existing gains setting in this field. We also consider the general setting of the FH and PV gains, for abbreviation, the FH-PV gains, in which the ratios are generally heterogeneous proportional coefficients, and which setting is still novel in this field. The PH and PV gains, that is, the PH-PV gains, are further special cases of the FH-FV gains.

Then, we provide the series of conditions that ensure exponential convergence to consensus, for the agents with the FH-FV gains, the general FH-PV gains, and the PH-PV gains, respectively. The conditions are provided in the vector or matrix form for conciseness. Moreover, the series of the conditions for each type of the gains setting has particular meaning for characterizing the heterogeneity of the gains, especially, when the digraph of the agents is far-from-balanced.

The remainder of this article is arranged as follows. Section II describes the problems. Section III is the preparations. Section IV provides consensus conditions for the FH-PV gains with generally heterogeneous proportional coefficients. Section V provides consensus conditions for the FH-PV gains with eigen-heterogeneous proportional coefficients. Section VI provides consensus conditions for the general FH-FV gains. Section VII provides consensus conditions for the PH-PV gains and comparisons with previous results. Section VIII is the conclusion. Table I lists the main abbreviations in this article.

*Notations:* Denote $\sigma_{\max}(\cdot)$ as the maximal singular value of a matrix. Denote $\lambda_{\max}(\cdot)$ as the maximal eigenvalue of a matrix if its eigenvalues are all real. $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors in the Euclidean space. $|\cdot|$ means an absolute value of a scalar parameter. $\|\cdot\|$ denotes the Euclidean norm of a vector, $\|x\| = \sqrt{x^T x}$. Vector $x > 0$ ($x \geq 0$) means that all entries of vector $x$ are positive (nonnegative). $C \succ 0$ ($C \succcurlyeq 0$) means that matrix $C$ is positive definite (semidefinite), $C \succ D$ ($C \succcurlyeq D$) means $C - D \succ 0$ ($C - D \succcurlyeq 0$). $I$ is the identity matrix, and $\mathbf{0}$ denotes a zero vector or matrix, with the dimensions determined in the subscript or in the context. Denote $\mathbf{1} := [1, 1, \ldots, 1]^T \in \mathbb{R}^n$. Notation diag$(v)$ for a vector parameter $v = [v_1, v_2, \ldots, v_n]^n$ means diag$(v_1, v_2, \ldots, v_n)$.

TABLE I
MAIN ABBREVIATIONS IN THIS ARTICLE

| Abbreviation | Meaning |
|---|---|
| the FH gains | the fully heterogeneous control gains (Definition 1) |
| the PH gains | the partially heterogeneous control gains |
| the FV gains | the fully-variable control gains (Definition 2) |
| the PV gains | the partially-variable control gains |
| the FH-FV gains | both the FH and FV control gains |
| the FH-PV gains | both the FH and PV control gains |
| the PH-PV gains | both the PH and PV control gains |
| the EB Laplacian | the eigen-balanced Laplacian matrix (Section III-B) |
| the OC-vector | the orthogonality-constraint vector (Section III-C) |

## II. PROBLEMS DESCRIPTION

This section describes the coupled agents with heterogeneous and variable control gains, the motivations, and the definitions of different types of control gains, and then the main concern of this article.

### A. Agents With Heterogeneous and Variable Control Gains

Consider $n$ agents in the $N$-dimensional Euclidean space, $N \geq 1$. Denote $x_i(t) \in \mathbb{R}^N$ (abbreviated as $x_i$) as the position of agent $i$. Consider the dynamics of the agents with implicit inertias while heterogeneous and variable control gains, that is

$$\ddot{x}_i = u_i,\ \ i = 1, 2, \ldots, n$$

where $u_i$ is the control input of agent $i$

$$u_i = -b_i(t)\dot{x}_i - \sum_{j\in\mathcal{N}_i} k_i(t)w_{ij}(x_i - x_j),\ \ i = 1, 2, \ldots, n \quad (1)$$

where $\mathcal{N}_i$ is the neighbor set of agent $i$, $w_{ij} > 0$ if $j \in \mathcal{N}_i$, otherwise, $w_{ij} = 0$; $w_{ii} = 0$ for all $i$; and

1) $b_i(t)$, $i = 1, 2, \ldots, n$, are the damping gains;
2) $k_i(t)$, $i = 1, 2, \ldots, n$, are the stiffness gains.

Here, the damping gains and the stiffness gains are allowed to be both heterogeneous and variables, the motivations are provided in Section II-B. The gains variables are general and can be even noncontinuous (i.e., no derivatives are required), for example, consider an FH-FV example for $n = 3$ as follows.

*Example 1:* An example of the FH-FV setting of the damping gains and the stiffness gains, for $n = 3$

$$\begin{cases} b_1(t) = 1 + e^{-t} \\ b_2(t) = 1 + e^{-2t} \\ b_3(t) = 1 + e^{\cos(t)} \\ k_1(t) = (1 + e^{-t})(2 + \sin(t)) \\ k_2(t) = (1 + 2e^{-2t})(1 + |\cos(t)|) \\ k_3(t) = \begin{cases} (1 + e^{\cos(t)}), & t \in [0, 1] \\ (1 + 2e^{-2t})(1 + |\cos(t)|), & t > 1. \end{cases} \end{cases}$$

The agents achieve asymptotic consensus (rendezvous), if

$$x_i \to x_j,\ \ \dot{x}_i \to \mathbf{0}\ \forall i, j = 1, 2, \ldots, n$$

where $\to$ means "asymptotically converges to."

### *B. Motivations on Heterogeneous and Variable Control Gains*

*1) Motivations on Variable Control Gains:* There are many motivations on variable control gains, not only theoretically but also practically. For example, consider the dynamics of the agents with explicitly variable inertias as

$$m_i(t)\ddot{x}_i = \tilde{u}_i,\ \ i = 1, 2, \ldots, n$$

where $m_i(t) > 0$ is the inertia of agent $i$ that can be a variable (e.g., due to the fuel consumption), $\tilde{u}_i$ is the control input

$$u_i = -\tilde{b}_i(t)\dot{x}_i - \sum_{j\in\mathcal{N}_i} \tilde{k}_i(t)w_{ij}(x_i - x_j),\ \ i = 1, 2, \ldots, n$$

where $\tilde{b}_i(t) > 0$ is the damping gain, and $\tilde{k}_i(t) > 0$ is the stiffness gain. This system is equivalent to the case that the agents have constant inertias while the control gains $\tilde{u}_i$ are the scaled variables; that is, the inertias can be viewed as being incorporated into the scaled gains as $b_i(t) := \tilde{b}_i(t)/m_i(t)$, $k_i(t) := \tilde{k}_i(t)/m_i(t)$. For conciseness, the dynamics of agents can be then described without explicit inertias while with variable control gains. Furthermore, the variance or inaccuracy (if with known bounds) of the agents' inertias can be also treated.

*2) Motivations on Heterogeneous Control Gains:* Considering heterogeneous settings of control gains is also meaningful for either theoretical merits or applications. For example, we have the following.

1) Even theoretically homogeneous gains will possibly become heterogeneous in applications, as such gains may have different values that are set by physical inaccurate actuators.
2) Also, if a few or all agents have the physical restrictions on their, respectively, maximum values of the control gains, then the gains setting would become heterogeneous.
3) A heterogeneous setting of control gains is flexible to design control systems with more decentralized features.

### *C. Definitions on Heterogeneous and Variable Control Gains*

Consider the set of the damping gains and the set of the stiffness gains. Then, we have the following definitions.

*Definition 1:* The FH setting of the damping and stiffness gains is defined as follows.

1) The damping gains $b_i(t)$, $i = 1, 2, \ldots, n$, are heterogeneous.
2) The stiffness gains $k_i(t)$, $i = 1, 2, \ldots, n$, are heterogeneous.
3) The ratios of the stiffness gains to the corresponding damping gains, that is, $k_i(t)/b_i(t)$, $i = 1, 2, \ldots, n$, are heterogeneous.

Otherwise, the gains are said PH.

Similar as heterogeneous classification of gains, the variable aspect of control gains can also be clarified as FV control gains and PV control gains.

