VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

Representation of uncertain multichannel digital signal spaces and study of pattern recognition based on metrics and difference values on fuzzy n-cell number spaces

Representation of Uncertain Multi-Channel Digital Signal Spaces and Study of Pattern Recognition Based on Metrics and Difference Values on Fuzzy $n-$cell Number Spaces

Guixiang Wang, Peng Shi, *Senior Member, IEEE*, and Paul Messenger

Abstract—In this paper, we discuss the problem of characterization for uncertain multi-channel digital signal spaces, propose using fuzzy $n-$cell number space to represent uncertain $n-$channel digital signal space, and put forward a method of constructing such fuzzy $n-$cell numbers. We introduce two new metrics and concepts of certain types of difference values on fuzzy $n-$cell number space, and study their properties. Further, based on the metrics or difference values appropriately defined we put forward an algorithmic version of pattern recognition in an imprecise or uncertain environment, and we also give practical examples to show the application and rationality of the proposed techniques.

Index Terms—Uncertain multi-channel digital signals, Fuzzy $n-$ cell numbers, $n-$ dimensional fuzzy vectors, Metrics, Difference values, Pattern recognition

I. INTRODUCTION

IT is known that in a precise or certain environment, multi-channel digital signals can be represented by elements of multi-dimensional Euclidean space, i.e., crisp multi-dimensional vectors. If however we wish to study multi channel digital signals in an imprecise or uncertain environment, then the signals themselves are imprecise or have no certain bound, and it becomes unwise to use crisp multidimensional vectors to represent them. In this paper, we recommend using fuzzy $n-$ cell numbers to represent imprecise or uncertain multi-channel digital signals, and put forward a method of constructing such fuzzy $n-$cell numbers.

The concept of general fuzzy numbers was introduced by Chang and Zadeh [2] in 1972 with the consideration of the properties of probability functions. Since then both the numbers and the problems in relation to them (see for example [3, 4, 5, 6, 11, 16, 19, 20, 21]) have been widely studied. With the development of theories and applications of fuzzy numbers, this concept becomes more and more important. In [7] Kaleva ever used a special type of $n-$ dimensional fuzzy number, whose sets of cuts are all hyper-rectangles. In 2002 we carefully studied the special type of $n-$ dimensional fuzzy number, and call it fuzzy $n-$ cell number in [14,15]. It has been demonstrated that fuzzy $n-$ cell number is used much more conveniently than general $n-$ dimensional fuzzy numbers in theoretical investigations and some fields of application in [14, 15, 17]. On the other hand, $n-$ dimensional fuzzy vector is also an important concept, which is the Cartesian product of n $1-$ dimensional fuzzy numbers. In 1985, Kaufmann and Gupta [8] already studied fuzzy vectors, soon afterwards, Miyakawa and Nakamura et al. [9,10,12] also studied the problems of theories and applications in relating to fuzzy vectors. In 1997, Butnariu [1] studied Methods of solving optimization problems and linear equations in the space of fuzzy vectors. Recently, we [14] showed that fuzzy $n-$ cell numbers and $n-$ dimensional fuzzy vectors can represent each other, and obtained the representations of the joint membership function and the edge membership functions of a fuzzy $n-$ cell number of each other.

In a previous paper [15], we defined a metric D_L on the fuzzy $n-$ cell number space, and studied its properties. And in paper [14], we again studied this type of metric in regard to two fuzzy $n-$ cell numbers as the form of $n-$ dimensional fuzzy vectors. Although metric D_L can be more conveniently used in applications and theoretical investigations, it has some shortcomings. That is, it has a tendency to be rougher, and can not really characterize the degree of difference of two fuzzy $n-$ cell numbers in some applications (see Example 3.1 in Section 3 of this paper). In this paper, in order to discuss the problem of pattern recognition in an imprecise or uncertain environment based on degree of difference, we define two new metrics and some concepts of difference values on fuzzy $n-$ cell number space, which may better characterize the

degree of difference of two fuzzy n-cell numbers in some applications, and study their properties.

It is well known that pattern recognition is an important field of research. In this aspect many research achievements have been obtained (for example, see [13]). In this paper, as applications of the metrics and difference values (defined by us), we also study the problem of pattern recognition in an imprecise or uncertain environment, put forward an algorithmic version of pattern recognition based on the metrics or difference values (defined by us) of fuzzy n-cell numbers, and also give examples to show the application and rationality of the method.

The organization of the paper is as follows. In Section 2, we give an example to show how to set up fuzzy n-cell numbers to represent imprecise or uncertain multi-channel digital signals. In Section 3, we define two new metrics, and study their properties. In Section 4, we introduce concepts of difference values of two fuzzy n-cell numbers, and examine their properties. In Section 5, an algorithmic version of pattern recognition is given based on the metrics or the difference values defined by us, and examples are also given to show the application and rationality of the method. Finally, in Section 6, we give a brief conclusion of this paper.

II. REPRESENTATIONS OF UNCERTAIN MULTI-CHANNEL DIGITAL SIGNALS

A fuzzy set of the Euclidean space R^n is a function $u : R^n \to [0,1]$. For fuzzy set u, we denote $[u]^r = \{x \in R^n : u(x) \geq r\}$ for $r \in [0,1]$, and $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$ (the closure of $\{x \in R^n : u(x) > 0\}$). If u is a normal and fuzzy convex fuzzy set of R^n, $u(x)$ is upper semi-continuous, and $[u]^0$ is compact, then we call u a n-dimensional fuzzy numbers, and denote the n-dimensional fuzzy number space by E^n. If $u \in E$, and for each $r \in [0,1]$, $[u]^r$ is a hyper rectangle, i.e., there exist $\underline{u_i}(r), \overline{u_i}(r) \in R$ with $\underline{u_i}(r) \leq \overline{u_i}(r)$, ($i = 1,2,\cdots,n$) such that $[u]^r = \prod_{i=1}^{n}[\underline{u_i}(r), \overline{u_i}(r)]$, then we call u is a fuzzy n-cell number, and denote the fuzzy n-cell number space by $L(E^n)$. A n-dimensional fuzzy vector is an ordered class $(u_1, u_2, \cdots, u_n)$, where $u_i \in E$ (i.e., E^1), $i = 1,2,\cdots,n$. In [14], We have shown that fuzzy n-cell numbers and n-dimensional fuzzy vectors can represent each other, and as the representation is unique, $L(E^n)$ and the n-dimensional fuzzy vector space (i.e., the Cartesian product $\overbrace{E \times E \times \cdots \times E}^{n}$) may be regarded as identical.

When exploring and discussing some quantity, properties or laws of movement of phenomena/objects in the physical world, it is essential for us to establish the description space of them. For instance, when the quantity in question is only the one with a single factor, we can take it as a dot in real number field R, that is, the space of quantities corresponding to single factor

can be described by 1-dimensional Euclidean space R. Similarly, we can describe the quantities with n factors, using n-dimensional Euclidean space R^n. However, in the physical world, many phenomena are imprecise or uncertain (such as, have no certain bound). When the quantity discussed by us possesses some imprecise or uncertain attributes, it is unsuitable that we use still R^n to represent the space of the quantities (see Remark 2.1). It is our opinion that using the fuzzy n-cell number space discussed in [14, 15] to describe the quantities with some uncertain factors and discuss these quantities in this n-dimensional fuzzy vector space is a more suitable method to reveal the objective laws of things in physical world (see Remark 2.1).

In the following example, we demonstrate how we construct a fuzzy n-cell number to represent a quantity that possesses some uncertain attributes based on statistical data. About the algorithmic version of such fuzzy n-cell numbers, we can see the first or second step of the algorithmic version in Section 5.

Example 2.1. It is well known that different kinds of terrain or landcover possess the different reflections of the electromagnetic spectrum. Based on this principle, one can set up a method to recognize the category of landcover, a challenging remote sensing classification problem, using spectral and terrain features for vegetation classification in some zone. In remote sensing classification, the colligation of all species covering a zone of 4500 m^2 can be boiled down to an element of remote sensing space. We use "Korean Pine accounts for the main part" to denote forest that mainly contains Korean Pines. Because in different "Korean Pine accounts for the main part" areas, there are many different factors such as the difference of the density of Korean Pines, of the species and quantity of other plants, of the physiognomy and so on, the values of reflections of the electromagnetic spectrum are also different. Therefore "Korean Pine accounts for the main part" should not be a certain crisp value but a fuzzy set without certain bound. So, using a fuzzy number to represent the spectral sensitivity level of the "Korean Pine accounts for the main part" is more suitable than using a crisp number. Suppose that we use 4 wave bands: MSS-4, MSS-5, MSS-6, MSS-7. We take 10 samples, and acquire the following data for some zone of "Korean Pine accounts for the main part":

	MSS -4	MSS -5	MSS -6	MSS -7
Sample 1	15.01	13.30	40.50	19.37
Sample 2	15.60	12.56	38.81	16.35
Sample 3	15.82	12.79	37.70	18.16
Sample 4	14.90	11.70	35.50	14.75
Sample 5	16.10	13.80	42.10	20.75
Sample 6	13.80	11.94	32.10	15.54
Sample 7	15.90	10.98	30.87	14.29
Sample 8	16.82	13.67	37.64	18.62
Sample 9	15.50	12.58	36.10	18.02
Sample 10	15.38	12.48	34.08	17.45

We can directly work out the following means μ_i ($i = 1,2,3,4$) and standard deviations σ_i ($i = 1,2,3,4$) from the data:

MSS-4   MSS-5   MSS-6   MSS-7

$\mu_i:$ $\mu_1 = 15.46$ $\mu_2 = 12.58$ $\mu_3 = 36.54$ $\mu_4 = 17.33$

$\sigma_i:$ $\sigma_1 = 1.22$ $\sigma_2 = 0.88$ $\sigma_3 = 3.55$ $\sigma_4 = 2.08$

