# Efficient algorithm for approximating Nash equilibrium of distributed aggregative games

Gehui Xu, Guanpu Chen, Hongsheng Qi, and Yiguang Hong, Fellow, IEEE

Abstract—In this paper, we aim to design a distributed approximate algorithm for seeking Nash equilibria of an aggregative game. Due to the local set constraints of each player, projectionbased algorithms have been widely employed for solving such problems actually. Since it may be quite hard to get the exact projection in practice, we utilize inscribed polyhedrons to approximate local set constraints, which yields a related approximate game model. We first prove that the Nash equilibrium of the approximate game is the  $\epsilon$ -Nash equilibrium of the original game, and then propose a distributed algorithm to seek the  $\epsilon$ -Nash equilibrium, where the projection is then of a standard form in quadratic programming. With the help of the existing developed methods for solving quadratic programming, we show the convergence of the proposed algorithm, and also discuss the computational cost issue related to the approximation. Furthermore, based on the exponential convergence of the algorithm, we estimate the approximation accuracy related to  $\epsilon$ . Additionally, we investigate the computational cost saved by approximation on numerical examples.

Index Terms— $\epsilon$ -Nash equilibrium; Approximation; Distributed algorithm; Aggregative game.

# I. INTRODUCTION

Seeking Nash equilibria (NE) in non-cooperative games has been widely investigated in social sciences and engineering. As one of the important non-cooperative games, the aggregative game has drawn much growing interest in many fields, such as demand response management [1] and multi-product enterprise oligopoly [2]. Particularly, because of complex topologies, or communication burdens, or privacy issues in large-scale networks, it is of great practical significance to seek NE in a distributed manner, where players achieve the NE with local data and communications through networks [3]–[6].

Since players' actions are usually constrained by local sets, projection-based distributed algorithms for NE or generalized Nash equilibria (GNE) seeking have been developed. For aggregative games, [7] studied projected distributed synchronous and asynchronous algorithms for NE computation over a network, while [8] investigated a projection-based distributed asymmetric algorithm for GNE seeking with affine coupling constraints. Then [9] designed a projected distributed continuous-time algorithm for non-smooth tracking dynamics with coupled constraints. Moreover, [10] proposed a projected distributed algorithm for NE seeking based on iterative Tikhonov regularization methods, while [11] discussed another projection-based algorithm on a time-varying communication network for seeking GNE with partial-decision information.

Various methods are usually adopted for projection operation, such as the sequential quadratic program (SOP) [12], the interior point method (IPM) [13], and the augmented Lagrangian method (ALM) [14]. However, the computational complexity may be exceptionally high for high-dimensional constraint sets, and the computational error may increase with the expansion of data and model scales. On the other hand, hyperplane approximation was widely employed in various practical situations such as multi-objective optimization problems [15], object tracking in video images [16], and feature categorization of machine learning [17]. With this inspiring idea, the players' feasible sets are approximated by constructing inscribed polyhedrons, which are thereby enclosed by a series of hyperplanes. Therefore, it is easier to obtain the projection on the hyperplanes than that on the boundaries of convex sets, because a general projection operation is converted into a standard quadratic program, and many developed methods for quadratic programming can be effectively adopted. Although the computational complexity can be reduced effectively in this way, the approximate process inevitably brings the loss of accuracy, related to the discussion of  $\epsilon$ -NE. However, considering the applications in distributed computing with large-scale models, it makes sense to sacrifice a little accuracy for time saving and complexity reduction.

The motivation of this paper is to explore efficient NE seeking of a distributed aggregative game, where we promote to use inscribed polyhedrons to approximate players' feasible sets.

The main contributions of this paper are listed in the following.

- We consider a distributed approximate NE seeking algorithm for aggregative games. Different from those in [6], [10], [11], we approximate players' local feasible sets with inscribed polyhedrons, which converts the general projection operation into a standard quadratic program. With the approximation, we study the seeking of an  $\epsilon$ -NE of the original game.
- We discuss the approximation procedure and analyze the approximation. To be specific, we provide an approximate method for constructing inscribed polyhedrons and

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Gehui Xu, Guanpu Chen, and Hongsheng Qi are with Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Beijing, China, and are also with School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China. (e-mail: xghapple@amss.ac.cn, chengp@amss.ac.cn, qihongsh@amss.ac.cn).

Yiguang Hong is with Department of Control Science and Engineering, Shanghai Research Institute for Intelligent Autonomous Systems, Tongji University, Shanghai, and is also with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China. (e-mail: yghong@iss.ac.cn).

discuss the computational cost saved by approximation. Then we prove that the NE of the approximate game is the  $\epsilon$ -NE of the original game and analyze the factors influencing the accuracy of  $\epsilon$ .

 We show that the proposed algorithm converges to the *ϵ*-NE with an exponential rate, and then give an upper bound of the value *ϵ*. Moreover, we discuss relationships between the computational cost and approximation from different viewpoints.

The remainder is organized as follows: Section II provides notations and preliminary knowledge as well as our problem formulation, while Section III discusses the approximation of players' local feasible sets with inscribed polyhedrons, and shows a relationship between the equilibria of the approximate game and the original one. Then Section IV obtains the convergence of a distributed approximate algorithm to seek the NE with treating the projection as a standard quadratic program and gives an upper bound of the value  $\epsilon$ , and Section V shows numerical examples for illustration of the proposed algorithm. Finally, Section VI concludes the paper.

## II. AGGREGATIVE GAME MODEL

In this section, we first give some basic notations and preliminary knowledge, and then formulate our problem.

## A. Notations and preliminaries

Denote  $\mathbb{R}^n$  (or  $\mathbb{R}^{m \times n}$ ) as the set of *n*-dimensional (or *m*by-*n*) real column vectors (or real matrices), and  $I_n$  as the  $n \times n$  identity matrix. Let  $\mathbf{1}_n$  (or  $\mathbf{0}_n$ ) be the *n*-dimensional column vector with all elements of 1 (or 0). Denote  $A \otimes$ B as the Kronecker product of matrices A and B. Take  $col(x_1, \dots, x_n) = (x_1^{\mathrm{T}}, \dots, x_n^{\mathrm{T}})^{\mathrm{T}}$  and  $\|\cdot\|$  as the Euclidean norm of vectors. Denote  $\nabla f$  as the gradient of function f. Denote  $\mathbf{B}_r(x) \subseteq \mathbb{R}^n$  as a ball with the center at point x and the radius r. Moreover, denote  $\mathbf{E}_v(c) \subseteq \mathbb{R}^n$  as an ellipsoid that

$$\sum_{i=1}^{n} \frac{(x_i - c_i)^2}{v_i^2} \le 1,$$

with the center at point  $c \triangleq (c_1, \dots, c_n)$  and the semiaxis  $v \triangleq (v_1, \dots, v_n)$ .

A set  $K \subseteq \mathbb{R}^n$  is convex if  $\omega x_1 + (1 - \omega)x_2 \in K$  for any  $x_1, x_2 \in K$  and  $0 \le \omega \le 1$ . For a closed convex set K, the projection map  $\Pi_K : \mathbb{R}^n \to K$  is defined as

$$\Pi_K(x) \triangleq \underset{y \in K}{\operatorname{argmin}} \|x - y\|.$$

The following basic property hold:

$$\|\Pi_K(x) - \Pi_K(y)\| \le \|x - y\|, \quad \forall x, y \in \mathbb{R}^n$$

A function  $f : \mathbb{R}^n \to \mathbb{R}$  is convex on K if

$$f(\omega x_1 + (1 - \omega)x_2) \le \omega f(x_1) + (1 - \omega)f(x_2),$$

for any  $x_1, x_2 \in K$  and  $0 \le \omega \le 1$ .

A mapping  $F : \mathbb{R}^n \to \mathbb{R}^n$  is said to be  $\kappa$ -strongly monotone on a set D if there exists a constant  $\kappa > 0$  such that

$$(F(x) - F(y))^{\mathrm{T}}(x - y) \ge \kappa ||x - y||^2, \quad \forall x, y \in D.$$

Given a set  $D \subseteq \mathbb{R}^n$  and a map  $F : D \to \mathbb{R}^n$ , the variational inequality problem VI(D, F) is defined to find a vector  $x^* \in D$  such that

$$(y - x^*)^{-1} F(x^*) \ge 0, \quad \forall y \in D,$$

whose solution is denoted by SOL(D, F).

The following lemma shows an equivalent relationship between the solution of VI(D, F) and the projection map, and discusses the properties of the solution of VI(D, F). Readers can find more details in [18, Proposition 1.5.8, Corollary 2.2.5, and Theorem 2.3.3].

**Lemma 1** Consider VI(D, F), where the set  $D \subseteq \mathbb{R}^n$  is convex and the map  $F : D \to \mathbb{R}^n$  is continuous. The following statements hold:

- (1)  $x \in \text{SOL}(D, F) \Leftrightarrow x = \prod_D (x \theta F(x)), \forall \theta > 0;$
- (2) if D is compact, then SOL(D, F) is nonempty and compact;
- (3) if D is closed and F(x) is strongly monotone, then VI(D, F) has at most one solution.

Take  $X, Z \subseteq \mathbb{R}^n$  as two non-empty sets. For  $y \in \mathbb{R}^n$ , denote dist(y, Z) as the distance between y and Z, i.e.,

$$dist(y, Z) = \inf_{z \in Z} \|y - z\|.$$

Define the Hausdorff metric of  $X, Z \subseteq \mathbb{R}^n$  by

$$H(X,Z) = \max\{\sup_{x \in X} dist(x,Z), \sup_{z \in Z} dist(z,X)\}.$$

The Hausdorff metric integrates all compact sets into a metric space.

A directed graph is defined as  $\mathcal{G} = (\mathcal{I}, \mathcal{E})$  with the node set  $\mathcal{I} = \{1, 2, \dots, N\}$  and the edge set  $\mathcal{E}$ .  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  is the adjacency matrix of  $\mathcal{G}$  such that if  $(j, i) \in \mathcal{E}$ , then  $a_{ij} > 0$ , which means that j belongs to i's neighbor set and i can receive the message sent from agent j, and  $a_{ij} = 0$  otherwise. A graph is said to be strongly connected if there is a sequence of intermediate vertices connected by edges for any pair of vertices. A graph is weight-balanced if  $\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji}$  for every  $i \in \mathcal{I}$ . The Laplacian matrix is  $L = \Delta - A$ , where  $\Delta = \text{diag} \{d_1, \dots, d_N\} \in \mathbb{R}^{N \times N}$  with  $d_i = \sum_{j=1}^{N} a_{ij}$ .