*Definition 2:* The FV setting of the damping and stiffness gains is defined as follows.

1) The damping gains $b_i(t)$, $i = 1, 2, \ldots, n$, are variables.
2) The stiffness gains $k_i(t)$, $i = 1, 2, \ldots, n$, are variables.
3) The ratios of the stiffness gains to the corresponding damping gains, that is, $k_i(t)/b_i(t)$, $i = 1, 2, \ldots, n$, are also variables (instead of the constant coefficients).

Otherwise, the gains are said PV.

### *D. Agents With Different Types of the Control Gains*

Consider the matrix formation, the diagonal matrices of the damping and stiffness control gains are defined, respectively, as

$$B(t) := \mathrm{diag}(b_1(t), b_2(t), \ldots, b_n(t))$$
$$K(t) := \mathrm{diag}(k_1(t), k_2(t), \ldots, k_n(t)).$$

In this article, we consider the different types of the damping and stiffness control gains.

1) *The FH-FV Gains:* The damping and stiffness control gains in (1) allow such type of control gains.
2) *The FH-PV Gains, With Generally Heterogeneous Proportional Coefficients Between the Set of the Damping Gains and the Set of the Stiffness Gains:* In this case, the control gains are FH while not FV, as here
$$k_i(t) = c_i b_i(t),\ \ i = 1, 2, \ldots, n$$
where $c_i$, $i = 1, 2, \ldots, n$, are constant coefficients while allowed to be *generally heterogeneous*, the control input becomes
$$u_i = -b_i(t)\left(\dot{x}_i + c_i \sum_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j)\right),\ \ i = 1, \ldots, n. \tag{2}$$
For convenience, consider the matrix formation of the control gains, define $C := \mathrm{diag}(c_1, c_2, \ldots, c_n)$, then
$$B(t)K^{-1}(t) = C^{-1}. \tag{3}$$
In (3), the heterogeneity of the entries of matrix $C$ characterizes the relative heterogeneity between the set of the damping gains and the set of the stiffness gains.
3) *The FH-PV Gains, With the Eigen-Heterogeneous Proportional Coefficients Between the Set of the Damping Gains and the Set of the Stiffness Gains:* In this case, for the proportion, $C = nc_oE$, that is
$$B(t)K^{-1}(t) = \frac{1}{nc_o}E^{-1}$$
or equivalently
$$\frac{\xi_1 b_1(t)}{k_1(t)} = \frac{\xi_2 b_2(t)}{k_2(t)} = \cdots = \frac{\xi_n b_n(t)}{k_n(t)} = \frac{1}{nc_o} \tag{4}$$
where $c_o$ is a constant coefficient, matrix $E$ and $\xi_1, \xi_2, \ldots, \xi_n$ will be defined in Section III-B that are determined by the coupling of the agents (if $\xi_1, \xi_2, \ldots, \xi_n$ are not FH for a certain coupling of the agents, then the control gains are called PH-PV gains).
4) *The PH-PV Gains With a Common Proportional Coefficient:* In this case, the control input becomes
$$u_i = -b_i(t)\left(\dot{x}_i + c_o \sum_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j)\right) \tag{5}$$

where the ratios of the heterogeneous damping gains to the respective stiffness gains are a common and constant proportional coefficient $c_o$, which is a special case of (2). For the comparison purpose, denote

$$c_o := \frac{c_1 + c_2 + \cdots + c_n}{n} > 0 \tag{6}$$

as the average value of the heterogeneous coefficients $c_i$, $i = 1, 2, \ldots, n$, in (2). Then, the vector form of the control gains is

$$B(t)K^{-1}(t) = \frac{1}{c_o} I. \tag{7}$$

*Remark 1:* In literature, for each agent, its damping gain is usually set to be identical to its stiffness gain, that is, $c_i = 1$, $i = 1, 2, \ldots, n$, for example, as in [25] and many other papers, which is a special case of (5).

*Remark 2:* In this article, no derivatives of the variable gains are required, for example, refer to Example 1 in Section II-A, whereas in [25], for those variable gains, their *derivatives* are assumed to exist and also constrained to be some continuous functions of the agents' states; also, no analytical bounds of those variable gains are provided for consensus.

### E. Main Concerns on Heterogeneity of the Gains

Denote $x := [x_1^T, x_2^T, \ldots, x_n^T]^T \in \mathbb{R}^{Nn}$, the vector form of system (1) is

$$\begin{pmatrix} \ddot{x} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -B(t) & -K(t)L \\ I & \mathbf{0} \end{pmatrix} \otimes I_N \begin{pmatrix} \dot{x} \\ x \end{pmatrix} := A(t) \otimes I_N \begin{pmatrix} \dot{x} \\ x \end{pmatrix}.$$

If the control gains $B$ and $K$ are constant, then the eigenvalues judgment is trivial. However, if the control gains are generally variables, then the eigenvalues judgment (for the necessary and sufficient conditions) fails [24, Ch. 4.6, p. 157].

Then, the main concern in this article is that: what are analytical conditions on such heterogeneous and variable control gains for exponential consensus?

## III. Preparations

This section is the preparations and preliminaries for derivation of the main results in the following sections in this article.

### A. Definitions and Preparations

Consider two matrices $A_1$ and $A_2 \in \mathbb{R}^{m\times m}$. Without loss of generality, assume that $\sigma_{\max}(A_1) \neq 0$ and $\sigma_{\max}(A_2) \neq 0$ in this article.

*Definition 3:* Define the positive semidefinite matrix function $\varphi : (\mathbb{R}^{m\times m})^2 \times \mathbb{R}^+ \mapsto \mathbb{R}^{m\times m}$ as

$$\varphi(A_1, A_2, c) := \frac{1}{c}(cA_1 + A_2)^T(cA_1 + A_2) \succcurlyeq 0$$

where $c > 0$ is a constant parameter. Define the positive semidefinite matrix function $\varphi_o : (\mathbb{R}^{m\times m})^2 \mapsto \mathbb{R}^{m\times m}$ as

$$\varphi_o(A_1, A_2) := \varphi(A_1, A_2, \tilde{c}) \succcurlyeq 0$$

where the positive scalar $\tilde{c} > 0$ is the optimal value that minimizes $\lambda_{\max}(\varphi(A_1, A_2, c))$, that is

$$\lambda_{\max}(\varphi(A_1, A_2, \tilde{c})) := \min_{\forall c>0} \lambda_{\max}(\varphi(A_1, A_2, c)).$$

*Definition 4:* Define the positive semidefinite matrix function $\varphi_1 : (\mathbb{R}^{m\times m})^2 \mapsto \mathbb{R}^{m\times m}$

$$\varphi_1(A_1, A_2) := \frac{\left(\frac{\sigma_{\max}(A_2)}{\sigma_{\max}(A_1)}A_1 + A_2\right)^T \left(\frac{\sigma_{\max}(A_2)}{\sigma_{\max}(A_1)}A_1 + A_2\right)}{\frac{\sigma_{\max}(A_2)}{\sigma_{\max}(A_1)}}.$$

From definition, we have

$$\varphi_1(A_1, A_2) \succcurlyeq 0$$
$$\varphi_1(A_1, A_2) = \varphi_1(A_2, A_1).$$

Define the non-negative function $\nu : (\mathbb{R}^{m\times m})^2 \mapsto \mathbb{R}$ as

$$\nu(A_1, A_2) := \min_{\forall c>0} \frac{\sigma_{\max}^2(cA_1 + A_2)}{c} \geq 0.$$

Define the nonnegative function $\nu_1 : (\mathbb{R}^{m\times m})^2 \mapsto \mathbb{R}$ as

$$\nu_1(A_1, A_2) := \frac{\sigma_{\max}^2\left(\frac{\sigma_{\max}(A_2)}{\sigma_{\max}(A_1)}A_1 + A_2\right)}{\frac{\sigma_{\max}(A_2)}{\sigma_{\max}(A_1)}} \geq 0.$$

From the definition, $\nu(A_1, A_2)$ has an upper bound

$$\nu(A_1, A_2) \leq \nu_1(A_1, A_2).$$

*Proposition 1:* From the definitions

$$\lambda_{\max}(\varphi_1(A_1, A_2)) = \nu_1(A_1, A_2). \tag{8}$$

That is

$$\nu_1(A_1, A_2)I \succcurlyeq \varphi_1(A_1, A_2). \tag{9}$$

*Proposition 2:* The following inequality holds:

$$\nu_1(A_1, A_2) \leq 4\sigma_{\max}(A_1)\sigma_{\max}(A_2).$$

If one matrix has all non-negative eigenvalues, while another matrix has all nonpositive eigenvalues, then the following inequality holds with obvious conservativeness:

$$\nu_1(A_1, A_2) < 4\sigma_{\max}(A_1)\sigma_{\max}(A_2). \tag{10}$$

*Proof:* Refer to the Appendix. ■

*Remark 3:* In this article, we only use (10) to derive a conservative condition *for the comparison purpose* with recent results (refer to Appendix C).

