FRIwE: Fuzzy Rule Identification With Exceptions

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***Abstract*—In this paper, the FRIwE method is proposed to identify fuzzy models from examples. Such a method has been developed trying to achieve a double goal:accuracy and interpretability. In order to do that, maximal structure fuzzy rules are firstly obtained based on a method proposed by Castro *et al.* In a second stage, the conflicts generated by the maximal rules are solved, thus increasing the model accuracy. The resolution of conflicts are carried out by including exceptions in the rules. This strategy has been identified by psychologists with the learning mechanism employed by the human being, thus improving the model interpretability. Besides, in order to improve the interpretability even more, several methods are presented based on reducing and merging rules and exceptions in the model. The exhaustive use of the training examples gives the method a special suitability for problems with small training sets or high dimensionality. Finally, the method is applied to an example in order to analyze the achievement of the goals.**

***Index Terms*—Conflicting rules, fuzzy model identification, interpretability, maximal rules, rule simplification.**

I. Introduction

System identification is a discipline dedicated to obtain a model as near a system as possible from a set of examples. These examples establish specific relations between the input and the output of the system.

Out of the different existing techniques, fuzzy model identification [2]–[5] is noted for representing the model by means of a collection of fuzzy rules. This representation technique, whose universal approximator property has been demonstrated by several authors (e.g., see [6] and [7]), also describes linguistically the existing relation between the input and the output of the system, thus taking care of the model interpretability as well [8].

In order to achieve a high level of interpretability, the identification of rules as general as possible must be tried, so that each rule covers the highest number of examples and, this way, the size of the rule base could be diminished. Nevertheless, obtaining those general rules can provoke that conflicting zones arise where rules with different consequents coexist, affecting negatively to the aforementioned interpretability.

In this paper, a strategy is proposed to solve these conflicts from the information available in the examples contained in the conflicting zone. This resolution will be carried out by including exceptions in the rules, which will allow to reduce the number of rules in the model and to increase, this way, its interpretability.

In Section II, the technique to obtain maximal rules to identify the system is expounded. In Section III we introduce a method for the conflict resolution and its representation by means of exceptions. Section IV describes several strategies to increase the model interpretability. Finally, the results obtained from applying the proposed FRIwE method to an example are analyzed in Section V.

II. Learning Maximal Structure Fuzzy Rules

Castro *et al.* present in [1] a strategy to learn multiple-input–single-output systems $(\Omega : X^n \to Y)$ from a set of m examples $E = \{e^1, \ldots, e^m\}$. Each example takes the form $e^i = ([x_1^i, \ldots, x_n^i], y^i)$, where x_j^i is the value of the jth input variable and y^i is the value of the output variable. Although the strategy in [1] has been applied into classification problems, the method can be extrapolated to other identification problems. The identified model is represented by means of a set of maximal rules of the form

$$R^i : \text{if } X_1 \text{ is } A_1^i \text{ and } \ldots \text{ and } X_n \text{ is } A_n^i \text{ then } Y \text{ is } LY^i \quad (1)$$

which, for short, we will sometimes denote as

$$R^i : A_1^i, \ldots, A_n^i \to LY^i.$$

Each A_j^i is a set of labels associated disjunctively with the jth input variable and taken from their respective fuzzy domain $DX_j = \{LX_{j,1}, \ldots, LX_{j,p_j}\}$, and LY^i is the label associated with the output variable and taken from its fuzzy domain $DY = \{LY_1, \ldots, LY_q\}$. The rules described in this form will be called *compound rules*. Furthermore, when using the notation presented in (1) a premise will be omitted if the labels associated with the variable equals its entire fuzzy domain, since this premise will be unnecessary.

The summary of the learning algorithm proposed in [1] is the following.

1. Transform the examples in initial rules.
2. For each initial rule:
    2.1. If the rule does not subsume in any definitive rule:
        2.1.1. For each label in each input variable:
            2.1.1.1. If the amplification of the rule is possible, amplify it.
        2.1.2. Store the amplified rule in the set of definitive rules.

In order to transform the examples into initial rules each value x_j^i and y^i is associated with the label having the highest membership degree out of all contained in the respective fuzzy domain.

Manuscript received October 1, 2001; revised May 30, 2003. This work was supported by Research Project PB98-1379-C02-01.

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Digital Object Identifier 10.1109/TFUZZ.2003.822685

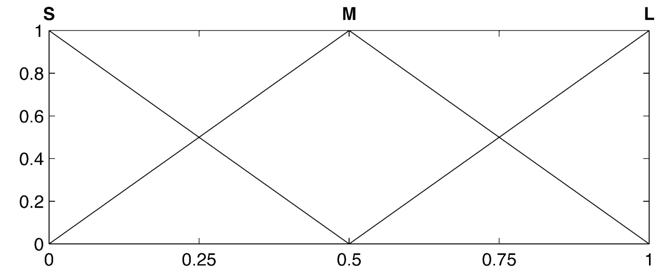

Fig. 1. Fuzzy domains of input and output variables.

TABLE I
R^2 AMPLIFICATION REACHES REGIONS NOT COVERED BY $R^{1'}$

X_2 \ X_1	S	M	L
S	$L^{(2)}$		
M			
L	$L^{(1)}$	$M^{(3)}$	$S^{(4)}$

X_2 \ X_1	S	M	L
S	$L^{(1')}$		
M	$L^{(1')}$		
L	$L^{(1')}$		

X_2 \ X_1	S	M	L
S	$L^{(1',2')}$	$L^{(2')}$	$L^{(2')}$
M	$L^{(1',2')}$	$L^{(2')}$	$L^{(2')}$
L	$L^{(1')}$		

Amplifying a rule consists in adding a label to one of its input variables in the antecedent. An amplification from R^i to $R^{i'}$ is possible if $R^{i'}$ does not conflict with any initial rule, i.e., if no initial rule R^j exists so that $A_k^j \in A_k^{i'}$, for all k, and $LY^j \neq LY^{i'}$.

As mentioned before, in this paper some modifications have been added to the original algorithm. One of them is introduced in the step 2.1 and consists in also trying to amplify the initial rules that subsume in other definitive rules. That way, some regions of the input space not covered by the existing definitive rules could be finally covered, since the amplification stems from a region different from the previous ones.

Example 1: Suppose a system with two inputs and one output whose domains have been partitioned as shown in Fig. 1, and a training set from which four initial rules have been derived [Table I, upper left section]: $R^1 : S, L \rightarrow L$, $R^2 : S, S \rightarrow L$, $R^3 : M, L \rightarrow M$, and $R^4 : L, L \rightarrow S$. After the amplification of the initial rule $R^1 : S, L \rightarrow L$, the definitive rule $R^{1'} : \{S\}, \{*\} \rightarrow L$ will be obtained, due to the initial rules with consequents different from L located in the input fuzzy regions (M, L) and (L, L) [see Table I, upper right section]. Next, the initial rule $R^2 : S, S \rightarrow L$, despite subsuming in the previous definitive rule, will reach regions of the input space not covered at this moment during its amplification, obtaining the definitive rule $R^{2'} : \{*\}, \{S, M\} \rightarrow L$ as illustrated in Table I, lower section.

However, it must be noticed that this improvement can cause that the amplification of an initial rule that subsumes in some definitive rule gives rise to a rule that also subsumes in it. Therefore, in order to delete possible redundancies, it will be verified if the previous situation occurs after every amplification, comparing the rule that has been just amplified with those rules in which it initially subsumed, that is, before the amplification.