From the means and the standard deviations, with

$$u_i(x_i) = \begin{cases} \dfrac{x_i - (\mu_i - 2\sigma_i)}{2\sigma_i} & \text{if} \quad x_i \in [\mu_i - 2\sigma_i, \mu_i] \cap (0, +\infty) \\[2mm] \dfrac{(\mu_i + 2\sigma_i) - x_i}{2\sigma_i} & \text{if} \quad x_i \in (\mu_i, \mu_i + 2\sigma_i] \cap (0, +\infty) \\[2mm] 0 & \text{if} \quad x_i \notin [\mu_i - 2\sigma_i, \mu_i + 2\sigma_i] \cap (0, +\infty) \end{cases}$$

$$(i = 1,2,3,4)$$

we can define 4 triangular model one-dimensional fuzzy numbers u_1, u_2, u_3 and u_4 that respectively correspond with MSS-4, MSS-5, MSS-6 and MSS-7:

$$u_1(x_1) = \begin{cases} \dfrac{x_1 - 13.02}{2.44} & \text{if} \quad x_1 \in [13.02, 15.46] \\[2mm] \dfrac{17.9 - x_1}{2.44} & \text{if} \quad x_1 \in (15.46, 17.9] \\[2mm] 0 & \text{if} \quad x_1 \notin [13.02, 17.9] \end{cases}$$

$$u_2(x_2) = \begin{cases} \dfrac{x_2 - 10.82}{1.76} & \text{if} \quad x_2 \in [10.82, 12.58] \\[2mm] \dfrac{14.34 - x_2}{1.76} & \text{if} \quad x_2 \in (12.58, 14.34] \\[2mm] 0 & \text{if} \quad x_2 \notin [10.82, 14.34] \end{cases}$$

$$u_3(x_3) = \begin{cases} \dfrac{x_3 - 29.44}{7.1} & \text{if} \quad x_3 \in [29.44, 36.54] \\[2mm] \dfrac{43.64 - x_3}{7.1} & \text{if} \quad x_3 \in (36.54, 43.64] \\[2mm] 0 & \text{if} \quad x_3 \notin [29.44, 43.64] \end{cases}$$

$$u_4(x_4) = \begin{cases} \dfrac{x_4 - 13.17}{4.16} & \text{if} \quad x_4 \in [13.17, 17.33] \\[2mm] \dfrac{21.49 - x_4}{4.16} & \text{if} \quad x_4 \in (17.33, 21.49] \\[2mm] 0 & \text{if} \quad x_4 \notin [13.17, 21.49] \end{cases}$$

By Theorem 3.1 and 3.2 in [14], we know that u_1, u_2, u_3 and u_4 determine a fuzzy $4-$cell number $u = (u_1, u_2, u_3, u_4)$, and the membership function of u is

$$u(x_1, x_2, x_3, x_4) = \min\{u_1(x_1), u_2(x_2), u_3(x_3), u_4(x_4)\}$$

$$(x_1, x_2, x_3, x_4) \in R^4$$

Then u can be used to represent "Korean Pine accounts for the main part".

Likewise, from the means and the standard deviations, according to

$$v_i(x_i) = \begin{cases} \exp(-\dfrac{(x_i - \mu_i)^2}{2\sigma_i^2}) & \text{if} \quad x_i \in (0, +\infty) \\[2mm] 0 & \text{if} \quad x_i \notin (0, +\infty) \end{cases} \quad i = 1,2,3,4$$

we can also define 4 Gaussian model one-dimensional fuzzy numbers v_1, v_2, v_3 and v_4 that respectively correspond with MSS-4, MSS-5, MSS-6 and MSS-7:

$$v_1(x_1) = \begin{cases} \exp(-\dfrac{(x_1 - 15.46)^2}{2.98}) & \text{if} \quad x_1 \in (0, +\infty) \\[2mm] 0 & \text{if} \quad x_1 \notin (0, +\infty) \end{cases}$$

$$v_2(x_2) = \begin{cases} \exp(-\dfrac{(x_2 - 12.58)^2}{1.56}) & \text{if} \quad x_2 \in (0, +\infty) \\[2mm] 0 & \text{if} \quad x_2 \notin (0, +\infty) \end{cases}$$

$$v_3(x_3) = \begin{cases} \exp(-\dfrac{(x_3 - 36.54)^2}{25.21}) & \text{if} \quad x_3 \in (0, +\infty) \\[2mm] 0 & \text{if} \quad x_3 \notin (0, +\infty) \end{cases}$$

$$v_4(x_4) = \begin{cases} \exp(-\dfrac{(x_4 - 17.33)^2}{8.65}) & \text{if} \quad x_4 \in (0, +\infty) \\[2mm] 0 & \text{if} \quad x_4 \notin (0, +\infty) \end{cases}$$

and obtain the membership function of the fuzzy $4-$cell number $v = (v_1, v_2, v_3, v_4)$ determined by v_1, v_2, v_3 and v_4 as $v(x_1, x_2, x_3, x_4) = \min\{v_1(x_1), v_2(x_2), v_3(x_3), v_4(x_4)\}$, $(x_1, x_2, x_3, x_4) \in R^4$. Then the fuzzy $4-$cell number v can also be used to represent the "Korean Pine accounts for the main part".

Remark 2.1. Of course, if the quantity to describe is precise and certain, we should use a crisp multi-dimensional vector to represent it. However, if the quantity to describe is imprecise and uncertain, such as "Korean Pine accounts for the main part", then using a fuzzy $n-$cell number to represent it is better than using a crisp $n-$dimensional vector. If we narrowly use a crisp multi-dimensional vector, such as $(15.46, 12.58, 36.54, 17.33)$ (i.e., the mean vector), to represent "Korean Pine accounts for the main part", then it can not clearly tell us the relationship of "Korean Pine accounts for the main part" and the zone whose value of reflection of electromagnetic spectrum is $(15.16, 12.80, 37.50, 16.79)$ since $(15.16, 12.80, 37.50, 16.79)$ $\neq (15.46, 12.58, 36.54, 17.33)$. If we use fuzzy $n-$cell number $v = (v_1, v_2, v_3, v_4)$ to represent it, then we can almost affirm that the zone whose value is $(15.16, 12.80, 37.50, 16.79)$ is part of "Korean Pine accounts for the main part" since $v(15.16, 12.80, 37.50, 16.79) = \min(0.94, 0.94, 0.93, 0.93) = 0.93$, i.e., the degree of the zone which is "Korean Pine accounts for the main part" is 0.93.

III. METRICS ON FUZZY $n-$CELL NUMBER SPACE

In [3], the authors studied the metric $d_p(\cdot, \cdot)$ (note that in this paper we rewrite $d_p(\cdot, \cdot)$ as $D_p(\cdot, \cdot)$) on general $n-$dimensional fuzzy number space E^n, which is defined by $D_p(u, v) = (\int_0^1 [d([u]^r, [v]^r)]^p \, dr)^{1/p}$ for any $u, v \in E^n$, and point out that the metric D_p is complete.

In [15], we studied the metrics D and D_L on $L(E^n)$, but the two metrics seem to be 'rough' in certain applications (see Example 3.1). In the following, other metrics are defined on $L(E^n)$, which better reveal the difference between two different uncertain quantities (see Example 3.1). Their properties are also discussed such that they may be used appropriately.

We denote $LC(R^n) = \{A :$ there exist $a_i \leq b_i$, $i = 1,2,\cdots,n$ such that $A = \prod_{i=1}^{n}[a_i, b_i]\}$, where $\prod_{i=1}^{n}[a_i, b_i]$ is the Cartesian product $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Theorem 3.1. We define mappings

$$\hat{d}_\alpha : LC(R^n) \times LC(R^n) \rightarrow [0,+\infty)$$

and

$$\tilde{d}_\alpha : LC(R^n) \times LC(R^n) \rightarrow [0,+\infty)$$

by

$$\hat{d}_\alpha(A,B) = \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{a_i} - \underline{b_i}|, |\overline{a_i} - \overline{b_i}|\}$$

and

$$\tilde{d}_\alpha(A,B) = \sum_{i=1}^{n} \alpha_i \frac{|\underline{a_i} - \underline{b_i}| + |\overline{a_i} - \overline{b_i}|}{2}$$

for any $A = \prod_{i=1}^{n}[\underline{a_i}, \overline{a_i}] \in LC(R^n)$ and $B = \prod_{i=1}^{n}[\underline{b_i}, \overline{b_i}] \in LC(R^n)$, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ satisfies $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0$, $i = 1,2,\cdots,n$. Then for any $A = \prod_{i=1}^{n}[\underline{a_i}, \overline{a_i}]$, $B = \prod_{i=1}^{n}[\underline{b_i}, \overline{b_i}]$, $C = \prod_{i=1}^{n}[\underline{c_i}, \overline{c_i}]$ in $LC(R^n)$ and each $k \in R$, $\hat{d}_\alpha$ and $\tilde{d}_\alpha$ satisfies

(1) $\hat{d}_\alpha(A,B) = \hat{d}_\alpha(B,A)$ and $\tilde{d}_\alpha(A,B) = \tilde{d}_\alpha(B,A)$;

(2) $\hat{d}_\alpha(A,B) \geq 0$ and $\tilde{d}_\alpha(A,B) \geq 0$;

(3) $\hat{d}_\alpha(A,B) = 0 \Leftrightarrow A = B \Leftrightarrow \tilde{d}_\alpha(A,B) = 0$;

(4) $\hat{d}_\alpha(A,B) \leq \hat{d}_\alpha(A,C) + \hat{d}_\alpha(C,B)$ and $\tilde{d}_\alpha(A,B) \leq \tilde{d}_\alpha(A,C) + \tilde{d}_\alpha(C,B)$;

(5) $\hat{d}_\alpha(A+C, B+C) = \hat{d}_\alpha(A,B)$ and $\tilde{d}_\alpha(A+C, B+C) = \tilde{d}_\alpha(A,B)$;

(6) $\hat{d}_\alpha(kA, kB) = |k| \hat{d}_\alpha(A,B)$ and $\tilde{d}_\alpha(kA, kB) = |k| \tilde{d}_\alpha(A,B)$.

