Improved Stability Criteria for Discrete-Time Delayed Neural Networks via Novel Lyapunov–Krasovskii Functionals

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***Abstract*—This article investigates the stability problem for discrete-time neural networks with a time-varying delay by focusing on developing new Lyapunov–Krasovskii (L–K) functionals. A novel L–K functional is deliberately tailored from two aspects: 1) the quadratic term and 2) the single-summation term. When the variation of the discrete-time delay is further considered, the constant matrix involved in the quadratic term is extended to be a delay-dependent one. All these innovations make a contribution to a quadratic function with respect to the delay from the forward differences of L–K functionals. Consequently, tractable stability criteria are derived that are shown to be more relaxed than existing results via numerical examples.**

***Index Terms*—Discrete-time neural network, Lyapunov–Krasovskii (L−K) functional, negative-definiteness lemma, stability analysis, time delay.**

I. Introduction

By MIMICKING the way interconnected neurons work, neural networks can manipulate information more efficiently than many other systems. Therefore, during the past several decades, neural networks have been widely used in various fields, such as speech recognition, image processing, and fault diagnosis [1], [2]. Compared to continuous-time counterparts, discrete-time neural networks seem to have greater practical application potentials because of extensive computer-based analysis and computation [3]. Besides, time delays are often present in various artificial systems, which may decrease the stability and reliability [4], [5]. So much effort has been spent on the stability study of discrete-time neural networks with time-varying delay [6]–[13].

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There is no doubt that stability is the prerequisite for applications. Regardless of what kind of stability problem is of concern, such as passivity [14], [15]; dissipativity [16], [17]; or robust stability [18], [19], the method used to analyze system stability is always the key concern [20]. These days, the powerful tool to address the stability problem for discrete-time delayed neural networks is the Lyapunov functional method [21]–[24]. Nevertheless, this method usually leads to a sufficient condition with more or less conservatism. This kind of conservatism comes from two resources: 1) the chosen L–K functional candidate and 2) related techniques to bound the forward difference of this candidate. Therefore, constructing a proper L–K functional and developing advanced techniques are two main ways to obtain a relaxed criterion [2].

As L–K functionals reported commonly include double-summation terms, how to handle summation terms that appear in their forward differences is essential for the reduction of conservatism. By taking the place of the free-weighting-matrix technique [4], summation inequalities become popular due to simplicity and straightforwardness [25]–[29], which are often applied together with the reciprocally convex combination lemma (RCCL) [30]–[32]. For instance, the Jensen summation inequality is often used in combination with the α-independent RCCL in early years [33]. Later, more relaxed combinations are popular [20]. However, as the gaps of these inequalities become smaller, the improvement of the inequality method may have little effect on improving the relaxation of stability criteria.

There is a common understanding that a chosen L–K functional candidate with more system information involved may result in a more relaxed condition. Therefore, besides double-summation terms, multiple-summation terms are now added into L–K functionals [33], [34]. In addition, many augmented L–K functionals are constructed, in which augmented vectors contain a number of state-related vectors. For instance, the augmented state vector involved in the quadratic term often contains the three vectors: 1) $x(k)$; 2) $\sum_{i=k-h_1}^{k-1} x(i)$; and 3) $\sum_{i=k-h_2}^{k-h_1-1} x(i)$ (see (1) and (4) for more details) [20]. The augmented state vector involved in the single-summation term often contains $x(i)$ and the activation function $f(x(i))$ [20], [33]. On the other hand, by partitioning the entire delay interval into more parts, delay-partitioning-based functionals are proposed in which more information of the discrete-time delay is considered [35], [36]. However, it is necessarily noted that the forward differences of the above-mentioned L–K functionals are all estimated to be *affine* with the delay. This kind of L–K functionals may lead to conservative stability conditions to some extent, which motivates this research.

This article focuses on developing new L–K functionals to study the stability problem for discrete-time neural networks with a time-varying delay. It aims to construct such an appropriate L–K functional that its forward difference can be estimated to be *quadratic* with the delay. To this end, two novel single-summation terms are involved in the L–K functional candidate. Meanwhile, a new augmented state vector is constructed in the quadratic term by adding two double-summation state vectors. Furthermore, when the variation of the discrete-time delay is considered, the quadratic term is further modified by changing the constant quadratic matrix to a delay-affine one. All these innovations are helpful to produce a quadratic function with respect to the delay. As a result, the quadratic function negative-definiteness lemma, recently reported, is employed to derive tractable LMI-based stability criteria. The proposed L–K functionals are shown to be very effective in reducing the conservatism of obtained criteria via two numerical examples.

Notations: Throughout this article, the notations are ordinary. For example, $\mathrm{Sym}\{X\}$ represents $X+X^T$. $Y \in \mathbb{S}_+^n$ means that Y is a symmetric and positive-definite matrix of $\mathbb{R}^{n\times n}$. $Z \in \mathbb{D}_+^n$ denotes that Z is a diagonal matrix of $\mathbb{S}_+^n$. $d \in \mathbb{N}$ implies that d is a non-negative integer.

II. Preliminary and Useful Lemmas

Let us consider the discrete-time neural network with a time-varying delay

$$x(k+1) = \mathrm{C}x(k) + \mathrm{A}f(x(k)) + A_d f(x(k-h(k))) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector associated with n neurons; $C := \mathrm{diag}\{c_1, c_2, \dots, c_n\}$ is the state feedback coefficient matrix; A and A_d are the connection weighting matrices; and $f(x(k)) := \mathrm{col}\{f_1(x_1(k)), f_2(x_2(k)), \dots, f_n(x_n(k))\}$ represents the neural activation function satisfying

$$f_i(0) = 0 \tag{2}$$

$$\sigma_i^- \le \frac{f_i(t_1) - f_i(t_2)}{t_1 - t_2} \le \sigma_i^+, \ \ t_1 \ne t_2 \tag{3}$$

where σ_i^+ and σ_i^- are known scalars, $i \in \{1, \dots, n\}$. The delay $h(k)$, abbreviated as h_k, satisfies the following constraint:

$$1 \le h_1 \le h_k \le h_2 \tag{4}$$

where h_1 and h_2 are known integers.