The following lemma is about the Laplacian matrix [19].

# **Lemma 2** Considering a directed graph G,

- (1) G is weight-balanced if and only if  $L + L^{T}$  is positive semidefinite;
- (2) G is strongly connected if and only if zero is a simple eigenvalue of L.

## B. Problem Formulation

Consider an N-player aggregative game, where the players are indexed by  $\mathcal{I} = \{1, \dots, N\}$ . For each  $i \in \mathcal{I}$ , the *i*th player has an action variable  $x_i$  in a local feasible set  $\Omega_i \subseteq \mathbb{R}^n$ . Denote  $\mathbf{\Omega} \triangleq \prod_{i=1}^N \Omega_i \subseteq \mathbb{R}^{nN}$ ,  $\mathbf{x} \triangleq col\{x_1, \dots, x_N\} \in \mathbf{\Omega}$  as the action profile for all players, and  $\mathbf{x}_{-i} \triangleq col\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N\}$  as the action profile for all players except player *i*. The *i*th player has a payoff function  $J_i(x_i, x_{-i}) : \mathbb{R}^{nN} \to \mathbb{R}$ . Define an aggregative term as

$$\mathcal{Q}(\boldsymbol{x}) \triangleq \frac{1}{N} \sum_{i=1}^{N} q_i(x_i).$$

Here  $q_i : \mathbb{R}^n \to \mathbb{R}^M$  is a map for the local contribution to the aggregation. Specifically,  $J_i(x_i, \boldsymbol{x}_{-i}) = f_i(x_i, \mathcal{Q}(\boldsymbol{x}))$  with the function  $f_i : \mathbb{R}^{n+M} \to \mathbb{R}$ . Given  $\boldsymbol{x}_{-i}$ , the *i*th player intends to solve

$$\min_{x_i \in \Omega_i} J_i\left(x_i, \boldsymbol{x}_{-i}\right). \tag{1}$$

**Definition 1 (Nash equilibrium)** A profile  $x^*$  is said to be a Nash equilibrium (NE) of game (1) if

$$J_i\left(x_i^*, \boldsymbol{x}_{-i}^*\right) \leq J_i\left(x_i, \boldsymbol{x}_{-i}^*\right), \; \forall i \in \mathcal{I}, \; \forall x_i \in \Omega_i.$$

In reality with various uncertainties, NE may not exist or be easily calculated. Therefore, we introduce the following definition.

**Definition 2** ( $\epsilon$ -Nash equilibrium) A profile  $x^*$  is said to be an  $\epsilon$ -Nash equilibrium of game (1) if

$$J_i\left(x_i^*, \boldsymbol{x}_{-i}^*\right) \le J_i\left(x_i, \boldsymbol{x}_{-i}^*\right) + \epsilon, \; \forall i \in \mathcal{I}, \; \forall x_i \in \Omega_i, \quad (2)$$

where the constant  $\epsilon > 0$ . Particularly,  $x^*$  is said to be a NE when  $\epsilon = 0$ .

The local payoff functions  $J_i$ , set constraints  $\Omega_i$ , and decision variables  $x_i$  are the private information. Moreover, the aggregative term Q(x) contains all the players' decisions, which cannot be observed by each player directly. Thus, player *i* generates an estimate  $\zeta_i$  of this aggregative term and exchange this information with its local neighbors through a network  $\mathcal{G}$ .

For clarification, we denote the pseudo-gradient by

$$F(\boldsymbol{x}) \triangleq col \left\{ \nabla_{x_1} J_1 \left( \cdot, \boldsymbol{x}_{-1} \right), \dots, \nabla_{x_N} J_N \left( \cdot, \boldsymbol{x}_{-N} \right) \right\}.$$

Define the map  $U_i : \mathbb{R}^n \times \mathbb{R}^M \to \mathbb{R}$  as

$$U_{i}(x_{i},\zeta_{i}) \triangleq \nabla_{\boldsymbol{x}_{i}} J_{i}(\cdot,\boldsymbol{x}_{-i})\Big|_{\mathcal{Q}(\boldsymbol{x})=\zeta_{i}}$$
(3)  
=  $\left(\nabla_{x_{i}} f_{i}(\cdot,\mathcal{Q}) + \frac{1}{N} \nabla_{\mathcal{Q}} f_{i}(x_{i},\cdot)^{\mathrm{T}} \nabla q_{i}\right)\Big|_{\mathcal{Q}=\zeta_{i}}.$ 

Let

$$U(\boldsymbol{x},\boldsymbol{\zeta}) \triangleq col\left(U_1\left(x_1,\zeta_1\right),\ldots,U_N\left(x_N,\zeta_N\right)\right).$$

Obviously,  $U(\boldsymbol{x}, \boldsymbol{1}_N \otimes \mathcal{Q}(\boldsymbol{x})) = F(\boldsymbol{x}).$ 

We give the following assumptions for game (1).

## Assumption 1

- For  $i \in \mathcal{I}$ ,  $\Omega_i$  is compact and convex.
- For  $i \in \mathcal{I}$ , the payoff function  $J_i(\cdot)$  is Lipschitz continuous in  $\boldsymbol{x}$ , while  $J_i(\cdot, \boldsymbol{x}_{-i})$  and the map  $q_i(x_i)$  are continuously differentiable in  $x_i$ . Moreover, the pseudogradient  $F(\boldsymbol{x})$  is  $\kappa$ -strongly monotone on the set  $\boldsymbol{\Omega}$ .
- The map  $U(x, \zeta)$  is  $c_1$ -Lipschitz continuous in  $x \in \Omega$ and  $c_2$ -Lipschitz continuous in  $\zeta$  for some constants  $c_1$ ,

 $c_2 > 0$ . Besides, for  $i \in \mathcal{I}$ ,  $q_i$  is  $c_3$ -Lipschitz continuous on  $\Omega_i$  for a constant  $c_3 > 0$ .

 The communication network G is strongly connected and weight-balanced.

The assumptions of convexity and differentiability about payoff functions are quite common and have been widely used in the literature. Besides, the strong monotonicity of the pseudo-gradient map F has been widely adopted to guarantee the uniqueness of NE [20]–[22]. Additionally, the assumption about the Lipschitz continuity of  $U(x, \zeta)$  and  $q_i$  is the same as that given in [23]. Moreover, the strongly connected and weight-balanced digraph is a generalization of connected undirected graphs in [24], [25], and is also employed in some other distributed algorithms [26], [27].

The following lemma reveals the relationship of a NE  $x^*$  and a solution to VI $(\Omega, F(x))$ , referring to [18, Proposition 1.4.2] and Lemma 1.

**Lemma 3** Under Assumption 1, a profile  $x^*$  is a NE if and only if

$$\boldsymbol{x}^* \in \mathrm{SOL}(\boldsymbol{\Omega}, F(\boldsymbol{x}))$$

Moreover, the game (1) admits a unique Nash equilibrium  $x^*$ .

Therefore, the main task of this paper is to design a distributed algorithm for seeking a NE of the aggregative game (1). Due to players' local feasible sets, projection-based methods have been widely used to solve related problems in the literature, e.g. [6], [10], [11]. Sometimes, it is not so easy to obtain the exact projection points in practice. In the following sections, we provide a scheme to reduce the complexity with an approximate solution.

#### **III. PROBLEM APPROXIMATION**

As we know, it is always easier to obtain the projection points on the hyperplanes than on the boundaries of general set constraints. Therefore, in this section, we use inscribed polyhedrons to approximate the players' local feasible sets.

An inscribed polyhedron of a closed convex set is defined as a polyhedron with all its vertices on the boundary of the convex set. These vertices construct a series of hyperplanes naturally, which enclose an inscribed polyhedron. Denote  $\mathcal{D}_s = \prod_{i=1}^N \mathcal{D}_{s_i}^i$ , where  $\mathcal{D}_{s_i}^i$  is an inscribed polyhedron of  $\Omega_i$  with  $s_i$  vertices, expressed as

$$\mathcal{D}_{s_i}^i = \left\{ x_i \in \mathbb{R}^n : B^i x_i \le b^i \right\}.$$
(4)

Here  $B^i \in \mathbb{R}^{p_i \times n}$  represent normal vectors of the hyperplanes enclosing  $\mathcal{D}_{s_i}^i$  with normalized rows,  $b_i \in \mathbb{R}^{p_i}$  are the distances from the hyperplanes to the origin point, and  $p_i$  is the number of hyperplanes for  $i \in \mathcal{I}$ .

The approximation of convex sets by inscribed polyhedrons has been studied in different problems [28], [29], which can be explicitly expressed by linear inequalities. Here our approximation of inscribed polyhedrons concentrates on players' local feasible sets, different from the approximate view angles [30], and the approximation for system parameters [31]. In fact, the approximate process with inscribed polyhedrons makes projection on a polyhedron easier than directly on a general set, because the projection of point  $x_0$  on a hyperplane  $D = \{x | B_0^T x = b_0\}$  can be written explicitly as  $\Pi_D(x_0) = x_0 + (b_0 - B_0^T x_0) B_0 / ||B_0||^2$ , which can save the corresponding computational cost.

