Fixed-Time Cooperative Tracking Control for Double-Integrator Multi-Agent Systems: A Time-Based Generator Approach

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Abstract—In this paper, both the fixed-time distributed consensus tracking and the fixed-time distributed average tracking problems for double-integrator-type multi-agent systems with bounded input disturbances are studied, respectively. Firstly, a new practical robust fixed-time sliding mode control method based on the time-based generator is proposed. Secondly, a fixed-time distributed consensus tracking observer for doubleintegrator-type multi-agent systems is designed to estimate the state disagreements between the leader and the followers under undirected and directed communication, respectively. Thirdly, a fixed-time distributed average tracking observer for doubleintegrator-type multi-agent systems is designed to measure the average value of reference signals under undirected communication. Note that both the observers for the distributed consensus tracking and the distributed average tracking are devised based on time-based generators and can be extended to that of highorder multi-agent systems trivially. Furthermore, by combing the fixed-time sliding mode control with the fixed-time observers, the fixed-time controllers are designed to solve the distributed consensus tracking and the distributed average tracking problems. Finally, a few numerical simulations are shown to verify the

Index Terms—Fixed-time, sliding mode control, time-based generator, distributed observer, consensus tracking, distributed average tracking.

I. INTRODUCTION

Distributed cooperation control has been a popular scientific research issue over the past decades owing to its significant value in reality such as distributed optimization [1], [2], tracking control [3]–[5], flocking and containment control [6]–[8].

In the distributed cooperation control of a flock of agents with local interactions, a premier task is to design an algorithm which makes each agent achieve consensus in position, velocity and so on. The consensus algorithm for single-integrator multi-agent systems was first developed in [9], and then some sufficient and necessary conditions for the consensus of double-integrator multi-agent systems were generalized in [10]. Distributed tracking control can be regarded

as an extension of generalized consensus control, in which the followers not only have to reach consensus, but also to follow with the specified trajectory. For example, in the distributed consensus tracking and distributed average tracking, the target trajectories are the states of the leader and the average value of multiple reference signals, respectively. However, in distributed algorithms, only a few or none of the agents can acquire the target information directly. Therefore, a frequently-used method to measure the target information is to establish a distributed observer. An observer-based algorithm for nonlinear agents was proposed in [11], [12] to achieve distributed consensus tracking. In [13]-[15], the distributed observers were designed to measure the average value of reference signals. However, these protocols are asymptotically stable, which implies that the upper-bounded convergence time is not guaranteed. So as to estimate the precise upper-bounded convergence time, the finite-time observers relying on initial conditions were proposed in [16]–[18]. Unfortunately, in some engineering practices, the initial states are not available or the convergent rate has to be faster. Therefore developing the fast converging algorithms without dependence on initial states is quite necessary. The fixed-time stability strategy was first investigated in [19], in which the prerequisite of initial conditions was eliminated. Some novel fixed-time algorithms for single-integrator multi-agent systems were developed in [20]. A fixed-time observer for double-integrator-type multi-agent systems to estimate the states of the leader was designed in [21] under undirected communication topology. Then A fixedtime directed edition for high-order integrator-type multi-agent systems was developed in [22]; Although, after the system states converging into the unit circle in the last step it is asymptotically stable, this method provides some inspirations for the protocol in this paper.

After estimating the task trajectory in a fixed-time, the next step is to devise the fixed-time controller for the agent to track the target trajectory. For double-integrator-type systems, sliding mode control is a type of classic nonlinear control protocol [23], which has the advantages of fast response, parameter change, insensitive to disturbance and simple physical implementation. Some finite-time sliding mode control methods were proposed in [24], [25]. However, the process of extending it to that of double-integrator-type systems is nontrivial due to the singular problem. An attempt to design the fixed-time sliding mode control protocol was made in [26] by utilizing a sinusoid function to offset the singularity in the neighborhood of zero, but it leads to a little uncertainty of the

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convergence time. The price of reducing the uncertainty is a sharp rise of the control input.

Besides the conventional fixed-time protocol that usually use two feedback terms, another popular technique in terminal control is the time-based generator technique, which was induced in [27] to induce the attraction of force fields. In [28], an finite-time sliding mode surface is designed, but it can't accurately track the trajectory and has no robustness. In [29], a predefined-time control method for single-input single-output controllable linear systems was proposed. A novel fixed-time consensus strategy for single-integrator multi-agent systems was developed in [30]. Furthermore, it proved that the fixed-time protocol based on the time-based generators had a less magnitude of control inputs.

As for the time synchronization between different agents, the clock synchronization device has been proposed in the existing paper [31], [32] to ensure the time synchronization. Therefore, it is not repeated here.

Motivated by the above results, by utilizing the time-based generator technique, five main contributions are made in this paper. Firstly, a new fixed-time nonsingular sliding mode control method is developed, which can precisely predesign the upper-bounded convergence time without dependence on initial states and has a less magnitude of control inputs compared with the conventional ones [21], [26]. Secondly, a new fixed-time distributed observer under undirected topology is proposed to evaluate the state disagreements between the leader and the followers. Thirdly, inspired by [22], the observer for undirected communication systems is extended to the systems with directed communication, but what's different is that the observer in this paper is a fully fixed-time protocol with the precise upper-bounded convergence time. Note that, all the observers proposed in this paper can be extended to that of high-order multi-agent systems trivially. Moreover, by combining the sliding mode control protocol with the distributed consensus tracking observers, two fixed-time controllers are developed which successfully extended the fixedtime distributed consensus tracking algorithms based on timebased generators for single-integrator-type multi-agent systems in [30] to the double ones. More importantly, the disturbance is considered in this paper, which is of great significance in practice. Finally, a controller is given to solve the fixed-time distributed average tracking problems for double-integratortype multi-agent systems. As far as I am concerned, there is no other fixed-time distributed average tracking algorithm for double-integrator-type multi-agent systems.

The rest of this paper is given as below. In section II, some mathematical preliminaries were given. In section III, the fixed-time sliding mode control protocol is investigated. Next, the observers for distributed consensus tracking under both undirected and directed graph are designed. Then the observer for distributed average tracking under undirected graph is designed. Furthermore, the distributed consensus tracking and the distributed average tracking algorithms are given. In section IV, several simulations are given. In section V, a few conclusions are made.

II. MATHEMATICAL PRELIMINARIES

A. Notations

The real number set and the N-dimensional real vector space are denoted by \mathbb{R} and \mathbb{R}^n , respectively. The signum function is represented by $\operatorname{sgn}(\cdot)$ and its vector form can be written as $\operatorname{sgn}(z) = [\operatorname{sgn}(z_1), \operatorname{sgn}(z_2), ..., \operatorname{sgn}(z_n)]^T$, where $z = [z_1, z_2, ..., z_n]^T$. Let $|\cdot|$ stand for the absolute value of a scalar. The vector q-norm can be written as $||z||_q = (|z_1|^q + |z_2|^q + ... + |z_n|^q)^{\frac{1}{q}}$. Let $\lambda_1(Q)$ and $\lambda_2(Q)$ represent the smallest and the second smallest eigenvalues of the matrix Q, respectively.