Proof. We only show Proofs (4), (5) and (6) (the other proofs are easy). From

$$\hat{d}_\alpha(A,B) = \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{a_i} - \underline{b_i}|, |\overline{a_i} - \overline{b_i}|\}$$
$$\leq \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{a_i} - \underline{c_i}| + |\underline{c_i} - \underline{b_i}|, |\overline{a_i} - \overline{c_i}| + |\overline{c_i} - \overline{b_i}|\}$$
$$\leq \sum_{i=1}^{n} \alpha_i \cdot (\max\{|\underline{a_i} - \underline{c_i}|, |\overline{a_i} - \overline{c_i}|\} + \max\{|\underline{c_i} - \underline{b_i}|, |\overline{c_i} - \overline{b_i}|\})$$
$$= \hat{d}_\alpha(A,C) + \hat{d}_\alpha(C,B)$$

$$\hat{d}_\alpha(A+C, B+C) = \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{a_i} + \underline{c_i} - \underline{b_i} + \underline{c_i}|, |\overline{a_i} + \overline{c_i} - \overline{b_i} + \overline{c_i}|\}$$
$$= \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{a_i} + \underline{c_i} - \underline{b_i} - \underline{c_i}|, |\overline{a_i} + \overline{c_i} - \overline{b_i} - \overline{c_i}|\}$$
$$= \hat{d}_\alpha(A,B)$$

$$\hat{d}_\alpha(kA, kB) = \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{ka_i} - \underline{kb_i}|, |\overline{ka_i} - \overline{kb_i}|\}$$
$$= \sum_{i=1}^{n} \alpha_i \cdot \max\{|k\underline{a_i} - k\underline{b_i}|, |k\overline{a_i} - k\overline{b_i}|\}$$
$$= |k| \hat{d}_\alpha(A,B)$$

we see that (4), (5) and (6) of the theorem hold for $\hat{d}_\alpha$. For $\tilde{d}_\alpha$, we can similarly prove that (4), (5) and (6) of the theorem also hold. □

Theorem 3.2. We define mappings

$$\hat{D}_{\alpha,p} : L(E^n) \times L(E^n) \rightarrow [0,+\infty)$$

and

$$\tilde{D}_{\alpha,p} : L(E^n) \times L(E^n) \rightarrow [0,+\infty)$$

by

$$\hat{D}_{\alpha,p}(u,v) = (\int_0^1 [r \cdot \hat{d}_\alpha([u]^r, [v]^r)]^p \, dr)^{1/p}$$

and

$$\tilde{D}_{\alpha,p}(u,v) = (\int_0^1 [r \cdot \tilde{d}_\alpha([u]^r, [v]^r)]^p \, dr)^{1/p}$$

i.e.,

$$\hat{D}_{\alpha,p}(u,v)$$
$$= (\int_0^1 [r \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{u_i}(r) - \underline{v_i}(r)|, |\overline{u_i}(r) - \overline{v_i}(r)|\}]^p \, dr)^{1/p}$$

and

$$\tilde{D}_{\alpha,p}(u,v)$$
$$= \frac{1}{2}(\int_0^1 [r \sum_{i=1}^{n} \alpha_i \cdot (|\underline{u_i}(r) - \underline{v_i}(r)| + |\overline{u_i}(r) - \overline{v_i}(r)|)]^p \, dr)^{1/p}$$

for any $(u,v) \in L(E^n) \times L(E^n)$, where $p \geq 1$, and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ satisfies $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i > 0$, $i = 1,2,\cdots,n$. Then for any $u, v, w \in L(E^n)$ and each $k \in R$, $\hat{D}_{\alpha,p}$ and $\tilde{D}_{\alpha,p}$ satisfy

(1) $\hat{D}_{\alpha,p}(u,v) = \hat{D}_{\alpha,p}(v,u)$ and $\tilde{D}_{\alpha,p}(u,v) = \tilde{D}_{\alpha,p}(v,u)$;

(2) $\hat{D}_{\alpha,p}(u,v) \geq 0$ and $\tilde{D}_{\alpha,p}(u,v) \geq 0$;

(3) $\hat{D}_{\alpha,p}(u,v) = 0 \Leftrightarrow u = v \Leftrightarrow \tilde{D}_{\alpha,p}(u,v) = 0$;

(4) $\hat{D}_{\alpha,p}(u,v) \leq \hat{D}_{\alpha,p}(u,w) + \hat{D}_{\alpha,p}(w,v)$ and $\tilde{D}_{\alpha,p}(u,v) \leq \tilde{D}_{\alpha,p}(u,w) + \tilde{D}_{\alpha,p}(w,v)$;

(5) $\hat{D}_{\alpha,p}(u+w, v+w) = \hat{D}_{\alpha,p}(u,v)$ and $\tilde{D}_{\alpha,p}(u+w, v+w) = \tilde{D}_{\alpha,p}(u,v)$;

(6) $\hat{D}_{\alpha,p}(ku, kv) = |k| \hat{D}_{\alpha,p}(u,v)$ and $\tilde{D}_{\alpha,p}(ku, kv) = |k| \tilde{D}_{\alpha,p}(u,v)$.

Proof. It is obvious that (1) and (2) of the theorem hold.

By the definition of $\hat{D}_{\alpha,p}$, it is obvious that $u = v \Rightarrow \hat{D}_{\alpha,p}(u,v) = 0$. Otherwise, let $\hat{D}_{\alpha,p}(u,v) = 0$. Then we have $\int_0^1 [r \sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{u_i}(r) - \underline{v_i}(r)|, |\overline{u_i}(r) - \overline{v_i}(r)|\}]^p \, dr = 0$. Taking note of $\alpha_i > 0$, we see that $r(\sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{u_i}(r) - \underline{v_i}(r)|, |\overline{u_i}(r) - \overline{v_i}(r)|\}) = 0$ holds for r almost everywhere on $[0,1]$. Further, we have that $\sum_{i=1}^{n} \alpha_i \cdot \max\{|\underline{u_i}(r) - \underline{v_i}(r)|, |\overline{u_i}(r) - \overline{v_i}(r)|\} = 0$ holds for r almost everywhere on $[0,1]$, so we can see that $\underline{u_i}(r) = \underline{v_i}(r)$ and $\overline{u_i}(r) = \overline{v_i}(r)$ holds for r almost everywhere on $[0,1]$ for $i = 1,2,\cdots,n$. Therefore, we obtain that $[u]^r = [v]^r$ holds for r almost everywhere on $[0,1]$, so we know that $u = v$ holds by Lemma 2.1 in [18]. Thus, $\hat{D}_{\alpha,p}(u,v) = 0 \Leftrightarrow u = v$ holds. Likewise, we can prove

$\widetilde{D}_{\alpha,p}(u,v)=0 \iff u=v$, so (3) of the theorem holds. From Theorem 3.1, we have

$$\hat{D}_{\alpha,p}(u,v)=(\int_0^1[r\sum_{i=1}^n\alpha_i\cdot\max\{|\underline{u_i}(r)-\underline{v_i}(r)|,|\overline{u_i}(r)-\overline{v_i}(r)|\}]^p\,dr)^{1/p}$$

$$\leq[\int_0^1(r\cdot[\hat{d}_\alpha([u]^r,[w]^r)+\hat{d}_\alpha([w]^r,[v]^r)])^p\,dr]^{1/p}$$

$$\leq(\int_0^1[r\cdot\hat{d}_\alpha([u]^r,[w]^r)]^p\,dr)^{1/p}+(\int_0^1[r\cdot\hat{d}_\alpha([w]^r,[v]^r)]^p\,dr)^{1/p}$$

$$=\hat{D}_{\alpha,p}(u,w)+\hat{D}_{\alpha,p}(w,v)$$

The proofs of $\widetilde{D}_{\alpha,p}(u,v)\leq\widetilde{D}_{\alpha,p}(u,w)+\widetilde{D}_{\alpha,p}(w,v)$, (5) and (6) can be similarly proved. $\square$

Remark 3.1. From Theorems 3.1 and 3.2, we know $\hat{d}_\alpha,\widetilde{d}_\alpha$ and $\hat{D}_{\alpha,p},\widetilde{D}_{\alpha,p}$ are metrics on $LC(R^n)$ and $L(E^n)$, respectively, and satisfy translation invariance, absolute homogeneity. Also, from the factor r of the integrands in the definitions of $\hat{D}_{\alpha,p}(u,v)$ and $\widetilde{D}_{\alpha,p}(u,v)$, we can see that the bigger the degrees of the points are, which belong to the fuzzy $n-$cell numbers u and v, the greater the effects on the metric of u and v. This is true in reality.

Example 3.1. Let u,v,w be the $2-$cell numbers defined by: $u=(u_1,u_2)$, $v=(v_1,v_2)$ and $w=(w_1,w_2)$, where,

$$u_i(x)=\begin{cases}x & \text{if } x\in[0,1]\\2-x & \text{if } x\in(1,2]\\0 & \text{if } x\notin[0,2]\end{cases},\ v_i(x)=\begin{cases}x-2 & \text{if } x\in[2,3]\\4-x & \text{if } x\in(3,4]\\0 & \text{if } x\notin[2,4]\end{cases}$$

$$(i=1,2)$$

$$w_1(x)=\begin{cases}x & \text{if } x\in[0,1]\\2-x & \text{if } x\in(1,2]\\0 & \text{if } x\notin[0,2]\end{cases},\ w_2(x)=\begin{cases}x-2 & \text{if } x\in[2,3]\\4-x & \text{if } x\in(3,4]\\0 & \text{if } x\notin[2,4]\end{cases}$$

Then we know that $\underline{u_i}(r)=r$, $\overline{u_i}(r)=2-r$, $\underline{v_i}(r)=2+r$, $\overline{v_i}(r)=4-r$ ($i=1,2$), $\underline{w_1}(r)=r$, $\overline{w_1}(r)=2-r$, $\underline{w_2}(r)=2+r$ and $\overline{w_2}(r)=4-r$ for $r\in[0,1]$. From the definitions of D_L, we have $D_L(u,v)=2=D_L(u,w)$, i.e., D_L can not tell us the difference of $D_L(u,v)$ and $D_L(u,w)$, so we say that D_L seems to be 'rough' (similar proof for D). However, as a matter of fact, $D_L(u,v)$ and $D_L(u,w)$ should have some difference. Taking $\alpha=(1/2,\ 1/2)$, from the definitions of $\widetilde{D}_{\alpha,p}$, we can obtain $\widetilde{D}_{\alpha,p}(u,v)=\dfrac{2}{(1+p)^{1/p}}\ >\ \dfrac{1}{(1+p)^{1/p}}=\widetilde{D}_{\alpha,p}(u,w)$, this accord with fact.