Thereby, with the help of inscribed polyhedrons, we consider a related approximate game,

$$\min_{x_i \in \mathcal{D}_{s_i}^i} J_i\left(x_i, \boldsymbol{x}_{-i}\right).$$
(5)

Before revealing the relationship between the approximate game (5) and the original game (1), we first discuss how the Hausdorff distance between two different inscribed polyhedrons influences the relationship of the normal vectors of their hyperplanes. Denote  $\mathcal{D}_{s_1}^1$  as an inscribed polyhedron of a convex and compact set  $\Omega \subseteq \mathbb{R}^n$  with  $W_1$  as the set of vertices on the boundary of  $\Omega$ , i.e.,

$$\mathcal{D}_{s_1}^1 = \left\{ x \in \mathbb{R}^n : B^1 x \le b^1 \right\}.$$
(6)

Similarly, denote  $\mathcal{D}_{s_2}^2$  as another inscribed polyhedron with  $W_2$  as the set of vertices on the boundary of  $\Omega$ , where  $W_2 = W_1 \cup \{w_0\}$  with  $w_0$  as an additional vertex, i.e.,

$$\mathcal{D}_{s_2}^2 = \left\{ x \in \mathbb{R}^n : B^2 x \le b^2 \right\}.$$
(7)

Suppose that there are  $p_1$  rows of  $B^1$  and  $b^1$ ,  $p_2$  rows of  $B^2$  and  $b^2$ , the first  $p_1 - 1$  rows of  $B^1$  are the same as the first  $p_1 - 1$  rows of  $B^2$ . As a result, the two matrices can be written by row as

$$B^{1} = \begin{bmatrix} B_{1} \\ \vdots \\ B_{p_{1}-1} \\ B_{p_{1}}^{1} \end{bmatrix}, \quad B^{2} = \begin{bmatrix} B_{1} \\ \vdots \\ B_{p_{1}-1} \\ B_{p_{1}}^{2} \\ \vdots \\ B_{p_{2}}^{2} \end{bmatrix}.$$
(8)

**Lemma 4** For  $B_i^2$  as any row of matrix  $B^2$ , there exists a corresponding row  $B_{i(i)}^1$  of matrix  $B^1$  such that

$$\left|B_i^2 - B_{j(i)}^1\right| \to 0, \quad \text{as } H(\mathcal{D}_{s_1}^1, \mathcal{D}_{s_2}^2) \to 0.$$

The proof of Lemma 4 can be found in Appendix A. In addition, the following lemma describes the Hausdorff metric between a convex set and its inscribed polyhedron, referring to [32].

**Lemma 5** For a convex set  $\Omega \subseteq \mathbb{R}^n$ , there exists an inscribed polyhedron  $\mathcal{D}_s$  of  $\Omega$  such that the upper bound of the Hausdorff metric between  $\Omega$  and  $\mathcal{D}_s$  satisfies

$$H(\mathcal{D}_s, \Omega) \le \frac{C_\Omega}{s^{2/(n-1)}}$$

where  $C_{\Omega}$  is a constant related with the curvature of  $\Omega$  and s is the number of vertices in  $\mathcal{D}_s$ .

Based on Lemmas 4 and 5, it is time to reveal the relationship between the approximate game (5) and the original game (1). Note that the Nash equilibrium  $x^*$  of game (5) is the unique solution to VI( $\mathcal{D}_s, F(x)$ ) by Lemma 3. If the payoff function  $J_i$  is fixed, then different polyhedron approximations result in different variational inequality solutions. Thereby, we write  $x^* = x^*(\mathcal{D}_s)$  for game (5). Moreover, denote the unique Nash equilibrium by  $x^*(\Omega)$  for game (1). Then we have the following result.

**Theorem 1** Under Assumption 1, the NE of the approximate game (5) is the  $\epsilon$ -NE of the original game (1).

Proof. Take

$$\mathcal{D}_{s_1} = \prod_{i=1}^N \mathcal{D}^i_{s_{1,i}}, \quad \mathcal{D}_{s_2} = \prod_{i=1}^N \mathcal{D}^i_{s_{2,i}}$$

as two arbitrarily inscribed polyhedrons of  $\Omega$ . Denote  $\mathcal{D}_{s_1+N} = \prod_{i=1}^{N} \mathcal{D}_{s_{1,i}+1}^i$ , where vertices in  $\mathcal{D}_{s_{1,i}+1}^i$  consist of all nodes in  $\mathcal{D}_{s_{1,i}}^i$  and one different vertex in  $\mathcal{D}_{s_{2,i}}^i$  for  $i \in \mathcal{I}$ .  $\mathcal{D}_{s_1+2N}, \mathcal{D}_{s_1+3N}, \cdots, \mathcal{D}_{s_1+s_2}$  are denoted in a similar way, where  $\mathcal{D}_{s_1+s_2} = \prod_{i=1}^{N} \mathcal{D}_{s_{1,i}+s_{2,i}}^i$  is the profile of polyhedrons whose vertices consist of all the vertices in both  $\mathcal{D}_{s_1}$  and  $\mathcal{D}_{s_2}$ . Without losing generality, consider  $s_{2,i} \leq s_{2,j}$ . If  $s_{2,i} < s_{2,j}$  and there is no additive point in  $\mathcal{D}_{s_{2,i}}^i$  for  $\mathcal{D}_{s_{1,i}+s_{2,i}}^i$ , then we keep  $\mathcal{D}_{s_{1,i}+s_{2,i}}^i$  unchanged and increase the vertices in  $\mathcal{D}_{s_{2,j}}^j$  successively. Continue this process until  $\mathcal{D}_{s_1+s_2}$  is reached.  $\mathcal{D}_{s_2+N}, \mathcal{D}_{s_2+2N}, \cdots, \mathcal{D}_{s_1+s_2}$  can be defined similarly.

Note that the difference between  $\|\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_1}) - \boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_2})\|$ can be decomposed into a series of similar structures such as  $\|\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_1}) - \boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_1+N})\|$ ,  $\|\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_1+N}) - \boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_1+2N})\|$ , and so on. Hence, we only need to investigate  $\|\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_1}) - \boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_{s_1})\|$ .

Assume that  $H(\mathcal{D}_{s_{1,i}}^i, \mathcal{D}_{s_{2,i}}^i) \leq \eta$  for  $i \in \mathcal{I}$  and a positive constant  $\eta$ . Due to the Hausdorff metric on convex and compact sets, there holds

$$H(\mathcal{D}_{s_{1,i}}^{i}, \mathcal{D}_{s_{1,i}+1}^{i}) \le H(\mathcal{D}_{s_{1,i}}^{i}, \mathcal{D}_{s_{2,i}}^{i}) \le \eta.$$

By Lemma 4, when  $H(\mathcal{D}_{s_{1,i}}^i, \mathcal{D}_{s_{1,i}+1}^i) \leq \eta$ , the *l*th row of  $B^{1,i}$  and  $B^{(1,i)+1}$  satisfy

$$\left\| B_l^{1,i} - B_{j(l)}^{(1,i)+1} \right\| \to 0, \text{ as } \eta \to 0.$$

Correspondingly,  $\|b^{1,i} - b^{(1,i)+1}\| \to 0$  as  $\eta \to 0$ . Then  $\mathcal{D}_{s_1} \to \mathcal{D}_{s_1+N}$  as  $\eta \to 0$ . Since  $\mathrm{SOL}(\mathcal{D}_s, F(\boldsymbol{x}))$  exists as an isolated solution, by [18, Proposition 5.4.1],

$$\operatorname{SOL}(\boldsymbol{\mathcal{D}}_{s_1}, F(\boldsymbol{x})) \to \operatorname{SOL}(\boldsymbol{\mathcal{D}}_{s_1+N}, F(\boldsymbol{x})), \quad \operatorname{as} \eta \to 0.$$

Therefore, for any  $\epsilon > 0$ , there exists  $\eta > 0$  such that if  $H(\mathcal{D}_{s_{1,i}}^{i}, \mathcal{D}_{s_{1,i}+1}^{i}) < \eta$ , then

$$\begin{aligned} & \|\boldsymbol{x}^{*}\left(\boldsymbol{\mathcal{D}}_{s_{1}}\right) - \boldsymbol{x}^{*}\left(\boldsymbol{\mathcal{D}}_{s_{1}+N}\right)\| \\ &= \|\operatorname{SOL}\left(\boldsymbol{\mathcal{D}}_{s_{1}}, F(\boldsymbol{x})\right) - \operatorname{SOL}\left(\boldsymbol{\mathcal{D}}_{s_{1}+N}, F(\boldsymbol{x})\right)| \\ &\leq \epsilon. \end{aligned}$$

Similarly,

$$egin{aligned} &\|m{x}^*\left(m{\mathcal{D}}_{s_1}
ight) - m{x}^*\left(m{\mathcal{D}}_{s_2}
ight)\|\ &\leq &\|m{x}^*\left(m{\mathcal{D}}_{s_1}
ight) - m{x}^*\left(m{\mathcal{D}}_{s_1+N}
ight)\|\ &+ \cdots + \|m{x}^*\left(m{\mathcal{D}}_{s_1+s_2-N}
ight) - m{x}^*\left(m{\mathcal{D}}_{s_1+s_2}
ight)\|\ &+ \|m{x}^*\left(m{\mathcal{D}}_{s_2}
ight) - m{x}^*\left(m{\mathcal{D}}_{s_2+N}
ight)\|\ &+ \cdots + \|m{x}^*\left(m{\mathcal{D}}_{s_1+s_2-N}
ight) - m{x}^*\left(m{\mathcal{D}}_{s_1+s_2}
ight)\|\ &\leq s\epsilon, \end{aligned}$$

which means that  $x^*(\mathcal{D}_s)$  is continuous in  $\mathcal{D}_s$  under the Hausdorff metric. Moreover, by  $\lim_{s\to\infty} H(\mathcal{D}_{s_1}, \Omega) = 0$  in Lemma 5, we have

$$\lim_{s
ightarrow\infty}oldsymbol{x}^{st}\left(oldsymbol{\mathcal{D}}_{s}
ight)=oldsymbol{x}^{st}(oldsymbol{\Omega}),$$

where  $x^*(\Omega)$  is the Nash equilibrium of game (1).

Finally, we analyze the difference between  $J_i(\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_s))$  and  $J_i(\boldsymbol{x}'_i, \boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{D}}_s))$ , where the *i*th player's equilibrium strategy is  $x_i^*(\boldsymbol{\mathcal{D}}_s)$  with respect to  $\boldsymbol{\mathcal{D}}_s$  and  $x'_i$  is arbitrarily chosen from  $\Omega_i$ , while other players' strategies remain the same  $\boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{D}}_s)$ . When  $H\left(D_{s_i}^i, \Omega_i\right) \leq \eta$  for  $i \in \mathcal{I}$ ,

$$\begin{aligned} &J_{i}\left(\boldsymbol{x}^{*}\left(\boldsymbol{\mathcal{D}}_{s}\right)\right) - J_{i}\left(\boldsymbol{x}_{i}', \boldsymbol{x}_{-i}^{*}\left(\boldsymbol{\mathcal{D}}_{s}\right)\right) \\ &\leq \left\|J_{i}\left(\boldsymbol{x}_{i}', \boldsymbol{x}_{-i}^{*}\left(\boldsymbol{\Omega}\right)\right) - J_{i}\left(\boldsymbol{x}_{i}', \boldsymbol{x}_{-i}^{*}\left(\boldsymbol{\mathcal{D}}_{s}\right)\right)\right\| \\ &+ \left\|J_{i}\left(\boldsymbol{x}^{*}\left(\boldsymbol{\mathcal{D}}_{s}\right)\right) - J_{i}\left(\boldsymbol{x}_{i}', \boldsymbol{x}_{-i}^{*}\left(\boldsymbol{\Omega}\right)\right)\right\| \\ &+ J_{i}\left(\boldsymbol{x}^{*}\left(\boldsymbol{\Omega}\right)\right) - J_{i}\left(\boldsymbol{x}_{i}', \boldsymbol{x}_{-i}^{*}\left(\boldsymbol{\Omega}\right)\right). \\ &\leq \varsigma_{i} \left\|\boldsymbol{x}^{*}\left(\boldsymbol{\mathcal{D}}_{s}\right) - \boldsymbol{x}^{*}\left(\boldsymbol{\Omega}\right)\right\| + \varsigma_{i} \left\|\boldsymbol{x}_{-i}^{*}\left(\boldsymbol{\Omega}\right) - \boldsymbol{x}_{-i}^{*}\left(\boldsymbol{\mathcal{D}}_{s}\right)\right\| + 0 \\ &\leq 2\varsigma_{i}\epsilon, \end{aligned}$$

where  $\varsigma_i$  is the Lipschitz constant of  $J_i$ . This completes the proof.