B. Graph Theory

The communication topology of a group of n+1 agents can be represented by a graph \mathcal{G} . If there is a leader in them, the other n agents can be expressed as a subgraph \mathcal{G}_s . The weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is constructed with a set of nodes $\mathcal{V} = \{v_1, v_2, ..., v_{n+1}\}$ and a set of edges $\mathcal{E} = \{e_1, e_2, ..., e_m\}$. A directed edge from v_i to v_i can be denoted as (v_i, v_i) , which means v_i can receive information from v_i . An directed path from v_i to v_i consists of a sequence of edges in the form of $\mathcal{E}_{ij} = \{(v_i, v_{i+1}), ..., (v_{j-1}, v_j)\}$, which means the information can flow from v_i to v_i . When replacing the directed edges by the undirected, it becomes undirected path and the information flow is bidirectional. It is said to contain a spanning tree if at leat there exists a node which has directed paths to all other nodes. The undirected graph is connected if and only if there at least exists an undirected path between any two notes. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times m}$ denote the adjacency matrix and the incidence matrix of the graph respectively, and $a_{ij} = 1$ if there exists a directed edge from v_j to v_i , else $a_{ij} = 0$. With regard to undirected graphs, $a_{ij} = a_{ji}$. let $O = [o_{ij}] \in \mathbb{R}^{n \times n}$ denote the degree matrix and $o_{ii} = \sum_{j=1}^{n} a_{ij}$, else $p_{ij} = 0$. Then the Laplacian matrix is written as $L \in \mathbb{R}^{n \times n} = O - A$. Set $a_{0i} = 0$ and $a_{i0} = 1$ if the agent i can acquire information from the leader, else $a_{i0} = 0$ and then set $B = \text{diag}\{a_{i0}, a_{i1}, ..., a_{in}\}.$

C. Time-Based Generator

The time-based generator $\xi(t)$ is a kind of time dependent function that can be seen as a termination function. Its general properties can be generalized as follows.

- 1) $\xi(t)$ is a non-decreasing and continuous function.
- 2) With time going by, $\xi(t)$ increases from the initial state $\xi(0) = 0$ to $\xi(t_s) = 1$, and when $t > t_s$, $\xi(t) \equiv 1$, where t_s can be predesigned arbitrarily.
- 3) $\xi(0) = 0$ and when $t \ge t_s$, $\dot{\xi}(t) \equiv 0$.

Remark 1. A typical time-based generator function $\xi(t)$ is presented as follows [30].

$$\begin{cases} \xi(t) = \frac{10}{t_s^6} t^6 - \frac{24}{t_s^5} t^5 + \frac{15}{t_s^4} t^4, \ 0 \le t \le t_s, \\ 1, & t > t_s, \end{cases}$$

Think about the differential equation as below.

$$\dot{z} = -h(t)z, z(0) = z_0, \tag{1}$$

where h(t) is constructed as

$$h(t) = k \frac{\dot{\xi}}{1 - \xi + \delta},\tag{2}$$

where $k \in \mathbb{R}$ and $\delta \in \mathbb{R}$ are two positive constants which satisfy k > 1 and $0 < \delta << 1$.

Solving the differential equation (1) one has

$$z = \left(\frac{1 - \xi + \delta}{1 + \delta}\right)^k z_0. \tag{3}$$

With t growing from 0 to t_s , ξ grows from 0 to 1 smoothly. Therefore, when $t \in [0,t_s)$, z gradually approaches $z_0(\frac{\delta}{1+\delta})^k$. When $t \geq t_s$, the result will remain the same. If let $\delta = 0.001$ and k = 3, at the terminal moment t_s , the solution of (1) will be $z = 10^{-9}z_0$. Thus we can nearly think that z reaches zero at t_s and the initial state z_0 has no effect on the convergence time.

D. Problem Description

1) Fixed-Time Distributed Consensus Tracking: Suppose that there is a double-integrator-type multi-agent system with a leader and n agents. The leader can be represented by

$$\begin{cases} \dot{x}_0 = v_0, \\ \dot{v}_0 = u_0, \end{cases} \tag{4}$$

where $x_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}$ represent the position and velocity of the leader, respectively. $u_0 \in \mathbb{R}$ represents the control input bounded by a positive constant u_{max} .

Then the followers can be modeled by

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = u_i + d_i, \ i = 1, 2, ..., n, \end{cases}$$
(5)

where $x_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ denote the position and the velocity of the agent i, respectively. $u_i \in \mathbb{R}$ and $d_i \in \mathbb{R}$ denote the control input and the uncertainty, where d_i takes the positive constant d_{max} as the boundary.

The objective of fixed-time distributed consensus tracking is to devise the control input only using local information for each follower, which enable the followers to achieve consensus with the leader in a fixed time independent of initial states.

Definition 1. (Fixed-time distributed consensus tracking) For the system described by (4) and (5), with the given observer and control input u_i , it is said to achieve fixed-time distributed consensus tracking if all the followers can achieve consensus with the leader in a fixed-time T_{max} independent of initial conditions, i.e.,

$$\begin{cases}
\lim_{t \to T_{max}} |x_i - x_0| + |v_i - v_0| \le c \\
\lim_{t \to \infty} |x_i - x_0| + |v_i - v_0| = 0,
\end{cases}$$
(6)

where T_{max} can be predesignated arbitrarily independent of initial conditions and c can be limited to the desired level.

2) Fixed-Time Distributed Average Tracking: Consider a double-integrator-type multi-agent system with n agents represented by (5), and each agent i has a reference signal $r_i \in \mathbb{R}$ described as follows.

$$\begin{cases} \dot{r}_i = f_i, \\ \dot{f}_i = a_i^r, \ i = 1, 2, ..., n, \end{cases}$$
 (7)

where f_i and a_i^r are the velocity and acceleration of reference signal r_i , respectively. Note that a_i^r is bounded by a positive constant a_{max} . Let $\overline{r} = \frac{1}{n} \sum_{i=1}^n r_i$, $\overline{f} = \frac{1}{n} \sum_{i=1}^n f_i$ and $\overline{a} = \frac{1}{n} \sum_{i=1}^n f_i$ be the average value of the reference signals.

The objective of fixed-time distributed average tracking is to devise control inputs only using local information for the agents, which enable them to achieve consensus with the average value of multiple reference signals in a fixed time without dependence on initial states.

Definition 2. (Fixed-time distributed average tracking) For the system described by (5) and (7), with the given observer and control input u_i , it is said to achieve fixed-time distributed average tracking if all the agents can achieve consensus with the average value of the multiple reference signals in a fixed-time T_{max} which can be predesigned arbitrarily and independent of initial states, i.e.,

$$\begin{cases}
\lim_{t \to T_{max}} |x_i - \overline{r}| + |v_i - \overline{f}| \le c \\
\lim_{t \to \infty} |x_i - \overline{r}| + |v_i - \overline{r}| = 0,
\end{cases}$$
(8)

III. MAIN RESULTS

A. Fixed-Time Sliding Mode Control

Lemma 1. [22] Suppose that $z(0) = z_0$ and V(z) is a positive definite Lyapunov candidate which satisfies the inequality as below.

$$\dot{V}(z) + \mu V^{\nu}(z) < 0,$$
 (9)

where $\mu \geq 0$ and $\nu \in (0,1)$. Then z will converge to zero in a finite time $T(z_0)$ such that

$$T(z_0) \le \frac{1}{\mu(1-\nu)} V^{1-\nu}(z_0).$$
 (10)

A typical double-integrator-type control system is given as follows.

$$\begin{cases} \dot{z_1} = z_2, \\ \dot{z_2} = u + \varrho, \end{cases} \tag{11}$$

where $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}$ are the system states. $\delta \in \mathbb{R}$ is a disturbance bounded by a positive constant ϱ_{max} .