### B. Eigenproperties of the EB Laplacian Matrix

Define $\mathcal{W} = [w_{ij}] \geq 0$ as the coupling matrix for the dynamics equations of the agents. Define the corresponding Laplacian matrix of the coupling matrix as

$$L = \mathrm{diag}(\mathcal{W}\mathbf{1}) - \mathcal{W}.$$

Define weighted digraph $\mathcal{G}$ of $L$ as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, matrix $\mathcal{A} = \mathcal{W}^T$ (not $\mathcal{W}$) is the weighted adjacency matrix of $\mathcal{G}$, that is,

$e_{ij} = (i, j) \in \mathcal{E}$ *iff* $w_{ji} > 0$, which represents the directed information flow from agent $i$ to agent $j$ in digraph $\mathcal{G}$ [11].

*Assumption 1:* Assume that digraph $\mathcal{G}$ is strongly connected, that is, $L$ is irreducible ($L$ is irreducible *iff* $\mathcal{G}$ is strongly connected [1]).

*Lemma 1:* With Assumption 1, Laplacian $L$ has only one zero eigenvalue. There exists a *positive* [1, Lemma 1] vector

$$\xi := [\xi_1, \xi_2, \ldots, \xi_n] \in \mathbb{R}^{1\times n},\ \ \xi > 0$$

as a *left eigenvector* of Laplacian $L$ corresponding to its only zero eigenvalue.

Without loss of generality, assume that

$$\langle \xi^T, \mathbf{1}\rangle = \xi_1 + \xi_2 + \cdots + \xi_n = 1$$

thus, with Lemma 1, $\xi$ is unique.

*Definition 5:* Define matrix

$$E := \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n) \succ 0.$$

Then, $EL$ is an eigen-balanced (EB) Laplacian matrix associated with $L$, since $EL\mathbf{1} = (EL)^T\mathbf{1}$ [1].

### C. Constrained Rayleigh Quotient

The following reviews the *constrained Rayleigh quotient* and its greatest lower bound (i.e., the infimum) [1] for self-containment in this article.

Denote the eigenvalues of the Laplacian $EL + L^TE$ as

$$0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots \le \mu_n$$

where $\mu_2 > 0$ (due to Assumption 1). Denote $\omega_1 \in \mathbb{R}^n$ as the right eigenvector (with $\|\omega_1\| = 1$) of $EL+L^TE$ corresponding to $\mu_1$, then

$$\omega_1 = \big(1/\sqrt{n}\big)\mathbf{1}.$$

*Lemma 2 (Constrained Rayleigh Quotient):* For any vector $\zeta \in \mathbb{R}^n$, satisfying $\langle \zeta, \mathbf{1}\rangle \neq 0$, variable $x$ is constrained to be *orthogonal* to vector $\zeta$ (which is called the orthogonality-constraint vector, or for abbreviation, the OC-vector), then [1]

$$\min_{\substack{\forall x\in\mathbb{R}^n,\, x\neq \mathbf{0},\, \langle x,\zeta\rangle=0\\ \forall \zeta\in\mathbb{R}^n,\, \langle \zeta,\mathbf{1}\rangle\neq 0}} \frac{x^T\big(EL+L^TE\big)x}{x^Tx} \ge \frac{\mu_2}{\langle \zeta, \omega_1\rangle^{-2}\|\zeta\|^2} := \mu_\zeta .$$

*Lemma 3:* For vector $\zeta \in \mathbb{R}^n$. $\forall \zeta > 0$, then [1, Lemma 2]

$$1 \le \langle \zeta, \omega_1\rangle^{-1}\|\zeta\| < \sqrt{n}$$

where the equal sign holds *iff* $\zeta$ parallels $\mathbf{1}$.

*Corollary 1:* For any *positive* OC-vector, that is, $\forall \zeta > 0$, one has

$$\min_{\forall x\in\mathbb{R}^n,\, x\neq\mathbf{0},\, \langle x,\zeta\rangle=0} \frac{x^T\big(EL+L^TE\big)x}{x^Tx} > \frac{\mu_2}{n}.$$

*Lemma 4:* In Lemma 2, let $\zeta = \xi^T$, then the greatest lower bound of the constrained Rayleigh quotient is

$$\mu_\xi := \frac{\mu_2}{n\|\xi\|^2}. \tag{11}$$

Then, from [1, Corollary 2]

$$\begin{cases} \frac{\mu_2}{n} < \mu_\xi \le \mu_2 \\ \mu_\xi \to \frac{\mu_2}{n} \Leftrightarrow \|\xi\| \to 1 \\ \mu_\xi = \mu_2 \Leftrightarrow \xi = (1/n)\mathbf{1}^T. \end{cases}$$

### D. Preliminaries on the Matrix Functions

*Definition 6:* Digraph $\mathcal{G}$ is called weakly symmetric, if $EL = L^TE$. Digraph $\mathcal{G}$ is balanced, if $\xi = (1/n)\mathbf{1}^T$. Otherwise, it is said unbalanced.

*Proposition 3:* If $\mathcal{G}$ is weakly symmetric, then the matrix function

$$\varphi_o\big(EL, -L^TE\big) = \mathbf{0}.$$

If $\mathcal{G}$ is balanced, then $\forall \ell = 0, 1, 2, \ldots$, we have the matrix function

$$\varphi_o\Big(EL, -L^TE^\ell\Big) = \varphi_o\bigg(\frac{1}{n}L, -\frac{1}{n^\ell}L^T\bigg) = \frac{1}{n^{1+\ell}}\varphi_o\big(L, -L^T\big).$$

*Proof:* Refer to the Appendix. ∎

*Proposition 4:* If $\mathcal{G}$ is weakly symmetric, then the matrix function

$$\varphi_1\big(EL, -L^TE\big) = \mathbf{0}.$$

If $\mathcal{G}$ is balanced, then $\forall \ell = 0, 1, 2, \ldots$, we have the matrix function

$$\varphi_1\Big(EL, -L^TE^\ell\Big) = \varphi_1\bigg(\frac{1}{n}L, -\frac{1}{n^\ell}L^T\bigg) = \frac{1}{n^{1+\ell}}\varphi_1\big(L, -L^T\big).$$

*Proof:* Refer to the Appendix. ∎

From definition, if $\mathcal{G}$ is weakly symmetric, then

$$\nu_1\big(EL, -L^TE\big) = 0.$$

## IV. Exponential Consensus Conditions for the FH-PV Gains With the Ratios as Generally Heterogeneous Proportional Coefficients

### A. Main Results

*Theorem 1:* Consider system (2), that is, the FH-PV gains with the ratios as generally heterogeneous proportional coefficients (3). Then, the agents achieve consensus exponentially, if

$$E^\ell C^{-1}B(t) \succ \frac{E^\ell L + L^TE^\ell}{2} + \frac{1}{2\mu_C}\varphi_o\Big(EL, -L^TE^\ell\Big) \tag{12}$$

for any one value of $\ell$ with $\ell \in \{0, 1, 2, \ldots\}$, in which

$$\mu_C := \frac{\mu_2}{\big\langle C^{-1}\xi^T, \omega_1\big\rangle^{-2}\big\|C^{-1}\xi^T\big\|^2}. \tag{13}$$

*Proof:* Refer to the Appendix. ∎

*Remark 4:* The gains $C^{-1}B(t)$ are called the heterogeneous combinatorial gains

$$C^{-1}B(t) = \Lambda(t) := B^2(t)K^{-1}(t)$$

the $i$th diagonal entry of $\Lambda(t)$ is $b_i^2(t)/k_i(t)$, which generalizes the homogeneous combinatorial gains.