Theorem 1, based on convex set geometry and metric spaces, transforms the considered game into a variational problem. The accuracy of  $\epsilon$ -NE is influenced by several factors, specifically, the vertices number of the approximate inscribed polyhedrons, the Lipschitz constants of payoff functions  $J_i(x)$ for  $i \in \mathcal{I}$ , and geometric structures of convex sets  $\Omega_i$  (referring to the constant  $C_{\Omega}$  with  $\Omega = \Omega_i$  in Lemma 5). Obviously, when constructing polyhedrons with more vertices, we obtain more hyperplanes enclosed the polyhedrons (more rows of matrix  $B^i$  and vectors  $b^i$ ), which results in lower  $H\left(\mathcal{D}_{s_i}^i, \Omega_i\right)$ (referring to  $\mathcal{D}_s = \mathcal{D}_{s_i}^i$  and  $\Omega = \Omega_i$  in Lemma 5) and higher accuracy of  $\epsilon$ . This conforms with the intuition.

Actually, there have been methods on how to construct a proper inscribed polyhedron such that its vertices or faces are approximate to the convex set in the best way. In other words, the Hausdorff metric between the convex set and the inscribed polyhedron can satisfy Lemma 5. Briefly, we introduce some methods for constructing an approximation polyhedron.

When the vertices or faces are constructed successively, we can design iterative algorithms to find the best inscribed polyhedron. The main idea of iterative algorithms is to construct a polyhedron  $\mathcal{D}^{k+1} = \operatorname{conv} \left( \mathcal{D}^k \cup \{w_{k+1}\} \right)$  every iteration, where  $w_{k+1}$  is a point from  $\partial \Omega$  (i.e., the boundary of  $\Omega$ ).

One of the methods of constructing point  $w_{k+1}$  is described as follows. For  $u \in \mathbb{R}^n$ , denote  $g_{\Omega}(u) = \max\{\langle u, x \rangle : x \in \Omega\}$ as the support function of  $\Omega$  on the unit sphere of directions  $S^{n-1} = \{u \in \mathbb{R}^n : ||u|| = 1\}$ . The additional point  $w_{k+1} \in$  $\partial \Omega$  belongs to the support plane parallel to the hyperplane in  $\mathcal{D}^k$ , for which the quantity  $g_{\Omega}(u) - g_{\mathcal{D}^k}(u)$  attains its maximum on the set of external normals  $u \in S^{n-1}$  to the hyperplanes of  $\mathcal{D}^k$  [32]. Meanwhile, the initial polyhedron could be constructed by the method [33].

Additionally, the efficiency of the algorithm in the class of ellipsoids was described in [34]. For sets with twice differentiable boundaries and positive curvatures, the improved approximation algorithms were proposed in [32], [34]. For sets with nonsmooth boundaries, the convergence velocity of algorithms was obtained in [35].

Since the set constraint of each player is private information to itself, different players can approximate their feasible sets through different construction methods separately, in advance and offline. Therefore, the computational cost and complexity of constructing vertices or faces of inscribed polyhedrons do not affect the computational efficiency of the distributed algorithm essentially.

#### IV. DISTRIBUTED ALGORITHM

In this section, we propose a distributed algorithm for the approximate game (5) and investigate its convergence performance.

In fact, each player has its own choices for approximation, with local objective function  $J_i(x_i, x_{-i})$ , local approximate set constraints  $\mathcal{D}_{s_i}^i$ ,  $B^i$ , and  $b^i$  is private knowledge of player *i*. In multi-agent frameworks, it is considered that player *i* can communicate with its neighbors through a network. Then we propose Algorithm 1 for seeking the  $\epsilon$ -NE.

Let  $\beta_1$ ,  $\beta_2 > 0$  be some constants satisfying

$$0 < \beta_1 < \frac{2\kappa}{c^2},\tag{9a}$$

$$\beta_2 > \frac{2c_2 \cdot c_3(2+\beta_1 \cdot \kappa + 2\beta_1 \cdot c)}{\lambda \left(2\kappa - \beta_1 \cdot c^2\right)},\tag{9b}$$

where  $c \triangleq c_1 + c_2 \cdot c_3$ , and  $\lambda$  is the smallest positive eigenvalue of  $\frac{1}{2}(L + L^T)$  (*L* is the Laplacian matrix). Actually, the information of the eigenvalue  $\lambda$  can be obtained by a distributed method given in [36] in advance. Thus, under Assumption 1, the value of  $\lambda$ , the strongly monotone parameter  $\kappa$ , and Lipschitz constants guarantee that the appropriate values of  $\beta_1$  and  $\beta_2$  can always be obtained.

Since the *i*th player's local feasible set  $\Omega_i$  is approximated by inscribed polyhedron  $\mathcal{D}_{s_i}^i$  offline, the algorithm contains a subproblem for solving a standard quadratic programming problem  $QP(x_i, \zeta_i)$  at each step [37], defined as

$$\min_{y} \| (x_i - \beta_1 U_i (x_i, \zeta_i)) - y \|^2, \text{ s.t. } B^i y \le b^i, \qquad (10)$$

where  $U_i$  was defined in (3),  $B^i y \leq b^i$  is equivalent to  $y \in \mathcal{D}_{s_i}^i$  in (4). Denote SOL-QP $(x_i, \zeta_i)$  as the solution to the QP problem (10). Thus, the distributed approximate algorithm to solve game (5) is designed as follows.

#### Algorithm 1 for each $i \in \mathcal{I}$

Initialization:  $x_i(0), y_i(0) \in \mathcal{D}_{s_i}^i, \ \phi_i(0) = \mathbf{0}_M, \ \zeta_i(0) = q_i(x_i(0)).$ Dynamics renewal:  $\dot{x}_i = y_i - x_i,$   $\dot{\phi}_i = \beta_2 \sum_{j=1}^N a_{ij} (\zeta_j - \zeta_i),$   $\zeta_i = \phi_i + q_i (x_i),$  $y_i = \text{SOL-QP}(x_i, \zeta_i),$ 

where  $a_{ij}$  is the (i, j)th element of the adjacency matrix.

In Algorithm 1, the *i*th player calculates the local decision variable  $x_i \in \mathcal{D}_{s_i}^i$  based on projected gradient play dynamics by solving a  $QP(x_i, \zeta_i)$  problem at each step. The local variable  $\zeta_i$  is to estimate the global aggregation  $\mathcal{Q}(\boldsymbol{x})$ . The design idea is improved based on [1], [23], in which the projection in our algorithm is obtained with quadratic programming, thus improving the computational efficiency.

**Remark 1** Quadratic programming in Algorithm 1 ensures that the projection is solvable in polynomial time, even with a large number of linear inequality constraints, while the general nonlinear programming corresponding to the highdimensional nonlinear constraints cannot guarantee this [38]. For example, the computational cost of the projection on ellipsoid constraints is  $O(n^4)$  [39], whereas it is  $O(n^{2.5})$  on linear constraints caused by approximation [40], especially O(n) if linear constraints are generalized bounded constraints [41]. More details about the computational cost saved by approximation are explained by numerical experiments in Section V.

A compact form of Algorithm 1 can be written as

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{y} - \boldsymbol{x}, & \boldsymbol{x}(0) \in \boldsymbol{\mathcal{D}}_s, \\ \dot{\boldsymbol{\zeta}} = -\beta_2 L \otimes I_M \boldsymbol{\zeta} + \frac{d}{dt} \boldsymbol{q}(\boldsymbol{x}), & \boldsymbol{\zeta}(0) = \boldsymbol{q}(\boldsymbol{x}(0)), \end{cases}$$
(11)

where  $\boldsymbol{\zeta} = col(\zeta_1, \dots, \zeta_N), \ \boldsymbol{q}(\boldsymbol{x}) = col(q_1(x_1), \dots, q_N(x_N)),$  $\boldsymbol{y} = col(y_1, \dots, y_N)$  with  $y_i = \text{SOL-QP}(x_i, \zeta_i)$  basically.

Then we first verify the equivalency between the equilibrium of dynamics (11) and the Nash equilibrium  $x^*(\mathcal{D}_s)$  of (5), whose proof is straightforward by Lemma 1 and Lemma 2.

Lemma 6 Under Assumption 1, the equilibrium of (11) is

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\zeta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^* \left( \boldsymbol{\mathcal{D}}_s \right) \\ \boldsymbol{\zeta}^* \left( \boldsymbol{\mathcal{D}}_s \right) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^* \left( \boldsymbol{\mathcal{D}}_s \right) \\ \boldsymbol{1}_N \otimes \mathcal{Q} \left( \boldsymbol{x}^* \left( \boldsymbol{\mathcal{D}}_s \right) \right) \end{bmatrix}, \quad (12)$$

where  $\boldsymbol{x}^{*}(\boldsymbol{\mathcal{D}}_{s})$  is the NE of approximate game (5).

From Lemma 2, the strong connectivity and weight balance of graph  $\mathcal{G}$  guarantee  $\zeta_1 = \zeta_2 = \cdots = \zeta_N$ , and  $\frac{1}{N} \sum_{i=1}^N \zeta_i = \mathcal{Q}(\boldsymbol{x})$ . Together with Lemma 1, the point given in (12) is the equilibrium of (11). Moreover, by Lemma 6, the convergence of Algorithm 1 is discussed in the following lemma, by easily extending [23, Theorem 2].