If we restrain the metric D_p (i.e. $d_p(\cdot,\cdot)$ defined by Diamond in [3], see paragraph 1 of this section) on general $n-$dimensional fuzzy number space E^n into on $L(E^n)$, then it also becomes a metric on $L(E^n)$. In the following, we give the relationships of the metrics $\hat{D}_{\alpha,p},\widetilde{D}_{\alpha,p}$ and D_p.

Theorem 3.3. Metrics $\hat{D}_{\alpha,p},\widetilde{D}_{\alpha,p}$ and D_p satisfy

(1) $\dfrac{1}{2}\hat{D}_{\alpha,p}\leq\widetilde{D}_{\alpha,p}\leq\hat{D}_{\alpha,p}\leq D_p\leq D$, i.e., $\dfrac{1}{2}\hat{D}_{\alpha,p}(u,v)\leq\widetilde{D}_{\alpha,p}(u,v)$

$\leq\hat{D}_{\alpha,p}(u,v)\leq D_p(u,v)\leq D(u,v)$ for any $u,v\in L(E^n)$ (D is discussed in [15]).

(2) $\hat{D}_{\alpha,p}\leq\dfrac{1}{(p+1)^{1/p}}D_L\leq\dfrac{1}{(p+1)^{1/p}}D$, i.e., $\hat{D}_{\alpha,p}(u,v)$

$\leq\dfrac{1}{(p+1)^{1/p}}D_L(u,v)\leq\dfrac{1}{(p+1)^{1/p}}D(u,v)$ for any $u,v\in L(E^n)$.

Proof. For any $u,v\in L(E^n)$ and $r\in[0,1]$, by the definitions of $\hat{d}_\alpha$ and $\widetilde{d}_\alpha$, we have

$$\frac{1}{2}\hat{d}_\alpha([u]^r,[v]^r)$$

$$=\frac{1}{2}\sum_{i=1}^n\alpha_i\cdot\max\{|\underline{u_i}(r)-\underline{v_i}(r)|,|\overline{u_i}(r)-\overline{v_i}(r)|\}$$

$$=\frac{1}{2}\sum_{i=1}^n\alpha_i\cdot\frac{|\underline{u_i}(r)-\underline{v_i}(r)|+|\overline{u_i}(r)-\overline{v_i}(r)|+\big||\underline{u_i}(r)-\underline{v_i}(r)|-|\overline{u_i}(r)-\overline{v_i}(r)|\big|}{2}$$

$$\leq\sum_{i=1}^n\alpha_i\cdot\frac{|\underline{u_i}(r)-\underline{v_i}(r)|+|\overline{u_i}(r)-\overline{v_i}(r)|+|\underline{u_i}(r)-\underline{v_i}(r)|+|\overline{u_i}(r)-\overline{v_i}(r)|}{4}$$

$$\leq\sum_{i=1}^n\alpha_i\cdot\frac{|\underline{u_i}(r)-\underline{v_i}(r)|+|\overline{u_i}(r)-\overline{v_i}(r)|}{2}\quad(=\widetilde{d}_\alpha([u]^r,[v]^r))$$

$$\leq\sum_{i=1}^n\alpha_i\cdot\frac{2\max\{|\underline{u_i}(r)-\underline{v_i}(r)|,|\overline{u_i}(r)-\overline{v_i}(r)|\}}{2}$$

$$=\hat{d}_\alpha([u]^r,[v]^r)$$

From this, we can directly obtain $\dfrac{1}{2}\hat{D}_{\alpha,p}(u,v)\leq\widetilde{D}_{\alpha,p}(u,v)$

$\leq\hat{D}_{\alpha,p}(u,v)$.

By Theorem 4.4 in [15], we know $d_L\leq d\leq\sqrt{n}d_L$, where, $d_L(A,B)=\max_{1\leq i\leq n}\{|\underline{a_i}-\underline{b_i}|,|\overline{a_i}-\overline{b_i}|\}$ for any $A=\prod_{i=1}^n[\underline{a_i},\overline{a_i}]\in LC(R^n)$ and $B=\prod_{i=1}^n[\underline{b_i},\overline{b_i}]\in LC(R^n)$. Therefore, for any $u,v\in L(E^n)$, we have

$$\hat{D}_{\alpha,p}(u,v)$$

$$=(\int_0^1[r\sum_{i=1}^n\alpha_i\cdot\max\{|\underline{u_i}(r)-\underline{v_i}(r)|,|\overline{u_i}(r)-\overline{v_i}(r)|\}]^p\,dr)^{1/p}$$

$$\leq(\int_0^1[r\sum_{i=1}^n\alpha_i\cdot\max_{1\leq j\leq n}\{|\underline{u_j}(r)-\underline{v_j}(r)|,|\overline{u_j}(r)-\overline{v_j}(r)|\}]^p\,dr)^{1/p}$$

$$=(\int_0^1[r\cdot d_L([u]^r,[v]^r)\sum_{i=1}^n\alpha_i]^p\,dr)^{1/p}$$

$$\leq(\int_0^1[d([u]^r,[v]^r)]^p\,dr)^{1/p}\ (=D_p(u,v))$$

$$\leq(\int_0^1[\sup_{r\in[0,1]}d([u]^r,[v]^r)]^p\,dr)^{1/p}$$

$$=(\int_0^1[D(u,v)]^p\,dr)^{1/p}$$

$$\leq D(u,v)$$

Thus, we also obtain $\hat{D}_{\alpha,p}(u,v)\leq D_p(u,v)\leq D(u,v)$ and the proof of (1) of the theorem is complete.

From

$$\hat{D}_{\alpha,p}(u,v)$$

$$=(\int_0^1[r\sum_{i=1}^n\alpha_i\cdot\max\{|\underline{u_i}(r)-\underline{v_i}(r)|,|\overline{u_i}(r)-\overline{v_i}(r)|\}]^p\,dr)^{1/p}$$

$$\leq (\int_0^1 [r \cdot d_L([u]^r,[v]^r)]^p \, dr)^{1/p}$$

$$\leq (\int_0^1 [r \cdot \sup_{r \in [0,1]} d_L([u]^r,[v]^r)]^p \, dr)^{1/p}$$

$$= (\int_0^1 [r D_L(u,v)]^p \, dr)^{1/p}$$

$$\leq D(u,v)(\int_0^1 r^p \, dr)^{1/p}$$

$$= \frac{1}{(p+1)^{1/p}} D(u,v)$$

and Theorem 4.5 in [15], we can see that (2) of the theorem holds. $\square$

Remark 3.2. From (1) of Theorem 3.3, we see $\frac{1}{2}\hat{D}_{\alpha,p} \leq \widetilde{D}_{\alpha,p} \leq \hat{D}_{\alpha,p}$, i.e., $\hat{D}_{\alpha,p}$ and $\widetilde{D}_{\alpha,p}$ are equivalent, so we know that $\hat{D}_{\alpha,p}$ and $\widetilde{D}_{\alpha,p}$ induce equivalent topologies on $L(E^n)$ by the knowledge of topological space.

IV. DIFFERENCE VALUES ON FUZZY $n-$CELL NUMBER SPACE

In Section 3, we discussed metrics on $L(E^n)$. But sometimes these have some shortcomings demonstrating the difference of two objects. For example, we consider that the degree of difference of 1 and 2 is bigger than the degree of difference of 10^{10} and $10^{10}+1$ though their metrics (Euclidean metric) measure both are 1. A mapping from the Cartesian product $X \times X$ of a set X into R needs to satisfy stronger conditions in order that it can become a metric, and this brings limitations in some applications. The measure used to characterize the differences does not need to satisfy all metric conditions, for example, when we set up a method of pattern recognition basing on the principle of minimal difference (i.e. the principle of maximal likelihood), the measure used to characterize the differences does not need to satisfy all metric conditions. To conveniently set up methods of pattern recognition using fuzzy $n-$cell numbers, we introduce the concepts of difference values on $L(E^n)$, and study the properties.

Let $u \in L(E^n)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in R^n$ satisfy $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$ ($i=1,2,\cdots,n$). We denote $M_\alpha(u) = \sum_{i=1}^n \alpha_i \int_0^1 r[\underline{u_i}(r) + \overline{u_i}(r)] dr$, denote $\overline{M}(u) = M_\alpha(u)$ as $\alpha = (\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$, and denote $M(u) = M_1(u)$ as $u \in E$.