The objective of fixed-time sliding mode control is to devise a control input u which drives the system (11) to the equilibrium point in a fixed time, i.e., $[z_1,z_2]=[0,0]$. The process of fixed-time double-integrator sliding mode control is generally divided into two sections. In the first section, the control input forces the system to arrive at the prescribed surface in a fixed time t_{a1} ; In the second section, the system will slide along the surface to the equilibrium point in a fixed time t_{a2} . Therefore, The whole convergence time is bounded by $T_a = t_{a1} + t_{a2}$. In order to converge in the fixed time in each stage, two time-based generators ξ_{a1} and

 ξ_{a2} are used sequentially. ξ_{a1} ensures the system to arrive at the prescribed surface in t_{a1} and then invalid. ξ_{a2} guarantees the fixed convergence time t_{a2} . Let $h_{a1}(t)=k\frac{\dot{\xi}_{a1}}{1-\xi_{a1}+\delta}$ and $h_{a2}(t)=k\frac{\dot{\xi}_{a2}}{1-\xi_{a2}+\delta}$. Then we have

$$h_1(t) = \begin{cases} h_{a1}(t), t \in [0, t_{a1}), \\ h_{a2}(t), t \in [t_{a1}, t_{a1} + t_{a2}), \\ 0, t \in [t_{a1} + t_{a2}, +\infty). \end{cases}$$
(12)

Remark 2. Since $\dot{\xi}_{a1}(0) = \dot{\xi}_{a1}(t_{a1}) = \dot{\xi}_{a2}(t_{a1}) = \dot{\xi}_{a2}(t_{a1} + t_{a2}) = 0$, one obtains $h_{a1}(0) = h_{a1}(t_{a1}) = h_{a2}(t_{a1}) = h_{a2}(t_{a1} + t_{a2}) = 0$, which shows the connectivity of $h_1(t)$. Furthermore, owing to the nonnegativity of $\dot{\xi}_{a1}$ and $\dot{\xi}_{a2}$, $h_1(t)$ is also nonnegative.

Remark 3. In order to clarify the idea, let ξ_{a1} , $h_{a1}(t)$ and ξ_{a2} , $h_{a2}(t)$ have the same structure. But, in simulation, owing to $\xi_{a1}(t_{a1}) = 1$ and $\xi_{a2}(t_{a1}) = 0$, which leads to a sharp decrease of the derivative and causes problems. By resetting $\hat{\xi}_{a2}(t) = \xi_{a2} + 1$ and $\hat{h}_{a2}(t) = k \frac{\hat{\xi}_{a2}}{2 - \xi_{a2} + \delta}$, the problem caused by discontinuity is solved. Moreover, in the different steps, k and k can be selected as different constants respectively.

In this paper, the fixed-time sliding mode surface is selected as

$$s = (\frac{1}{2}h_1(t) + 1)z_1 + z_2. \tag{13}$$

If s=0, the system arrives at the sliding mode surface and has the form as below.

$$z_2 = \dot{z}_1 = -(\frac{1}{2}h_1(t) + 1)z_1. \tag{14}$$

The control input is devised as follows.

$$u = -\frac{1}{2}\dot{h}_1(t)z_1 - (\frac{1}{2}h_1(t) + 1)z_2 - \frac{1}{2}h_1(t)s - \rho \operatorname{sgn}(s),$$
(15)

where ρ is a positive constant satisfying $\rho \geq |\varrho_{max}| + 1$.

Theorem 1. With the given control input (15), the system (11) will arrive at the sliding mode surface (s = 0) in a fixed-time t_{a1} , and then slide along the surface (s = 0) to the equilibrium point $[z_1, z_2] = [0, 0]$ in a fixed-time t_{a2} . Thus the final upper-bounded convergence time is $T_a = t_{a1} + t_{a2}$.

Proof. The Lyapunov candidate is constructed as $V_1 = \frac{1}{2}s^2$. Differentiating (13) against time one has

$$\dot{s} = \frac{1}{2}\dot{h}_1(t)z_1 + (\frac{1}{2}h_1(t) + 1)z_2 + u + \varrho. \tag{16}$$

Substituting the control input (15) in to (16) one has

$$\dot{s} = -\frac{1}{2}h_1(t)s - \rho \operatorname{sgn}(s) + \varrho. \tag{17}$$

Differentiating the Lyapunov candidate V_1 against time and then substituting (17) into it one has

$$\dot{V}_{1} = s\dot{s}
= -\frac{1}{2}h_{1}(t)s^{2} - \rho|s| + \varrho s
\leq -\frac{1}{2}h_{1}(t)s^{2} - (\rho - \varrho_{max})|s|
\leq -\frac{1}{2}h_{1}(t)s^{2}
= -h_{1}(t)V_{1}.$$
(18)

When $t \in [0, t_{a1}), h_1(t) = h_{a1}(t)$. According to (1) one obtains

$$\lim_{t \to t_{a1}} V_1 \le \left(\frac{1 - \xi_{a1} + \delta}{1 + \delta}\right)^k V_1(0) = \left(\frac{\delta}{1 + \delta}\right)^k V_1(0), \quad (19)$$

where, according to (3), $(\frac{\delta}{1+\delta})^k V_1(0)$ is in the near region of zero.

When $t \ge t_{a1}$, $h_1(t) = h_{a2}(t)$ and one obtains

$$\dot{V}_1 = -\frac{1}{2}h_{a2}(t)s^2 - \rho|s| + \delta s$$

$$\leq -(\rho - \delta_{max})|s|$$

$$\leq -|s|$$

$$= -\sqrt{2V_1} \tag{20}$$

According to Lemma 1 one has that V_1 will converge to zero after t_{a1} within a finite time \hat{t}_{a1} , i.e., $\hat{t}_{a1} \leq \sqrt{2V_1(t_{a1})} \leq \sqrt{2V_1(0)(\frac{\delta}{1+\delta})^k}$. Although V_1 doesn't converge to zero perfectly at t_{a1} , which means that the system states are in the near region of the sliding mode surface (s=0), the system states will still converge to zero along the sliding surface in the fixed-time. In order to clarify the idea clearly, at first, it is assumed that V_1 converges to zero at t_{a1} , which means s=0 as well. Then the case that there is a small error between the sliding surface and the system states at t_{a1} is investigated.

A Lyapunov candidate is constructed as $V_2 = \frac{1}{2}z_1^2$. Differentiating V_2 along (14) one has

$$V_{2} = z_{1}\dot{z}_{1}$$

$$= -\frac{1}{2}h_{1}(t)z_{1}^{2} - z_{1}^{2}$$

$$\leq -\frac{1}{2}h_{1}(t)z_{1}^{2}$$

$$= -h_{1}(t)V_{2}$$
(21)

From (1) one obtains that $\lim_{t\to t_{a1}+t_{a2}}V_2=(\frac{\delta}{1+\delta})^kV_2(t_{a1})$. When $t\geq t_{a1}+t_{a2}$, owing to $h_1(t)=0$ and $\dot{V}_2=-z_1^2=-2V_2$ one concludes that V_2 will converge to zero exponentially. Since $z_2=-(h_1(t)+1)z_1=-z_1$, z_2 will converge to zero with the same rate of z_1 as well.