This article aims to derive relaxed stability criteria for neural network (1) with h_k satisfying (4). Meanwhile, the computation burden should not significantly increase. To do so, we focus on developing a new L–K functional that is *fundamental* to a *good* stability condition.

Before presenting useful lemmas, we define two functions $s_1(\tau) := \tau + 1$ and $s_2(\tau) := (\tau+1)(\tau+2)$ for $\tau \in \mathbb{N}$ and the following notations:

$$K_1 := \mathrm{diag}\{\sigma_1^+, \dots, \sigma_n^+\}, \quad K_2 := \mathrm{diag}\{\sigma_1^-, \dots, \sigma_n^-\}$$
$$h_{2k} := h_2 - h_k, \quad h_{k1} := h_k - h_1, \quad h_{21} := h_2 - h_1.$$

Lemma 1 [37]: For constant integers α and β satisfying $\alpha \le \beta$ and a function $f(k,i)$ with $k \in \mathbb{N}$, the equation

$$\Delta F(k) = f(k, k+\beta+1) - f(k, k+\alpha) + \sum_{i=k+\alpha+1}^{k+\beta+1} \Delta f(k,i)$$

holds, where $F(k) := \sum_{i=k+\alpha}^{k+\beta} f(k,i)$, $\Delta F(k) := F(k+1) - F(k)$, and $\Delta f(k,i) := f(k+1,i) - f(k,i)$.

Lemma 2 [37]: For a matrix function $M(h_k) = \Phi_2 h_k^2 + \Phi_1 h_k + \Phi_0$, where Φ_2, Φ_1, and Φ_0 are coefficient matrices, if the following inequalities:

$$M(h_1) < 0, \quad -\Phi_2 h_{21}^2 + 4M(h_1) < 0 \tag{5}$$
$$M(h_2) < 0, \quad -\Phi_2 h_{21}^2 + M(h_1) + M(h_2) < 0 \tag{6}$$

hold, one has $M(h_k) < 0$ for $\forall h_k \in [h_1, h_2]$.

Lemma 3 [25], [28]: For a matrix $R \in \mathbb{S}_+^n$ and a function $\{\omega(i) \in \mathbb{R}^n | i \in [\alpha, \beta]\}$, the inequality

$$(\beta - \alpha) \sum_{i=\alpha}^{\beta-1} \Delta\omega^T(i) R \Delta\omega(i) \ge \sum_{l=0}^{2} (2l+1) \vartheta_l^T R \vartheta_l \tag{7}$$

holds, where $\Delta\omega(i) := \omega(i+1) - \omega(i)$, $\vartheta_0 := \omega(\beta) - \omega(\alpha)$,

$$\vartheta_1 := \omega(\beta) + \omega(\alpha) - 2\sum_{i=\alpha}^{\beta} \frac{\omega(i)}{s_1(\beta-\alpha)}$$
$$\vartheta_2 := \omega(\beta) - \omega(\alpha) + 6\sum_{i=\alpha}^{\beta} \frac{\omega(i)}{s_1(\beta-\alpha)} - 12 \sum_{i=\alpha}^{\beta}\sum_{j=i}^{\beta} \frac{\omega(j)}{s_2(\beta-\alpha)}.$$

Lemma 4 [30]: For $R_1, R_2 \in \mathbb{S}_+^n$ and $T \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} R_1 & T \\ * & R_2 \end{bmatrix} \ge 0$, the matrix inequality

$$\begin{bmatrix} \frac{1}{\alpha}R_1 & 0 \\ * & \frac{1}{1-\alpha}R_2 \end{bmatrix} \ge \begin{bmatrix} R_1 & T \\ * & R_2 \end{bmatrix}$$

holds for any $\alpha \in (0,1)$.

Lemma 5: For integers h_1 and h_2 satisfying $h_1 < h_2$ and a vector function $\{x(i) | i \in [k-h_2, k-h_1]\}$, the following two equations hold:

$$\sum_{i=k-h_2}^{k-h_1} x(i) = \sum_{i=k-h_k}^{k-h_1} x(i) + \sum_{i=k-h_2}^{k-h_k} x(i) - x(k-h_k) \tag{8}$$

$$\sum_{i=k-h_2}^{k-h_1}\sum_{j=i}^{k-h_1} x(j) = \sum_{i=k-h_k}^{k-h_1}\sum_{j=i}^{k-h_1} x(j) + \sum_{i=k-h_2}^{k-h_k}\sum_{j=i}^{k-h_k} x(j) + h_{2k}\sum_{j=k-h_k}^{k-h_1} x(j) - (h_{2k}+1)x(k-h_k). \tag{9}$$

Proof: From

$$\begin{aligned}\sum_{i=k-h_2}^{k-h_1} x(i) &= \left(\sum_{i=k-h_k}^{k-h_1} + \sum_{i=k-h_2}^{k-h_k} -(i=k-h_k)\right)x(i)\\ &= \sum_{i=k-h_k}^{k-h_1} x(i) + \sum_{i=k-h_2}^{k-h_k} x(i) - x(k-h_k)\end{aligned}$$

we obtain (8).

It follows from (8) that:

$$\begin{aligned}\sum_{i=k-h_2}^{k-h_1}\sum_{j=i}^{k-h_1} x(j) &= \sum_{i=k-h_k}^{k-h_1}\sum_{j=i}^{k-h_1} x(j)\\ &\quad + \sum_{i=k-h_2}^{k-h_k}\sum_{j=i}^{k-h_1} x(j) - \sum_{j=k-h_k}^{k-h_1} x(j).\end{aligned}$$

Hence, (9) holds due to the following fact:

$$\begin{aligned}&\sum_{i=k-h_2}^{k-h_k}\sum_{j=i}^{k-h_1} x(j)\\ &= \sum_{i=k-h_2}^{k-h_k}\left(\sum_{j=i}^{k-h_k} + \sum_{j=k-h_k}^{k-h_1} -(j=k-h_k)\right)x(j)\\ &= \sum_{i=k-h_2}^{k-h_k}\sum_{j=i}^{k-h_k} x(j) + s_1(h_{2k})\left(\sum_{j=k-h_k}^{k-h_1} x(j) - x(k-h_k)\right).\end{aligned}$$

■

Remark 1: For convenient application, the matrix version of the quadratic function negative-definiteness lemma is directly presented in Lemma 2. In order to highlight the effectiveness of proposed L–K functionals on reducing conservatism, the α-independent RCCL shown in Lemma 4 is employed. Compared to others, the original RCCL is easier to deal with, with only one free matrix introduced. Two summation equations are collected in Lemma 5 that are useful in summation calculations.