Definition 4.1. Let $u,v \in L(E^n)$ with $M_\alpha(u)$, $M_\alpha(v) \geq 0$ and $M_\alpha(u) + M_\alpha(v) \neq 0$. We denote

$$L_{\alpha,a}(u,v) = \frac{\sum_{i=1}^n \alpha_i \int_0^1 2r|\underline{u_i}(r) - \underline{v_i}(r)| \, dr}{[M_\alpha(u) + M_\alpha(v)]^a}$$

and

$$R_{\alpha,a}(u,v) = \frac{\sum_{i=1}^n \alpha_i \int_0^1 2r|\overline{u_i}(r) - \overline{v_i}(r)| \, dr}{[M_\alpha(u) + M_\alpha(v)]^a}$$

and call $L_{\alpha,a}(u,v)$ and $R_{\alpha,a}(u,v)$ a left difference value and a right difference value of u and v (with respect to the weight α and parameter a), respectively. And we denote

$$\Delta_{\alpha,a}(u,v) = \frac{1}{2}[L_{\alpha,a}(u,v) + R_{\alpha,a}(u,v)]$$

i.e.,

$$\Delta_{\alpha,a}(u,v) = \frac{\sum_{i=1}^n \alpha_i \int_0^1 r|(\underline{u_i}(r) - \underline{v_i}(r)| + |\overline{u_i}(r) - \overline{v_i}(r)|]dr}{\left(\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{u_i}(r) + \underline{v_i}(r) + \overline{u_i}(r) + \overline{v_i}(r)]dr\right)^a}$$

and call $\Delta_{\alpha,a}(u,v)$ a difference value of u and v (with respect to the weight α and parameter a), where, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in R^n$ with $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$ ($i=1,2,\cdots,n$), and $a \in (0,+\infty)$. Specially, we denote $\overline{\Delta}_a(u,v) = \Delta_{\alpha,a}(u,v)$ as $\alpha = (\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$, and $\Delta_a(u,v) = \Delta_{1,a}(u,v)$ as $u,v \in E$.

Remark 4.1. (1) Generally speaking, we consider that the degree of the difference of two numbers is related not only by the metric of them but also by the sizes of them. As the metrics are the same, the bigger the sizes of the two numbers are, the smaller the degree of their difference is. The denominator $[M_\alpha(u) + M_\alpha(v)]^a$ in the definition of $\Delta_{\alpha,a}$ just plays the action (see Example 4.1), and the exponent a in $[M_\alpha(u) + M_\alpha(v)]^a$ can be properly chosen accordingly to the case in question. (2) Taking the note of that $\sum_{i=1}^n \alpha_i \left(\frac{M(u_i) + M(v_i)}{M_\alpha(u) + M_\alpha(v)}\right)^a = 1$ holds as $\alpha = (\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$ and $a = 1$, and $\sum_{i=1}^n \alpha_i \left(\frac{M(u_i) + M(v_i)}{M_\alpha(u) + M_\alpha(v)}\right)^a = 1$ does not necessarily hold as $\alpha \neq (\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$ or $a \neq 1$, from

$$\Delta_{\alpha,a}(u,v)$$

$$= \frac{\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r) - \underline{v_i}(r)| + |\overline{u_i}(r) - \overline{v_i}(r)|]dr}{\left(\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{u_i}(r) + \underline{v_i}(r) + \overline{u_i}(r) + \overline{v_i}(r)]dr\right)^a}$$

$$= \sum_{i=1}^n \alpha_i \left(\frac{\int_0^1 r[\underline{u_i}(r) + \underline{v_i}(r) + \overline{u_i}(r) + \overline{v_i}(r)]dr}{\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{u_i}(r) + \underline{v_i}(r) + \overline{u_i}(r) + \overline{v_i}(r)]dr}\right)^a$$

$$\cdot \frac{\int_0^1 r[|\underline{u_i}(r) - \underline{v_i}(r)| + |\overline{u_i}(r) - \overline{v_i}(r)|]dr}{\left(\int_0^1 r[\underline{u_i}(r) + \underline{v_i}(r) + \overline{u_i}(r) + \overline{v_i}(r)]dr\right)^a}$$

$$= \sum_{i=1}^n \alpha_i \left(\frac{M(u_i) + M(v_i)}{M_\alpha(u) + M_\alpha(v)}\right)^a \Delta_a(u_i, v_i)$$

we can directly see that $\overline{\Delta}_1(u,v)$ is a convex combination of $\Delta_1(u_i,v_i)$, $i=1,2,\cdots,n$, but $\Delta_{\alpha,a}(u,v)$ is not necessarily a convex combination of $\Delta_a(u_i,v_i)$, $i=1,2,\cdots,n$ as $\alpha \neq (\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$ or $a \neq 1$.

Example 4.1. Let u,v,u',v' be the $2-$ cell numbers defined by: $u=(u_1,u_2)$, $v=(v_1,v_2)$, $u'=(u'_1,u'_2)$ and $v'=(v'_1,v'_2)$, where,

$$u_i(x_i) = \begin{cases} x_i & \text{if} & x_i \in [0,1] \\ 2-x_i & \text{if} & x_i \in (1,2] \\ 0 & \text{if} & x_i \notin [0,2] \end{cases}$$

$$v_i(x_i) = \begin{cases} -10+x_i & \text{if} & x_i \in [10,11] \\ 12-x_i & \text{if} & x_i \in (11,12] \\ 0 & \text{if} & x_i \notin [10,12] \end{cases}$$

$$u'_i(x_i) = \begin{cases} -100+x_i & \text{if} & x_i \in [100,101] \\ 102-x_i & \text{if} & x_i \in (101,102] \\ 0 & \text{if} & x_i \notin [100,102] \end{cases}$$

$$v'_i(x_i) = \begin{cases} -110+x_i & \text{if} & x_i \in [10,11] \\ 112-x_i & \text{if} & x_i \in (11,12] \\ 0 & \text{if} & x_i \notin [10,12] \end{cases}$$

and $i=1,2$. Then we know that $\underline{u_i}(r)=r$, $\overline{u_i}(r)=2-r$, $\underline{v_i}(r)=10+r$, $\overline{v_i}(r)=12-r$, $\underline{u'_i}(r)=100+r$, $\overline{u'_i}(r)=102-r$, $\underline{v'_i}(r)=110+r$, $\overline{v'_i}(r)=112-r$ ($r \in [0,1]$, $i=1,2$). From the definitions of $\widetilde{D}_{\alpha,p}$ and $\Delta_{\alpha,a}$, we can obtain

$$\widetilde{D}_{\alpha,p}(u,v)$$
$$= \frac{1}{2}\Big[\int_0^1 [r(\alpha_1(|r-10-r|+|2-r-12+r|) + \alpha_2(|r-10-r|+|2-r-12+r|))]^p dr\Big]^{1/p}$$
$$= \frac{1}{2}\Big(\frac{20^p}{(1+p)}\Big)^{1/p}$$

$$\widetilde{D}_{\alpha,p}(u',v')$$
$$= \frac{1}{2}\Big[\int_0^1 [r(\alpha_1(|100+r-110-r|+|102-r-112+r|) + \alpha_2(|100+r-110-r|+|102-r-112+r|))]^p dr\Big]^{1/p}$$
$$= \frac{1}{2}\Big(\frac{20^p}{(1+p)}\Big)^{1/p}$$

$$\Delta_{\alpha,a}(u,v)$$
$$= \frac{\alpha_1\int_0^1 r(|r-10-r|+|2-r-12+r|)dr + \alpha_2\int_0^1 r(|r-10-r|+|2-r-12+r|)dr}{\Big(\alpha_1\int_0^1 r(r+10+r+2-r+12-r)dr + \alpha_2\int_0^1 r(r+10+r+2-r+12-r)dr\Big)^a}$$
$$= \frac{10}{12^a}$$

$$\Delta_{\alpha,a}(u',v')$$
$$= \frac{\alpha_1\int_0^1 r(|100+r-110-r|+|102-r-112+r|)dr + \alpha_2\int_0^1 r(|100+r-110-r|+|102-r-112+r|)dr}{\Big(\alpha_1\int_0^1 r(100+r+110+r+102-r+112-r)dr + \alpha_2\int_0^1 r(100+r+110+r+102-r+112-r)dr\Big)^a}$$
$$= \frac{10}{212^a}$$

so we have $\widetilde{D}_{\alpha,p}(u,v)=\widetilde{D}_{\alpha,p}(u',v')$ and $\Delta_{\alpha,a}(u,v)>\Delta_{\alpha,a}(u',v')$.

Property 4.1. Let $u,v \in L(E^n)$ with $M_\alpha(u)$, $M_\alpha(v) \geq 0$ and $M_\alpha(u)+M_\alpha(v) \neq 0$, $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n) \in R^n$ with $\sum_{i=1}^n \alpha_i=1$ and $\alpha_i > 0$ ($i=1,2,\cdots,n$), and $a \in (0,+\infty)$. Then

(1) $\Delta_{\alpha,a}(u,v) \geq 0$;

(2) $\Delta_{\alpha,a}(u,v)=0$ if and only if $u=v$;

(3) $\Delta_{\alpha,a}(u,v)=\Delta_{\alpha,a}(v,u)$;

(4) $\Delta_{\alpha,a}(u,w) \geq \Delta_{\alpha,a}(v,w)$ for any $w \in L(E^n)$ and $u \leq v \leq w$;

(5) $\Delta_{\alpha,a}(u+w,v+w) \leq \Delta_{\alpha,a}(u,v)$ for any $w \in L(E^n)$ and $w \geq (\hat{0},\hat{0},\cdots,\hat{0})$;

(6) $\Delta_{\alpha,a}(ku,kv)=k^{1-a}\Delta_{\alpha,a}(u,v)$ for any $k > 0$.

Proof. It is obvious that the conclusions (1) and (3) hold. The proof of conclusion (2) can also be completed by imitating the proof of (3) of Theorem 3.2 by using Lemma 2.1 in [18].