In the following proof, the influence of $V_1(t_{a1}) \neq 0$ is analysed. According to the relationship between V_1 and s, suppose that the system states converge to the adjacent region of the sliding surface at t_{a1} and there exists an error e, i.e., $s = \{e|t \geq t_{a1}\}$. When $t \geq t_{a1}$, $\dot{V}_1 \leq -|s|$, which means V_1 as well as |e| are non-increasing functions and bounded by

 $\hat{V}_1=(\frac{\delta}{1+\delta})^kV_1(0)$ and $|\hat{e}|=\sqrt{2(\frac{\delta}{1+\delta})^kV_1(0)}$, respectively. Then (13) can be rewritten as

$$e = (\frac{1}{2}h_1(t) + 1)z_1 + z_2.$$
(22)

The derivative of z_1 can be obtained as

$$\dot{z}_1 = -(\frac{1}{2}h_1(t) + 1)z_1 + e. \tag{23}$$

Substituting (23) into \dot{V}_2 one has

$$\dot{V}_2 = -\frac{1}{2}h_1(t)z_1^2 - z_1^2 + ez_1
= -h_1(t)V_2 - z_1^2 + ez_1.$$
(24)

Note that |e| is very small and non-increasing, and the convergence time is bounded by \hat{t}_{a1} . If $|z_1|<|e|$, which means that $|z_1|$ has been in the near region of zero. Meanwhile, $|z_1|$ is bounded by $|\hat{e}|$. After $t_{a1}+\hat{t}_{a1}$, due to |e|=0, $|z_1|$ will at least converge exponentially. If $|z_1|\geq |e|$, one has $V_2\leq -h_1(t)V_2$. From (1), V_2 will nearly converge to zero within a fixed time t_{a2} or converge to $|\hat{e}|$ in \hat{t}_{a1} , Whatever the case may be, V_2 will converge nearly to zero in a fixed time $T_a=t_{a1}+t_{a2}$. Then the proof has been completed.

B. Fixed-Time Distributed Consensus Tracking Observer Under Undirected Communication

Assumption 1. The topology subgraph G_s for the followers is undirected and connected; There at least exists a follower which can acquire information from the leader.

Lemma 2. [21] If $L \in \mathbb{R}^{n \times n}$ is the Laplacian matrix of a undirected connected graph, and the nonnegative diagonal matrix $B = \text{diag}\{a_{10},...,a_{n0}\}$ with at least one element greater than zero, then Q = L + B is a positive definite matrix.

In the subsection, a fixed-time distributed observer based on time-based generators is designed for each follower to measure the relative position and velocity disagreements between the leader and itself under undirected communication. set two time-based generators as ξ_{b1} and ξ_{b2} , and then $h_{b1}(t) = \frac{k\dot{\xi}_{b1}}{1-\xi_{b1}+\delta}$ and $h_{b2}(t) = \frac{k\dot{\xi}_{b2}}{1-\xi_{b2}+\delta}$. Let $h_2(t)$ has the same structure as $h_1(t)$. Denote the real tracking errors as $\tilde{x}_i = x_i - x_0$ and $\tilde{v}_i = v_i - v_0$. Then fixed-time distributed the observers α_i and β_i of estimating \tilde{x}_i and \tilde{v}_i are proposed as below.

$$\dot{\alpha}_{i} = \beta_{i} - b_{1}h_{2}(t) \left\{ \sum_{j=0}^{n} a_{ij} [(\alpha_{i} - \alpha_{j}) - (x_{i} - x_{j})] \right\}$$

$$- b_{2} \operatorname{sgn} \left\{ \sum_{j=0}^{n} a_{ij} [(\alpha_{i} - \alpha_{j}) - (x_{i} - x_{j})] \right\},$$

$$\dot{\beta}_{i} = u_{i} - c_{1}h_{2}(t) \left\{ \sum_{j=0}^{n} a_{ij} [(\beta_{i} - \beta_{j}) - (v_{i} - v_{j})] \right\}$$

$$- c_{2} \operatorname{sgn} \left\{ \sum_{j=0}^{n} a_{ij} [(\beta_{i} - \beta_{j}) - (v_{i} - v_{j})] \right\},$$
 (25)

where $i=1,...,n,\ \alpha_0=0,\ \beta_0=0.\ b_1,\ b_2,\ c_1$ and c_2 are positive constants satisfying $b_1=c_1\geq \frac{1}{2\lambda_1(Q)},\ b_2\geq 1$ and $c_2>u_{max}+d_{max}.$

Let $\tilde{\alpha}_i = \alpha_i - \tilde{x}_i$ and $\tilde{\beta}_i = \beta_i - \tilde{v}_i$ be the errors between the observing disagreements and the real disagreements. If all the errors converge to zero in $t_{b1} + t_{b2}$, the observer is designed successfully.

Theorem 2. With the given dynamics (4), (5) and observer (25), under Assumption 1, α_i and β_i converges to \tilde{x}_i and \tilde{v}_i within a fixed-time $T_b = t_{b1} + t_{b2}$.

Proof. Following from (25), $\dot{\tilde{\alpha}}_i$ and $\dot{\tilde{\beta}}_i$ can be written as

$$\dot{\tilde{\alpha}}_{i} = \tilde{\beta}_{i} - b_{1}h_{2}(t) \sum_{j=0}^{n} a_{ij}(\tilde{\alpha}_{i} - \tilde{\alpha}_{j})$$

$$- b_{2}\operatorname{sgn}\left[\sum_{j=0}^{n} a_{ij}(\tilde{\alpha}_{i} - \tilde{\alpha}_{j})\right],$$

$$\dot{\tilde{\beta}}_{i} = -c_{1}h_{2}(t) \sum_{j=0}^{n} a_{ij}(\tilde{\beta}_{i} - \tilde{\beta}_{j})$$

$$- c_{2}\operatorname{sgn}\left[\sum_{i=0}^{n} a_{ij}(\tilde{\beta}_{i} - \tilde{\beta}_{j})\right] - d_{i} + u_{0}. \tag{26}$$

Let $\tilde{\alpha} = [\tilde{\alpha}_1, ..., \tilde{\alpha}_n]^T$, $\tilde{\beta} = [\tilde{\beta}_1, ..., \tilde{\beta}_n]^T$, $d = [d_1, ..., d_n]^T$ and $u \in \mathbb{R}^n = [u_0, ..., u_0]^T$. The vector form of (26) can be written as

$$\dot{\tilde{\alpha}} = \tilde{\beta} - b_1 h_2(t) Q \tilde{\alpha} - b_2 \operatorname{sgn}(Q \tilde{\alpha}),
\dot{\tilde{\beta}} = -c_1 h_2(t) Q \tilde{\beta} - c_2 \operatorname{sgn}(Q \tilde{\beta}) - d + u,$$
(27)

where Q = L + B is a positive definite matrix according to Lemma 2.