III. Main Results

A. New L–K Functional

Let us define the notations for the sake of clarity

$$\begin{aligned}\xi_1(k) &:= \mathrm{col}\{x(k), x(k-h_1), x(k-h_k), x(k-h_2)\}\\ \xi_2(k) &:= \mathrm{col}\{f(x(k)), f(x(k-h_1)), f(x(k-h_k))\\ &\qquad \times f(x(k-h_2))\}\\ \xi_3(k) &:= \mathrm{col}\left\{\sum_{i=k-h_1}^{k}\frac{x(i)}{s_1(h_1)}, \sum_{i=k-h_k}^{k-h_1}\frac{x(i)}{s_1(h_{k1})}\right.\\ &\qquad \times \sum_{i=k-h_2}^{k-h_k}\frac{x(i)}{s_1(h_{2k})}, \sum_{i=k-h_1}^{k}\sum_{j=i}^{k}\frac{x(j)}{s_2(h_1)}\\ &\qquad \times \left.\sum_{i=k-h_k}^{k-h_1}\sum_{j=i}^{k-h_1}\frac{x(j)}{s_2(h_{k1})}, \sum_{i=k-h_2}^{k-h_k}\sum_{j=i}^{k-h_k}\frac{x(j)}{s_2(h_{2k})}\right\}\end{aligned}$$

$$\xi(k) := \mathrm{col}\left\{\xi_1(k), \xi_2(k), \xi_3(k), \sum_{i=k-h_k}^{k-h_1} x(i), \sum_{i=k-h_2}^{k-h_k} x(i)\right\}.$$

Now, a new L–K functional is constructed as follows:

$$V(k) := \sum_{i=0}^{3} V_i(k) \tag{10}$$

where

$$\begin{aligned}V_0(k) &:= \chi_0^T(k) P \chi_0(k)\\ V_1(k) &:= \sum_{i=k-h_1}^{k-1}\chi_1^T(i) Q_1 \chi_1(i) + \sum_{i=k-h_2}^{k-h_1-1}\chi_1^T(i) Q_2 \chi_1(i)\\ V_2(k) &:= \sum_{i=k-h_1}^{k-1}\eta_1^T(k,i) Q_3 \eta_1(k,i)\\ &\quad + \sum_{i=k-h_2}^{k-h_1-1}\eta_2^T(k,i) Q_4 \eta_2(k,i)\\ V_3(k) &:= h_1 \sum_{i=k-h_1}^{k-1}\sum_{j=i}^{k-1} y^T(j) R_1 y(j)\\ &\quad + h_{21}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-1} y^T(j) R_2 y(j)\end{aligned}$$

with the augmented vectors

$$\begin{aligned}\chi_0(k) &:= \mathrm{col}\left\{\eta_0(k), \sum_{i=k-h_1}^{k-1}\sum_{j=i}^{k-1} x(j)\right.\\ &\qquad \left.\times \sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} x(j)\right\}\\ \chi_1(i) &:= \mathrm{col}\{x(i), f(x(i))\}\\ y(k) &:= x(k+1) - x(k)\\ \eta_0(k) &:= \mathrm{col}\left\{x(k), \sum_{i=k-h_1}^{k-1} x(i), \sum_{i=k-h_2}^{k-h_1-1} x(i)\right\}\\ \eta_1(k,i) &:= \mathrm{col}\left\{x(i), x(k), \sum_{j=i}^{k-1} x(j), \sum_{j=k-h_1}^{i} x(j)\right\}\\ \eta_2(k,i) &:= \mathrm{col}\left\{x(i), x(k), \sum_{j=i}^{k-h_1-1} x(j), \sum_{j=k-h_2}^{i} x(j)\right\}.\end{aligned}$$

Remark 2: In existing L–K functionals, the augmented vector $\eta_0(k)$ defined in the quadratic term $\eta_0^T(k)P\eta_0(k)$ is usually composed of the following three vectors: 1) $x(k)$; 2) $\sum_{i=k-h_1}^{k-1} x(i)$; and $\sum_{i=k-h_2}^{k-h_1-1} x(i)$ [20], [33], [34]. However, in the new L–K functional (10), two double-summation vectors $\sum_{i=k-h_1}^{k-1}\sum_{j=i}^{k-1} x(j)$ and $\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} x(j)$ are added into the augmented state vector $\chi_0(k)$, which can lead to more information among various state vectors considered. In addition, inspired by our previous work [37], the two complementary summation couples $\{\sum_{j=i}^{k-1} x(j), \sum_{j=k-h_1}^{i} x(j)\}$ and $\{\sum_{j=i}^{k-h_1-1} x(j), \sum_{j=k-h_2}^{i} x(j)\}$ are, respectively, encompassed in the two augmented state vectors $\eta_1(k,i)$ and $\eta_2(k,i)$, which make all state vectors among $\xi(k)$ appear in the forward difference of $V_2(k)$. To the best of our knowledge, it is the first time for such single-summation terms to be used to analyze the stability of the delayed neural network. All these innovations lead to a quadratic function with respect to the delay from the forward difference of $V(k)$. This has the potential to achieve a more relaxed stability condition via the negative-definiteness lemma recently reported [37], compared to existing results via the convex optimization method.