**Lemma 7** Under Assumption 1, the algorithm (11) converges at an exponential rate. Moreover, x in (11) exponentially converges to the NE of (5).

Furthermore, from Lemma 6, take

$$\boldsymbol{\sigma} \triangleq \boldsymbol{\zeta} - \mathbf{1}_N \otimes \mathcal{Q}(\boldsymbol{x}).$$

The distributed algorithm (11) of the approximate game (5) can be written via a general distributed projected gradient dynamics, by  $L\mathbf{1}_N = \mathbf{0}_N$  and (11), as follows:

$$\dot{\boldsymbol{x}} = \Pi_{\boldsymbol{\mathcal{D}}_s}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{1}_N \otimes \mathcal{Q}(\boldsymbol{x}) + \boldsymbol{\sigma})) - \boldsymbol{x}, \quad (13a)$$

$$\dot{\boldsymbol{\sigma}} = -\beta_2 L \otimes I_M \boldsymbol{\sigma} + \frac{a}{dt} \left( \boldsymbol{q}(\boldsymbol{x}) - \mathbf{1}_N \otimes \mathcal{Q}(\boldsymbol{x}) \right)$$
(13b)

$$egin{aligned} &= -eta_2 L \otimes I_M oldsymbol{\sigma} + (
abla oldsymbol{q}(oldsymbol{x}) - oldsymbol{1}_N \otimes 
abla oldsymbol{\mathcal{Q}}(oldsymbol{x}))^{ ext{T}} \cdot \ & \left(\Pi_{oldsymbol{\mathcal{D}}_s}\left(oldsymbol{x} - eta_1 U\left(oldsymbol{x}, oldsymbol{1}_N \otimes oldsymbol{\mathcal{Q}}(oldsymbol{x}) + oldsymbol{\sigma})
ight) - oldsymbol{x}
ight), \end{aligned}$$

where  $\boldsymbol{x}(0) \in \boldsymbol{\mathcal{D}}_s$  and  $\boldsymbol{\sigma}(0) = \boldsymbol{q}(\boldsymbol{x}(0)) - \boldsymbol{1}_N \otimes \boldsymbol{\mathcal{Q}}(\boldsymbol{x}(0)).$ 

Analogously, the distributed algorithm for original game (1) (without any approximation) can be written as

$$\dot{\boldsymbol{x}} = \Pi_{\boldsymbol{\Omega}} (\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{1}_N \otimes \mathcal{Q}(\boldsymbol{x}) + \boldsymbol{\sigma})) - \boldsymbol{x}, \dot{\boldsymbol{\sigma}} = -\beta_2 L \otimes I_M \boldsymbol{\sigma} + (\nabla \boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{1}_N \otimes \nabla \mathcal{Q}(\boldsymbol{x}))^{\mathrm{T}} \cdot (\Pi_{\boldsymbol{\Omega}} (\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{1}_N \otimes \mathcal{Q}(\boldsymbol{x}) + \boldsymbol{\sigma})) - \boldsymbol{x}),$$
(14)

where  $\boldsymbol{x}(0) \in \boldsymbol{\Omega}$  and  $\boldsymbol{\sigma}(0) = \boldsymbol{q}(\boldsymbol{x}(0)) - \boldsymbol{1}_N \otimes \mathcal{Q}(\boldsymbol{x}(0)).$ 

For clarification, let  $\boldsymbol{m} \triangleq -\beta_2 L \otimes I_M \boldsymbol{\sigma}$ ,  $\boldsymbol{\rho} \triangleq \nabla \boldsymbol{q}(\boldsymbol{x}) - \mathbf{1}_N \otimes \nabla \mathcal{Q}(\boldsymbol{x})$  in (13) and (14). Denote  $\boldsymbol{z} = col \{\boldsymbol{x}, \boldsymbol{\sigma}\} \in \mathbb{R}^{nN+MN}$ . Then a compact form of (14) can be written as

$$\dot{\boldsymbol{z}} = G_{\boldsymbol{\Omega}}(\boldsymbol{z}),\tag{15}$$

where

$$G_{\Omega}(\boldsymbol{z}) = \left[ \begin{array}{c} \Pi_{\Omega}(\boldsymbol{x} - \beta_{1}U(\boldsymbol{x}, \boldsymbol{1}_{N} \otimes \mathcal{Q}(\boldsymbol{x}) + \boldsymbol{\sigma})) - \boldsymbol{x} \\ \boldsymbol{m} + \boldsymbol{\rho}^{\mathrm{T}} (\Pi_{\Omega}(\boldsymbol{x} - \beta_{1}U(\boldsymbol{x}, \boldsymbol{1}_{N} \otimes \mathcal{Q}(\boldsymbol{x}) + \boldsymbol{\sigma})) - \boldsymbol{x}) \end{array} \right].$$

In essence, from Lemma 7, the conclusion of exponential convergence is also applicable to (15). According to this property, it follows from the converse theorem for exponentially stable systems [42, Theorem 4.14] that there exists a Lyapunov function  $V_{\Omega}(z)$  of (15) satisfying the following inequalities,

$$a_{1} \|\boldsymbol{z} - \boldsymbol{z}^{*}(\boldsymbol{\Omega})\|^{2} \leq V_{\boldsymbol{\Omega}}(\boldsymbol{z}) \leq a_{2} \|\boldsymbol{z} - \boldsymbol{z}^{*}(\boldsymbol{\Omega})\|^{2},$$
  
$$\dot{V}_{\boldsymbol{\Omega}} \leq -a_{3} \|\boldsymbol{z} - \boldsymbol{z}^{*}(\boldsymbol{\Omega})\|^{2},$$
  
$$\|\frac{\partial V_{\boldsymbol{\Omega}}}{\partial \boldsymbol{z}}\| \leq a_{4} \|\boldsymbol{z} - \boldsymbol{z}^{*}(\boldsymbol{\Omega})\|,$$
  
(16)

where  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  are positive constants, and  $\boldsymbol{z}^*(\boldsymbol{\Omega}) = col \{\boldsymbol{x}^*(\boldsymbol{\Omega}), \boldsymbol{\sigma}^*(\boldsymbol{\Omega})\}\$  is the exponentially stable equilibrium point of system (15).

Moreover, (13) can be rewritten as

$$\dot{\boldsymbol{z}} = G_{\boldsymbol{\mathcal{D}}_s}(\boldsymbol{z}),\tag{17}$$

with

$$G_{\mathcal{D}_s}(\boldsymbol{z}) = \begin{bmatrix} \Pi_{\mathcal{D}_s}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{1}_N \otimes \mathcal{Q}(\boldsymbol{x}) + \boldsymbol{\sigma})) - \boldsymbol{x} \\ \boldsymbol{m} + \boldsymbol{\rho}^{\mathrm{T}} (\Pi_{\mathcal{D}_s}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{1}_N \otimes \mathcal{Q}(\boldsymbol{x}) + \boldsymbol{\sigma})) - \boldsymbol{x}) \end{bmatrix}$$

which can be regarded as a perturbed system of (15). Denote

$$e(\boldsymbol{z}) \triangleq G_{\boldsymbol{\mathcal{D}}_s}(\boldsymbol{z}) - G_{\boldsymbol{\Omega}}(\boldsymbol{z}).$$

Consequently, system (17) can be rewritten as

$$\dot{\boldsymbol{z}} = G_{\boldsymbol{\Omega}}(\boldsymbol{z}) + e(\boldsymbol{z}), \tag{18}$$

with the perturbation term as

$$e(\boldsymbol{z}) = \begin{bmatrix} \Pi_{\boldsymbol{\mathcal{D}}_s}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{\zeta})) - \Pi_{\boldsymbol{\Omega}}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{\zeta})) \\ \boldsymbol{\rho}^{\mathrm{T}} (\Pi_{\boldsymbol{\mathcal{D}}_s}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{\zeta})) - \Pi_{\boldsymbol{\Omega}}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{\zeta}))) \end{bmatrix}.$$

Then we investigate the upper bound of  $\epsilon$ . Note that e(z) reflects the difference in projected dynamics on inscribed polyhedrons  $\mathcal{D}_s$  and original action sets  $\Omega$ , respectively. It is essentially caused by the approximation of game (1). Consider an arbitrary approximate construction based on Hausdorff distances  $H = col(h_1, \dots, h_N)$ , where  $h_i = H(\Omega_i, \mathcal{D}_{s_i}^i)$  represents the Hausdorff distance between the original set  $\Omega_i$  and its inscribed polyhedron  $\mathcal{D}_{s_i}^i$  for  $i \in \mathcal{I}$ . Then the following lemma shows an upper bound of ||e(z)||, whose proof is given in Appendix B.

#### Lemma 8 Under Assumption 1, given the Hausdorff distances

H, we have

$$\|e(\boldsymbol{z})\| \le \delta(\boldsymbol{H})$$
(19)  
=  $(1+c_3) \sqrt{\sum_{i=1}^{N} \left(\frac{2}{\nu_i} \arccos(1-\nu_i h_i) + h_i\right)^2},$ 

where  $c_3$  is the Lipschitz constant of  $q_i$ , and  $\nu_i$  is a constructive curvature related merely to the structure of  $\Omega_i$  for  $i \in \mathcal{I}$ .

From Lemma 8, since  $\nu_i$  is independent of any approximation of  $\Omega_i$ , the bound of e(z) is explicitly affected by Hausdorff distances H. Clearly, a lower metric yields a lower bounds of e(z). Furthermore, the next lemma investigates the influence of e(z) on perturbed system (18), whose proof is shown in Appendix C.