Construct a Lyapunov candidate as $V_3 = \frac{1}{2}\tilde{\beta}^TQ\tilde{\beta}$. Because Q is a positive definite matrix, V_3 is well defined. Differentiate V_3 against time such that

$$\dot{V}_{3} = \tilde{\beta}^{T} Q[-c_{1}h_{2}(t)Q\tilde{\beta} - c_{2}\operatorname{sgn}(Q\tilde{\beta}) - d + u]
\leq -c_{1}h_{2}(t)(Q^{\frac{1}{2}}\tilde{\beta})^{T}Q(Q^{\frac{1}{2}}\tilde{\beta}) - (c_{2} - u_{max} - d_{max})||Q\tilde{\beta}||_{1}
\leq -c_{1}\lambda_{1}(Q)h_{2}(t)\tilde{\beta}^{T}Q\tilde{\beta}
\leq -h_{2}(t)V_{3}.$$
(28)

When $t\in[0,t_{b1})$, $h_2(t)=h_{b1}(t)$. According to (1) one concludes that $\lim_{t\to t_{b1}}V_3\leq(\frac{\delta}{1+\delta})^kV_3(0)<<1$. Then due to

$$V_{3} = \frac{1}{2}\tilde{\beta}^{T}(L+B)\tilde{\beta}$$

$$= \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\tilde{\beta}_{i}-\tilde{\beta}_{j})^{2} + \frac{1}{2}\sum_{i=1}^{n}a_{i0}\tilde{\beta}_{i}^{2}, \quad (29)$$

one has that $\lim_{t\to t_{b1}} |\tilde{\beta}_i| \le 2\sqrt{(\frac{\delta}{1+\delta})^k V_3(0)} << 1$. When $t > t_{b1}$, one has

$$\dot{V}_3 = -(\theta_2 - u_{max} - d_{max})||Q\tilde{\beta}||_1. \tag{30}$$

Owing to

$$||Q\tilde{\beta}||_1 \ge ||Q\tilde{\beta}||_2 = \sqrt{(Q\tilde{\beta})^T Q\tilde{\beta}} \ge \sqrt{\lambda_1(Q)\tilde{\beta}^T Q\tilde{\beta}}, \quad (31)$$

one obtains

$$\dot{V}_3 \le -(\theta_2 - u_{max} - d_{max})\sqrt{2\lambda_1(Q)V_3} \le 0.$$
 (32)

Therefore, V_3 will keep decreasing and $|\tilde{\beta}_i| << 1$ is ensured. Following from Lemma 1, when $t \geq t_{b1}$, V_3 will converge to zero within a finite time \hat{t}_{b1} , i.e., $\hat{t}_{b1} \leq (\theta_2 - u_{max} - d_{max})\sqrt{(\frac{\delta}{1+\delta})^k \frac{2V_3(0)}{\lambda_1(Q)}}$.

Construct a Lyapunov candidate as $V_4 = \frac{1}{2}\tilde{\alpha}^T Q\tilde{\alpha}$. Differentiate it against time and then one has

$$\dot{V}_4 = \tilde{\alpha}^T Q \dot{\tilde{\alpha}}
= \tilde{\alpha}^T Q \tilde{\beta} - b_1 h_2(t) \tilde{\alpha}^T Q Q \tilde{\alpha} - b_2 \tilde{\alpha}^T Q \operatorname{sgn}(Q \tilde{\alpha})
= -b_1 h_2(t) \tilde{\alpha}^T Q Q \tilde{\alpha} - b_2 ||Q \tilde{\alpha}||_1 + (Q \tilde{\alpha})^T \tilde{\beta}.$$
(33)

Since when $t \geq t_{b1}$, $|\tilde{\beta}_i| \ll 1$, one has

$$\dot{V}_{4} \le -b_{1}\lambda_{1}(Q)h_{2}(t)\tilde{\alpha}^{T}Q\tilde{\alpha} - (b_{2}-1)||Q\tilde{\alpha}||_{1}
\le -h_{2}(t)V_{4}.$$
(34)

When $t \in [t_{b1}, t_{b1} + t_{b2})$, $h_2(t) = h_{b2}(t)$. According to (1), one concludes that $\lim_{t \to t_{b1} + t_{b2}} V_4 \le (\frac{\delta}{1+\delta})^k V_4(t_{b1}) \approx 0$. When $t \ge t_{b1} + t_{b2}$, due to $h_2(t) = 0$ and $\tilde{\beta} = 0$, from (33) one has $\dot{V}_4 = -b_2 ||Q\tilde{\alpha}||_1$. Compared with (32), one concludes that V_4 will converge to zero within finite time after $t_{b1} + t_{b2}$. That also means that the observer successfully complete the observing task in a fixed time $T_b = t_{b1} + t_{b2}$. Thus the whole proof has been finished.

C. Fixed-Time Distributed Consensus Tracking Observer Under Directed Communication

Assumption 2. There is a spanning tree in the directed graph \mathcal{G} , where the leader is set as the root node. Note that, the subgraph \mathcal{G}_s doesn't need to be strongly connected or contain a spanning tree.

Lemma 3. [33] Under Assumption 2, define H = L + B, $p = [p_1, ..., p_n]^T = H^{-T}1_n$, $P = \text{diag}\{p_i\}$, $Q = \frac{H^TP + PH}{2}$. Then we have that P and Q are both positive definite.

Let $\overline{d} = |\max_{i} \{ \sum_{j=0}^{n} a_{ij} (d_i - d_j) \}|$ and $p_{max} = \max_{i} \{ p_i \}$, where $d_0 = 0$. Then the observer is given as below.

$$\dot{\alpha}_{i} = \beta_{i} - 2b_{1}[h_{2}(t) + 2] \left\{ \sum_{j=0}^{n} a_{ij}[(\alpha_{i} - \alpha_{j}) - (x_{i} - x_{j})] \right\}$$

$$- b_{2}[h_{2}(t) + 2]\operatorname{sgn} \left\{ \sum_{j=0}^{n} a_{ij}[(\alpha_{i} - \alpha_{j}) - (x_{i} - x_{j})] \right\},$$

$$\dot{\beta}_{i} = u_{i} - 2c_{1}[h_{2}(t) + 2] \left\{ \sum_{j=0}^{n} a_{ij}[(\beta_{i} - \beta_{j}) - (v_{i} - v_{j})] \right\},$$

$$- c_{2}[h_{2}(t) + 2]\operatorname{sgn} \left\{ \sum_{j=0}^{n} a_{ij}[(\beta_{i} - \beta_{j}) - (v_{i} - v_{j})] \right\},$$
(35)

where $\alpha_0 = \beta_0 = 0$, and b_1 , b_2 , c_1 , c_2 are positive constants satisfying $b_1 = c_1 \ge \frac{p_{max}}{4\lambda_1(Q)}$, $b_2 \ge \frac{p_{max}}{\lambda_1(Q)}$, $c_2 \ge \frac{p_{max}(\overline{d} + u_{max})}{\lambda_1(Q)}$

Theorem 3. With the given dynamics (4), (5) and observer (35), under Assumption 2, α_i and β_i converges to \tilde{x}_i and \tilde{v}_i within a fixed-time $T_b = t_{b1} + t_{b2}$.