B. Improved Stability Criteria

Theorem 1: For given h_1 and h_2, neural network (1) with the delay h_k satisfying (4) is asymptotically stable, if there exist matrices $P \in \mathbb{S}_+^{5n}$, $Q_1, Q_2 \in \mathbb{S}_+^{2n}$, $Q_3, Q_4 \in \mathbb{S}_+^{4n}$, $R_1, R_2 \in \mathbb{S}_+^{n}$, $H_i \in \mathbb{D}_+^{n}$, $i \in \{1,\ldots,7\}$, $S \in \mathbb{R}^{3n\times 3n}$, and $L_1, L_2 \in \mathbb{R}^{n\times 16n}$ such that the following inequalities:

$$\begin{bmatrix} \tilde{R}_2 & S \\ (*) & \tilde{R}_2 \end{bmatrix} \geq 0, \tag{11}$$

$$\Xi(h_1) < 0, \quad -\Phi_2 h_{21}^2/4 + \Xi(h_1) < 0 \tag{12}$$

$$\Xi(h_2) < 0, \quad -\Phi_2 h_{21}^2 + \Xi(h_1) + \Xi(h_2) < 0 \tag{13}$$

hold, where

$$\Xi(h_k) = \Xi_0(h_k) + \Xi_1 + \Xi_2(h_k) + \Xi_3 + \Omega_1 + \Omega_2 \tag{14}$$

$$\begin{aligned}\Xi_0(h_k) &= \Pi_2^T P \Pi_2 - \Pi_1^T P \Pi_1 \\ &\quad + \mathrm{Sym}\{(\Pi_2 - \Pi_1)^T P \Pi_0(h_k)\}\end{aligned} \tag{15}$$

$$\begin{aligned}\Xi_1 &= \begin{bmatrix} e_1 \\ e_5 \end{bmatrix}^T Q_1 \begin{bmatrix} e_1 \\ e_5 \end{bmatrix} - \begin{bmatrix} e_2 \\ e_6 \end{bmatrix}^T (Q_1 - Q_2) \begin{bmatrix} e_2 \\ e_6 \end{bmatrix} \\ &\quad - \begin{bmatrix} e_4 \\ e_8 \end{bmatrix}^T Q_2 \begin{bmatrix} e_4 \\ e_8 \end{bmatrix}\end{aligned} \tag{16}$$

$$\begin{aligned}\Xi_2(h_k) &= c_{10}^T Q_3 c_{10} - c_{20}^T Q_3 c_{20} + h_1 c_{30}^T Q_3 c_{30} \\ &\quad + \mathrm{Sym}\{c_{40}^T Q_3 c_{30}\} + c_5^T(h_k) Q_4 c_5(h_k) \\ &\quad - c_6^T(h_k) Q_4 c_6(h_k) + h_{21} c_{70}^T Q_4 c_{70} \\ &\quad + \mathrm{Sym}\{c_8^T(h_k) Q_4 c_{70}\}\end{aligned} \tag{17}$$

$$\Xi_3 = e_{s1}^T\left(h_1^2 R_1 + h_{21}^2 R_2\right) e_{s1} - \Gamma_0^T \tilde{R}_1 \Gamma_0 - \Xi_{31} \tag{18}$$

$$\begin{aligned}\Omega_1 &= \sum_{i=1}^{4} \mathrm{Sym}\{\pi_{1i}^T H_i \pi_{2i}\} \\ &\quad + \sum_{i=1}^{3} \mathrm{Sym}\Big\{\left(\pi_{1i} - \pi_{1(i+1)}\right)^T H_{i+4} \\ &\qquad \times \left(\pi_{2i} - \pi_{2(i+1)}\right)\Big\}\end{aligned} \tag{19}$$

$$\begin{aligned}\Omega_2 &= \mathrm{Sym}\{L_1^T(s_1(h_{k1})e_{10} - e_{15})\} \\ &\quad + \mathrm{Sym}\{L_2^T(s_1(h_{2k})e_{11} - e_{16})\}\end{aligned} \tag{20}$$

$$\begin{aligned}\Phi_{21} &= (c_{511} - c_{512})^T Q_4 (c_{511} - c_{512}) \\ &\quad - (c_{611} - c_{612})^T Q_4 (c_{611} - c_{612}) \\ &\quad + \mathrm{Sym}\{(-c_{821} + c_{822} + c_{823})^T Q_4 c_{70}\}\end{aligned}$$

$$\Phi_2 = \Phi_{21} + \mathrm{Sym}\Big\{(\Pi_2 - \Pi_1)^T P \tilde{\Pi}_0\Big\} \tag{21}$$