Furthermore, it must be regarded that the initial rules obtained from the examples could come into conflict if they have the same antecedents but different consequents. Therefore, it is necessary to design a mechanism to determine which of those conflicting initial rules must be amplified. The method consists in working out the certainty degree of every initial rule from the positive and negative examples that the rule presents in the training set and processing all the initial rules in descendant order according to their certainty degrees, rejecting any rule with the same antecedent as another rule previously processed. Besides, this order introduces a heuristic criterion for processing the initial rules, amplifying the rules with highest certainty degree first.

In this paper, a certainty degree measure is proposed which extends the one presented in [9], adapted to rules with fuzzy consequents.

Definition 1: Given a rule $R_{LY^i}^{i_1 \ldots i_n} : LX_{1,i_1}, \ldots, LX_{n,i_n} \rightarrow LY^i$ ($LX_{j,\cdot} \in DX_j$, $LY^i \in DY$) and a set of examples $E = \{e^1, e^2, \ldots, e^m\}$ where each example takes the form $e^j = ([x_1^j, x_2^j, \ldots, x_n^j], y^j)$, the certainty degree of R^i over E is defined as

$$\omega_I\left(R_{LY^i}^{i_1 \ldots i_n}\right) = \frac{\beta\left(R_{LY^i}^{i_1 \ldots i_n}\right) - \overline{\beta}\left(R_{LY^i}^{i_1 \ldots i_n}\right)}{\sum_{k=1}^{q} \beta\left(R_{LY_k}^{i_1 \ldots i_n}\right)} \tag{2}$$

where q is the number of labels in the output fuzzy domain DY, and

$$\beta\left(R_{LY}^{i_1 \ldots i_n}\right) = \sum_{e^j \in E} \mu_{LX_{1,i_1}}\left(x_1^j\right) \times \cdots \times \mu_{LX_{n,i_n}}\left(x_n^j\right) \times \mu_{LY}(y^j)$$

$$\overline{\beta}\left(R_{LY^i}^{i_1 \ldots i_n}\right) = \sum_{\substack{k=1 \\ LY_k \neq LY^i}}^{q} \frac{\beta\left(R_{LY_k}^{i_1 \ldots i_n}\right)}{q-1}.$$

Thus, this measure is based on the q covering degrees $\beta(R_{LY_1}^{i_1 \ldots i_n}), \ldots, \beta(R_{LY_q}^{i_1 \ldots i_n})$ presented over the training set by the q rules that can be defined over the input fuzzy region $(LX_{1,i_1}, \ldots, LX_{n,i_n})$. A rule $R_{LY^i}^{i_1 \ldots i_n}$ with covering degree above the mean of the rest of the rules (rules with consequent different from LY^i) will take a positive certainty degree, whereas the rule will take a negative certainty degree in the opposite situation. If no examples cover the region delimited by the antecedent a certainty degree equal to 0 will be assigned to the rule, thus indicating a neutral certainty degree.

It must be noticed that this measure is defined on single rules (i.e., with only one label associated to each input variable). Nevertheless, this is not a problem, because the certainty degree will only be necessary for this type of rules.

Thus, the identification algorithm over which we will work in this article is the following.

LMSFR algorithm

1. Transform the examples in initial rules, removing

TABLE II
TWO-INPUT, ONE-OUTPUT FUZZY MODEL

		X_1		
		S	**M**	**L**
X_2	**S**	L	L	L
	M	L	L	M
	L	L	M	S

redundant rules and ordering the rest in descending order according to their certainty degrees.

2. For each initial rule which does not conflict with any initial rule previously processed:
   2.1. For each label in each input variable:
      2.1.1. If the amplification of the rule is possible, amplify it.
   2.2. If the amplified rule does not subsume in any definitive rule, store it in the set of definitive rules.

Example 2: Suppose a system with two inputs and one output consisting in the fuzzy relation generated by the model in Table II. Next, it will be tried to identify this model stemming from the following training set:

$$\begin{array}{ll} ((0.09, 0.80), 0.87) & ((0.29, 0.83), 0.71) \\ ((0.18, 0.20), 1.00) & ((0.49, 0.80), 0.71) \\ ((0.19, 0.91), 0.79) & ((0.72, 0.78), 0.50) \\ ((0.20, 0.23), 1.00) & ((0.82, 0.95), 0.29) \\ ((0.21, 0.76), 0.77) & ((0.91, 1.00), 0.11). \end{array}$$

In order to do that, suppose that the model will use the fuzzy partitions shown in Fig. 1 during the identification process. First, the identification algorithm will obtain the following set of initial rules and their associated certainty degrees:

$$\begin{array}{ll} R_L^{1,3} : S, L \to L\ (0.39) & R_M^{2,3} : M, L \to M\ (0.34) \\ R_L^{1,1} : S, S \to L\ (1.00) & R_M^{2,3} : M, L \to M\ (0.34) \\ R_L^{1,3} : S, L \to L\ (0.39) & R_M^{2,3} : M, L \to M\ (0.34) \\ R_L^{1,1} : S, S \to L\ (1.00) & R_M^{3,3} : L, L \to M\ (0.22) \\ R_L^{1,3} : S, L \to L\ (0.39) & R_S^{3,3} : L, L \to S\ (0.28). \end{array}$$

This set will be reduced to the set of rules shown here once the redundancies are removed and the rest of rules are ordered

$$\begin{array}{l} R_L^{1,1} : S, S \to L\ (1.00) \\ R_L^{1,3} : S, L \to L\ (0.39) \\ R_M^{2,3} : M, L \to M\ (0.34) \\ R_S^{3,3} : L, L \to S\ (0.28) \\ R_M^{3,3} : L, L \to M\ (0.22). \end{array}$$

As described in Example 1, the amplifications of the first and the second initial rules give the definitive rules:

$$\begin{array}{l} R^1 : \{*\}, \{S, M\} \to L \\ R^2 : \{S\}, \{*\} \to L. \end{array}$$

Due to the initial rules $R_L^{1,3}$ and $R_S^{3,3}$, the rule $R_M^{2,3}$ can not be amplified through the dimension of variable X_1 but it will extend all over the domain of variable X_2. Therefore, the amplified rule will be

$$R^3 : \{M\}, \{*\} \to M.$$

Similarly, due to the initial rule $R_M^{2,3}$, the rule $R_S^{3,3}$ will produce the amplified rule

$$R^4 : \{L\}, \{*\} \to S.$$

The last rule $R_M^{3,3}$ will not be amplified, since it comes into conflict with the previous initial rule.

III. ADDING EXCEPTIONS TO FUZZY RULES

In the previous algorithm, the search of rules as general as possible causes that different consequents can coexist in some fuzzy regions of the input space. Next, a strategy is proposed to solve these conflicts.

During the learning process presented before, the information contained in the examples is used only for the extraction and ordering of the initial rules. From that moment, the amplification of a rule to a certain input fuzzy region only depends on whether this region is or not occupied by any other initial rule. Therefore, this process ignores the information that could be contained in the training set about such a region. The basis that will support the approach proposed here to solve the conflicts consists in taking advantage of this information.