**Lemma 9** Let  $V_{\Omega}(z)$  be a Lyapunov function satisfying (16) in  $\Xi = \{z \in \Omega \times \mathbb{R}^{MN} | ||z - z^*(\Omega)|| < r\}$ , which is a compact set. Suppose

$$\|e(\boldsymbol{z})\| \le \delta(\boldsymbol{H}) < \frac{a_3}{a_4} \sqrt{\frac{a_1}{a_2}} \mu r, \tag{20}$$

for all  $z \in \Xi$  and a positive constant  $\mu < 1$ . Then, for all  $||z(t_0) - z^*(\Omega)|| \le \sqrt{a_1/a_2}r$ , the solution z(t) of the perturbed system (18) satisfies

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^{*}(\boldsymbol{\Omega})\| \leq \sqrt{\frac{a_{2}}{a_{1}}} e^{-\omega(t-t_{0})} \|\boldsymbol{z}(t_{0}) - \boldsymbol{z}^{*}(\boldsymbol{\Omega})\|, \quad (21)$$

for  $t_0 \le t < t_0 + T$ , and

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^*(\boldsymbol{\Omega})\| \le R,\tag{22}$$

for  $t \ge t_0 + T$ , where T is a finite positive scalar,

$$\omega = \frac{(1-\mu)a_3}{2a_2}, \ R = \frac{a_4}{\mu a_3} \sqrt{\frac{a_2}{a_1}} \delta(\mathbf{H}).$$

Lemma 9 explains that if e(z) is small enough, then  $\|\boldsymbol{z}(t) - \boldsymbol{z}^*(\boldsymbol{\Omega})\|$  of (18) is ultimately bounded by a small bound, where  $z^*(\Omega) = col \{x^*(\Omega), \sigma^*(\Omega)\}$  is the exponentially stable equilibrium of the nominal system (15). Moreover, since  $a_1$ ,  $a_2$ ,  $a_3$ ,  $a_4$  and  $\mu$  are constants, the global exponential convergence of (15) guarantees that for any  $\delta(\mathbf{H})$  and  $\|\mathbf{z}(t_0) - \mathbf{z}^*(\mathbf{\Omega})\|$ , we can choose r large enough to satisfy (20) and the initial condition. Therefore, by the exponential convergence of (15), we can analyze the accuracy of  $\epsilon$  based on continuous-time dynamics and bounded stability of perturbed systems. Obviously, from (22),  $\|\boldsymbol{z}^*(\boldsymbol{\mathcal{D}}_s) - \boldsymbol{z}^*(\boldsymbol{\Omega})\|$  and  $\|\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_s) - \boldsymbol{x}^*(\boldsymbol{\Omega})\|$  are bounded, where  $\boldsymbol{z}^*(\boldsymbol{\mathcal{D}}_s) = col \{\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_s), \boldsymbol{\sigma}^*(\boldsymbol{\mathcal{D}}_s)\}$  is the equilibrium of (18). Recalling the definition of  $\epsilon$ -NE, this upper bound of  $\|x^*(\mathcal{D}_s) - x^*(\Omega)\|$  can be regarded as a discrepancy proportional to the upper bound of  $\epsilon$ .

Together with Lemma 8 and Lemma 9, the conclusion about the approximation accuracy is shown in the following theorem.

**Theorem 2** Under Assumption 1,

$$\epsilon \le \frac{2a_4}{a_3} \sqrt{\frac{a_2}{a_1}} \frac{\varsigma_i}{\mu} \delta(\boldsymbol{H}), \tag{23}$$

where the constant  $\mu \in (0, 1)$ ,  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  are positive

constants in (16),  $\varsigma_i$  is the Lipschitz constant of  $J_i$ , and  $\delta(\mathbf{H})$  is defined in (19).

**Proof.** Similar to the last part in the proof of Theorem 1, recalling the definition of  $\epsilon$ -NE, the difference between  $J_i(\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_s))$  and  $J_i(x'_i, \boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{D}}_s))$  satisfies

$$\begin{split} &J_i\left(\boldsymbol{x}^*\left(\boldsymbol{\mathcal{D}}_s\right)\right) - J_i\left(x_i', \boldsymbol{x}_{-i}^*\left(\boldsymbol{\mathcal{D}}_s\right)\right) \\ &\leq \left\|J_i\left(x_i', \boldsymbol{x}_{-i}^*(\boldsymbol{\Omega})\right) - J_i\left(x_i', \boldsymbol{x}_{-i}^*\left(\boldsymbol{\mathcal{D}}_s\right)\right)\right\| \\ &+ \left\|J_i\left(\boldsymbol{x}^*\left(\boldsymbol{\mathcal{D}}_s\right)\right) - J_i\left(\boldsymbol{x}^*(\boldsymbol{\Omega})\right)\right\| \\ &+ J_i\left(\boldsymbol{x}^*(\boldsymbol{\Omega})\right) - J_i\left(x_i', \boldsymbol{x}_{-i}^*(\boldsymbol{\Omega})\right) \\ &\leq_{\varsigma_i}\left\|\boldsymbol{x}^*\left(\boldsymbol{\mathcal{D}}_s\right) - \boldsymbol{x}^*(\boldsymbol{\Omega})\right\| + \varsigma_i\left\|\boldsymbol{x}_{-i}^*(\boldsymbol{\Omega}) - \boldsymbol{x}_{-i}^*\left(\boldsymbol{\mathcal{D}}_s\right)\right\|, \end{split}$$

where the *i*th player's equilibrium strategy is  $x_i^*(\mathcal{D}_s)$  with respect to  $\mathcal{D}_s$  and  $x_i'$  is arbitrarily chosen from  $\Omega_i$ . Meanwhile, other players' strategies remain the same  $x_{-i}^*(\mathcal{D}_s)$ .

Due to Lemma 9, with  $\boldsymbol{z}^*(\boldsymbol{\mathcal{D}}_s) = col \{\boldsymbol{x}^*(\boldsymbol{\mathcal{D}}_s), \boldsymbol{\sigma}^*(\boldsymbol{\mathcal{D}}_s)\}\$ as the equilibrium of (18), it follows from (22) that  $\|\boldsymbol{z}^*(\boldsymbol{\mathcal{D}}_s) - \boldsymbol{z}^*(\boldsymbol{\Omega})\| \leq R$ . Then

$$\left\| oldsymbol{x}^{*}\left( oldsymbol{\mathcal{D}}_{s}
ight) - oldsymbol{x}^{*}(oldsymbol{\Omega}) 
ight\| \leq R.$$

As a result,

$$egin{aligned} &arsigma_{i} \left\| oldsymbol{x}^{*}\left(oldsymbol{\mathcal{D}}_{s}
ight) - oldsymbol{x}^{*}(oldsymbol{\Omega}) 
ight\| + arsigma_{i} \left\| oldsymbol{x}^{*}_{-i}(oldsymbol{\Omega}) - oldsymbol{x}^{*}_{-i}\left(oldsymbol{\mathcal{D}}_{s}
ight) 
ight\| \ &\leq rac{2a_{4}}{a_{3}} \sqrt{rac{a_{2}}{a_{1}}} rac{arsigma_{i}}{\mu} \delta(oldsymbol{H}), \end{aligned}$$

which completes the proof.

**Remark 2** From (23), the upper bound of  $\epsilon$  is proportional to the bound of e(z), which indicates that arbitrarily small perturbations will not cause a significant deviation. Moreover, it can be regarded as the robustness of the nominal system with an exponentially stable equilibrium. Thus, with the help of the analysis in Section III, we show the accuracy of  $\epsilon$  based on bounded stability of perturbed systems, and give an estimation of the upper bound.

#### V. NUMERICAL EXPERIMENTS

We examine the computational efficiency and approximation accuracy of Algorithm 1 on Nash-Cournot games and demand response management models in the following two subsections.

#### A. For approximation accuracy

To illustrate the convergence and approximation, we consider a classical Cournot game played by N = 4 competitive players over a network as in [7] and [23]. For  $i \in \mathcal{I} = \{1, \dots, N\}$ , the action set  $\Omega_i$  is an elliptical region that

$$\Omega_i = \mathbf{E}_{4,3}(0,0) = \left\{ x_i \in \mathbb{R}^2 : \frac{x_{i1}^2}{4^2} + \frac{x_{i2}^2}{3^2} \le 1 \right\}.$$

The payoff function  $f_i(x_i, Q(\boldsymbol{x}))$  is

$$f_i(x_i, \mathcal{Q}(\boldsymbol{x})) = x_i^{\mathrm{T}}(d_i(x_i) - p(\mathcal{Q}(\boldsymbol{x}))), \qquad (24)$$

where  $d_i(x_i) = 0.5(x_i + (13-i)\mathbf{1}_2)$  and  $p = N\mathbf{1}_2 - 0.01\mathcal{Q}(\mathbf{x})$ with  $\mathcal{Q}(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^N x_j$ .



Fig. 1. Trajectories of all players' strategies.



Fig. 2. Trajectories of approximation by different inscribed polyhedrons.

Clearly, the game model satisfies Assumption 1 with constants  $\kappa = 1$ ,  $c_1 = 1.0025$ ,  $c_2 = 0.01$ , and  $c_3 = 1$ . We adopt the following ring graph as the network  $\mathcal{G}$ ,

$$1 \to 2 \to 3 \to 4 \to 1.$$

To render condition (9), assign  $\beta_1 = 0.1$  and  $\beta_2 = 1$ . Also, set tolerance  $t_{tol} = 10^{-3}$  and the terminal criterions

$$\|\dot{\boldsymbol{x}}(t)\| \le t_{tol}, \quad \|\boldsymbol{\zeta}(t)\| \le t_{tol},$$

where  $\dot{\boldsymbol{x}}(t)$  and  $\boldsymbol{\zeta}(t)$  were given in (11).

We present trajectories by approximating  $\mathbf{E}_{4,3}(0,0)$  with inscribed octagons. The trajectories of one dimension of each strategy  $x_i$  are shown in Fig. 1. The strategies of all players converge to their corresponding equilibrium points with an exponential rate, which verifies the correctness of our algorithm. Fig. 2 shows different strategy trajectories of one fixed player with inscribed triangles, rectangles, hexagons, octagons, decagons, and dodecagons to approximate  $\mathbf{E}_{4,3}(0,0)$ , respectively. The vertical axis represents the value of the convergent  $\epsilon$ -NE and the horizontal axis represents the iteration time of Algorithm 1. As can be seen from Fig. 2, equilibria with different polyhedrons get closer to the exact solution with more



Fig. 3. Strategy trajectories of player 1 with different algorithms.

accurate approximations.

Moreover, the numerical values of  $\epsilon$  under different types of approximation are listed in Table 1. Obviously, the value of  $\epsilon$  decreases with the increase of the edges of polyhedrons and the decrease of Hausdorff distances, which is consistent with the approximation results in the previous sections.

#### B. For computational efficiency

Here, we show the computational efficiency of Algorithm 1 on a class of demand response management problems under various network scales and parameter settings.