$$\begin{aligned}
\tilde{\Pi}_0 &= \mathrm{col}\{0, 0, 0, 0, e_{13} + e_{14} - e_{10}\} \\
\Pi_0(h_k) &= \mathrm{col}\{0, s_1(h_1)e_9, s_1(h_{k1})e_{10} + s_1(h_{2k})e_{11} \\
&\qquad \times s_2(h_1)e_{12}, \beta_1(h_k)\} \\
\Pi_1 &= \mathrm{col}\{e_1, -e_1, -e_2 - e_3, -s_1(h_1)e_1, -s_1(h_{21})e_2\} \\
\Pi_2 &= \mathrm{col}\{e_s, -e_2, -e_3 - e_4, -s_1(h_1)e_9, -\beta_0\}, \\
\beta_1(h_k) &= s_2(h_{k1})e_{13} + s_2(h_{2k})e_{14} + h_{2k}s_1(h_{k1})e_{10} \\
&\quad - s_1(h_{2k})e_3 \\
\beta_0 &= e_{15} + e_{16} - e_3 \\
\pi_{1i} &= K_1 e_i - e_{i+4}, \quad \pi_{2i} = e_{i+4} - K_2 e_i \\
e_i &= \left[0_{n\times(i-1)n} \quad I_{n\times n} \quad 0_{n\times(16-i)n}\right], \; i \in \{1,\ldots,16\} \\
e_s &= Ce_1 + Ae_5 + A_d e_7, \quad e_{s1} = e_s - e_1 \\
c_{10} &= \mathrm{col}\{e_1, e_1, 0, s_1(h_1)e_9\} \\
c_{20} &= \mathrm{col}\{e_2, e_1, -e_1 + s_1(h_1)e_9, e_2\} \\
c_{30} &= \mathrm{col}\{0, e_{s1}, e_1, -e_2\} \\
c_{40} &= \mathrm{col}\{-e_2 + s_1(h_1)e_9, h_1 e_1, c_{403}, c_{404}\} \\
c_{403} &= -h_1 e_1 - s_1(h_1)e_9 + s_2(h_1)e_{12} \\
c_{404} &= -e_2 + s_2(h_1)(e_9 - e_{12}) \\
c_5(h_k) &= c_{50} + s_1(h_{k1})c_{511} + s_1(h_{2k})c_{512} \\
c_6(h_k) &= c_{60} + s_1(h_{k1})c_{611} + s_1(h_{2k})c_{612} \\
c_{50} &= \mathrm{col}\{e_2, e_1, 0, -e_3\}, \quad c_{511} = \mathrm{col}\{0, 0, 0, e_{10}\} \\
c_{512} &= \mathrm{col}\{0, 0, 0, e_{11}\}, \quad c_{60} = \mathrm{col}\{e_4, e_1, -e_2 - e_3, e_4\} \\
c_{611} &= \mathrm{col}\{0, 0, e_{10}, 0\}, \quad c_{612} = \mathrm{col}\{0, 0, e_{11}, 0\} \\
c_{70} &= \mathrm{col}\{0, e_{s1}, e_2, -e_4\} \\
c_8(h_k) &= c_{80} + s_1(h_{k1})c_{811} + s_1(h_{2k})c_{812} \\
&\quad + (h_{2k} - 1)s_1(h_{k1})c_{821} + s_2(h_{k1})c_{822} + s_2(h_{2k})c_{823} \\
c_{80} &= \mathrm{col}\{-e_3 - e_4, h_{21}e_1, -h_{21}e_2 + e_3, -e_4\} \\
c_{811} &= \mathrm{col}\{e_{10}, 0, 0, -e_3\} \\
c_{812} &= \mathrm{col}\{e_{11}, 0, -e_3 - e_{11}, (h_{21} + 2)e_{11}\} \\
c_{821} &= \mathrm{col}\{0, 0, e_{10}, 0\} \\
c_{822} &= \mathrm{col}\{0, 0, e_{13}, e_{10} - e_{13}\} \\
c_{823} &= \mathrm{col}\{0, 0, e_{14}, -e_{14}\} \\
\tilde{R}_i &= \mathrm{diag}\{R_i, 3R_i, 5R_i\}, \quad i = 1, 2
\end{aligned}$$

$$\Xi_{31} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}^T \begin{bmatrix} \tilde{R}_2 & S \\ (*) & \tilde{R}_2 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \tag{22}$$

$$\begin{aligned}
\Gamma_0 &= \mathrm{col}\{e_1 - e_2, e_1 + e_2 - 2e_9, e_1 - e_2 + 6e_9 - 12e_{12}\} \\
\Gamma_1 &= \mathrm{col}\{e_2 - e_3, e_2 + e_3 - 2e_{10}, e_2 \\
&\qquad - e_3 + 6e_{10} - 12e_{13}\} \\
\Gamma_2 &= \mathrm{col}\{e_3 - e_4, e_3 + e_4 - 2e_{11}, e_3 \\
&\qquad - e_4 + 6e_{11} - 12e_{14}\}.
\end{aligned}$$

Proof: Define $\Delta V_i(k) := V_i(k+1) - V_i(k)$. Along the trajectory of (1), the forward differences of $V_0(k)$ and $V_1(k)$ are, respectively, calculated

$$\begin{aligned}\Delta V_0(k) &= \chi_0^T(k+1) P \chi_0(k+1) - \chi_0^T(k) P \chi_0(k) \\ &= \xi^T(k) \Xi_0(h_k) \xi(k)\end{aligned}$$

$$\begin{aligned}\Delta V_1(k) &= \chi_1^T(k)Q_1\chi_1(k) - \chi_1^T(k-h_1)Q_1\chi_1(k-h_1)\\ &\quad + \chi_1^T(k-h_1)Q_2\chi_1(k-h_1)\\ &\quad - \chi_1^T(k-h_2)Q_2\chi_1(k-h_2)\\ &= \xi^T(k)\Xi_1\xi(k)\end{aligned}$$

where $\chi_0(k+1) = (\Pi_0(h_k) + \Pi_2)\xi(k)$, $\chi_0(k) = (\Pi_0(h_k) + \Pi_1)\xi(k)$, and $\Xi_0(h_k)$ and Ξ_1 are, respectively, defined in (15) and (16). Note that during calculation, Lemma 5 and the following two equations have been considered:

$$\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} x(j) = (\beta_1(h_k) - s_1(h_{21})e_2)\xi(k)$$

$$\sum_{i=k-h_2+1}^{k-h_1}\sum_{j=i}^{k-h_1} x(j) = (\beta_1(h_k) - \beta_0)\xi(k).$$

In the same way, $\Delta V_2(k)$ is obtained via Lemma 1 as follows:

$$\begin{aligned}\Delta V_2(k) &= \eta_1^T(k,k)Q_3\eta_1(k,k)\\ &\quad - \eta_1^T(k,k-h_1)Q_3\eta_1(k,k-h_1)\\ &\quad + \sum_{i=k-h_1+1}^{k}\Delta\big(\eta_1^T(k,i)Q_3\eta_1(k,i)\big)\\ &\quad + \eta_2^T(k,k-h_1)Q_4\eta_2(k,k-h_1)\\ &\quad - \eta_2^T(k,k-h_2)Q_4\eta_2(k,k-h_2)\\ &\quad + \sum_{i=k-h_2+1}^{k-h_1}\Delta\big(\eta_2^T(k,i)Q_4\eta_2(k,i)\big)\\ &= \xi^T(k)\Xi_2(h_k)\xi(k)\end{aligned}$$

where $\Xi_2(h_k)$ is defined in (17).