A compound rule of the form presented in (1) is equivalent to a conjunction of single rules with just one label associated to each input variable. For example, the compound rule

$$R^i : \text{if } X_1 \text{ is } \{S, L\} \text{ and } X_2 \text{ is } \{M\} \text{ then } Y \text{ is } M$$

is equivalent to the single rules

$$\begin{array}{c} R_M^{1,2} : \text{if } X_1 \text{ is } S \text{ and } X_2 \text{ is } M \text{ then } Y \text{ is } M \\ \text{and} \\ R_M^{3,2} : \text{if } X_1 \text{ is } L \text{ and } X_2 \text{ is } M \text{ then } Y \text{ is } M. \end{array}$$

Therefore, the set of single rules involved in a conflict can be isolated in order to select one of them according to a certain criterion. For this goal, a certainty degree for each single rule involved in the conflict will be calculated from the number of positive and negative examples that each rule presents in the training set, using the same measure described in (2).

However, regarding the strategy to solve conflicts, it must be noted that the main goal of the amplification is for the amplified rules to be as general as possible linguistically. Thus, the finally obtained consequents in an input subspace do not mean to be the best, since their values are determined by initial rules that can be far away from the subspace under consideration. For example, if a rule located in a vertex of the input space reaches the opposite vertex in the amplification, it does not seem that the single rule obtained in this last vertex has to be better than any other single rule located in that place.

Therefore, when solving a conflict, although it must be tried to restrict the selection of the best consequent to those involved in the conflicting rules in order to obtain maximal rules, it seems desirable to extend the space of selection if none of those rules has a sufficient degree of certainty. For that reason, a threshold μ

is established on the certainty degree in order to decide whether the search of the best rule must be extended to all the possible rules for the fuzzy conflicting region.

Next, trying to delimit the range of possible values to that threshold, two properties fulfilled by the measure in (2) are enumerated.

Property 1: The certainty degrees for all the possible rules with the same antecedent add up to zero. That is

$$\sum_{j=1}^{q} \omega\left(R_{LY_j}^{i_1\ldots i_n}\right) = 0.$$

Proof: For clearness, we denote $\beta(R_{LY_j}^{i_1\ldots i_n})$ and $\omega(R_{LY_j}^{i_1\ldots i_n})$ by means of β_j and ω_j, respectively. Thus, since

$$\begin{aligned}\sum_j \bar{\beta}_j &= \sum_j \frac{\beta_1 + \cdots + \beta_{j-1} + \beta_{j+1} + \cdots + \beta_q}{q-1}\\ &= \frac{(q-1)\beta_1 + \cdots + (q-1)\beta_q}{q-1}\\ &= \sum_j \beta_j\end{aligned}$$

it results that

$$\begin{aligned}\sum_j \omega_j &= \sum_j \frac{\beta_j - \bar{\beta}_j}{\sum_k \beta_k}\\ &= \frac{\sum_j \beta_j - \sum_j \bar{\beta}_j}{\sum_k \beta_k}.\\ &= \frac{\sum_j \beta_j - \sum_j \beta_j}{\sum_k \beta_k}\\ &= 0\end{aligned}$$

□

Property 2: Each rule having a consequent that is not covered by the output of any example covering its antecedent will take a certainty degree equal to $-1/(q-1)$. That is

$$\omega\left(R_{LY}^{i_1\ldots i_n}\right) = -\frac{1}{q-1} \qquad \forall LY \in DY | \sum_{e^k \in E^*} \mu_{LY}(y^k) = 0$$

where $E^* = \{e^k \in E | \mu_{LX_{1,i_1}}(x_1^k) \times \cdots \times \mu_{LX_{n,i_n}}(x_n^k) > 0\}$.

Proof: Using the same notation adopted before, since

$$\beta_j = \sum_{e^k \in E} \mu_{LX_{1,i_1}}\left(x_1^k\right) \times \cdots \times \mu_{LX_{n,i_n}}\left(x_n^k\right) \times \mu_{LY_j}(y^k) = 0$$

then

$$\begin{aligned}\omega_j &= \frac{\beta_j - \bar{\beta}_j}{\sum_k \beta_k} = \frac{0 - \bar{\beta}_j}{\sum_k \beta_k}\\ &= \frac{-\frac{\beta_1 + \cdots + \beta_{j-1} + \beta_{j+1} + \cdots + \beta_q}{q-1}}{\beta_1 + \cdots + \beta_{j-1} + 0 + \beta_{j+1} + \cdots + \beta_q}\\ &= -\frac{1}{q-1}.\end{aligned}$$

□

Proposition 1: Given the example set E, a rule with a certainty degree higher or equal to $(q-2)/[2(q-1)]$ is the rule with the highest certainty degree among the q possible rules with the same antecedent. That is

$$\text{if } \omega\left(R_{LY_*}^{i_1\ldots i_n}\right) \geq \frac{q-2}{2(q-1)}, \text{ then } \omega\left(R_{LY_*}^{i_1\ldots i_n}\right) = \max_j \omega\left(R_{LY_j}^{i_1\ldots i_n}\right).$$

Proof: Let r be the number of consequents covered by the examples in the input fuzzy region under consideration. Because of Property 2, $(q-r)$ rules will take a certainty degree equal to $-1/(q-1)$, and due to Property 1 the remaining r rules will comply with

$$\omega_{j_1} + \cdots + \omega_{j_r} = \frac{q-r}{q-1}.$$

i. For $r = 2$: Since

$$\omega_{j_1} + \omega_{j_2} = \frac{q-2}{q-1}$$

clearly if $\omega_{j_1} \geq (q-2)/2(q-1)$ (respectively, $\omega_{j_2} \geq (q-2)/2(q-1)$), then $\omega_{j_2} \leq (q-2)/2(q-1)$ (respectively, $\omega_{j_1} \leq (q-2)/2(q-1)$).

ii. Suppose the above proposition being true for r. Given a set $\{\omega_{j_1}, \ldots, \omega_{j_r}, \omega_{j_{r+1}}\}$ where

$$\omega_{j_1} + \cdots + \omega_{j_r} + \omega_{j_{r+1}} = \frac{q-(r+1)}{q-1}$$

and so that a $\omega_{j_*} \geq (q-2)/2(q-1)$ exists, without the loss of generality we can assume $\max\{\omega_{j_r}, \omega_{j_{r+1}}\} \leq \min\{\omega_{j_1}, \ldots, \omega_{j_{r-1}}\}$, and thus $\max\{\omega_{j_1}, \ldots, \omega_{j_{r-1}}\} \geq (q-2)/2(q-1)$. Now, denote

$$\omega_{j_{new}} = \omega_{j_r} + \omega_{j_{r+1}} - \frac{q-(r+1)}{q-1} - \frac{q-r}{q-1}.$$

Then

$$\omega_{j_1} + \cdots + \omega_{j_{r-1}} + \omega_{j_{new}} = \frac{q-r}{q-1}$$

and, provided that the proposition is true for r and there exists $\omega_{j_*} = \max\{\omega_{j_1}, \ldots, \omega_{j_{r-1}}\} \geq (q-2)/2(q-1)$, then $R_{LY_*}^{i_1\ldots i_n}$ is the rule with the highest certainty degree. Therefore, the proposition is true for $(r+1)$.

□

Based on these results, the interval $[0, (q-2)/2(q-1)]$ will be considered as the range of possible values for the threshold τ. On the one hand, a threshold below 0 would be insufficient, since it would allow the selection of rules with a weight below the mean. On the other hand, a threshold over $(q-2)/2(q-1)$ would be inadequate, since it would not yield to a better result when selecting the best rule and it would cause an increase in the number of rules to be considered during the conflict resolution, thus increasing the computational cost.

Once the best rule is selected, it will be necessary to modify the rest of compound rules involved in the conflict. In this respect, when several conflicting rules have the highest certainty degree, the rule having the consequent with the highest number of occurrences in the conflicting region will be selected, since it can exist more than one rule with the same consequent between the rules in conflict. This strategy tries to reduce the number of compound rules to be modified as much as possible.