From $u \leq v \leq w$, we know that $\underline{u_i}(r) \leq \underline{v_i}(r) \leq \underline{w_i}(r)$ and $\overline{u_i}(r) \leq \overline{v_i}(r) \leq \overline{w_i}(r)$ ($i=1,2,\cdots,n$), so $|\underline{u_i}(r)-\underline{w_i}(r)|$ $\geq|\underline{v_i}(r)-\underline{w_i}(r)|$ and $|\overline{u_i}(r)-\overline{w_i}(r)|\geq|\overline{v_i}(r)-\overline{w_i}(r)|$. Therefore, we have

$$\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)-\underline{w_i}(r)|+|\overline{u_i}(r)-\overline{w_i}(r)|]dr$$
$$\geq \sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{v_i}(r)-\underline{w_i}(r)|+|\overline{v_i}(r)-\overline{w_i}(r)|]dr$$

On the other hand, from $\underline{u_i}(r) \leq \underline{v_i}(r) \leq \underline{w_i}(r)$ and $\overline{u_i}(r) \leq \overline{v_i}(r) \leq \overline{w_i}(r)$ ($i=1,2,\cdots,n$), we can also see that $M_\alpha(u)$ $= \sum_{i=1}^n \alpha_i \int_0^1 r[\underline{u_i}(r)+\overline{u_i}(r)]dr \leq \sum_{i=1}^n \alpha_i \int_0^1 r[\underline{v_i}(r)+\overline{v_i}(r)]dr=M_\alpha(v)$, so we can obtain $0 \leq M_\alpha(u) \leq M_\alpha(v) \leq M_\alpha(w)$. Thus, we have

$$\Delta_{\alpha,a}(u,w) = \frac{\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)-\underline{w_i}(r)|+|\overline{u_i}(r)-\overline{w_i}(r)|]dr}{(M_\alpha(u)+M_\alpha(w))^a}$$
$$\geq \frac{\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{v_i}(r)-\underline{w_i}(r)|+|\overline{v_i}(r)-\overline{w_i}(r)|]dr}{(M_\alpha(v)+M_\alpha(w))^a}$$
$$= \Delta_{\alpha,a}(v,w)$$

so conclusion (4) holds.

From $w \geq \hat{0}$, we know $M_\alpha(w) \geq 0$, so we have
$$\Delta_{\alpha,a}(u+w,v+w)$$
$$= \frac{\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{(u+w)_i}(r)-\underline{(v+w)_i}(r)|+|\overline{(u+w)_i}(r)-\overline{(v+w)_i}(r)|]dr}{\Big(\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{(u+w)_i}(r)+\underline{(v+w)_i}(r)+\overline{(u+w)_i}(r)+\overline{(v+w)_i}(r)]dr\Big)^a}$$
$$= \frac{\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)+\underline{w_i}(r)-\underline{v_i}(r)-\underline{w_i}(r)|+|\overline{u_i}(r)+\overline{w_i}(r)-\overline{v_i}(r)-\overline{w_i}(r)|]dr}{\Big(\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{u_i}(r)+\underline{w_i}(r)+\underline{v_i}(r)+\underline{w_i}(r)+\overline{u_i}(r)+\overline{w_i}(r)+\overline{v_i}(r)+\overline{w_i}(r)]dr\Big)^a}$$
$$= \frac{\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)-\underline{v_i}(r)|+|\overline{u_i}(r)-\overline{v_i}(r)|]dr}{(M_\alpha(u)+M_\alpha(v)+2M_\alpha(w))^a}$$

$$\leq \frac{\sum_{i=1}^{n}\alpha_i\int_0^1 r[|\underline{u}_i(r)-\underline{v}_i(r)|+|\overline{u}_i(r)-\overline{v}_i(r)|]dr}{\left(M_\alpha(u)+M_\alpha(v)\right)^a}$$

$$=\Delta_{\alpha,a}(u,v)$$

i.e., conclusion (5) holds.

For any $k>0$, we have

$$\Delta_{\alpha,a}(ku,kv)$$

$$=\frac{\sum_{i=1}^{n}\alpha_i\int_0^1 r[|\underline{(ku)}_i(r)-\underline{(kv)}_i(r)|+|\overline{(ku)}_i(r)-\overline{(kv)}_i(r)|]dr}{\left(\sum_{i=1}^{n}\alpha_i\int_0^1 r[\underline{(ku)}_i(r)+\underline{(kv)}_i(r)+\overline{(ku)}_i(r)+\overline{(kv)}_i(r)]dr\right)^a}$$

$$=\frac{k\sum_{i=1}^{n}\alpha_i\int_0^1 r[|\underline{u}_i(r)-\underline{v}_i(r)|+|\overline{u}_i(r)-\overline{v}_i(r)|]dr}{k^a\left(\sum_{i=1}^{n}\alpha_i\int_0^1 r[\underline{u}_i(r)+\underline{v}_i(r)+\overline{u}_i(r)+\overline{v}_i(r)]dr\right)^a}$$

$$=k^{1-a}\Delta_{\alpha,a}(u,v)$$

so conclusion (6) holds. Therefore, the proof of the theorem is completed. □

Remark 4.2. (1) Although the conclusion (4) of Theorem 4.1 holds, $u\leq v\leq w$ does not imply $\Delta_{\alpha,a}(u,w)\geq\Delta_{\alpha,a}(u,v)$ (see Example 4.2). Comparing

$$\Delta_{\alpha,a}(u,w)=\frac{\sum_{i=1}^{n}\alpha_i\int_0^1 r[|\underline{u}_i(r)-\underline{w}_i(r)|+|\overline{u}_i(r)-\overline{w}_i(r)|]dr}{\left(M_\alpha(u)+M_\alpha(w)\right)^a}$$

to

$$\Delta_{\alpha,a}(u,v)=\frac{\sum_{i=1}^{n}\alpha_i\int_0^1 r[|\underline{u}_i(r)-\underline{v}_i(r)|+|\overline{u}_i(r)-\overline{v}_i(r)|]dr}{\left(M_\alpha(u)+M_\alpha(v)\right)^a}$$

we can see that the untruth of $\Delta_{\alpha,a}(u,w)\geq\Delta_{\alpha,a}(u,v)$ is caused only by $\left(M_\alpha(u)+M_\alpha(w)\right)^a\geq\left(M_\alpha(u)+M_\alpha(v)\right)^a$, and when $\left(M_\alpha(u)+M_\alpha(w)\right)^a$ and $\left(M_\alpha(u)+M_\alpha(v)\right)^a$ are properly smaller, $u\leq v\leq w$ can imply $\Delta_{\alpha,a}(u,w)\geq\Delta_{\alpha,a}(u,v)$. So, in general, we may choose a in (0,1] such that $\Delta_{\alpha,a}$ can reasonably characterize the degree of the difference of two fuzzy $n-$cell numbers.

(2) Generally speaking, the difference value $\Delta_{\alpha,a}$ does not satisfy the property of the triangular inequality, i.e., the inequality $\Delta_{\alpha,a}(u,v)\leq\Delta_{\alpha,a}(u,w)+\Delta_{\alpha,a}(w,v)$ does not necessarily hold for $u,v,w\in L(E^n)$. Example 4.3 can show it.

Example 4.2. Let u,v,w be the $2-$cell numbers defined by: $u=(\hat{0},\hat{0})$, $v=(v_1,v_2)$ and $w=(w_1,w_2)$, where

$$v_i(x_i)=\begin{cases}-1+x_i & \text{if} \quad x_i\in[1,2]\\3-x_i & \text{if} \quad x_i\in(2,3]\\0 & \text{if} \quad x_i\notin[1,3]\end{cases}, \quad w_i(x_i)=\begin{cases}-2+x_i & \text{if} \quad x_i\in[2,3]\\4-x_i & \text{if} \quad x_i\in(3,4]\\0 & \text{if} \quad x_i\notin[2,4]\end{cases}$$

and $i=1,2$. Then we know that $\underline{u}_i(r)=0$, $\overline{u}_i(r)=0$, $\underline{v}_i(r)=1+r$, $\overline{v}_i(r)=3-r$, $\underline{w}_i(r)=2+r$, and $\overline{w}_i(r)=4-r$ ($r\in[0,1]$, $i=1,2$), so we have $u\leq v\leq w$, but we can see $\Delta_{\alpha,a}(u,w)\leq\Delta_{\alpha,a}(u,v)$, from

$$\Delta_{\alpha,2}(u,w)$$

$$=\frac{\alpha_1\int_0^1 r[|0-2-r|+|0-4+r|]dr+\alpha_2\int_0^1 r[|0-2-r|+|0-4+r|]dr}{\left(\alpha_1\int_0^1 r[0+2+r+0+4-r]dr+\alpha_2\int_0^1 r[0+2+r+0+4-r]dr\right)^2}$$

$$=\frac{1}{3}$$

and

$$\Delta_{\alpha,2}(u,v)$$

$$=\frac{\alpha_1\int_0^1 r(|0-1-r|+|0-3+r|)dr+\alpha_2\int_0^1 r(|0-1-r|+|0-3+r|)dr}{\left(\alpha_1\int_0^1 r(0+1+r+0+3-r)dr+\alpha_2\int_0^1 r(0+1+r+0+3-r)dr\right)^2}$$

$$=\frac{1}{2}$$

Example 4.3. Let u,v,w be the $2-$cell numbers defined by: $u=(u_1,u_2)$, $v=(\hat{1},\hat{1})$ and $w=(\hat{2},\hat{2})$, where

$$u_i(x_i)=\begin{cases}\dfrac{19}{10}+x_i & \text{if} \quad x_i\in[-\dfrac{19}{10},-\dfrac{9}{10}]\\[2mm]\dfrac{1}{10}-x_i & \text{if} \quad x_i\in(-\dfrac{9}{10},\dfrac{1}{10}]\\[2mm]0 & \text{if} \quad x_i\notin[-\dfrac{19}{10},\dfrac{1}{10}]\end{cases}$$

and $i=1,2$. Then we know that $\underline{u}_i(r)=-\dfrac{19}{10}+r$, $\overline{u}_i(r)=\dfrac{1}{10}-r$, $\underline{v}_i(r)=1$, $\overline{v}_i(r)=1$, $\underline{w}_i(r)=2$, $\overline{w}_i(r)=2$ ($r\in[0,1]$, $i=1,2$), so we have $\Delta_{\alpha,1}(u,v)=19$, $\Delta_{\alpha,1}(u,w)=\dfrac{29}{11}$ and $\Delta_{\alpha,1}(v,w)=\dfrac{1}{3}$, it implies $\Delta_{\alpha,1}(u,v)>\Delta_{\alpha,1}(u,w)+\Delta_{\alpha,1}(v,w)$.