Now, along the trajectory of (1), we calculate the forward difference of $V_3(k)$

$$\Delta V_3(k) = y^T(k)(h_1^2R_1 + h_{21}^2R_2)y(k) - \sum_{i=1}^{2}\delta_i(k) \tag{23}$$

where $\delta_1(k) := h_1\sum_{i=k-h_1}^{k-1} y^T(i)R_1y(i)$ and $\delta_2(k) := h_{21}\sum_{i=k-h_2}^{k-h_1-1} y^T(i)R_2y(i)$. Applying Lemmas 3 and 4 to $\delta_1(k)$ and $\delta_2(k)$ leads to

$$\delta_1(k) \ge \xi^T(k)\Gamma_0^T\tilde{R}_1\Gamma_0\xi(k) \tag{24}$$

$$\begin{aligned}\delta_2(k) &\ge \xi^T(k)\begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix}^T\begin{bmatrix}\frac{1}{\alpha}\tilde{R}_2 & 0\\ 0 & \frac{1}{1-\alpha}\tilde{R}_2\end{bmatrix}\begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix}\xi(k)\\ &\ge \xi^T(k)\Xi_{31}\xi(k)\end{aligned} \tag{25}$$

where $\alpha = h_{k1}/h_{21}$ and Ξ_{31} is defined in (22). Note that the inequality $\begin{bmatrix}\tilde{R}_2 & S\\ (*) & \tilde{R}_2\end{bmatrix} \ge 0$ is the prerequisite for the second estimation of (25). By combining (23) with (24) and (25), we obtain

$$\Delta V_3(k) \le \xi^T(k)\Xi_3\xi(k) \tag{26}$$

where Ξ_3 is defined in (18).

According to (3), the following inequalities:

$$\rho_1(k,H) \ge 0, \quad \rho_2(k_1,k_2,H) \ge 0 \tag{27}$$

hold for any $k, k_1, k_2 \in \mathbb{N}$ and $H \in \mathbb{D}_+^n$, where

$$\begin{aligned}\rho_1(k,H) &:= 2\pi_1^T(k)H\pi_2(k)\\ \pi_1(k) &:= K_1x(k) - f(x(k)),\ \pi_2(k) := f(x(k)) - K_2x(k)\\ \rho_2(k_1,k_2,H) &:= 2(\pi_1(k_1) - \pi_1(k_2))^TH(\pi_2(k_1) - \pi_2(k_2)).\end{aligned}$$

As a result, it follows from (27) that:

$$\begin{aligned}0 &\le \rho_1(k,H_1) + \rho_1(k-h_1,H_2) + \rho_1(k-h_k,H_3)\\ &\quad + \rho_1(k-h_2,H_4) + \rho_2(k,k-h_1,H_5)\\ &\quad + \rho_2(k-h_1,k-h_k,H_6) + \rho_2(k-h_k,k-h_2,H_7)\\ &= \xi^T(k)\Omega_1\xi(k)\end{aligned}$$

where Ω_1 is defined in (19).

In addition, by considering relations among elements of $\xi(k)$, the following equations:

$$2\xi^T(k)L_1^T(s_1(h_{k1})e_{10} - e_{15})\xi(k) = 0$$
$$2\xi^T(k)L_2^T(s_1(h_{2k})e_{11} - e_{16})\xi(k) = 0$$

always hold, which leads to

$$\xi^T(k)\Omega_2\xi(k) = 0 \tag{28}$$

where Ω_2 is defined in (20).

As discussed above, the forward difference of $V(k)$ is estimated as follows:

$$\Delta V(k) \le \xi^T(k)\Xi(h_k)\xi(k)$$

where $\Xi(h_k)$ is defined in (14). It is noted that $\Xi(h_k)$ is quadratic with the delay h_k, which can be rewritten as

$$\Xi(h_k) = \Phi_2h_k^2 + \Phi_1h_k + \Phi_0$$

where Φ_2 [defined in (21)], Φ_1, and Φ_0 are matrix coefficients, irrespectively of h_k. Based on Lemma 2, $\Xi(h_k) < 0$ for any $h_k \in [h_1, h_2]$ is ensured by inequalities (12) and (13), which implies the asymptotic stability of neural network (1). This completes this proof. ■

Remark 3: A tractable LMI-based stability criterion is derived in Theorem 1 via Lemma 2. It is expected to be less conservative than those based on the convex optimization method [20] as well as those based on the original negative-definiteness lemma [38].

Remark 4: To avoid a cubic function with respect to the delay arising, the state vector $\xi(k)$ is augmented by adding two summation terms $\sum_{i=k-h_k}^{k-h_1} x(i)$ and $\sum_{i=k-h_2}^{k-h_k} x(i)$. As a result, the relations among elements of $\xi(k)$ should be considered by constructing the zero (28). Although the order of the function is reduced from three to two, two free matrices L_1 and L_2 are introduced that contain lots of decision variables. If the computation burden is of great concern, we can simplify the L–K functional $V(k)$ and obtain the following stability condition.

Corollary 1: For given h_1 and h_2, neural network (1) with h_k satisfying (4) is asymptotically stable, if there exist matrices $P \in \mathbb{S}_+^{3n}$, $Q_1, Q_2 \in \mathbb{S}_+^{2n}$, $Q_3, Q_4 \in \mathbb{S}_+^{4n}$, $R_1, R_2 \in \mathbb{S}_+^n$, $H_i \in \mathbb{D}_+^n$, $i \in \{1, \ldots, 7\}$, and $S \in \mathbb{R}^{3n\times 3n}$ such that (11)–(13) hold, where $\Xi(h_k)$ is defined in Theorem 1 with $\Omega_2 = 0$, and

$$\begin{aligned}\Pi_0(h_k) &= \mathrm{col}\{0, s_1(h_1)e_9, s_1(h_{k1})e_{10} + s_1(h_{2k})e_{11}\}\\ \Pi_1 &= \mathrm{col}\{e_1, -e_1, -e_2 - e_3\}\end{aligned}$$

$$\Pi_2 = \mathrm{col}\{e_s, -e_2, -e_3 - e_4\}, \quad \Phi_2 = \Phi_{21}$$
$$e_i = \left[0_{n\times(i-1)n} \quad I_{n\times n} \quad 0_{n\times(14-i)n}\right], \quad i \in \{1, \ldots, 14\}.$$

Proof: Let us consider the simplified L–K functional

$$V_s(k) := V_{s0}(k) + \sum_{i=1}^{3} V_i(k) \tag{29}$$

where $V_1(k)$, $V_2(k)$, and $V_3(k)$ are all defined in (10), and

$$V_{s0}(k) := \eta_0^T(k) P \eta_0(k)$$

where $\eta_0(k)$ is defined in (10). The remainder of the proof refers to Theorem 1 and is omitted here for brevity. ■

C. Delay-Variation-Dependent Stability Criterion

It is well recognized that the more information considered, the less conservatism achieved. Inspired by [20], we further study the effect of the variation of the discrete-time delay on the stability of neural network (1). To do so, the quadratic term $V_{s0}(k)$ defined in (29) is replaced by an updated quadratic term $\tilde{V}_{s0}(k)$ in which the quadratic matrix $P(h_k)$ is not constant but affine with the delay h_k. Thus, the information on the variation of the discrete-time delay is involved in the new stability criterion.