*Remark 5:* Condition (12), as many other conditions in this article, is provided in the vector form for conciseness.

*Remark 6:* Note that $E^\ell C^{-1}B(t)$ is a *diagonal matrix*, and the right-hand side of condition (12) is a constant matrix. Thus, it is not difficult to derive the condition for each gain in the scalar form, which condition is omitted for the limited space.

*Remark 7:* Although the conditions in this article are the centralized conditions instead of decentralized conditions, as they use the eigenvalues and eigenvectors of the digraph. It is known, however, that the eigenvalues and eigenvectors can be derived by *distributed estimations*, for example, [42] and [41].

*Remark 8:* The flexible choice of condition (12) with any one $\ell \in \{0, 1, 2, \ldots\}$ has particular meaning for characterizing heterogeneity of the gains, especially, when digraph $\mathcal{G}$ is non-balanced or even far-from-balanced. Condition (12) provides the lower bounds of the control gains and is easy to design.

*Corollary 2:* Consider system (2), that is, the FH-PV gains with the ratios as generally heterogeneous proportional coefficients (3).

1) Digraph $\mathcal{G}$ is balanced. Then, condition (12), with any one value of $\ell \in \{0, 1, 2, \ldots\}$, reduces to be the same as
$$C^{-1}B(t) \succ \frac{L+L^T}{2} + \frac{1}{2n\mu_C}\varphi_o\big(L, -L^T\big).$$
2) Digraph $\mathcal{G}$ is weakly symmetric. Then, the agents achieve consensus exponentially, if
$$EC^{-1}B(t) \succ EL.$$
3) Digraph $\mathcal{G}$ is symmetric. Then, the agents achieve consensus exponentially, if
$$C^{-1}B(t) \succ L.$$

*Proof:* The first result holds from Proposition 3. The second result holds since $\varphi_o(EL, -L^T E^\ell) = \mathbf{0}$ with $\ell = 1$ in (12). ∎

The function $\varphi_o(EL, -L^T E^\ell)$ for the optimal value $\tilde{c}$ can be solved numerically, but it is not easy to derive the analytical value in general. Here, we can also use value $\varphi_1(EL, -L^T E^\ell)$ replacing $\varphi_o(EL, -L^T E^\ell)$, and provide conservative but more calculable analytical conditions.

*Theorem 2:* Consider system (2), that is, the FH-PV gains with the ratios as generally heterogeneous proportional coefficients (3). Then, the agents achieve consensus exponentially, if
$$E^\ell C^{-1}B(t) \succ \frac{E^\ell L + L^T E^\ell}{2} + \frac{1}{2\mu_C}\varphi_1\Big(EL, -L^T E^\ell\Big) \qquad (14)$$
for any one value of $\ell$ with $\ell \in \{0, 1, 2, \ldots\}$. Furthermore, if digraph $\mathcal{G}$ is balanced, condition (14), with any $\ell \in \{0, 1, 2, \ldots\}$, reduces to be the same as
$$C^{-1}B(t) \succ \frac{L+L^T}{2} + \frac{1}{2n\mu_C}\varphi_1\big(L, -L^T\big).$$

*Proof:* The result can be derived from Theorem 1 and the definition of $\varphi_1$. ∎

## B. Easily Calculable Corollaries and Interpretations

The following is a more calculable corollary as the supplementary result of Section IV-A, as well as an interpretation for Remark 8.

*Proposition 5:* Consider system (2), that is, the FH-PV gains with the ratios as generally heterogeneous proportional coefficients (3). Then, the agents achieve consensus exponentially, if
$$E^\ell C^{-1}B(t) \succ \frac{E^\ell L + L^T E^\ell}{2} + \frac{1}{2\mu_C}\nu_1\Big(EL, -L^T E^\ell\Big) \qquad (15)$$
for any one value of $\ell$ with $\ell \in \{0, 1, 2, \ldots\}$. For example, for $\ell = 0, 1, 2$, condition (15) becomes, respectively, as
$$C^{-1}B(t) \succ \frac{L+L^T}{2} + \frac{\nu_1\big(EL, -L^T\big)}{2\mu_C}I$$
$$EC^{-1}B(t) \succ \frac{EL+L^T E}{2} + \frac{\nu_1\big(EL, -L^T E\big)}{2\mu_C}I$$
$$E^2C^{-1}B(t) \succ \frac{E^2L+L^T E^2}{2} + \frac{\nu_1\big(EL, -L^T E^2\big)}{2\mu_C}I.$$
The more conservative but explicit solutions (in the scalar form), corresponding to the above results (in the vector form), respectively, are that
$$\begin{aligned} b_i(t) &> c_i\left(\frac{\lambda_{\max(L+L^T)}}{2} + \frac{\nu_1\big(EL, -L^T\big)}{2\mu_C}\right), \\ &\quad i = 1, 2, \ldots, n; \\ b_i(t) &> \frac{c_i}{\xi_i}\left(\frac{\mu_n}{2} + \frac{\nu_1\big(EL, -L^T E\big)}{2\mu_C}\right), \quad i = 1, 2, \ldots, n; \\ b_i(t) &> \frac{c_i}{\xi_i^2}\left(\frac{\lambda_{\max(E^2L+L^T E^2)}}{2} + \frac{\nu_1\big(EL, -L^T E^2\big)}{2\mu_C}\right), \\ &\quad i = 1, 2, \ldots, n. \end{aligned} \qquad (16)$$

For interpretation of Remark 8 with, for example, (16), we have the following.

If digraph $\mathcal{G}$ is unbalanced, then $\xi \neq (1/n)\mathbf{1}^T$, the heterogeneity of the gains mainly comes from two aspects.

1) The heterogeneity of $c_i$, $i = 1, 2, \ldots, n$.
2) The heterogeneity of $\xi_i$, $i = 1, 2, \ldots, n$, especially, digraph $\mathcal{G}$ is far-from-balanced. For example, if one $\xi_i$ among others is bigger, then the corresponding $b_i(t)$ [refer to (16)] can be relatively smaller; while if one $\xi_i$ is very small, then the corresponding $b_i(t)$ is expected to be much bigger.

# V. Exponential Consensus Conditions for the FH-PV Gains With the Ratios as the Eigen-Heterogeneous Proportional Coefficients

For a given coupling, that is, a given Laplacian $L$, the value of $\mu_\xi$ in (11) is determined, which has the maximum value only if digraph $\mathcal{G}$ of $L$ is balanced (i.e., the entries of $\xi$ are homogeneous), refer to Lemma 4. That is, one has no flexibility to adjust it.

As a comparison, for $\mu_C$ defined in (13), $C$ is a parametrized and thus adjustable matrix. We have $\mu_C = \mu_2$ if $C\mathbf{1}$ is parallel to $E\mathbf{1}$, and $\mu_C < \mu_2$, otherwise. That is
$$\max_{\forall C \succ 0} \mu_C = \mu_2$$
the maximum value of $\mu_C$ achieves, if $C^{-1}\xi^T \| \mathbf{1}$ (i.e., if the entries of $C^{-1}\xi^T$ are homogeneous). The physical meaning for the maximum value of $\mu_C$ is that, the particular parametrized matrix $C$ can be selected just to cancel the heterogeneity of $\zeta$ for $\mu_\zeta$, as $\zeta = C^{-1}\xi^T$ satisfies here (Lemma 2). This leads to the eigen-heterogeneous proportional coefficients for the PH gains as follows.

In the following, we provide the eigen-heterogeneous proportional coefficients between the heterogeneous damping gains and the corresponding heterogeneous stiffness gains, which gains with such correlation have the least magnitudes that are provable to ensure the exponential consensus of the agents. Such coefficients (i.e., which ensure $\mu_C = \mu_2$) just coincide with the positive left eigenvector $\xi$ of Laplacian $L$ corresponding to the zero eigenvalue.