The procedure to modify the compound rules consists in adding exceptions to them. An exception is an n-tuple of labels $(LX_{1,i_1}, \ldots, LX_{n,i_n})$ that defines the fuzzy region of the input space where the compound rule is not applied.

The use of exceptions entails an improvement in the model expressiveness with respect to the traditional description methods, since it allows to decrease the number of rules necessary to describe the model.

Example 3: This fact can be observed in the example in Table II. The number of single rules describing the model is $3 \times 3 = 9$ rules. A description using the usual technique, which associates an input subspace having the same output (consequent) with the antecedent of each rule gives at least six fuzzy rules. For example

R^1 : if X_2 is $\{S\}$ then Y is L
R^2 : if X_1 is $\{S, M\}$ and X_2 is $\{M\}$ then Y is L
R^3 : if X_1 is $\{S\}$ and X_2 is $\{L\}$ then Y is L
R^4 : if X_1 is $\{L\}$ and X_2 is $\{M\}$ then Y is M
R^5 : if X_1 is $\{M\}$ and X_2 is $\{L\}$ then Y is M
R^6 : if X_1 is $\{L\}$ and X_2 is $\{L\}$ then Y is S.

However, the same model can be described with only five rules using exceptions

R^1 : if X_2 is $\{S, M\}$ then Y is L
excepting if X_1 is L and X_2 is M
R^2 : if X_1 is $\{S\}$ and X_2 is $\{L\}$ then Y is L
R^3 : if X_1 is $\{M\}$ and X_2 is $\{L\}$ then Y is M
R^4 : if X_1 is $\{L\}$ and X_2 is $\{L\}$ then Y is S
R^5 : if X_1 is $\{L\}$ and X_2 is $\{M\}$ then Y is M

equivalent to

R^1 : if X_2 is $\{S, M\}$ then Y is L
excepting if X_1 is L and X_2 is M then Y is M
R^2 : if X_1 is $\{S\}$ and X_2 is $\{L\}$ then Y is L
R^3 : if X_1 is $\{M\}$ and X_2 is $\{L\}$ then Y is M
R^4 : if X_1 is $\{L\}$ and X_2 is $\{L\}$ then Y is S

improving the interpretability of the model.

Moreover, several authors from the field of psychology [10], [11] have supported that the human mental model for the classification process (a type of identification) is based on extracting a set of imperfect rules to which occasional exceptions are added. This fact strengthens the belief that a model describing the system by means of rules with exceptions will be easier to interpret for the human being.

The following algorithm describes the proposed method to solve conflicts.

FRIwE algorithm

1. For each fuzzy region of the input space where two or more different consequents coexist do:
   1.1. Work out the certainty degrees of the single rules involved and select the highest (w_1).
   1.2. If w_1 reaches a threshold τ, go to step 1.5. Otherwise, continue on step 1.3.
   1.3. Search among the rest of possible rules for one with a certainty degree higher than w_1.
   1.4. If that rule exists, select it as the best rule (adding a new compound rule) and go to step 1.6. Otherwise, continue on step 1.5.
   1.5. If there are more than one different single rule with the highest certainty degree (w_1) between the conflicting rules, select the one appearing more times in the conflicting region. If all appear the same times, selectone of them (for example, the first one).
   1.6. Delete each single rule different from the selected one.
   1.7. For each deleted simple rule, form the appropriate exception and add it to its respective compound rule.

TABLE III
CONFLICTS IN THE IDENTIFIED RULE BASE

		X_1: S	M	L
X_2	S	L	L/M	L/S
	M	L	L/M	L/S
	L	L	M	S

Example 4: Consider the rule base from Example 2 obtained by means of the original identification algorithm without solving conflicts. This rule base presents several conflicts, as can be observed in the Table III.

In order to describe the procedure for solving conflicts, a maximum threshold $\tau = (3-2)/2(3-1) = 0.25$ is selected in this example. The algorithm proposed here begins solving the conflict located in the fuzzy region (M, S), where the single rules

$$R_L^{2,1} : M, S \rightarrow L$$
$$R_M^{2,1} : M, S \rightarrow M$$

coexist, corresponding to the amplified rules R^1 and R^3, respectively. Once their certainty degrees have been calculated ($\omega(R_L^{2,1}) = 1.00$ and $\omega(R_M^{2,1}) = -0.5$) the rule $R_L^{2,1}$ with the highest degree surpassing the threshold τ will be selected, therefore removing the single rule $R_M^{2,1}$ by adding an exception to the rule R^3

$$R^3 : \{M\}, \{*\} \rightarrow M \text{ excepting } \{(M, S)\}.$$

The next conflict (M, M) arises from the rules

$$R_L^{2,3} : M, M \rightarrow L \ (0.26)$$
$$R_M^{2,3} : M, M \rightarrow M \ (0.23)$$

and, thus, the rule $M, M \rightarrow L$ is selected and the amplified rule R^3 adds a new exception

$$R^3 : \{M\}, \{*\} \rightarrow M \text{ excepting } \{(M, S), (M, M)\}.$$

The next conflicting fuzzy region (L, S) is not covered by any example. Then, the conflicting rules

$$R_L^{3,1} : L, S \rightarrow L$$
$$R_S^{3,1} : L, S \rightarrow S$$

take a certainty degree equals to 0. This causes a search expansion to the rule $R_M^{3,1} : L, S \rightarrow M$, that will also present a certainty degree 0 for the same reason. Since the available information does not allow to establish a preference among the rules, the first rule initially in conflict is selected, giving rise to the following modification:

$$R^4 : \{L\}, \{*\} \rightarrow S \text{ excepting } \{(L, S)\}.$$

Finally, the conflict between the rules

$$R_L^{3,2} : L, M \rightarrow L\ (-0.50)$$
$$R_S^{3,2} : L, M \rightarrow S\ (-0.39)$$

leads to a search expansion again, and now the rule $R_M^{3,2} : L, M \rightarrow M$ (0.89) is selected, since its certainty degree surpasses -0.39, the highest degree between the previous ones. This causes the inclusion of new exceptions into the rules R^4 and R^1 and the addition of a new rule R^5. This way, the final rule base will be

$$R^1 : \{*\}, \{S, M\} \rightarrow L \text{ excepting } \{(L, M)\}$$
$$R^2 : \{S\}, \{*\} \rightarrow L$$
$$R^3 : \{M\}, \{*\} \rightarrow M \text{ excepting } \{(M, S), (M, M)\}$$
$$R^4 : \{L\}, \{*\} \rightarrow S \text{ excepting } \{(L, S), (L, M)\}$$
$$R^5 : \{L\}, \{M\} \rightarrow M$$

obtaining a model as the one trying to be identified.

IV. Improving the Interpretability

The model generated using the algorithm described in the previous section can still improve its interpretability in different ways. Next, several strategies are described for that goal.

A. Reducing Fuzzy Rules

The interpretability of the compound rules can increase if the set of exceptions of a rule is reduced by deleting labels from its antecedent. This happens when the $(n-1)$-dimensional subspace delimited by a label in the antecedent is totally excluded due to a subset of rule exceptions. For example, the rule

if X_1 is $\{S, L\}$ and X_2 is $\{S, M\}$ then Y is M
excepting if X_1 is S and X_2 is M also
excepting if X_1 is L and X_2 is M

could be reduced to the rule

if X_1 is $\{S, L\}$ and X_2 is $\{S, M\}$ then Y is M

since the subspace $\{S, L\}$ delimited in X_1 by the label M of X_2 is totally excluded due to the rule exceptions.