Define $\Delta h_k := h(k+1) - h(k)$ and assume that h_k satisfies (4) and that Δh_k satisfies

$$d_1 \le \Delta h_k \le d_2 \tag{30}$$

where $d_1, d_2 \in \mathbb{N}$ are known constants. Now, we construct the following new L–K functional:

$$\tilde{V}_s(k) := \tilde{V}_{s0}(k) + \sum_{j=1}^{3} V_j(k) \tag{31}$$

where $V_1(k)$, $V_2(k)$, and $V_3(k)$ are all defined in (10) and

$$\tilde{V}_{s0}(k) := \eta_0^T(k) P(h_k) \eta_0(k) \tag{32}$$

where $\eta_0(k)$ is defined in (10) and $P(h_k) := P_0 + h_k P_1$ with $P_0, P_1 \in \mathbb{S}^{3n}$.

Remark 5: To avoid the quadratic function with respect to h_k appearing in the forward difference of L–K functional [20], the augmented state vector in the added quadratic term is chosen as $\mathrm{col}\{x(k), \sum_{i=k-h_1}^{k-1} x(i)\}$, just the part of $\eta_0(k)$. This inevitably leads to some conservatism. However, in $\tilde{V}_s(k)$, the entire $\eta_0(k)$ remains unchanged, which can further contribute to a final quadratic function with the delay h_k.

Theorem 2: For given h_1 and h_2, neural network (1) with h_k satisfying (4) and with Δh_k satisfying (30) is asymptotically stable, if there exist matrices $P_0, P_1 \in \mathbb{S}^{3n}$, $Q_1, Q_2 \in \mathbb{S}_+^{2n}$, $Q_3, Q_4 \in \mathbb{S}_+^{4n}$, $R_1, R_2 \in \mathbb{S}_+^{n}$, $H_i \in \mathbb{D}_+^{n}$, $i \in \{1, \ldots, 7\}$, and $S \in \mathbb{R}^{3n\times 3n}$ such that (11) and the following inequalities:

$$P_0 + h_1 P_1 > 0, \quad P_0 + h_2 P_1 > 0 \tag{33}$$
$$\Xi(h_1, \Delta h_k) < 0, \quad \Xi(h_2, \Delta h_k) < 0 \tag{34}$$
$$-\tilde{\Phi}_2 h_{21}^2/4 + \Xi(h_1, \Delta h_k) < 0 \tag{35}$$
$$-\tilde{\Phi}_2 h_{21}^2 + \Xi(h_1, \Delta h_k) + \Xi(h_2, \Delta h_k) < 0 \tag{36}$$

hold for $\Delta h_k \in \{d_1, d_2\}$, where

$$\begin{aligned}
\Xi(h_k, \Delta h_k) &= \tilde{\Xi}_0(h_k) + \tilde{\Xi}_{01}(h_k, \Delta h_k) \\
&\quad + \Xi_1 + \Xi_2(h_k) + \Xi_3 + \Omega_1 \qquad (37)\\
\tilde{\Xi}_0(h_k) &= \Pi_2^T P(h_k)\Pi_2 - \Pi_1^T P(h_k)\Pi_1 \\
&\quad + \mathrm{Sym}\{(\Pi_2 - \Pi_1)^T P(h_k)\Pi_0(h_k)\} \\
\tilde{\Xi}_{01}(h_k, \Delta h_k) &= \Delta h_k(\Pi_0(h_k) + \Pi_2)^T P_1 (\Pi_0(h_k) + \Pi_2) \\
\tilde{\Phi}_2 &= \Phi_{21} + \mathrm{Sym}\left\{(\Pi_2 - \Pi_1)^T P_1 \tilde{\Pi}_0\right\} \\
&\quad + \Delta h_k \tilde{\Pi}_0^T P_1 \tilde{\Pi}_0 \\
\tilde{\Pi}_0 &= \mathrm{col}\{0, 0, e_{10} - e_{11}\}
\end{aligned}$$

where all other notations, such as Ξ_1, $\Xi_2(h_k)$, Ξ_3, Ω_1, and Φ_{21} are defined in Corollary 1.

Proof: Since $P(h_k)$ is affine with the delay h_k, $P(h_k) > 0$ for $h_k \in [h_1, h_2]$ is guaranteed by $P_0 + h_1 P_1 > 0$ and $P_0 + h_2 P_1 > 0$.