Proof. Let $\tilde{x}_i = x_i - x_0$, $\tilde{v}_i = v_i - v_0$ and $\tilde{\alpha}_i = \alpha_i - \tilde{x}_i$, $\tilde{\beta}_i = \beta_i - \tilde{v}_i$. Then we have

$$\dot{\tilde{\alpha}}_{i} = \tilde{\beta}_{i} - 2b_{1}[h_{2}(t) + 2] \sum_{j=0}^{n} a_{ij}(\tilde{\alpha}_{i} - \tilde{\alpha}_{j})
- b_{2}[h_{2}(t) + 2] \operatorname{sgn} \left[\sum_{j=0}^{n} a_{ij}(\tilde{\alpha}_{i} - \tilde{\alpha}_{j}) \right],
\dot{\tilde{\beta}}_{i} = -2c_{1}[h_{2}(t) + 2] \sum_{j=0}^{n} a_{ij}(\tilde{\beta}_{i} - \tilde{\beta}_{j})
- c_{2}[h_{2}(t) + 2] \operatorname{sgn} \left[\sum_{j=0}^{n} a_{ij}(\tilde{\beta}_{i} - \tilde{\beta}_{j}) \right] - d_{i} + u_{0}.$$
(36)

Let $z_i = \sum_{j=0}^n a_{ij} (\tilde{\beta}_i - \tilde{\beta}_j)$ and then one obtains

$$\dot{\tilde{\beta}}_i = -2c_1[h_2(t) + 2]z_i - c_2[h_2(t) + 2]\operatorname{sgn}(z_i) - d_i + u_0.$$
(37)

Differentiating z_i against time one has

$$\dot{z}_{i} = -2c_{1}[h_{2}(t) + 2] \sum_{j=0}^{n} a_{ij}(z_{i} - z_{j})
- c_{2}[h_{2}(t) + 2] \left\{ \sum_{j=0}^{n} a_{ij}[\operatorname{sgn}(z_{i}) - \operatorname{sgn}(z_{j})] \right\}
- \sum_{j=0}^{n} a_{ij}(d_{i} - d_{j}) + a_{i0}u_{0}.$$
(38)

According to $z=H\hat{\beta}$, where H is a nonsingular matrix. Thus if z converges to zero, $\tilde{\beta}$ converges as well. Construct a Lyapunov candidate as

$$V_5 = \sum_{i=1}^{n} p_i [c_1 z_i^2 + c_2 |z_i|].$$
 (39)

Then, one has

$$\begin{split} \dot{V}_5 &= \sum_{i=1}^n p_i [2c_1z_i + c_2 \mathrm{sgn}(z_i)] \times \\ & \left\{ -2c_1[h_2(t) + 2] \sum_{j=0}^n a_{ij} (z_i - z_j) \right. \\ & \left. - c_2[h_2(t) + 2] \sum_{j=0}^n a_{ij} [\mathrm{sgn}(z_i) - \mathrm{sgn}(z_j)] \right. \\ & \left. - \sum_{j=0}^n a_{ij} (d_i - d_j) + a_{i0} u_0 \right\} \\ &= - \left[h_2(t) + 2 \right] \sum_{i=1}^n p_i [2c_1z_i + c_2 \mathrm{sgn}(z_i)] \times \\ & \left\{ 2c_1 \sum_{j=0}^n a_{ij} (z_i - z_j) + c_2 \sum_{j=0}^n a_{ij} [\mathrm{sgn}(z_i) - \mathrm{sgn}(z_j)] \right\} \\ & \left. - \sum_{i=1}^n p_i [2c_1z_i + c_2 \mathrm{sgn}(z_i)] \left[\sum_{j=0}^n a_{ij} (d_i - d_j) - a_{i0} u_0 \right] \right. \\ &= - \left[h_2(t) + 2 \right] [2c_1z + c_2 \mathrm{sgn}(z)]^T P H [2c_1z + c_2 \mathrm{sgn}(z)] \end{split}$$

$$-\sum_{i=1}^{n} p_{i}[2c_{1}z_{i} + c_{2}\operatorname{sgn}(z_{i})] \left[\sum_{j=0}^{n} a_{ij}(d_{i} - d_{j}) - a_{i0}u_{0} \right]$$

$$= -[h_{2}(t) + 2][2c_{1}z + c_{2}\operatorname{sgn}(z)]^{T} Q[2c_{1}z + c_{2}\operatorname{sgn}(z)]$$

$$-\sum_{i=1}^{n} p_{i}[2c_{1}z_{i} + c_{2}\operatorname{sgn}(z_{i})] \left[\sum_{j=0}^{n} a_{ij}(d_{i} - d_{j}) - a_{i0}u_{0} \right]$$

$$\leq -\lambda_{1}(Q)[h_{2}(t) + 2][2c_{1}z + c_{2}\operatorname{sgn}(z)]^{T}[2c_{1}z + c_{2}\operatorname{sgn}(z)]$$

$$-\sum_{i=1}^{n} p_{i}[2c_{1}z_{i} + c_{2}\operatorname{sgn}(z_{i})] \left[\sum_{j=0}^{n} a_{ij}(d_{i} - d_{j}) - a_{i0}u_{0} \right]$$

$$= -\lambda_{1}(Q)[h_{2}(t) + 1] \left\{ \sum_{i=1}^{n} [4c_{1}^{2}z_{i}^{2} + 4c_{1}c_{2}|z_{i}| + c_{2}^{2}] \right\}$$

$$-\lambda_{1}(Q) \left\{ \sum_{i=1}^{n} [4c_{1}^{2}z_{i}^{2} + 4c_{1}c_{2}|z_{i}| + c_{2}^{2}] \right\}$$

$$-\sum_{i=1}^{n} p_{i}[2c_{1}z_{i} + c_{2}\operatorname{sgn}(z_{i})] \left[\sum_{j=0}^{n} a_{ij}(d_{i} - d_{j}) - a_{i0}u_{0} \right]$$

$$\leq -4c_{1}\lambda_{1}(Q)[h_{2}(t) + 1] \left\{ \sum_{i=1}^{n} [c_{1}z_{i}^{2} + c_{2}|z_{i}|] \right\}$$

$$-c_{2}\lambda_{1}(Q) \left\{ \sum_{i=1}^{n} [4c_{1}|z_{i}| + c_{2}] \right\}$$

$$+p_{max}(\overline{d} + u_{max}) \left\{ \sum_{i=1}^{n} [2c_{1}|z_{i}| + c_{2}] \right\}$$

$$\leq -\frac{4c_{1}\lambda_{1}(Q)}{p_{max}} [h_{2}(t) + 1] \left\{ \sum_{i=1}^{n} p_{max}[c_{1}z_{i}^{2} + c_{2}|z_{i}|] \right\}$$

$$\leq -h_{2}(t)V_{5} - V_{5}$$

$$\leq -h_{2}(t)V_{5}.$$

$$(40)$$

When $t \in [0,t_{b1})$, $h_2(t) = h_{b1}(t)$. Using differential equation (1) one has $\lim_{t \to t_{b1}} V_5 \le (\frac{\delta}{1+\delta})^k V_5(0) \approx 0$. When $t \ge t_{b1}$, $\dot{V}_5 \le -V_5$. Then V_5 will converge exponentially and $|z_i| << 1$ is ensured. Therefore, the observer can successfully estimate the velocity disagreements between the leader and the followers in t_{b1} .

In the following proof, let $w_i = \sum_{j=0}^n a_{ij} (\tilde{\alpha}_i - \tilde{\alpha}_j)$. Then substituting w_i into (36) one obtains

$$\dot{\tilde{\alpha}}_i = \tilde{\beta}_i - 2b_1[h_2(t) + 2]w_i - b_2[h_2(t) + 2]\operatorname{sgn}(w_i). \tag{41}$$

Differentiating w_i against time one has

$$\dot{w}_i = z_i - 2b_1[h_2(t) + 2] \sum_{j=0}^n a_{ij}(w_i - w_j) - b_2[h_2(t) + 2] \left\{ \sum_{j=0}^n a_{ij}[\operatorname{sgn}(w_i) - \operatorname{sgn}(w_j)] \right\}.$$
(42)

When $t \in [t_{b1}, t_{b1} + t_{b2})$, due to $|z_i| << 1$, z_i in (42) can be seen as a bounded disturbance in (38). The following proof is the same as before and not restated. Until now, the proof of Theorem 3 has been finished.