It must be taken into account that compound rules that initially do not subsume in other definitive rules could finally subsume due to the rule reduction. Thus, this fact must be verified after the reduction and, if it is the case, the redundant rule must be removed.

Next, an algorithm is described that will be called whenever an exception is added to a rule during the conflict resolution. This algorithm will decide if a reduction can be carried out and, if that is the case, will accomplish it.

Rule reduction algorithm

1. Given the compound rule
   $R^i : A_1^i, \ldots, A_n^i \rightarrow LY^i$ with exceptions
   $E^i = \{E_1^i, \ldots, E_l^i\}$, where $E_l^i = (LX_{1,l_1}, \ldots, LX_{n,l_n})$
   is the new exception added to that rule.
2. For each d from 1 to n do:
   2.1. Set up a set of exceptions E^* taking LX_{d,l_d} in the dth element of every exception and taking the different combinations of the labels from $A_1^i, \ldots, A_{d-1}^i, A_{d+1}^i, \ldots, A_n^i$ in the rest of elements. That is, $E^* = A_1^i \times \cdots \times A_{d-1}^i \times LX_{d,l_d} \times A_{d+1}^i \times \cdots \times A_n^i$.
   2.2. If $E^* \subseteq E^i$ then set $E^i = E^i - E^*$ and $A_d^i = A_d^i - \{LX_{d,l_d}\}$, and go to step 3.
3. If the reduced rule subsumes in some other compound rule, delete it.

Although generally a rule could lose more than one label in the reduction process, due to the order for solving conflicts established by the algorithm in the previous section, this will never happen. Hence, a jump to step 3 occurs when a reduction has taken place.

Example 5: Suppose that the rule $R : \{S, M\}, \{S, L\}, \{S, M, L\} \rightarrow M$ with exceptions $E = \{(S, S, S), (S, L, S), (S, L, M), (S, L, L), (M, S, M), (M, L, S), (M, L, M)\}$ adds a new exception (M, L, L) during the conflict resolution, resulting in $E = E \cup (M, L, L)$.

The spatial representation of this rule is shown in Fig. 2, where the latest exception is marked with a star and the rest with crosses.

Reduction begins taking the first label M from the new exception and setting up a set of exceptions containing M in the first position and containing the different combinations from the sets $\{S, L\}$ and $\{S, M, L\}$ in the rest of positions, that is, $E_1^* = M \times \{S, L\} \times \{S, M, L\} = \{(M, S, S), (M, S, M), (M, S, L), (M, L, S), (M, L, M), (M, L, L)\}$. Since E_1^* is not included in E, the label M associated with X_1 is not removed.

Next, the second label from the exception L is taken and the set $E_2^* = \{S, M\} \times L \times \{S, M, L\} = \{(S, L, S), (S, L, M), (S, L, L), (M, L, S), (M, L, M), (M, L, L)\}$ is set up. Since $E_2^* \subset E$, the set E_2^* is subtracted from E and the label L from X_2 is removed, giving the reduced rule $R : \{S, M\}, \{S\}, \{S, M, L\} \rightarrow M$ with exceptions $E = \{(S, S, S), (M, S, M)\}$.

Before ending the reduction process, it will be verified if the reduced rule subsumes in some other rule, in order to remove it if that is the case.

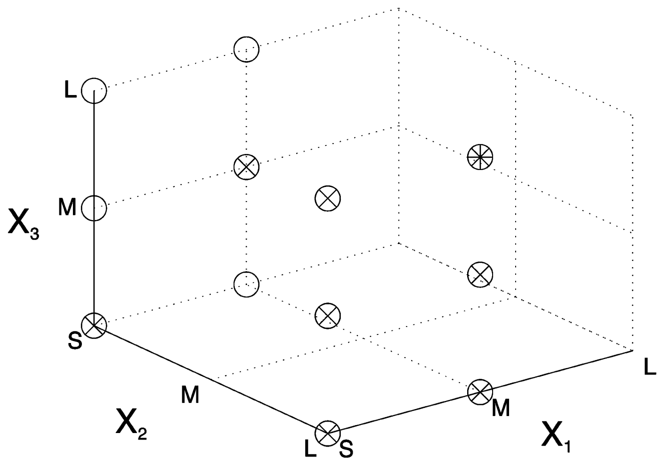

Fig. 2. R rule before the reduction in Example 5.

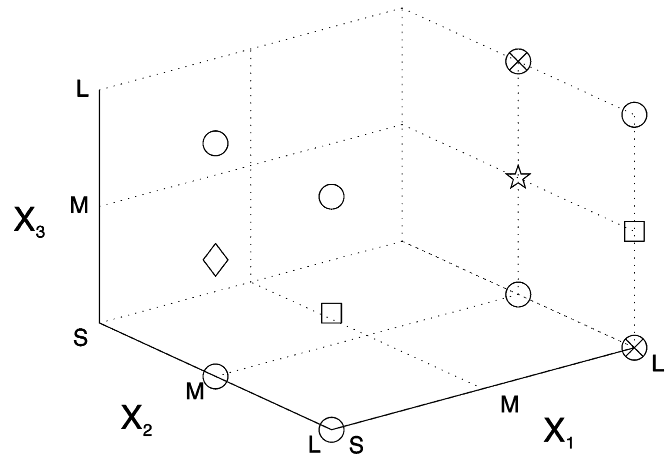

Fig. 3. Rules to be merged: R^1 (circles), R^2 (squares), R^3 (diamonds), and R^4 (pentagram).

Example 6: The rule base obtained from Example 4 can also benefit from the reduction process, yielding to the rule base

$$R^1 : \{*\}, \{S, M\} \rightarrow L \text{ excepting}\{(L, M)\}$$
$$R^2 : \{S\}, \{*\} \rightarrow L$$
$$R^3 : \{M\}, \{L\} \rightarrow M$$
$$R^4 : \{L\}, \{L\} \rightarrow S$$
$$R^5 : \{L\}, \{M\} \rightarrow M$$

easier to be interpreted.

B. Merging Fuzzy Rules

In the algorithm presented in Section III, a rule is added to the set of definitive rules when the selected rule is not one of the conflicting ones (step 1.4). This can lead to a considerable increase in the number of rules with respect to the one obtained by the original identification algorithm. In order to minimize this increase, it must be tried to merge that rule with any of the existing compound rules after the addition of a new rule.

Proposition 2: A rule $R^i : A_1^i, A_2^i, \dots, A_n^i \rightarrow LY^i$ with exceptions $E^i = \{E_1^i, \dots, E_{l_i}^i\}$ could be merged with another rule $R^j : A_1^j, A_2^j, \dots, A_n^j \rightarrow LY^j$ with exceptions $E^j = \{E_1^j, \dots, E_{l_j}^j\}$ if the following are fulfilled.

1) $LY^i = LY^j$.
2) There exists an r so that $A_r^i \neq A_r^j$.
3) $A_s^i = A_s^j$, for all $s \neq r$.

The result will be a new rule $R^* : A_1^i, \dots, A_r^i \cup A_r^j, \dots, A_n^i \rightarrow LY^i$ with exceptions $E^* = E^i \cup E^j$.

Rule merging must also be tried after rule reduction, since the merging condition could be satisfied if the rule loses a label in the antecedent along the reduction process. Because of that, this merging will be tried in step 3 of the algorithm presented in the previous section, once it has been verified that the rule does not subsume in other rules.