*Corollary 3:* Consider system (2) and the FH-PV gains with the ratios as the eigen-heterogeneous proportional coefficients (4). Then, the agents achieve consensus exponentially, if

$$\frac{1}{nc_o}E^{\ell-1}B(t) \succ \frac{E^{\ell}L + L^T E^{\ell}}{2} + \frac{1}{2\mu_2}\varphi_o\Big(EL, -L^T E^{\ell}\Big) \quad (17)$$

for any one value of $\ell$ with $\ell \in \{0, 1, 2, \ldots\}$.

*Proof:* From Theorem 1, let $C = nc_o E$, then $\mu_C = \mu_2$. The result holds. ∎

This result has the particular meaning for a large $n$ of agents with nonweakly symmetric or nonsymmetric digraph $\mathcal{G}$, which digraph means that $\varphi_o(EL, -L^T E)$ is positive definite.

The corresponding corollaries from Corollary 2 and Theorem 2 can also be derived similarly and are omitted here for the limited space.

## VI. Exponential Consensus Conditions for the FH-FV Gains

The section provides two main theorems of exponential consensus for the FH-FV gains.

*Theorem 3:* Consider system (1) with the FH-FV gains. The agents achieve consensus exponentially, if there exist positive constant scalars $c, r > 0$ such that the control gains satisfy

$$B(t) - \frac{r}{2}\Big(EL + L^T E\Big) - \frac{1}{2r\mu_2}\varphi(rEL, A_2, c) \succ 0 \quad (18)$$

where $A_2 := B(t) - r^{-1}E^{-1}K(t) - rL^T E$.

*Proof:* Refer to the Appendix. ∎

*Corollary 4:* Consider system (1) with the FH-FV gains, and digraph $\mathcal{G}$ is weakly symmetric. Then, the agents achieve consensus exponentially:

1) if

$$B(t) - \frac{r}{2}(EL + L^T E) - \frac{1}{2r\mu_2}\Big(B(t) - r^{-1}E^{-1}K(t)\Big)^2 \succ 0$$

2) or more conservatively but more concisely, if

$$\begin{cases} B(t) \succ rEL \\ K(t) = rEB(t). \end{cases}$$

*Proof:* For the first item, let $c = 1$, condition (18) becomes this condition. The second item can be derived from the first item. ∎

*Theorem 4:* Consider system (1) with the FH-FV gains. The agents achieve consensus exponentially, if there exist positive constant scalars $c, r > 0$ such that the gains satisfy

$$EB(t) - \frac{r}{2}\Big(EL + L^T E\Big) - \frac{1}{2r\mu_\xi}\varphi(A_1, rEL, c) \succ 0$$

where $A_1 := EB(t) - r^{-1}EK(t) - rL^T E$.

*Proof:* Refer to the Appendix. ∎

## VII. Exponential Consensus Conditions for the PH-PV Gains With the Ratios as a Common Proportional Coefficient and Comparisons With Previous Results

This section provides the exponential consensus conditions for the PH-PV gains with the ratios as a common proportional coefficient, and then compares them with previous results.

### A. PH-PV Gains With a Common Proportional Coefficient

*Corollary 5:* Consider system (5), that is, the PH-PV gains with the ratios as a common proportional coefficient. The agents achieve consensus exponentially, if

$$\frac{1}{c_o}E^{\ell}B(t) \succ \frac{E^{\ell}L + L^T E^{\ell}}{2} + \frac{1}{2\mu_\xi}\varphi_o\Big(EL, -L^T E^{\ell}\Big) \quad (19)$$

for any one nonnegative integer $\ell \in \{0, 1, 2, \ldots\}$.

*Proof:* From Theorem 1, let $C = c_o I$, then $\mu_C = \mu_\xi$. The result holds. ∎

The corresponding corollary from Theorem 2 can also be derived and is omitted here for the limited space.

*Proposition 6:* Consider system (5), that is, the PH-PV gains with the ratios as a common proportional coefficient. The agents achieve consensus exponentially, if

$$\frac{1}{c_o}E^{\ell}B(t) \succ \frac{E^{\ell}L + L^T E^{\ell}}{2} + \frac{1}{2\mu_\xi}\varphi_o\Big(E - \xi^T\xi, -L^T L^T E^{\ell}\Big)$$

for any one value of $\ell$ with $\ell \in \{0, 1, 2, \ldots\}$. For example, the condition for $\ell = 0$ is

$$\frac{1}{c_o}B(t) \succ \frac{L + L^T}{2} + \frac{1}{2\mu_\xi}\varphi_o\big(E - \xi^T\xi, -L^T L^T\big). \quad (20)$$

*Proof:* Refer to the Appendix. ∎

### B. Eigen-Heterogeneous Coefficients versus a Common Coefficient

For the left side terms of (17) and (19), we have

$$\mathbf{1}^T\left(\frac{1}{nc_o}E^{\ell-1}B(t)\right)\mathbf{1} = \mathbf{1}^T\left(\frac{1}{c_o}E^{\ell}B(t)\right)\mathbf{1}$$

that is, the physical meaning is that the overall cost of gains $B(t)$ is the same for each case.

However, please note the difference between $\mu_\xi$ of condition (19) and $\mu_2$ of condition (17), which relation is shown in Lemma 4. That is, the common proportional coefficient (7) (and as a result, the gains setting in [25], refer to Remark 1) is not a good proportional correlation *for a general weighted digraph* $\mathcal{G}$, especially, when $\mathcal{G}$ is far-from-balanced, which will induce $\mu_\xi \to \mu_2/n$ (Lemma 4) and thus will make (19) more conservative, especially, for a large $n$.

### C. Comparisons With Previous Results

The heterogeneous combinatorial gains $\Lambda(t)$ generalize the homogeneous combinatorial gain [11], [12] (the combinatorial gain is defined as the ratio of the square of the damping gain to the stiffness gain).

This article provides the general FH-FV gains for consensus, the gains setting and its derivations are general than, for example, [11], [12], and [25].

The following compares our results with [25, Ths. 3.1, 3.2, and 3.3], which use the special PH gains.

1) *Comparison on the Settings of Heterogeneous Gains:* In this article, we consider the FH control gains, and show that the damping gains and the stiffness gains play the *distinct* roles. As a comparison, Mei *et al.* [25] considered only a special case of the PH control gains, in which for each agent, its damping gain is just identical to its stiffness gain, that is, $b_i \equiv k_i$.
2) *Comparison With [25, Theor. 3.1]:* First, our result (Section IV) is for the variable FH gains, while [25, Theor. 3.1] is for the constant PH gains. Also, our result has less conservative bounds of the gains, for the details, refer to Appendix D.
3) *Comparison With [25, Theor. 3.3]:*
   a) These results prove asymptotic (not exponential convergence) consensus with growing variable gains, which values (that are available only via *numerical simulations*, *not analytically*) can be less conservative than our result in Section VI. While our result provides *analytically* lower bounds of *variables gains* for the *exponential convergence* of consensus.
   b) Moreover, [25, Theor. 3.3] is for the specially growing variable gains, while our result is for general variable gains *without the requirements of the derivatives*.

## VIII. Conclusion

In this article, we investigate the double-integrator agents with position coupling and the FH-FV gains, and provide exponential consensus conditions for the FH-FV gains, the general FH-PV gains, and the PH-PV gains, respectively, which demonstrate the relations of the control gains with respect to the eigenproperties of the agents' digraph. The results are possibly insightful to derive still less conservative lower bounds for such control gains to ensure the exponential consensus of the agents.

There are many interesting consensus problems for coupled double-integrator agents with the general setting of the FH-FV gains, which will be considered in the future. For example, what are the results if some of the heterogeneous damping or stiffness gains are allowed to be zero or even negative (negative values represent for anti-consensus effects between agents, for example, as the interactions of the agents with signed digraphs in [28] and [29])? (The FH setting can allow some nonpositive gains; whereas the PH setting [25] is impossible to allow nonpositive gains.) What is the result of a general digraph without strong connectivity? What is the result of a designated convergence rate on the exponential consensus of the agents? The possible extension of the proposed method to the cases of some general heterogeneous affine nonlinear multiagent systems (e.g., as the systems in [31], [32], [35], and [36]) will also be considered in the future.