D. Fixed-Time Distributed Averaging Tracking Observer

Assumption 3. The topology graph for the n agents is undirected and connected. Each agent can only receive the information form one reference signal.

Lemma 4. [34] If $L \in \mathbb{R}^{n \times n}$ is a Laplacian matrix of a connected undirected graph and $D \in \mathbb{R}^{n \times m}$ is its relative incidence matrix. Then for any vector $z \in \mathbb{R}^n$ one has

$$z^T L D \operatorname{sgn}(D^T z) \ge \lambda_2(L) z^T D \operatorname{sgn}(D^T z).$$
 (43)

The distributed observer is given as below.

$$\dot{\alpha}_{i} = -b_{1}h_{2}(t)\sum_{j=1}^{n} a_{ij}(\alpha_{i} - \alpha_{j})$$

$$-b_{2}\sum_{j=1}^{n} a_{ij}\operatorname{sgn}(\alpha_{i} - \alpha_{j}) + \beta_{i},$$

$$\dot{\beta}_{i} = -c_{1}h_{2}(t)\sum_{j=1}^{n} a_{ij}(\beta_{i} - \beta_{j})$$

$$-c_{2}\sum_{j=1}^{n} a_{ij}\operatorname{sgn}(\beta_{i} - \beta_{j}) + a_{i}^{r},$$
(44)

where b_1, b_2, c_1, c_2 are positive constants satisfying $b_1 = c_1 \ge \frac{1}{2\lambda_2(L)}, b_2 \ge 1, c_2 > 2a_{max}$; Moreover the initial states satisfy $\sum_{i=1}^n \alpha_i(0) = \sum_{i=1}^n x_i(0)$ and $\sum_{i=1}^n \beta_i(0) = \sum_{i=1}^n v_i(0)$. Note that $\sum_{i=1}^n \dot{\alpha}_i = \sum_{i=1}^n \beta_i$ and $\sum_{i=1}^n \dot{\beta}_i = \sum_{i=1}^n a_i^r$. Therefore under the given initial states, we have $\sum_{i=1}^n \beta_i = \sum_{i=1}^n \beta$

Note that $\sum_{i=1}^{n} \dot{\alpha}_i = \sum_{i=1}^{n} \beta_i$ and $\sum_{i=1}^{n} \dot{\beta}_i = \sum_{i=1}^{n} a_i^T$. Therefore under the given initial states, we have $\sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} v_i$ and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} x_i$ all the time. Following from this, if all the observers achieve consensus in a fixed time, the average value of the multiple reference signals is obtained successfully.

Theorem 4. With the given dynamics (5), (7) and observer (44), under Assumption 3, α_i and β_i converges to \overline{r} and \overline{f} within a fixed-time $T_b = t_{b1} + t_{b2}$.

Proof. Construct a Lyapunov candidate as $V_6 = \frac{1}{2}\beta^T L\beta$ and (44) can be written in the vector form as

$$\dot{\alpha} = -b_1 h_2(t) L \alpha - b_2 D \operatorname{sgn}(D^T \alpha)$$

$$\dot{\beta} = -c_1 h_2(t) L \beta - c_2 D \operatorname{sgn}(D^T \beta) + a. \tag{45}$$

Then we have

$$\dot{V}_{6} = \beta^{T} L \dot{\beta}
= \beta^{T} L [-c_{1}h_{2}(t)L\beta - c_{2}D\operatorname{sgn}(D^{T}\beta) + a]
\leq -c_{1}h_{2}(t)\beta^{T} LL\beta - c_{2}\beta^{T} LD\operatorname{sgn}(D^{T}\beta) + \beta^{T} La
\leq -c_{1}h_{2}(t)\lambda_{2}(L)\beta^{T} L\beta - c_{2}\beta^{T} D\operatorname{sgn}(D^{T}\beta) + \beta^{T} La
\leq -h_{2}(t)V_{6} - c_{2}||D^{T}\beta||_{1} + (D^{T}\beta)^{T} D^{T}a
\leq -h_{2}(t)V_{6} - (c_{2} - 2a_{max})||D^{T}\beta||_{1}
\leq -h_{2}(t)V_{6}.$$
(46)

When $t \in [0, t_{b1})$, $h_2(t) = h_{b1}(t)$. Using differential equation (1) one has $\lim_{t \to t_{b1}} V_6 \le (\frac{\delta}{1+\delta})^k V_6(0) \approx 0$. When $t \ge t_{b1}$, $\dot{V}_6 \le -(c_2 - 2a_{max})||D^T\beta||_1$. Then compared with (32) one

has V_6 will converge in finite time after t_{b1} and $\beta_i << 1$ is ensured. Construct the Lyapunov candidate as $V_6 = \frac{1}{2}\alpha^T L\alpha$

$$\dot{V}_7 = \alpha^T L \dot{\alpha}
= \alpha^T L [-b_1 h_2(t) L \alpha - c_2 D \operatorname{sgn}(D^T \alpha) + \beta]
= -b_1 h_2(t) \alpha^T L L \alpha - c_2 \alpha^T L D \operatorname{sgn}(D^T \alpha) + \alpha L \beta]
\leq -b_1 \lambda_2(L) h_2(t) \alpha^T L \alpha - c_2 ||D^T \alpha||_1 + (D^T \alpha)^T D^T \beta
\leq -h_2(t) V.$$
(47)

Due to $\beta_i << 1$, $|\beta_i - \beta_j| << 1$ is ensured. Then one has $c_2||D^T\alpha||_1 > (D^T\alpha)^TD^T\beta$. The following is the same as before and hence omitted. Until now, the proof of Theorem 4 has been finished.

Remark 4. All the observers proposed in this paper can be extended to that of high-order multi-agent systems by using more time-based generators and more integrators.

E. Distributed Consensus Tracking and Distributed Average Tracking Control

After designing the observers, the next step is to design the controllers by using the information provided by the observers. In this subsection, the control inputs for the distributed consensus tracking and the distributed average tracking will be given.

Theorem 5. Under dynamics (4), (5) and Assumption 1 (Assumption 2), with the observer (25) (observer (35)) and the control input

$$u_{i} = \begin{cases} = 0, t \in [0, T_{c}), \\ = -\frac{1}{2}\dot{h}_{1}(t)\alpha_{i} - (\frac{1}{2}h_{1}(t) + 1)\beta_{i} \\ -\frac{1}{2}h_{1}(t)s_{i} - \rho \operatorname{sgn}(s_{i}), t \geq T_{b}, \end{cases}$$
(48)

where $T_c \geq T_b$, $s_i = (\frac{1}{2}h_1(t) + 1)\alpha_i + \beta_i$, $\rho \geq d_{max} + u_{max} + 1$, the fixed-time distributed consensus tracking for double-integrator-type multi-agent systems is solved. Further more, the upper-bounded convergence time is $T_a + T_c$.