Example 4.4. Let u,v,w be the $2-$cell numbers (see Fig.1) defined by: $u=(u_1,u_2)$, $v=(\hat{2},\hat{2})$ and $w=(w_1,w_2)$, where

$$u_i(x_i)=\begin{cases}x_i & \text{if} \quad x_i\in[0,1]\\2-x_i & \text{if} \quad x_i\in(1,2]\\0 & \text{if} \quad x_i\notin[0,2]\end{cases}$$

$$w_i(x_i)=\begin{cases}-1+x_i & \text{if} \quad x_i\in[1,2]\\3-x_i & \text{if} \quad x_i\in(2,3]\\0 & \text{if} \quad x_i\notin[1,3]\end{cases}$$

and $i=1,2$. Then we know that $\underline{u}_i(r)=r$, $\overline{u}_i(r)=2-r$, $\underline{v}_i(r)=2$, $\overline{v}_i(r)=2$, $\underline{w}_i(r)=1+r$, $\overline{w}_i(r)=3-r$ ($r\in[0,1]$, $i=1,2$), so we can obtain that $\widetilde{D}_{\alpha,p}(u,v)=\dfrac{1}{2}=\widetilde{D}_{\alpha,p}(u,w)$ and $\Delta_{\alpha,a}(u,v)=\dfrac{2}{3^a}=\Delta_{\alpha,a}(u,w)$. However, it is obvious (see Fig. 4.1) that the degree of the difference of u and v is different from the degree of the difference of u and w.
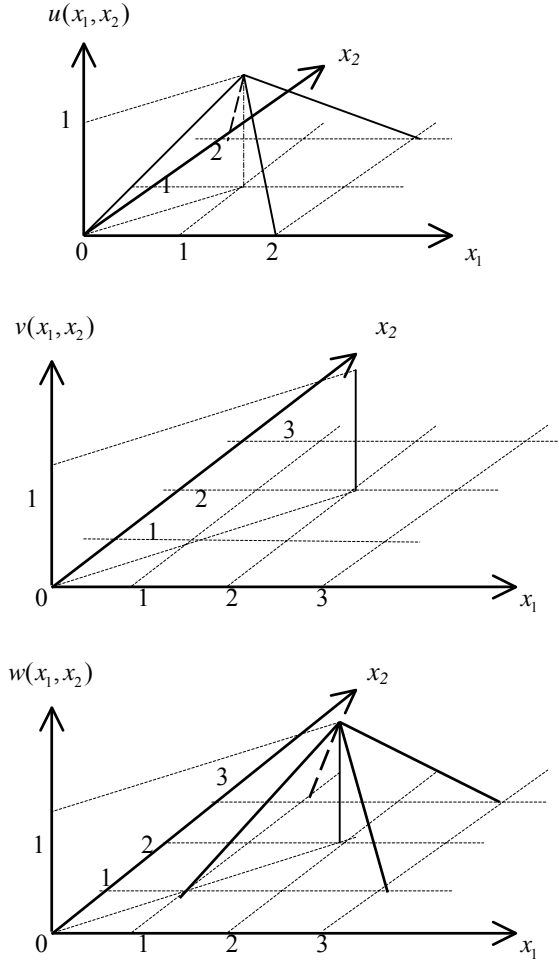
Fig. 1. Fuzzy $2-$ cell numbers u, v and w in Example 4.4

In fact, sometimes, the degree of the difference of two fuzzy numbers is not only related with the metric and the sizes of them, but also related with the degree of fuzzy (we call it fuzzy degree) of them. Example 4.4 shows that for the u,v,w given, the metric $\widetilde{D}_{\alpha,p}$ and the difference value $\Delta_{\alpha,a}$ can not tell us the difference of the degree of the difference u and v with the degree of the difference of u and w since $\widetilde{D}_{\alpha,p}(u,v)$ $=\widetilde{D}_{\alpha,p}(u,w)$ and $\Delta_{\alpha,a}(u,v)=\Delta_{\alpha,a}(u,w)$, but we see that the two degrees of the differences indeed have some differences. In order to overcome the defect, we introduce the following concept.

Definition 4.2. Let $u,v\in L(E^n)$. We denote

$\Lambda_{\alpha,a}(u,v)$

$=\left(\sum_{i=1}^{n}\alpha_i\int_0^1 r[|\,\underline{u_i}(r)-\underline{v_i}(r)\,|+|\,\overline{u_i}(r)-\overline{v_i}(r)\,|]dr\right)$

$\cdot\exp\left(a\sum_{i=1}^{n}\alpha_i(|\int_{\underline{u_i}(0)}^{u_i{}^*(1)}u_i(t)dt-\int_{\underline{v_i}(0)}^{v_i{}^*(1)}v_i(t)dt\,|+|\int_{u_i{}^*(1)}^{\overline{u_i}(0)}u_i(t)dt-\int_{v_i{}^*(1)}^{\overline{v_i}(0)}v_i(t)dt\,|)\right)$

and call $\Lambda_{\alpha,a}(u,v)$ a difference value of u and v (with respect to the weight α and parameter a), where, $u_i^*(1)=\frac{\underline{u_i}(1)+\overline{u_i}(1)}{2}$, $v_i^*(1)=\frac{\underline{v_i}(1)+\overline{v_i}(1)}{2}$, and $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$

$\in R^n$ with $\sum_{i=1}^{n}\alpha_i=1$ and $\alpha_i\geq 0$ ($i=1,2,\cdots,n$), and $a\in(0,+\infty)$. We denote $\overline{\Lambda}_a(u,v)=\Lambda_{\alpha,a}(u,v)$ as $\alpha=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$, and $\Lambda_a(u,v)=\Lambda_{1,a}(u,v)$ as $u,v\in E$.

Example 4.5. Let u,v,w be the $2-$ cell numbers defined in Example 4.4. Then

$\Lambda_{\alpha,1}(u,v)$

$=\left(\sum_{i=1}^{2}\alpha_i\int_0^1 r(|\,r-2\,|+|\,2-r-2\,|)dr\right)$

$\cdot\exp\left(\sum_{i=1}^{2}\alpha_i(|\int_0^1 tdt-\int_2^2 1dt\,|+|\int_1^2(2-t)dt-\int_2^2 1dt\,|)\right)$

$=\int_0^1 2rdr\cdot\exp(1)$

$=e$

and

$\Lambda_{\alpha,1}(u,w)$

$=\left(\sum_{i=1}^{2}\alpha_i\int_0^1 r(|\,r-1-r\,|+|\,2-r-3+r\,|)dr\right)$

$\cdot\exp\left(\sum_{i=1}^{2}\alpha_i(|\int_0^1 tdt-\int_1^2(t-1)dt\,|+|\int_1^2(2-t)dt-\int_2^3(3-t)dt\,|)\right)$

$=\int_0^1 2rdr\cdot\exp(0)$

$=1$

so we see $\Lambda_{\alpha,1}(u,v)>\Lambda_{\alpha,1}(u,w)$. Therefore, in this case, the difference value $\Lambda_{\alpha,a}$ is more suitable than metrics and difference value $\Delta_{\alpha,a}$ to characterize the degree of the difference of two fuzzy $n-$ cell numbers.

Property 4.2. Let $u,v\in L(E^n)$, $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in R^n$ with $\sum_{i=1}^{n}\alpha_i=1$ and $\alpha_i>0$ ($i=1,2,\cdots,n$), and $a\in(0,+\infty)$. Then

(1) $\Lambda_{\alpha,a}(u,v)\geq 0$;

(2) $\Lambda_{\alpha,a}(u,v)=0$ if and only if $u=v$;

(3) $\Lambda_{\alpha,a}(u,v)=\Lambda_{\alpha,a}(v,u)$;

(4) $\Lambda_{\alpha,a}(u+\hat{b},v+\hat{b})=\Lambda_{\alpha,a}(u,v)$ for any $b\in R$;

(5) $\Lambda_{\alpha,a}(ku,kv)=|\,k\,|\Lambda_{\alpha,|k|a}(u,v)$ for any $k\in R$.

Proof. It is obvious that the conclusions (1) and (3) hold. The proof of conclusion (2) can also be completed by imitating the proof of (3) of Theorem 3.2 by using Lemma 2.1 in [18].

For any $b\in R$ and $i=1,2,\cdots,n$, we have

$(u_i+\hat{b})(t)=\sup\{r\in[0,1]:t\in[\underline{(u_i+\hat{b})}(r),\overline{(u_i+\hat{b})}(r)]\}$

$=\sup\{r\in[0,1]:t-b\in[\underline{u_i}(r),\overline{u_i}(r)]\}$

$=u_i(t-b)$

hence

$\Lambda_{\alpha,a}(u+\hat{b},v+\hat{b})$

$=\left(\sum_{i=1}^{n}\alpha_i\int_0^1 r\,|\,(\underline{u_i}(r)+b-\underline{v_i}(r)-b\,|+|\,\overline{u_i}(r)+b-\overline{v_i}(r)-b\,|]dr\right)$

$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i (|\int_{\underline{u_i}(0)+b}^{u_i^*(1)+b} (u_i+\hat{b})(t)dt - \int_{\underline{v_i}(0)+b}^{v_i^*(1)+b}(v_i+\hat{b})(t)dt | \right.$$
$$\left. +|\int_{u_i^*(1)+b}^{\overline{u_i}(0)+b}(u_i+\hat{b})(t)dt - \int_{v_i^*(1)+b}^{\overline{v_i}(0)+b}(v_i+\hat{b})(t)dt |) \right)$$

$$=\left(\sum_{i=1}^n \alpha_i \int_0^1 r\,|\,(\underline{u_i}(r)-\underline{v_i}(r)| + |\overline{u_i}(r)-\overline{v_i}(r)|]dr \right)$$
$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i(|\int_{\underline{u_i}(0)}^{u_i^*(1)}(u_i+\hat{b})(s+b)ds - \int_{\underline{v_i}(0)}^{v_i^*(1)}(v_i+\hat{b})(s+b)ds | \right.$$
$$\left. +|\int_{u_i^*(1)}^{\overline{u_i}(0)}(u_i+\hat{b})(s+b)ds - \int_{v_i^*(1)}^{\overline{v_i}(0)}(v_i+\hat{b})(s+b)ds |) \right)$$

$$=\left(\sum_{i=1}^n \alpha_i \int_0^1 r\,|\,(\underline{u_i}(r)-\underline{v_i}(r)| + |\overline{u_i}(r)-\overline{v_i}(r)|]dr \right)$$
$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i(|\int_{\underline{u_i}(0)}^{u_i^*(1)}(u_i)(s)ds - \int_{\underline{v_i}(0)}^{v_i^*(1)}(v_i)(s)ds | \right.$$
$$\left. +|\int_{u_i^*(1)}^{\overline{u_i}(0)}(u_i)(s)ds - \int_{v_i^*(1)}^{\overline{v_i}(0)}(v_i)(s)ds |) \right)$$

$$=\Lambda_{\alpha,a}(u,v)$$

i.e., conclusion (4) holds.