On the other hand, along the trajectory of (1), the forward difference of $\tilde{V}_{s0}(k)$ is computed

$$\begin{aligned}
\Delta \tilde{V}_{s0}(k) &= \eta_0^T(k+1)(P(h_k) + \Delta h_k P_1)\eta_0(k+1) \\
&\quad - \eta_0^T(k) P(h_k) \eta_0(k) \\
&= \xi^T(k) \tilde{\Xi}_0(h_k) \xi(k) + \Delta h_k \eta_0^T(k+1) P_1 \eta_0(k+1) \\
&= \xi^T(k)\left(\tilde{\Xi}_0(h_k) + \tilde{\Xi}_{01}(h_k, \Delta h_k)\right)\xi(k)
\end{aligned}$$

where $\tilde{\Xi}_0(h_k)$ and $\tilde{\Xi}_{01}(h_k, \Delta h_k)$ are both defined in Theorem 2. Hence, the forward difference of $\tilde{V}_s(k)$ defined in (31) is estimated as follows:

$$\Delta \tilde{V}_s(k) \le \xi^T(k) \Xi(h_k, \Delta h_k) \xi(k)$$

where $\Xi(h_k, \Delta h_k)$ is defined in (37). It is seen that $\Xi(h_k, \Delta h_k)$ is quadratic with h_k and affine with Δh_k. Therefore, $\Xi(h_k, \Delta h_k) < 0$ for $h_k \times \Delta h_k \in [h_1, h_2] \times [d_1, d_2]$ is guaranteed by inequalities (34)–(36). This completes the proof. ■

IV. Numerical Examples

Two numerical examples are given in this section to compare the proposed stability criteria with some of existing results from two aspects: 1) conservatism and 2) computation burden. They are, respectively, indicated by two indexes: 1) maximum allowable upper bound (MAUB) and 2) the number of decision variables (NDVs).

Example 1: Consider the delayed neural network (1) with the parameters

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad A = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.004 \end{bmatrix}$$
$$A_d = \begin{bmatrix} -0.01 & 0.01 \\ -0.02 & -0.01 \end{bmatrix}$$
$$K_1 = \mathrm{diag}\{1, 1\}, \quad K_2 = \mathrm{diag}\{0, 0\}.$$

In this example, MAUBs h_2 for different h_1 are computed by Corollary 1 and Theorem 1 in this article and other conditions reported recently. From Table I, we find that MAUBs obtained by Corollary 1 are all larger than those obtained by conditions proposed in [6]–[9], and [20] as h_1 takes any value

TABLE I
MAUBs h_2 FOR DIFFERENT h_1 IN EXAMPLE 1

h_1	2	4	6	8	10	20	NDVs
Th.1 [6]	12	14	16	18	20	30	$4.5n^2 + 7.5n$
Co.3.2 [7]	13	15	17	19	21	31	$68n^2 + 10n$
Th.1 [8]	15	17	18	20	23	32	$29n^2 + 12n$
Co.3.3 [9]	32	34	36	38	40	52	$20n^2 + 14n$
Th.1 [20]	99	101	103	105	107	117	$13.5n^2 + 11.5n$
Co. 1	102	104	106	108	110	120	$34.5n^2 + 15.5n$
Th. 1	104	106	108	110	112	122	$74.5n^2 + 16.5n$

TABLE II
MAUBs h_2 FOR DIFFERENT h_1 IN EXAMPLE 2

h_1	4	6	8	10	12	15	NDVs
Th.1 [6]	18	18	20	20	20	23	$4.5n^2 + 7.5n$
Th.1 [10]	20	20	21	21	21	23	$61.5n^2 + 17.5n$
Co.1 [33]	20	20	21	21	22	23	$44n^2 + 13n$
Co.2 [39]	20	20	21	21	22	24	$46n^2 + 6n$
Th.1 [20]	20	20	21	22	22	24	$13.5n^2 + 11.5n$
Co. 1	20	21	21	22	23	24	$34.5n^2 + 15.5n$
Th. 1	20	21	21	22	23	24	$74.5n^2 + 16.5n$

TABLE III
MAUBs h_2 FOR DIFFERENT h_1 WITH THE VARIATION OF DELAY CONSIDERED

h_1	4	6	8	10	12	15	NDVs
Th.1 [20] $_{d\geq 2}$	20	20	21	22	22	24	$15.5n^2 + 12.5n$
Th.1 [20] $_{d=1}$	20	21	21	22	23	24	$15.5n^2 + 12.5n$
Th.1 [20] $_{d=0}$	21	21	22	23	23	25	$15.5n^2 + 12.5n$
Th. 2 $_{d\geq 2}$	20	21	21	22	23	24	$39n^2 + 17n$
Th. 2 $_{d=1}$	21	21	22	22	23	24	$39n^2 + 17n$
Th. 2 $_{d=0}$	23	23	24	25	25	27	$39n^2 + 17n$

from the set {2, 4, 6, 8, 10, 20}. This clearly validates the effectiveness of the new L–K functional $V_s(k)$ defined in (29). As expected, Theorem 1 produces the largest MAUBs among all conditions, including Corollary 1, which implies that the two double-summation state vectors added into $V_0(k)$ are helpful to reduce the conservatism further.

In addition, NDVs involved in different conditions are also listed. It is not difficult to find that the NDV involved in Corollary 1 is very competitive, which is even smaller than those involved in some other conditions.

Example 2: Consider the delayed neural network (1) with the following parameters:

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad A = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}$$
$$A_d = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}$$
$$K_1 = \text{diag}\{1, 1\}, \quad K_2 = \text{diag}\{0, 0\}.$$

As done in Example 1, both MAUBs and NDVs are computed for different conditions in Table II. The listed MAUBs shows that both Corollary 1 and Theorem 1 produce relaxed results than other conditions proposed in [6], [10], [20], and [33]. Especially, it is observed that the MAUB (i.e., the largest h_2) is improved from 20 to 21 by Corollary 1 and Theorem 1 when $h_1 = 6$.

In Table III, MAUBs are carefully compared between Theorem 2 in this article and [20, Th. 1], which are both derived by considering the information of the variation of h_k. It is seen that, no matter $d = 0$, 1 or $d \geq 2$ (let $d = d_2 = -d_1$), Theorem 2 always achieves less conservative results than Theorem 1 [20]. This clearly shows the effectiveness of the new L–K functional $\tilde{V}_s(k)$ defined in (31). In addition, it is noted that if we let $P_1 = 0$ and $P_0 = P$, Theorem 2 is reduced to Corollary 1.

V. CONCLUSION

This article has studied the stability problem for discrete-time delayed neural networks. New L–K functionals have been tailored by taking into account more information of state vectors, which could lead to quadratic functions with respect to the delay from their forward differences, as well as when the variation of the delay is considered. Relaxed stability criteria have been consequently obtained by applying the newly-developed negative-definiteness lemma. Two numerical examples have demonstrated that they are more relaxed than existing results recently reported.

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