Consider N electricity users with the demand of energy consumption as in [1], [9]. For  $i \in \mathcal{I} = \{1, ..., N\}$ , the action set  $\Omega_i$  is the energy consumption of the *i*th user and  $f_i(x_i, \mathcal{Q}(\boldsymbol{x}))$  is the cost function in the following form,

$$f_i(x_i, \mathcal{Q}(\boldsymbol{x})) = \iota_i(x_i - \pi_i)^{\mathrm{T}}(x_i - \pi_i) + x_i^{\mathrm{T}} P(\mathcal{Q}(\boldsymbol{x})), \quad (25)$$

where  $\iota_i$  is constant and  $\pi_i$  is the nominal value of energy consumption for  $i = \{1, ..., N\}$ , with  $P(\mathcal{Q}(\boldsymbol{x})) = \omega_i N \mathcal{Q}(\boldsymbol{x}) + p_0$  and

$$\mathcal{Q}(\boldsymbol{x}) = \frac{1}{N} \sum_{j=1}^{N} x_j.$$
(26)

Set N = 10,  $\iota_i = 0.05$ ,  $\omega_i = 0.001$ ,  $p_0 = \mathbf{1}_3$ , and  $\pi_i = 0.5(10 - i)\mathbf{1}_3 \in \mathbb{R}^3$ . Then the action set  $\Omega_i$  of each player is an elliptical region denoted by  $\mathbf{E}_{7.6.5}(0, 0, 0)$ .

Take a ring graph as the communication network  $\mathcal{G}$ ,

 $1 \to 2 \to \dots \to 10 \to 1,$ 

and assign  $\beta_1 = 0.5$  and  $\beta_2 = 2$  to meet the condition (9). Besides, we set tolerance  $t_{tol} = 10^{-3}$ .

Here, we use numerical optimization to directly process the projections on nonlinear constraints  $\mathbf{E}_{7,6,5}(0,0,0)$  for comparison. Fig. 3 shows the different strategy trajectories of one fixed player in dynamics (14) by Algorithm 1 (i.e., with approximation), the algorithm based on sequential quadratic program (Algorithm-SQP) and the algorithm based on the interior point method (Algorithm-IPM) (i.e., without approximation). Algorithm 1 makes projections on the inscribed

TABLE I PERFORMANCE OF DIFFERENT APPROXIMATIONS.

Polyhedrons	Triangle	Rectangle	Hexagon	Octagon	Decagonal	Dodecagonal
Values of $\epsilon$	1.3470	0.8491	0.5187	0.2261	0.1069	0.0473

 TABLE II

 PERFORMANCE OF DIFFERENT ALGORITHMS ON ELLIPSOID CONSTRAINTS

Algorith	ım	Iteration	Time (sec)	
	s = 8	236	0.082	
Algorithm 1	s = 12	248	0.085	
	s = 24	257	0.086	
Algorithm	-SQP	306	0.131	
Algorithm	-IPM	472	0.237	

polyhedrons of  $\mathbf{E}_{7,6,5}(0,0,0)$  with the number of vertice s = 12, while Algorithm-SQP and Algorithm-IPM make projections on  $\mathbf{E}_{7,6,5}(0,0,0)$  directly. In Fig. 3, the vertical axis represents the value of the convergent equilibria, and the horizontal axis represents the real running time in seconds. Clearly, Algorithm 1 converges faster, although it does not converge to the exact equilibrium point. However, from the error shown in Fig. 3, this sacrifice is tolerable.

Moreover, according to Remark 1, the complexity of Algorithm 1 can be roughly characterized as  $O(Nn^{2.5})$ , while is  $O(Nn^4)$  for Algorithm-SQP and Algorithm-IPM on ellipsoid constraints. To further illustrate the computational cost saved by approximation, we report the performance of the three algorithms in Table II. Algorithm 1 is based on different polyhedrons for  $\mathbf{E}_{7,6,5}(0,0,0)$ , where the number of vertices are s = 8, 12, 24 separately. Table 1 lists the average running time of solving the one-stage projection subproblem and the total number of iterations for the computational complexity of these algorithms. It shows that Algorithm 1 has fewer iterations and faster velocity because obtaining a projection on the boundary of linear constraints (to solve a standard quadratic program) is faster than doing that for general constraint sets. Besides, the increase of the number of vertices (i.e., linear constraints) has no significant impact on the computational cost of Algorithm 1.

Note that the complexity is mainly affected by the dimension of decision variables and the number of players. For further comparison, we consider Algorithm 1, Algorithm-SQP, and Algorithm-IPM for  $\epsilon$ -NE (NE) seeking under different network configurations. The payoff functions and the aggregative terms coincide with (25) and (26). Table III reflects the real running time of these algorithms under different dimensions of decision variables. Take n = 4, 10, 20, 50, 100. Here  $\Omega_i$  is a high-dimensional ball  $\mathbf{B}_r(q)$  in the corresponding spaces. On the other hand, Table IV reflects the real running time of these algorithms under directed ring graphs, undirected complete graphs, and Erdős-Rényi (ER) graphs with various network sizes, respectively. Take  $\mathcal{I} = 4, 20, 50, 100$ .  $\Omega_i$  is a corresponding ellipsoid ball in the three-dimensional space. Numerical results in both Table III and Table IV show that Algorithm 1 achieves a faster convergence speed than Algorithm-SQP and Algorithm-IPM. Moreover, with the expansion of the network size and the range of set constraints, our algorithm significantly reduces the computational cost.

## VI. CONCLUSION

A distributed approximate algorithm has been proposed for NE seeking of aggregative games, with the players' actions constrained by local constraint sets and a weight-balanced network digraph. By employing inscribed polyhedrons to approximate players' local feasible sets, the projection operation has been transformed into a standard quadratic program. The equilibrium point of the algorithm has been proved to be the  $\epsilon$ -NE of the original game, and the exponential convergence of the algorithm has been guaranteed. Moreover, an upper bound of the value of  $\epsilon$  has been estimated by analyzing a perturbed system. Finally, the computational efficiency and the approximation accuracy of our algorithm have been illustrated by numerical examples.

# APPENDIX A Proof of Lemma 4

With  $\mathcal{D}_{s_1}^1$  and  $\mathcal{D}_{s_2}^2$  defined in (6) and (7) as two profiles of inscribed polyhedrons of  $\Omega$ , we assume that  $W_0$  consists of the vertices constructing the hyperplanes together with  $w_0$  in  $\mathcal{D}_{s_2}^2$ . Denote  $\mathcal{M}_0$  as a hyperplane constructed by vertices in  $W_0$  without loss of generality. Denote  $\eta = H(\mathcal{D}_{s_1}^1, \mathcal{D}_{s_2}^2)$ , and  $v_0$  as the projection point of  $w_0$  on  $\mathcal{M}_0$ . Since  $\mathcal{D}_{s_2}^2$  is convex and  $w_0$  is on the boundary of the convex set  $\Omega$ ,

$$\eta = dist(\mathcal{D}_{s_1}^1, w_0) = \inf_{v \in \mathcal{M}_0} \|w_0 - v\| = \|w_0 - v_0\|.$$

Denote  $k_0$  as the projection point of  $w_0$  on the relative boundary of  $\mathcal{M}_0$ . Then

$$||w_0 - k_0|| = \inf_{k \in rbd(\mathcal{M}_0)} ||w_0 - k||.$$

Because the normalized vectors  $B_i^1$  (or  $B_i^2$ ) represent the normal vectors of hyperplanes enclosing  $\mathcal{D}_{s_1}^1$  (or  $\mathcal{D}_{s_2}^2$ ), as defined in (8), we only need to investigate the difference between  $B_{p_1}^1$ and the last  $p_2 - p_1 + 1$  rows of  $B^2$ .

Note that the dimension of each hyperplane is n-1. By the definition of the gap metric in [43] and [44], the angle between two hyperplanes uniquely equals to that between their normal vectors. Then there exists a derived angular metric  $\psi$  and a corresponding scalar  $\tau_i \in [0, \pi/2)$  for  $p_1 \le i \le p_2$  such that

$$\sin \tau_i = \psi(B_i^2, B_{p_1}^1) = \frac{\eta}{\|w_0 - k_0\|}$$

Dimensions	n = 4	n = 10	n = 20	n = 50	n = 100
Algorithm 1	0.05	0.11	1.04	2.45	5.14
Algorithm-SQP	0.17	0.36	3.09	8.31	17.78
Algorithm-IPM	0.36	0.64	5.94	10.51	24.83

 TABLE IV

 Real running time of different algorithms over different types of graphs and various network sizes.

Players	Feasible set constraints	Graph types	Real running time (min)		
		Graph types	Algorithm 1	Algorithm-SQP	Algorithm-IPM
N = 4		ER	0.03	0.10	0.18
	$\mathbf{E}_{5,4,3}(0,0,0)$	ring	0.02	0.10	0.23
		complete	0.03	0.09	0.14
N = 20	$\mathbf{E}_{9,8,7}(0,0,0)$	ER	0.17	0.51	1.06
		ring	0.19	0.65	1.20
		complete	0.17	0.51	0.92
N = 50	$\mathbf{E}_{14,13,12}(0,0,0)$	ER	0.86	1.47	2.35
		ring	1.43	3.48	5.26
		complete	1.35	3.45	6.07
N = 100	$\mathbf{E}_{23,22,21}(0,0,0)$	ER	3.26	8.20	15.58
		ring	3.67	8.76	13.23
		complete	3.76	14.69	24.30

On this basis, we investigate the plane containing the vectors  $\overrightarrow{v_0w_0}$  and  $\overrightarrow{k_0w_0}$ . We can always construct a circular arc through the point  $w_0$  on this plane, where it satisfies the following conditions,

- (a) its center falls on the vector containing the points  $w_0$  and  $v_0$ ;
- (b) all its ending points are on the hyperplane M<sub>0</sub>, and in the interior of Ω;
- (c) its diameter is larger than  $\eta$ .