The method is recursive, since the merged rule could satisfy the merging condition with respect to some other existing in the rule base. The following algorithm describes the method for merging rules.

Rule merging algorithm

1. Given the compound rule $R^i : A_1^i, \dots, A_n^i \rightarrow LY^i$ with exceptions $E^i = \{E_1^i, \dots, E_{l_i}^i\}$ trying to be merged.
2. If there is another rule $R^j : A_1^j, \dots, A_n^j \rightarrow LY^j$ with exceptions $E^j = \{E_1^j, \dots, E_{l_j}^j\}$ in the set of definitive rules so that it is possible to be merged with R^i:
   2.1. Replace the rules R^i and R^j by the rule $R^* : A_1^i, \dots, A_r^i \cup A_r^j, \dots, A_n^i \rightarrow LY^i$ with exceptions $E^* = E^i \cup E^j$.
   2.2. Try to merge R^*.

Example 7: Suppose a rule base containing the rules

$$R^1 : \{S, L\}, \{M, L\}, \{S, L\} \rightarrow L \quad \text{excepting } \{(L, L, S), (L, M, L)\}$$
$$R^2 : \{S, L\}, \{L\}, \{M\} \rightarrow L$$
$$R^3 : \{S\}, \{M\}, \{M\} \rightarrow L$$

and that the rule $R^4 : \{L\}, \{M\}, \{M\} \rightarrow L$ is added during the conflict resolution (see Fig. 3).

The rules R^4 and R^3 can be merged, since they only differ in the set of labels related to X_1, resulting in $R^{3'} : \{S, L\}, \{M\}, \{M\} \rightarrow L$. Next, this rule can be merged with R^2, since they only differ in the set of labels associated with X_2, giving the rule $R^{2'} : \{S, L\}, \{M, L\}, \{M\} \rightarrow L$. Finally, this rule merges with R^1, from which it only differs in the set of labels related to X_3, finally merging the four rules in

$$R^{1'} : \{S, L\}, \{M, L\}, \{S, M, L\} \rightarrow L \quad \text{excepting } \{(L, L, S), (L, M, L)\}.$$

C. Merging Exceptions

Until now, exceptions have been described as n-tuples of labels that define fuzzy regions in the input space similar to the ones defined by the antecedents of single rules. Therefore, the exceptions expressed in that way can be considered *single exceptions.*

Trying to increase the model interpretability, the concept of compound rule can be translated to the representation of exceptions, giving rise to *compound exceptions*. Thus, a compound exception can be defined as an n-tuple $E_i = (E_{i,1}, \ldots, E_{i,n})$, where $E_{i,k} \subseteq DX_k$.

In order to obtain a description as compact as possible by means of exceptions, it is necessary to state a mechanism for merging exceptions in a similar way to that for merging rules explained in the previous subsection. Nevertheless, whereas rule merging runs *on line* (i.e., it is done during conflict resolution), exception merging will run *off line* (i.e., once the final exceptions of every rule have been obtained). This is due to the use of single exceptions in the rule reduction procedure.

Proposition 3: An exception $E_i = (E_{i,1}, \ldots, E_{i,n})$ could merge with another one $E_j = (E_{j,1}, \ldots, E_{j,n})$ if the following are fulfilled.

1) There exists an r so that $E_{i,r} \neq E_{j,r}$.
2) $E_{i,s} = E_{j,s}$, for all $s \neq r$.

The result of the merger will be a new exception $E_* = (E_{i,1}, \ldots, E_{i,r} \cup E_{j,r}, \ldots, E_{i,n})$.

The following algorithm describes the method for merging exceptions.

Exception merging algorithm

1. Given the set of exceptions $E = \{E_1, \ldots, E_l\}$ and the exception trying to be merged $E_i = (E_{i,1}, \ldots, E_{i,n})$.
2. If there exists a $j \neq i$, so that it is possible to merge E_i and E_j:
   2.1. Replace the exceptions E_i and E_j by the exception $E_* = (E_{i,1}, \ldots, E_{i,r} \cup E_{j,r}, \ldots, E_{i,n})$.
   2.2. Try to merge E_*.

This recursive algorithm will be called for each rule while any of its exceptions can be merged.

Example 8: Suppose that a rule with the set of the following exceptions results from the conflict resolution:

$$\begin{aligned} E_1 &= (\{S\},\{S\}) \qquad & E_4 &= (\{M\},\{M\}) \\ E_2 &= (\{S\},\{M\}) \qquad & E_5 &= (\{M\},\{L\}) \\ E_3 &= (\{M\},\{S\}). \end{aligned}$$

The first run of the previous algorithm tries to merge E_1. As a result, this exception is with E_2, producing the set of merged exceptions:

$$\begin{aligned} E_1' &= (\{S\},\{S,M\}) \quad & E_4 &= (\{M\},\{M\}) \\ E_3 &= (\{M\},\{S\}) \quad & E_5 &= (\{M\},\{L\}). \end{aligned}$$

In the second run, the exceptions E_3 and E_4 will be merged firstly and, due to the recursive characteristic of the algorithm, the resulting exception will be merged with E_1', giving the set

$$\begin{aligned} E_1'' &= (\{S,M\},\{S,M\}) \\ E_5 &= (\{M\},\{L\}). \end{aligned}$$

Since the exceptions can not be merged anymore, this set will be the final set of exceptions.

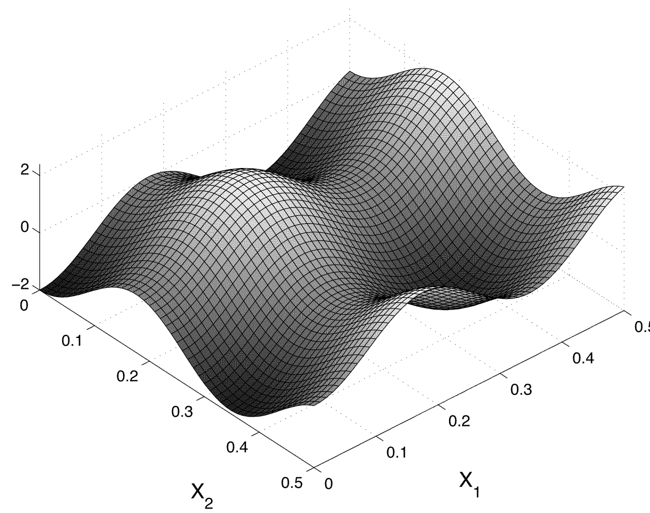

Fig. 4. Output surface for the generalized Rastrigin's function.

V. Experimental Results

In order to analyze the interpretability results of the FRIwE method proposed in this paper, it was applied to the approximation of the two-dimensional system described by the generalized Rastrigin's function [12], and the possible gain obtained when using exceptions will be illustrated. Furthermore, the approximation capacity of the method will be evaluated applying it to the identification of an example designed by Friedman to evaluate MARS algorithm [13], an identification method based on regression techniques. The example consists in a system with four inputs and one output described by an alternating current series circuit. The reason for selecting this example lies in the considerable dimensionality of the input space, in order to analyze the approximation ability of the method when applied to systems that are difficult to identify. In order to check the accuracy and interpretability levels achieved with respect to other identification methods, the proposed method has been compared with the identification method proposed by Wang and Mendel [14], which is traditionally used to compare results and, thus, allows to establish comparisons with other fuzzy model identification methods through it. Besides, the results from MARS algorithm presented in [13] are shown for the second example, although, in this case, the comparison is only feasible with respect to the accuracy achieved by the model, due to the different nature of the techniques.