## Appendix A
## Preparations for Proofs

For nonzero vectors $a$ and $b \in \mathbb{R}^N$, the Cauchy–Schwarz inequality

$$|\langle a, b\rangle| \le \langle a, a\rangle^{1/2}\langle b, b\rangle^{1/2}$$

with the equality holds if $a$ and $b$ are linearly dependent [23].

The maximal singular value $\sigma_{\max}(A)$ of matrix $A$ is defined as the square root of the maximal eigenvalue of $A^TA$, that is, $\sigma_{\max}(A) := \sqrt{\lambda_{\max}(A^TA)}$.

From definition, if $A$ is symmetric, then $\sigma_{\max}(A) = |\lambda_{\max}(A)|$.

From definition, for a general matrix $A$

$$\begin{aligned}\sigma_{\max}\left(A^2\right) &= \sqrt{\lambda_{\max}\left(A^TA^TAA\right)} = \sqrt{\lambda_{\max}\left(A^TAA^TA\right)} \\ &= \lambda_{\max}\left(A^TA\right) = \sigma^2_{\max}(A). \\ \|Ax\| &= \sqrt{(Ax)^T(Ax)} = \sqrt{x^TA^TAx} \le \sigma_{\max}(A)\|x\|.\end{aligned}$$

*Lemma A1:* For two matrices $A_1$ and $A_2$ and a scalar $c$, then

$$\begin{aligned}\sigma_{\max}(A_1 + A_2) &\le \sigma_{\max}(A_1) + \sigma_{\max}(A_2) \\ \sigma_{\max}(A_1A_2) &\le \sigma_{\max}(A_1)\sigma_{\max}(A_2) \\ \sigma_{\max}(cA_1) &= c\sigma_{\max}(A_1).\end{aligned}$$

Define the weighted centroid of $x_i \in \mathbb{R}^N$ as

$$\bar{x}_c := \sum_{i=1}^{n} \xi_i x_i \in \mathbb{R}^N.$$

Define $\bar{x} := \mathbf{1} \otimes \bar{x}_c = (\mathbf{1}\xi \otimes I_N)x \in \mathbb{R}^{Nn}$, and

$$e := x - \bar{x} = (I_{Nn} - \mathbf{1}\xi \otimes I_N)x = (I - \mathbf{1}\xi) \otimes I_N x. \quad (21)$$

*Lemma A2:* Denote $e = [e_1^T, e_2^T, \ldots, e_n^T]^T$ and $e_i = x_i - \bar{x}_c$. Then

$$\sum_{i=1}^{n} \xi_i e_i = \mathbf{0} \in \mathbb{R}^N. \quad (22)$$

That is, $(\xi \otimes \mathbf{1}_N)e = (\xi \otimes \mathbf{1}_N)((I - \mathbf{1}\xi) \otimes I_N)x = \mathbf{0} \in \mathbb{R}^N$.

For the scalars $a_1, a_2 > 0$ and $x > 0$, then

$$\min_{\forall x>0} \frac{(a_1x + a_2)^2}{x} = 4a_1a_2, \text{ at } x = a_2/a_1. \quad (23)$$

For verification, one can verify the derivative about $x$ as

$$\frac{d\left((a_1x + a_2)^2/x\right)}{dx} = \frac{2a_1(a_1x + a_2)x - (a_1x + a_2)^2}{x^2}.$$

Let the derivative be zero, then $x = a_2/a_1$.

### A. Proofs of Propositions 2–4

*Proof of Proposition 3:* From definition, we have

$$\varphi_o\left(\frac{1}{n}L, -\frac{1}{n^\ell}L^T\right) = \frac{1}{\tilde{c}}\left(\tilde{c}\frac{1}{n}L - \frac{1}{n^\ell}L^T\right)^T\left(\tilde{c}\frac{1}{n}L - \frac{1}{n^\ell}L^T\right).$$

Then, this optimal value $\tilde{c}$ is also the optimal value for

$$\begin{aligned}\frac{1}{\tilde{c}n^{\ell}}&\left(\tilde{c}n^{\ell-1}L-L^{T}\right)^{T}\left(\tilde{c}n^{\ell-1}L-L^{T}\right)\\&=\frac{1}{n}\frac{1}{\tilde{c}n^{\ell-1}}\left(\tilde{c}n^{\ell-1}L-L^{T}\right)^{T}\left(\tilde{c}n^{\ell-1}L-L^{T}\right)\\&=\frac{1}{n}\varphi_{o}\left(L,-L^{T}\right).\end{aligned}$$

That is

$$\frac{1}{n^{1+\ell}}\varphi_{o}\left(L,-L^{T}\right)=\varphi_{o}\left(\frac{1}{n}L,-\frac{1}{n^{\ell}}L^{T}\right).$$

Then, the result holds. ■

*Proof of Proposition 4:* If $\mathcal{G}$ is balanced, $\xi=(1/n)\mathbf{1}^{T}$, then

$$\begin{aligned}&\varphi_{1}\left(EL,-L^{T}E^{\ell}\right)\\&=\varphi_{1}\left(\frac{1}{n}L,-\frac{1}{n^{\ell}}L^{T}\right)\\&=\frac{\left(\frac{\sigma_{\max}\left(\frac{1}{n^{\ell}}L^{T}\right)}{\sigma_{\max}\left(\frac{1}{n}L\right)}\frac{1}{n}L-\frac{1}{n^{\ell}}L^{T}\right)^{T}\left(\frac{\sigma_{\max}\left(\frac{1}{n^{\ell}}L^{T}\right)}{\sigma_{\max}\left(\frac{1}{n}L\right)}\frac{1}{n}L-\frac{1}{n^{\ell}}L^{T}\right)}{\frac{\sigma_{\max}\left(\frac{1}{n^{\ell}}L^{T}\right)}{\sigma_{\max}\left(\frac{1}{n}L\right)}}\\&=\frac{1}{n^{1+\ell}}\varphi_{1}\left(L,-L^{T}\right).\end{aligned}$$

Then the result holds. ■

*Proof of Proposition 2:* Note that

$$\begin{aligned}\min_{\forall c>0}&\frac{\sigma_{\max}^{2}(cA_{1}+A_{2})}{c}\\&\leq\min_{\forall c>0}\frac{(c\sigma_{\max}(A_{1})+\sigma_{\max}(A_{2}))^{2}}{c}\\&=\frac{\left(\frac{\sigma_{\max}(A_{2})}{\sigma_{\max}(A_{1})}\sigma_{\max}(A_{1})+\sigma_{\max}(A_{2})\right)^{2}}{\frac{\sigma_{\max}(A_{2})}{\sigma_{\max}(A_{1})}}\\&=4\sigma_{\max}(A_{1})\sigma_{\max}(A_{2})\end{aligned}$$

where the first inequality holds form Lemma A1, the first equal sign holds from (23) with $c=\sigma_{\max}(A_{2})/\sigma_{\max}(A_{1})$. For (10), note that

$$\sigma_{\max}\left(\frac{\sigma_{\max}(A_{2})}{\sigma_{\max}(A_{1})}A_{1}+A_{2}\right)\leq\frac{\sigma_{\max}(A_{2})}{\sigma_{\max}(A_{1})}\sigma_{\max}(A_{1})+\sigma_{\max}(A_{2}).$$

If one matrix has all nonnegative eigenvalues, another has all nonpositive eigenvalues, the inequality has much conservativeness

$$\sigma_{\max}\left(\frac{\sigma_{\max}(A_{2})}{\sigma_{\max}(A_{1})}A_{1}+A_{2}\right)<\frac{\sigma_{\max}(A_{2})}{\sigma_{\max}(A_{1})}\sigma_{\max}(A_{1})+\sigma_{\max}(A_{2}).$$

Then, (10) holds. ■

## Appendix B
## Proofs

In the following, without loss of generality, assume $N=1$. The results with $N\geq 2$ can be derived using the Kronecker product. For conciseness, $t$ in all variables is omitted.