Particularly, when such an arc is constructed, its curvature is also determined and does not change with the decrease of  $\eta$  in the sequel, since it is only dependent on the relative location of point  $w_0$  on the boundary of  $\Omega_i$ . Denote this curvature by  $\gamma_i$ , and one of the ending points of this arc by  $r_0$ . Similarly, by the gap metric in [43] and [44], denote the angle between the vector  $\overline{r_0w_0}$  and the hyperplane  $\mathcal{M}_0$ by  $\alpha_i \in [0, \pi/2)$ . Accordingly, with the angular metric  $\psi$ ,  $\sin \alpha_i = \psi(\overline{r_0w_0}, \mathcal{M}_0)$ , which eventually leads to

$$\cos \alpha_i = \frac{\langle w_0 - r_0, v_0 - r_0 \rangle}{\|w_0 - r_0\| \cdot \|v_0 - r_0\|},$$

space. Furthermore,

$$\tan \alpha_i = \frac{\eta}{\sqrt{(1/\gamma_i)^2 - (1/\gamma_i - \eta)^2}} = \frac{1}{\sqrt{\frac{2}{\eta\gamma_i} - 1}}$$

Since  $\overrightarrow{v_0r_0}$  and  $\overrightarrow{v_0k_0}$  are collinear vectors, with  $||v_0 - r_0|| \le ||v_0 - k_0||$ , there should be  $\tau_i \le \alpha_i$ . Recalling  $\alpha_i \le \tan \alpha_i$ , we have

$$\tau_i \le \alpha_i \le \frac{1}{\sqrt{\frac{2}{\eta\gamma_i} - 1}}.$$
(27)

Additionally, for  $p_1 \leq i \leq p_2$ , there is  $P_i \in SO(n)$  such that  $B_i^2 = B_{p_1}^1 P_i$ . From [45, Theorem 2.21],

$$P_i = I + \tau_i \mathcal{V}_i + o(\tau_i), \quad \mathcal{V}_i \in \mathfrak{g}(SO(n)),$$

where  $\mathcal{V}_i$  is a constant matrix and  $\mathfrak{g}(\cdot)$  represents its Lie algebra. Since  $B^1$  and  $B^2$  are normalized rows,

$$\left\| B_{i}^{2} - B_{p_{1}}^{1} \right\| = \left\| (P_{i} - I) B_{p_{1}}^{1} \right\| = \left\| \tau_{i} \mathcal{V}_{i} + o(\tau_{i}) \right\|.$$

Note that  $\lim_{\eta \to 0} \frac{1}{\sqrt{\frac{2}{\eta \gamma_i} - 1}} = 0$ . Together with (27), we have

$$||B_i^2 - B_{p_1}^1|| = ||\tau_i \mathcal{V}_i + o(\tau_i)|| \to 0, \quad \text{as } \eta \to 0.$$

Thus, the conclusion follows.

where  $v_0$ ,  $r_0$ , and  $w_0$  are certain points in the high-dimensional

# APPENDIX B Proof of Lemma 8

Note that e(z) is related to the difference caused by projection on the inscribed polyhedron  $\mathcal{D}_s$  and the original action set  $\Omega$ . For  $i \in \mathcal{I}$ , denote  $x^1$  and  $x^2$  as two projection points on  $\Omega_i$  and  $\mathcal{D}_{s_i}^i$ , respectively, i.e.,

$$x^{1} = \Pi_{\Omega_{i}}(x_{i} - \beta_{1}U_{i}(x_{i}, \zeta_{i})),$$
  
$$x^{2} = \Pi_{\mathcal{D}_{s_{i}}^{i}}(x_{i} - \beta_{1}U_{i}(x_{i}, \zeta_{i})).$$

Then

 $\|x^{1} - x^{2}\| = \|\Pi_{\Omega_{i}}(x_{i} - \beta_{1}U_{i}(x_{i}, \zeta_{i})) - \Pi_{\mathcal{D}_{s_{i}}^{i}}(x_{i} - \beta_{1}U_{i}(x_{i}, \zeta_{i}))\|.$ 

Recalling the definition of the inscribed polyhedron,  $x^2$  is the point projected onto a hyperplane basically, where we can construct a vector perpendicular to this hyperplane and passing through  $x^2$ . Denote the intersection point between this vector and the boundary of  $\Omega_i$  by  $x^3$ . Eventually,  $x^1$ ,  $x^2$ , and  $x^3$  form a triangle. Therefore,

$$||x^{1} - x^{2}|| < ||x^{1} - x^{3}|| + ||x^{2} - x^{3}||.$$

Recall the definition of H, for the *i*th player,

$$h_{i} = \|x_{\diamond} - y_{\diamond}\|$$
  
=  $\max \left\{ \sup_{x \in \mathcal{D}_{s_{i}}^{i}} \operatorname{dist}(x, \Omega_{i}), \sup_{y \in \Omega_{i}} \operatorname{dist}(y, \mathcal{D}_{s_{i}}^{i}) \right\}.$ 

Accordingly, we investigate a plane containing the vector  $\overrightarrow{x \diamond y} \diamond$ . We try to find a curvature related with  $\Omega_i$  and then construct a piece of a circular arc with this curvature on this plane. Similar to Lemma 4, the constructed arc needs to satisfy

- (a) it passes through the point y<sub>◊</sub> and its center falls on the vector containing the points of x<sub>◊</sub> and y<sub>◊</sub>;
- (b) its ending points are on the hyperplane perpendicular to the vector  $\overrightarrow{x_{\diamond}y_{\diamond}}$ ;
- (c) its ending points are outside  $\Omega_i$ .

Since the found curvature is related with  $\Omega_i$  rather than any approximation information, its value can be regarded as a constant. It is obvious that there always exists such an arc. Denote this curvature by  $\nu_i$ , i.e., the circular radius by  $1/\nu_i$ , the center point of the arc by  $c_0$ , the angle corresponding to the arc by  $\theta_i$ , an ending point of the arc by  $d_0$ , and the length of the arc by  $l_i$ . Clearly,  $l_i = \theta_i/\nu_i$ . Then, recalling the gap metric and the derived angular metric in [43] and [44], we have

$$\theta_i = 2 \arccos \frac{\|c_0 - x_\diamond\|}{\|d_0 - c_0\|} = 2 \arccos(1 - \nu_i h_i).$$

Moreover,  $||x^2 - x^3||$  can be bounded by the Hausdorff distance  $h_i$  and  $||x^1 - x^3||$  can be bounded by the length of arc  $l_i$  intuitively. Consequently, the bound of the difference between the projection dynamics on  $\mathcal{D}_{s_i}^i$  and  $\Omega_i$  of the *i*th player can be expressed by

$$\begin{aligned} \|\Pi_{\Omega_i}(x_i - \beta_1 U_i(x_i, \zeta_i)) - \Pi_{\mathcal{D}_{s_i}^i}(x_i - \beta_1 U_i(x_i, \zeta_i))\| \\ &\leq l_i + h_i \\ &= \frac{2}{\nu_i} \arccos(1 - \nu_i h_i) + h_i. \end{aligned}$$

The analysis of other players is similar to that of player *i*. To sum up, ||e(z)|| can be bounded by  $\delta(H)$ , that is,

$$\begin{aligned} \|e(\boldsymbol{z})\| &\leq (1 + \|\nabla \boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{1}_N \otimes \nabla \mathcal{Q}(\boldsymbol{x})\|) \cdot \\ \|\Pi_{\boldsymbol{\mathcal{D}}_s}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{\zeta})) - \Pi_{\boldsymbol{\Omega}}(\boldsymbol{x} - \beta_1 U(\boldsymbol{x}, \boldsymbol{\zeta}))\| \\ &= (1 + c_3) \sqrt{\sum_{i=1}^N \left(\frac{2}{\nu_i} \arccos(1 - \nu_i h_i) + h_i\right)^2} \\ &\triangleq \delta(\boldsymbol{H}), \end{aligned}$$

where  $c_3$  is a Lipschitz constant of  $q_i$  for  $i \in \mathcal{I}$ . This yields the conclusion.

# APPENDIX C Proof of Lemma 9

Take  $V_{\Omega}(z)$  as a Lyapunov function of (15) that satisfies (16). Then the derivative of  $V_{\Omega}(z)$  along the trajectories of (18) satisfies

$$egin{aligned} \dot{V}_{\mathbf{\Omega}}(oldsymbol{z}) &\leq -a_3 \|oldsymbol{z} - oldsymbol{z}^*\|^2 + \left\|rac{\partial V_{\mathbf{\Omega}}}{\partial oldsymbol{z}}
ight\| \|e(oldsymbol{z})\| \ &\leq -a_3 \|oldsymbol{z} - oldsymbol{z}^*\|^2 + a_4 \delta \|oldsymbol{z} - oldsymbol{z}^*\|, \end{aligned}$$

where  $\delta = \delta(\boldsymbol{H}), \boldsymbol{z}^* = \boldsymbol{z}^*(\boldsymbol{\Omega})$ . For a positive constant  $\mu < 1$ and  $\|\boldsymbol{z} - \boldsymbol{z}^*\| \ge \delta a_4/\mu a_3$ , it satisfies

$$\begin{split} \dot{V}_{\Omega}(\boldsymbol{z}) &\leq -(1-\mu)a_3 \|\boldsymbol{z} - \boldsymbol{z}^*\|^2 - \mu a_3 \|\boldsymbol{z} - \boldsymbol{z}^*\|^2 \\ &+ a_4 \delta \|\boldsymbol{z} - \boldsymbol{z}^*\| \\ &\leq -(1-\mu)a_3 \|\boldsymbol{z} - \boldsymbol{z}^*\|^2. \end{split}$$

Denote  $K = \delta a_4/\mu a_3$ . Once  $V_{\Omega} \ge a_2 K^2$ ,  $\|\boldsymbol{z} - \boldsymbol{z}^*\| \ge K$  and  $\dot{V}_{\Omega} \le -(1-\mu)a_3/a_2 V_{\Omega}$ , which implies

$$V_{\mathbf{\Omega}}(\boldsymbol{z}) \leq e^{-(1-\mu)a_3/a_2(t-t_0)}V_{\mathbf{\Omega}}(\boldsymbol{z}(t_0)).$$

Hence,

$$\begin{aligned} \|\boldsymbol{z}(t) - \boldsymbol{z}^*\| &\leq \left(\frac{V_{\boldsymbol{\Omega}}(\boldsymbol{z})}{a_1}\right)^{1/2} \\ &\leq \left(\frac{1}{a_1} e^{-(1-\mu)a_3/a_2(t-t_0)} V_{\boldsymbol{\Omega}}\left(\boldsymbol{z}(t_0)\right)\right)^{1/2} \\ &= \sqrt{\frac{a_2}{a_1}} e^{-\omega(t-t_0)} \|\boldsymbol{z}(t_0)\|, \end{aligned}$$

which holds over the interval  $[t_0, t_0 + T)$  when  $V_{\Omega} \ge a_2 K^2$ . For  $t \ge t_0 + T$ , we have

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^*\| \leq \sqrt{\frac{V_{\boldsymbol{\Omega}}(\boldsymbol{z})}{a_1}} \leq \sqrt{\frac{a_2}{a_1}}K = R.$$

This yields the conclusion.

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