A. Example 1: Generalized Rastrigin's Function

The function to be identified is described by the following equation:

$$\begin{aligned} &f : [0, 0.5] \times [0, 0.5] \to [-2, 2.4] \\ &f(x_1, x_2) = x_1 + x_2 - \cos(18x_1) - \cos(18x_2). \end{aligned} \tag{3}$$

This function is a multimodal function whose output surface is shown in Fig. 4. The low dimensionality of the system will help us to analyze the interpretability of the resulting models.

1) Data Base: The fuzzy model has two input and one output fuzzy variables. The fuzzy domains for all the variables

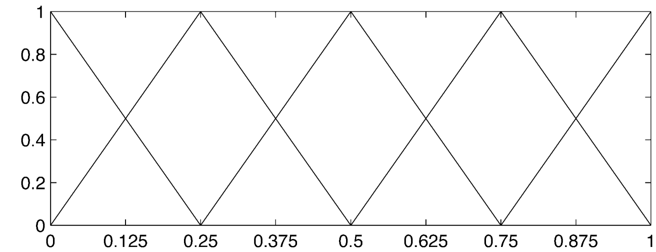

Fig. 5. Fuzzy domains of the input and output variables.

were defined using five linguistic labels with triangular membership functions, as it is shown in Fig. 5. The real domains of the input variables were normalized by means of the appropriate scale factors $sf_{x_1} = 0.5$, $sf_{x_2} = 0.5$, and $sf_y = 4.4$, and the shift constants $sh_{x_1} = 0$, $sh_{x_2} = 0$, and $sh_y = -2$.

2) Experiments: The FRIwE method were compared with the Wang and Mendel method [14]. The approximation capacity of both was evaluated through the mean square error

$$MSE = \frac{\sum_{i-1}^{N} \left[f(x_1) - \hat{f}(x_i)\right]^2}{N.}$$

In order to do that, five different test sets where used for each model, each of them with 500 examples ($N = 500$). With this strategy, it can be provided, not only the global error as the mean of those five errors, but the reliability degree of such an error by means of the standard deviation.

The experiments were carried out using three training set sizes: 10, 20, and 50 examples. For each size, 100 simulations of the FRIwE and the Wang and Mendel method were run, using a different training set in each of them.

In order to complete the rule base generated from the Wang and Mendel method and, thus, to estimate the model error, an average value M was assigned to all the fuzzy regions to which the identification algorithm did not assign a fuzzy output.

3) Results: In Table IV(top) the average global errors of the models for each training set size are shown, along with the standard deviations for the different simulations (in parentheses). The standard deviations represent the dependence of the results from the specific training sets. Although results from the FRIwE method were obtained for both minimum and maximum thresholds ($\tau = 0$ and $\tau = 0.25$, respectively), the table only includes the former since it always performs better.

It can be observed that the FRIwE method presents the best accuracy results for all the training set sizes. Furthermore, the decreasing standard deviation of FRIwE method shows the largest independence from the training set.

Table IV (bottom) shows the average number of rules representing the models obtained from Wang and Mendel's and FRIwE methods. Again, the FRIwE method achieves the best results, besides presenting a smoother increase in the number of rules along the training set size. It provides a degree of interpretability better than the model by Wang and Mendel and roughly independent from the training set size.

In order to illustrate the best interpretability of the FRIwE method, Table V shows an instance of the fuzzy models achieved with the FRIwE method one using a training set with 50 examples. The model appears described by means of rules with exceptions (Table V (top)) and in tabular form (Table V (bottom)). Fig. 6 shows the output surface of the model. The model is described with only seven rules and one exception and provides an MSE equal to 0.0234. Moreover, as can be observed, the antecedent of rule R^4 coincides with the exception in rule R^7 and, therefore, the former could be integrated as an extension of the exception in R^7:

$$R^7 : \{S, M, XL\}, \{S, M, XL\} \rightarrow PS$$
$$\text{excepting}\{XL\}, \{XL\} \rightarrow NS$$

diminishing the number of input subspaces to be considered for the model description.

TABLE IV
RESULTS FOR THE GENERALIZED RASTRIGIN'S FUNCTION

	Training set size		
	10	**20**	**50**
W&M	.074(.016)	.053 (.013)	.031(.0083)
FRIwE ($\tau = 0$)	**.054**(.013)	**.039**(.0097)	**.028**(.0057)

	Training set size		
	10	**20**	**50**
W&M	8.2	13.0	20.3
FRIwE ($\tau = 0$)	**6.3**	**8.6**	**10.4**

TABLE V
DESCRIPTION OF A MODEL OBTAINED FROM THE FRIWE METHOD USING RULES WITH EXCEPTIONS (TOP) AND IN TABULAR FORM (BOTTOM)

$R^1 : \{XS, L\}, \{XS\} \rightarrow NL$
$R^2 : \{XS, L\}, \{M, L\} \rightarrow NS$
$R^3 : \{L\}, \{S, M, L, XL\} \rightarrow NS$
$R^4 : \{XL\}, \{XL\} \rightarrow NS$
$R^5 : \{S, M, XL\}, \{XS, L\} \rightarrow Z$
$R^6 : \{XS\}, \{S, XL\} \rightarrow Z$
$R^7 : \{S, M, XL\}, \{S, M, XL\} \rightarrow PS$
excepting $\{XL\}, \{XL\}$

		X_1				
		XS	**S**	**M**	**L**	**XL**
	XS	*NL*	*Z*	*Z*	*NL*	*Z*
	S	*Z*	*PS*	*PS*	*NS*	*PS*
X_2	**M**	*NS*	*PS*	*PS*	*NS*	*PS*
	L	*NS*	*Z*	*Z*	*NS*	*Z*
	XL	*Z*	*PS*	*PS*	*NS*	*NS*

B. Example 2: An Alternating Current Series Circuit

In this example, the system consists in the alternating current series circuit represented schematically in Fig. 7. This system involves a resistor R, an inductor L, and a capacitor C. Besides, a generator places a voltage

$$V_{ab} = V_o \sin \omega$$

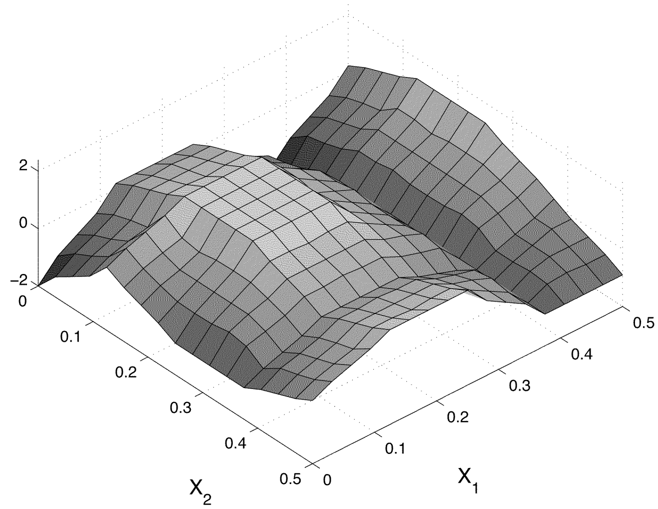

Fig. 6. Output surface of a model obtained from the FRIwE method.