### A. Proofs for Results in Sections IV–VI

*Proof of Theorem 1:* For $\ddot{x}=-B(\dot{x}+CLx)$, where $B$ is variable. Denote $q:=CLx$, $v:=\dot{x}+CLx=\dot{x}+q$. Then

$$\begin{aligned}\dot{q}&=CL\dot{x}=CL(v-q)\\\dot{v}&=\ddot{x}+\dot{q}=-Bv+CL(v-q).\end{aligned}$$

Then, the system can be expressed equivalently as

$$\begin{pmatrix}\dot{q}\\\dot{v}\end{pmatrix}=\begin{pmatrix}-CL&CL\\-CL&-B+CL\end{pmatrix}\begin{pmatrix}q\\v\end{pmatrix}:=\tilde{A}\begin{pmatrix}q\\v\end{pmatrix}.$$

Define the quadratic function

$$V:=\left(q^{T},v^{T}\right)P\left(q^{T},v^{T}\right)^{T}$$

where matrix $P$ is *constant and positive definite*, as

$$P:=\begin{pmatrix}cEC^{-1}&\mathbf{0}\\\mathbf{0}&E^{\ell}C^{-1}\end{pmatrix}$$

in which, $c>0$ is a constant parameter. Then

$$\begin{aligned}P\tilde{A}&=\begin{pmatrix}cEC^{-1}&\mathbf{0}\\\mathbf{0}&E^{\ell}C^{-1}\end{pmatrix}\begin{pmatrix}-CL&CL\\-CL&-B+CL\end{pmatrix}\\&=\begin{pmatrix}-cEL&cEL\\-E^{\ell}L&-E^{\ell}C^{-1}B+E^{\ell}L\end{pmatrix}.\end{aligned}$$

Note that

$$\left\langle\xi C^{-1},q\right\rangle=\left\langle\xi C^{-1},CLx\right\rangle=0$$

then, from Lemma 2

$$\begin{aligned}q^{T}(EL+L^{T}E)q&\geq\frac{\mu_{2}}{\left\langle\xi C^{-1},\omega_{1}\right\rangle^{-2}\left\|\xi C^{-1}\right\|^{2}}\|q\|^{2}\\&:=\mu_{C}\|q\|^{2}.\end{aligned}$$

Then

$$\dot{V}\leq-\left(q^{T},v^{T}\right)Q_{1}\left(q^{T},v^{T}\right)^{T}$$

where

$$Q_{1}=\begin{pmatrix}c\mu_{C}I&-cEL+L^{T}E^{\ell}\\-cL^{T}E+E^{\ell}L&2E^{\ell}C^{-1}B-E^{\ell}L-L^{T}E^{\ell}\end{pmatrix}.$$

Then, $Q_{1}$ is positive definite, if

$$\begin{aligned}&2E^{\ell}C^{-1}B-E^{\ell}L-L^{T}E^{\ell}\\&\quad-\frac{1}{c\mu_{C}}\left(cL^{T}E-E^{\ell}L\right)\left(cEL-L^{T}E^{\ell}\right)\\&=2E^{\ell}C^{-1}B-E^{\ell}L-L^{T}E^{\ell}-\frac{1}{\mu_{C}}\varphi\left(EL,-L^{T}E^{\ell},c\right)\succ 0.\end{aligned}$$

Then, the result holds. ■

*Remark 9:* In the following, if we consider another set of the vector variables: $\tilde{e}:=CLe$, $v:=\dot{x}+CLx=\dot{x}+\tilde{e}$. Then

$$\dot{\tilde{e}}=CL(I-\mathbf{1}\xi)(v-\tilde{e})=CLv-CL\tilde{e}$$

and $\dot{v}=(CL-B)v-CL\tilde{e}$, that is

$$\begin{pmatrix}\dot{v}\\\dot{\tilde{e}}\end{pmatrix}=\begin{pmatrix}CL-B&-CL\\CL&-CL\end{pmatrix}\begin{pmatrix}v\\\tilde{e}\end{pmatrix}.$$

Consider the quadratic function

$$V:=(v^{T},\tilde{e}^{T})\begin{pmatrix}E^{\ell}C^{-1}&\mathbf{0}\\\mathbf{0}&cEC^{-1}\end{pmatrix}\begin{pmatrix}v\\\tilde{e}\end{pmatrix}$$

in which, $E$ and $C$ are constants, $c > 0$ is a constant parameter

$$\begin{pmatrix} E^\ell C^{-1} & \mathbf{0} \\ \mathbf{0} & cEC^{-1} \end{pmatrix}\begin{pmatrix} CL - B & -CL \\ CL & -CL \end{pmatrix} = \begin{pmatrix} E^\ell L - E^\ell C^{-1}B & -E^\ell L \\ cEL & -cEL \end{pmatrix}.$$

Then, we will obtain Theorem 1, and the details are omitted here.

*Remark 10:* Define $v := \dot{x} + CLx$, define $e$ as (21). Then

$$\begin{aligned} \dot{e} &= (I - \mathbf{1}\xi)\dot{x} = (I - \mathbf{1}\xi)(v - CLe) \\ &= (I - \mathbf{1}\xi)v - (I - \mathbf{1}\xi)CLe \\ \dot{v} &= \ddot{x} + CL\dot{x} = -Bv + CL(v - CLe) \\ &= (CL - B)v - CLCLe. \end{aligned}$$

Define the quadratic function $V = (e^T, v^T)P(e^T, v^T)^T$, where matrix $P$ is *constant and positive definite*, as

$$P := \begin{pmatrix} cE & \mathbf{0} \\ \mathbf{0} & E^\ell C^{-1} \end{pmatrix}$$

then consider the derivative of $V$. Note that

$$\begin{aligned} &\begin{pmatrix} cE & \mathbf{0} \\ \mathbf{0} & E^\ell C^{-1} \end{pmatrix}\begin{pmatrix} -(I - \mathbf{1}\xi)CL & I - \mathbf{1}\xi \\ -CLCL & CL - B \end{pmatrix} \\ &= -\begin{pmatrix} c(E - \xi^T\xi)CL & -c\big(E - \xi^T\xi\big) \\ E^\ell LCL & E^\ell C^{-1}B - E^\ell L \end{pmatrix} \end{aligned} \tag{24}$$

then it is difficult to derive a consensus result, since $(E - \xi^T\xi)$ has zero eigenvalue.

*Proof of Proposition 6:* From (24), if (7) holds, that is, if $C = c_o I$, then

$$\dot{V} = -\big(e^T, v^T\big)Q\big(e^T, v^T\big)^T$$

where

$$Q = \begin{pmatrix} cc_o\big(EL + L^TE\big) & c_oL^TL^TE^\ell - c\big(E - \xi^T\xi\big) \\ c_oE^\ell LL - c\big(E - \xi^T\xi\big) & 2E^\ell B/c_o - E^\ell L - L^TE^\ell \end{pmatrix}.$$

Note that (22) always holds for $e$, from Lemma 2

$$e^T(EL + L^TE)e \ge \mu_\xi \|e\|^2.$$

Then, $\dot{V} \le -(e^T, v^T)Q_1(e^T, v^T)^T$, where $Q_1$ is

$$Q_1 = \begin{pmatrix} cc_o\mu_\xi I & c_oL^TL^TE^\ell - c\big(E - \xi^T\xi\big) \\ c_oE^\ell LL - c\big(E - \xi^T\xi\big) & 2E^\ell B/c_o - E^\ell L - L^TE^\ell \end{pmatrix}.$$

Then $Q_1 \succ 0$ if

$$\begin{aligned} &\frac{2}{c_o}E^\ell B - E^\ell L - L^TE^\ell - \frac{1}{cc_o\mu_\xi}\Big(c(E - \xi^T\xi) - c_oE^\ell LL\Big) \\ &\quad\times\Big(c\big(E - \xi^T\xi\big) - c_oL^TL^TE^\ell\Big) \\ &= \frac{2}{c_o}E^\ell B - E^\ell L - L^TE^\ell - \frac{1}{\mu_\xi}\varphi\Big(E - \xi^T\xi, -L^TL^TE^\ell, \frac{c}{c_o}\Big) \\ &\succ 0. \end{aligned}$$