Proof. When $t \ge T_b$, one has $\alpha_i = x_i - x_0$ and $\beta_i = v_i - v_0$. Then we have

$$\dot{s}_{i} = \frac{1}{2}\dot{h}_{1}(t)\alpha_{i} + (\frac{1}{2}h_{1}(t) + 1)\dot{\alpha}_{i} + \dot{\beta}_{i}
= \frac{1}{2}\dot{h}_{1}(t)\alpha_{i} + (\frac{1}{2}h_{1}(t) + 1)\beta_{i} + \dot{v}_{i} - \dot{v}_{0}
= \frac{1}{2}\dot{h}_{1}(t)\alpha_{i} + (\frac{1}{2}h_{1}(t) + 1)\beta_{i} + u_{i} + d_{i} - u_{0}.$$
(49)

Then substitute (48) into (49), one has

$$\dot{s}_i = -\frac{1}{2}h_1(t)s_i - \rho \operatorname{sgn}(s_i) + d_i - u_0.$$
 (50)

The other part of the proof is the same as Theorem 1 and omitted. Thus the whole proof has been finished. \Box

Theorem 6. Under dynamics (5), (7) and Assumption 3, with the observer (44) and the control input

$$u_i = \begin{cases} = 0, \in [0, T_c), \\ = -\frac{1}{2}\dot{h}_1(t)(x_i - \alpha_i) - (\frac{1}{2}h_1(t) + 1)(v_i - \beta_i) \\ -\frac{1}{2}h_1(t)s_i - \rho \operatorname{sgn}(s_i), t \ge T_b, \end{cases}$$

where $s_i = (\frac{1}{2}h_1(t) + 1)(x_i - \alpha_i) + (v_i - \beta_i)$, $\rho \ge d_{max} + a_{max} + 1$, the fixed-time distributed average tracking problems for double-integrator-type multi-agent systems is solved. Further more, the upper-bounded convergence time is $T_a + T_c$.

Proof. When $t \geq T_b$, one has $\alpha_i = \overline{r}$ and $\beta_i = \overline{f}$. Then we have

$$\dot{s}_i = \frac{1}{2}\dot{h}_1(t)(x_i - \alpha_i) + (\frac{1}{2}h_1(t) + 1)(v_i - \beta_i) + u_i + d_i - \dot{\beta}_i.$$
(52)

Then substitute (51) into (52), one has

$$\dot{s}_i = -\frac{1}{2}h_1(t)s_i - \rho \operatorname{sgn}(s_i) + d_i - \overline{a}. \tag{53}$$

The other part of the proof is the same as Theorem 1 and omitted. Thus the whole proof has been completed. \Box

IV. NUMERICAL SIMULATIONS

Example 1. A simulation for Theorem 1 is given as follows. Set k=2, $\delta=0.01$, $\varrho=\sin(t)$, $\rho=2$, $t_{a1}=t_{a2}=3$, $z_1(0)=200$ and $z_2(0)=100$. The results with upper-bounded convergence time $T_a=6s$ are shown in Fig. 1.

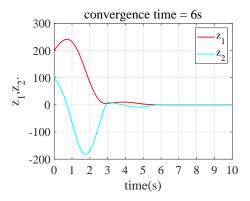


Fig. 1. The results of the sliding mode control in Example 1.

Example 2. A simulation for Theorem 5 under Assumption 1 is given as follows. Consider a multi-agent system described by (4) and (5) with the undirected communication topology in Fig. 2. Set k=2, $\delta=0.01$, $\rho=8$, $b_1=c_1=4$, $b_2=1$, $c_2=8$, $t_{a1}=t_{a2}=3$, $t_{b1}=t_{b2}=1.5$, $T_b=T_c$, $u_0=1+5\sin(t)$ and $x_i(0)$, $v_i(0)$ with random states. The results with upperbounded convergence time $T_a+T_c=9s$ are shown in Fig. 3 and Fig. 4.

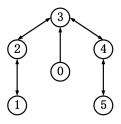


Fig. 2. The communication topology in Example 2.

(51)

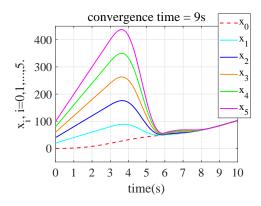


Fig. 3. The positions of the agents in Example 2.

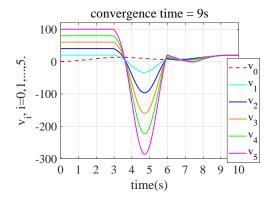


Fig. 4. The velocities of the agents in Example 2.

Example 3. A simulation for Theorem 5 under Assumption 2 is given as follows. Consider a multi-agent system described by (4) and (5) with the directed communication topology in Fig. 5. Set k=2, $\delta=0.01$, $\rho=21$, $b_1=c_1=2$, $b_2=7$, $c_2=34$, $t_{a1}=t_{a2}=2$, $t_{b1}=t_{b2}=1$, $T_b=T_c$, $u_0=2+18\sin(t)$ and $x_i(0)$, $v_i(0)$ with random states. The results with upper-bounded convergence time $T_a+T_c=6s$ are shown in Fig. 6 and Fig. 7.

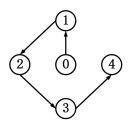


Fig. 5. The communication topology in Example 3.

Example 4. A simulation for Theorem 6 under Assumption 3 is given as follows. Consider a multi-agent system described by (5) and (7) with the undirected communication topology in Fig. 8. Set k=2, $\delta=0.01$, $\rho=63$, $b_1=c_1=0.25$, $b_2=1$, $c_2=123$, $t_{a1}=t_{a2}=4$, $t_{b1}=t_{b2}=2$, $T_b=T_c$, $a_1^r=41+20\sin(5t)$, $a_2^r=51+10\sin(5t)$, $a_3^r=30+30\sin(5t)$, $a_4^r=40+20\sin(5t)$ and $x_i(0)$, $v_i(0)$ with random states. The results with upper-bounded convergence time $T_a+T_c=12s$ are shown in Fig. 9 and Fig. 10.

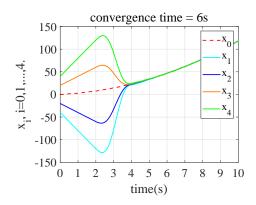


Fig. 6. The positions of the agents in Example 3.

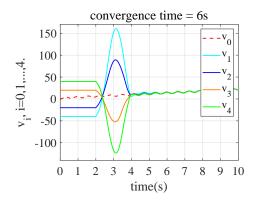


Fig. 7. The velocities of the agents in Example 3.

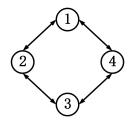


Fig. 8. The communication topology in Example 4.

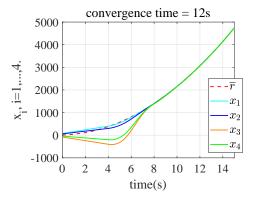


Fig. 9. The positions of the agents in Example 4.

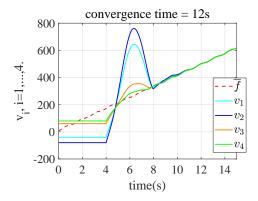


Fig. 10. The velocities of the agents in Example 4.

V. Conclusions

In this paper, both the fixed-time distributed consensus tracking and the fixed-time distributed average tracking problems for double-integrator-type multi-agent systems are solved by using time-based generators. Different from traditional fixed-time methods, the time-based generator approach can directly predesign the fixed time, which is of great significance in reality. But the tradeoff is the introduce of time dependent function. Moreover, it is trivial to extend the fixed-time sliding mode control method in this article to Euler-Lagrange systems. By combining the fixed-time sliding mode control method of Euler-Lagrange systems and the observers in this article, the fixed-time distributed consensus tracking and distributed average tracking for multiple Euler-Lagrange systems can be achieved. Also, the fixed-time distributed consensus tracking problem for single-integrator multi-agent systems under directed graph can be solved by devising a controller similar to the velocity observer in (44).

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