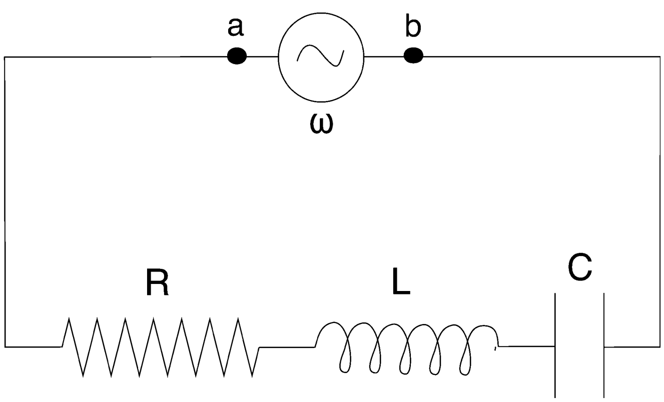

Fig. 7. Alternating current series circuit.

across the terminals a and b, where ω is the angular frequency. The amplitude of the electric current I_{ab} is governed by the impedance Z of the circuit, which can be obtained from the component values in the circuit through the equation

$$Z(R,\omega,L,C) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \quad (4)$$

The identification problem consists in approximating this function in the input range

$$0 \le R \le 100\ \Omega$$
$$20 \le f \le 280\ \text{Hzs}$$
$$0 \le L \le 1\ \text{Hs}$$
$$1 \le C \le 11\ \mu\text{Fs}$$

where $\omega = 2\pi f$.

1) Data Base: The fuzzy model used to describe the system has four fuzzy input variables (R, f, L, and C) and one output variable (Z). The fuzzy domains for all the variables were the same as in the previous example (Fig. 5). The real domains of the input variables were normalized by means of the scale factors $sf_R = 100$, $sf_f = 260$, $sf_L = 1$, and $sf_C = 10$ and the shift constants $sh_R = 0$, $sh_f = 20$, $sh_L = 0$, and $sh_C = 1$. The varying range of the impedance for the input subspace under consideration could be determined from (4) and it was normalized by means of a scale factor $sf_Z = 1763$ and a shift constant $sh_Z = 0$.

2) Experiments: As mentioned previously, in this experiments the results from FRIwE method were compared additionally with the ones presented in [13] generated by the MARS algorithm. The approximation capacity of the methods was evaluated in this case through the scaled mean square error:

$$\text{SMSE} = \frac{\sum_{i=1}^{N}\left[f(\mathbf{x}_i) - \hat{f}(\mathbf{x}_i)\right]^2}{N \cdot Var\ f(\mathbf{x})}$$

where Var is the variance, because it was the index used by Friedman in his paper. As in the previous example, five different test sets where used for each model, but now with 1000 examples each ($N = 1000$) due to the higher dimensionality.

Training sets with 50, 100, 200, and 400 examples were considered and 20 simulations were run for each size. The results obtained from MARS were extracted directly from [13], available for training sets with 100, 200, and 400 examples.

Again, for error estimation purposes, the models from the Wang and Mendel's method were completed with rules having as consequents the central value of the output fuzzy domain.

3) Results: Table VI (top) shows the accuracy results obtained for this example, using again a minimum threshold ($\tau = 0$) in the FRIwE method. The results were similar to the ones obtained in the previous example, achieving the FRIwE method the best fitting with the system for all the training set sizes (results are not available for MARS algorithm with $N = 50$).

Table VI (bottom) shows the interpretability results. The MARS method is not compared, since it does not provide the model as a set of rules. As in the previous example, the FRIwE method achieves higher compactness in the description of the model.

In order to assert the best fitting of the FRIwE method, Fig. 8 shows the global error obtained for each simulation with $N = 400$ and the ones obtained from the Wang and Mendel's methodm, along with its standard deviations. In all the simulations, the proposed method results in a better performance, and the standard deviations provide a high degree of credibility for the calculated errors. Similar results were obtained for the rest of training set sizes.

TABLE VI
RESULTS FOR THE ALTERNATING CURRENT SERIES CIRCUIT

	Training set size			
	50	**100**	**200**	**400**
W&M	1.26(.06)	.97(.05)	.69 (.04)	.48 (.02)
MARS	—	.28(.17)	.12 (.06)	.067(.015)
FRIwE ($\tau = 0$)	**.30 (.08)**	**.17(.04)**	**.090(.010)**	**.062(.008)**

	Training set size			
	50	**100**	**200**	**400**
W&M	47	89	162	270
FRIwE ($\tau = 0$)	**43**	**55**	**68**	**74**

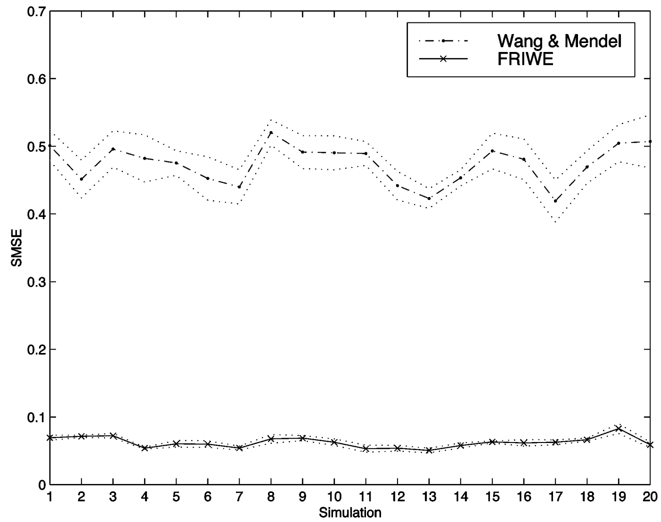

Fig. 8. SMSE variations ($N = 400$).

VI. Conclusion

In this paper, a method is proposed for fuzzy model identification with maximal rules. On the one hand, this method tries to extract as much as possible information from the training examples. For that aim, it makes use of them, firstly, for the extraction of a set of initial rules and, secondly, for solving the conflicts caused by the amplification of the initial rules. This maximum exploitation of the training examples makes the method suitable in situations where the training set reveals insufficient for other methods, either due to the lack of examples, or because of the high dimensionality of the input space.

On the other hand, the model interpretability must be considered essential in the fuzzy model identification framework, since that linguistic interpretability is the feature that distinguishes these techniques as opposed to others. In this sense, the inclusion of exceptions into the rules is proposed as the method for representing the resolution of conflicts, which leads to a more compact model description. Furthermore, several strategies are proposed that increase the model interpretability even more, such as rule reduction, rule merging, and exception merging.

VII. Future Research

One of the main research lines currently under consideration is directed toward reducing the computational cost of the method, mainly in systems with a high dimensionality (more than five or six input variables). As possible solutions currently under consideration, the authors propose a preprocessing that allows to select the most significant input variables, or the use of low-dimensional expansion techniques to decompose the system in a set of simpler subsystems, each of them establishing a relation between a subset of the inputs and the output.

The limitation in the number of exceptions that a rule have associated is another issue to be considered. Rules with an excessive number of exceptions could conflict with the interpretability goal. With this aim, the inclusion of an exception to a rule could be represented either by adding a new exception to the rule or by dissecting the rule in two new rules with a few exceptions attached, depending on some interpretability criterium.

Other alternatives consist in either optimizing the proposed method, perhaps achieving a compromise between the model interpretability and the computational cost, or looking for certainty measures more efficient than the proposed by Ishibuchi *et al.*, which has a considerable computational cost when applied to large training sets.

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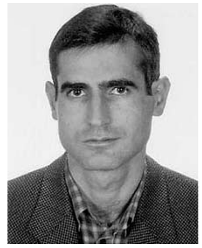

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