Then, the result holds. ■

### B. Proofs for Results in Section VII

*Proof of Theorem 3:* For $\ddot{x} = -(B\dot{x} + KLx)$, define $q := rELx$, where $r > 0$ is a constant coefficient, $v := \dot{x} + q$. Then

$$\ddot{x} = -B(v - q) - r^{-1}E^{-1}Kq = -Bv + (B - r^{-1}E^{-1}K)q.$$

Then, $\dot{q} = rEL(v - q)$, and

$$\begin{aligned} \dot{v} &= -Bv + \Big(B - r^{-1}E^{-1}K\Big)q + rEL(v - q) \\ &= (rEL - B)v + \Big(B - r^{-1}E^{-1}K - rEL\Big)q. \end{aligned}$$

That is

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -rEL & rEL \\ B - r^{-1}E^{-1}K - rEL & rEL - B \end{pmatrix}\begin{pmatrix} q \\ v \end{pmatrix}.$$

Define the quadratic function

$$V := \big(q^T, v^T\big)P\big(q^T, v^T\big)^T$$

where matrix $P$ is *constant and positive definite*, as

$$P := \begin{pmatrix} cI & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix},$$

in which, $c > 0$ is a constant parameter. Then

$$\begin{aligned} &\begin{pmatrix} cI & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}\begin{pmatrix} -rEL & rEL \\ B - r^{-1}E^{-1}K - rEL & rEL - B \end{pmatrix} \\ &= \begin{pmatrix} -crEL & crEL \\ B - r^{-1}E^{-1}K - rEL & rEL - B \end{pmatrix}. \end{aligned}$$

Then, $\dot{V} = -(q^T, v^T)Q(q^T, v^T)^T$, where

$$Q = \begin{pmatrix} cr\big(EL + L^TE\big) & -A_2 - crEL \\ -A_2^T - crL^TE & 2B - r\big(EL + L^TE\big) \end{pmatrix}$$

where $A_2 = B - r^{-1}E^{-1}K - rL^TE$.

Note that $\langle \mathbf{1}, ELx \rangle = 0$, then, from Lemma 2

$$q^T(EL + L^TE)q > \mu_2\|q\|^2.$$

Then

$$Q_1 = \begin{pmatrix} cr\mu_2 I & -crEL - A_2 \\ -crL^TE - A_2^T & 2B - r\big(EL + L^TE\big) \end{pmatrix}.$$

If $Q_1 \succ 0$, that is, if

$$\begin{aligned} &2B - r\big(EL + L^TE\big) - \frac{1}{cr\mu_2}\big(crL^TE + A_2^T\big)(crEL + A_2) \\ &= 2B - r\big(EL + L^TE\big) - \frac{1}{r\mu_2}\varphi(rEL, A_2, c) \succ 0 \end{aligned}$$

then, $Q \succ 0$, the result holds. ■

*Proof of Theorem 4:* Define $q := rLx$, $v := \dot{x} + q$. Then

$$\ddot{x} = -(B\dot{x} + KLx) = -B(v - q) - r^{-1}Kq.$$

Then, $\dot{q} = rL(v - q)$

$$\dot{v} = (rL - B)v + \Big(B - r^{-1}K - rL\Big)q.$$

Then

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -rL & rL \\ B - r^{-1}K - rL & rL - B \end{pmatrix}\begin{pmatrix} q \\ v \end{pmatrix}.$$

Define the quadratic function

$$V := \big(q^T, v^T\big)P\big(q^T, v^T\big)^T.$$

where matrix $P$ is *constant and positive definite*, as

$$P := \begin{pmatrix} E^\ell & \mathbf{0} \\ \mathbf{0} & cE \end{pmatrix}$$

in which, $c > 0$ is a constant parameter. Then

$$\begin{pmatrix} E^{\ell} & \mathbf{0} \\ \mathbf{0} & cE \end{pmatrix}\begin{pmatrix} -rL & rL \\ B - r^{-1}K - rL & rL - B \end{pmatrix} = \begin{pmatrix} -rE^{\ell}L & rE^{\ell}L \\ cE\big(B - r^{-1}K - rL\big) & crEL - cEB \end{pmatrix}.$$

Then, $\dot{V} = -(q^T, v^T)Q(q^T, v^T)^T$, where

$$Q = \begin{pmatrix} r\big(E^{\ell}L + L^T E^{\ell}\big) & -cA_1 - rE^{\ell}L \\ -cA_1^T - rL^T E^{\ell} & 2cEB - cr\big(EL + L^T E\big) \end{pmatrix}$$

where $A_1 = (EB - r^{-1}EK - rL^T E)$.

Note that $\langle \xi^T, Lx\rangle = 0$, then, from Lemma 2

$$q^T\big(EL + L^T E\big)q > \mu_\xi \|q\|^2.$$

Then, for $\ell = 1$

$$Q_1 = \begin{pmatrix} r\mu_\xi & -cA_1 - rEL \\ -cA_1^T - rL^T E & 2cEB - cr\big(EL + L^T E\big) \end{pmatrix}.$$

If $Q_1 \succ 0$, that is, if

$$2EB - r\big(EL + L^T E\big) - \frac{1}{cr\mu_\xi}(cA_1 + rEL)^T(cA_1 + rEL) \succ 0$$

then, $Q \succ 0$, the result holds. ■

*Remark 11:* The following variables are infeasible for FH gains. Define $q := Lx$, $v := \dot{x}$. Then, $\dot{v} = -Bv - Kq$ and $\dot{q} = Lv$. Then

$$\begin{pmatrix} \dot{v} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -B & -K \\ L & \mathbf{0} \end{pmatrix}\begin{pmatrix} v \\ q \end{pmatrix}.$$

Define the quadratic function

$$V := \big(v^T, q^T\big)P\big(v^T, q^T\big)^T.$$

where matrix $P$ is *constant and positive definite*, as

$$P := \begin{pmatrix} I & I \\ I & DI \end{pmatrix},$$

in which, $D \succ I$ is a constant symmetric matrix. Then, consider the derivative of $V$, note that

$$\begin{pmatrix} I & I \\ I & DI \end{pmatrix}\begin{pmatrix} -B & -K \\ L & \mathbf{0} \end{pmatrix} = \begin{pmatrix} -B + L & -K \\ -B + DL & -K \end{pmatrix}.$$

Then, $\dot{V} = -(v^T, q^T)Q(v^T, q^T)^T$, where

$$Q = \begin{pmatrix} 2B - L - L^T & B + K - L^T D \\ B + K - DL & 2K \end{pmatrix}.$$

Then, $Q \succ 0$ means

$$2B - L - L^T - \frac{1}{2}\big(B + K - L^T D\big)K^{-1}(B + K - DL) \succ 0$$

which, however, cannot be positive definite; since

$$\begin{aligned} &\big(B + K - L^T D\big)K^{-1}(B + K - DL) \\ &= \Big((B + K)K^{-1} - L^T DK^{-1}\Big)(B + K - DL) \\ &= K^{-1}(B + K)^2 - (B + K)K^{-1}DL - L^T DK^{-1}(B + K) \\ &\quad + L^T DK^{-1}DL \end{aligned}$$

and $2B - (1/2)K^{-1}(B + K)^2$ is negative semidefinite.

## APPENDIX C
## COMPARISON WITH PREVIOUS RESULTS

For comparison, we first derive a conservative condition than (20), then show that this conservative condition is still much less conservative than [25, Theor. 3.1]. Using (9) and (10), the condition conservative than (20) is

$$B(t) \succ \frac{L + L^T}{2} + \frac{2\sigma_{\max}\big(E - \xi^T\xi\big)\sigma_{\max}\big(L^2\big)}{\mu_\xi}I$$

(here, $c_o = 1$ for comparison), a still more conservative condition is

$$b_i(t) > \left(\sigma_{\max}(L) + \frac{2\sigma_{\max}\big(E - \xi^T\xi\big)\sigma_{\max}\big(L^2\big)}{\mu_\xi}\right). \quad (25)$$

Note that $\sigma_{\max}(E - \xi^T\xi) < 1$.

Here, (25) is still less conservative than [25, Theor. 3.1] in two aspects.

1) For the lower bounds, our results use $\mu_\xi$ (Lemmas 2 and 4) instead of using the conservative term $\mu_2/n$.
2) For the order of $\sigma_{\max}(L)$: condition (25) has order $\sigma^2_{\max}(L)$, while [25, Theor. 3.1] has the highest order $\sigma^4_{\max}(L)$.

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