For any $k>0$, we have
$$\Lambda_{\alpha,a}(ku,kv)$$
$$=\left(\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{(ku)_i}(r)-\underline{(kv)_i}(r)| + |\overline{(ku)_i}(r)-\overline{(kv)_i}(r)|]dr \right)$$
$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i(|\int_{\underline{(ku)_i}(0)}^{(ku)_i^*(1)}(ku)_i(t)dt - \int_{\underline{(kv)_i}(0)}^{(kv)_i^*(1)}(kv)_i(t)dt | \right.$$
$$\left. +|\int_{(ku)_i^*(1)}^{\overline{(ku)_i}(0)}(ku)_i(t)dt - \int_{(kv)_i^*(1)}^{\overline{(kv)_i}(0)}(kv)_i(t)dt |) \right)$$

$$=\left(\sum_{i=1}^n \alpha_i \int_0^1 r[|\,k\underline{u_i}(r)-k\underline{v_i}(r)| + |\,k\overline{u_i}(r)-k\overline{v_i}(r)|]dr \right)$$
$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i(|\int_{k\underline{u_i}(0)}^{ku_i^*(1)}u_i(t/k)dt - \int_{k\underline{v_i}(0)}^{kv_i^*(1)}v_i(t/k)dt | \right.$$
$$\left. +|\int_{ku_i^*(1)}^{k\overline{u_i}(0)}u_i(t/k)dt - \int_{kv_i^*(1)}^{k\overline{v_i}(0)}v_i(t/k)dt |) \right)$$

$$=\left(k\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)-\underline{v_i}(r)| + |\overline{u_i}(r)-\overline{v_i}(r)|]dr \right)$$
$$\cdot \exp\left(ka\sum_{i=1}^n \alpha_i(|\int_{\underline{u_i}(0)}^{u_i^*(1)}u_i(s)ds - \int_{\underline{v_i}(0)}^{v_i^*(1)}v_i(s)ds | \right.$$
$$\left. +|\int_{u_i^*(1)}^{\overline{u_i}(0)}u_i(s)ds - \int_{v_i^*(1)}^{\overline{v_i}(0)}v_i(s)ds |) \right)$$

$$=|k|\cdot[\Lambda_{\alpha,|k|a}(u,v)]$$

For any $k<0$, we have
$$\Lambda_{\alpha,a}(ku,kv)$$
$$=\left(\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{(ku)_i}(r)-\underline{(kv)_i}(r)| + |\overline{(ku)_i}(r)-\overline{(kv)_i}(r)|]dr \right)$$
$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i(|\int_{\underline{(ku)_i}(0)}^{(ku)_i^*(1)}(ku)_i(t)dt - \int_{\underline{(kv)_i}(0)}^{(kv)_i^*(1)}(kv)_i(t)dt | \right.$$
$$\left. +|\int_{(ku)_i^*(1)}^{\overline{(ku)_i}(0)}(ku)_i(t)dt - \int_{(kv)_i^*(1)}^{\overline{(kv)_i}(0)}(kv)_i(t)dt |) \right)$$

$$=\left(\sum_{i=1}^n \alpha_i \int_0^1 r[|\,k\overline{u_i}(r)-k\overline{v_i}(r)| + |\,k\underline{u_i}(r)-k\underline{v_i}(r)|]dr \right)$$

$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i(|\int_{k\overline{u_i}(0)}^{ku_i^*(1)}u_i(t/k)dt - \int_{k\overline{v_i}(0)}^{kv_i^*(1)}v_i(t/k)dt | \right.$$
$$\left. +|\int_{ku_i^*(1)}^{k\underline{u_i}(0)}u_i(t/k)dt - \int_{kv_i^*(1)}^{k\underline{v_i}(0)}v_i(t/k)dt |) \right)$$

$$=\left(|k|\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)-\underline{v_i}(r)| + |\overline{u_i}(r)-\overline{v_i}(r)|]dr \right)$$
$$\cdot \exp\left(a\sum_{i=1}^n \alpha_i(|\int_{\underline{u_i}(0)}^{u_i^*(1)}ku_i(s)ds - \int_{\underline{v_i}(0)}^{v_i^*(1)}kv_i(s)ds | \right.$$
$$\left. +|\int_{u_i^*(1)}^{\overline{u_i}(0)}ku_i(s)ds - \int_{v_i^*(1)}^{\overline{v_i}(0)}kv_i(s)ds |) \right)$$

$$=\left(|k|\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)-\underline{v_i}(r)| + |\overline{u_i}(r)-\overline{v_i}(r)|]dr \right)$$
$$\cdot \exp\left(|k|\cdot a\sum_{i=1}^n \alpha_i(|\int_{\underline{u_i}(0)}^{u_i^*(1)}u_i(s)ds - \int_{\underline{v_i}(0)}^{v_i^*(1)}v_i(s)ds | \right.$$
$$\left. +|\int_{u_i^*(1)}^{\overline{u_i}(0)}u_i(s)ds - \int_{v_i^*(1)}^{\overline{v_i}(0)}v_i(s)ds |) \right)$$

$$=|k|\cdot[\Lambda_{\alpha,|k|a}(u,v)]$$

If $k=0$, it is obvious that $\Lambda_{\alpha,a}(ku,kv)=|k|\Lambda_{\alpha,|k|a}(u,v)$ holds, so conclusion (5) holds. Therefore, the proof of the theorem is completed. $\square$

At the end of the section, combining the definitions of $\Delta_{\alpha,a}$ and $\Lambda_{\alpha,a}$, we give the following definition of difference value $\Gamma_{\alpha,a}$.

Definition 4.3. Let $u,v \in L(E^n)$ with $M_\alpha(u)$, $M_\alpha(v) \geq 0$ and $M_\alpha(u)+M_\alpha(v) \neq 0$. We denote
$$\Gamma_{\alpha,a}(u,v)$$
$$=\frac{\exp\left(a_2\sum_{i=1}^n \alpha_i(|\int_{\underline{u_i}(0)}^{u_i^*(1)}u_i(t)dt - \int_{\underline{v_i}(0)}^{v_i^*(1)}v_i(t)dt | + |\int_{u_i^*(1)}^{\overline{u_i}(0)}u_i(t)dt - \int_{v_i^*(1)}^{\overline{v_i}(0)}v_i(t)dt |) \right)}{\left(\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{u_i}(r)+\underline{v_i}(r)+\overline{u_i}(r)+\overline{v_i}(r)]dr \right)^{a_1}}$$
$$\cdot \left(\sum_{i=1}^n \alpha_i \int_0^1 r[|\underline{u_i}(r)-\underline{v_i}(r)| + |\overline{u_i}(r)-\overline{v_i}(r)|]dr \right)$$

and call $\Gamma_{\alpha,a}(u,v)$ a difference value of u and v (with respect to α and $a=(a_1,a_2)$), where, $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n) \in R^n$ with $\sum_{i=1}^n \alpha_i=1$ and $\alpha_i \geq 0$ ($i=1,2,\cdots,n$), and $a=(a_1,a_2) \in (0,+\infty) \times (0,+\infty)$. We denote $\overline{\Gamma}_a(u,v)=\Gamma_{\alpha,a}(u,v)$ as $\alpha=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})$, and $\Gamma_a(u,v)=\Gamma_{1,a}(u,v)$ as $u,v \in E$.

Likewise, we have the following properties about the difference value $\Gamma_{\alpha,a}$.

Property 4.3. Let $u,v \in L(E^n)$ with $M_\alpha(u)$, $M_\alpha(v) \geq 0$ and $M_\alpha(u)+M_\alpha(v) \neq 0$, $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n) \in R^n$ with $\sum_{i=1}^n \alpha_i=1$ and $\alpha_i > 0$ ($i=1,2,\cdots,n$), and $a=(a_1,a_2) \in (0,+\infty) \times (0,+\infty)$. Then

(1) $\Gamma_{\alpha,a}(u,v) \geq 0$;

(2) $\Gamma_{\alpha,a}(u,v)=0$ if and only if $u=v$;

(3) $\Gamma_{\alpha,a}(u,v)=\Gamma_{\alpha,a}(v,u)$;

(4) $\Gamma_{\alpha,a}(u+\hat{b},v+\hat{b}) \le \Gamma_{\alpha,a}(u,v)$ for any $b \in [0,+\infty)$;

(5) $\Gamma_{\alpha,a}(ku,kv) = k^{2-a_1}\Gamma_{\alpha,b}(u,v)$ for any $k \in [0,+\infty)$, where $b = (a_1, ka_2)$.

Proof. The proofs of the properties can be completed similarly with the proofs of Property 4.1 and 4.2, respectively, so we omit it. □

V. PATTERN RECOGNITION BASED ON METRICS AND DIFFERENCE VALUES

In Section 3 and 4, we discussed metrics and difference values on $L(E^n)$. In this section, we put forward an algorithmic version of pattern recognition in an imprecise or uncertain environment based on the metrics and difference values defined by us, and give examples to show the application (see Example 5.1) and rationality (see Example 5.2) of the method.

Consider a problem to identify an object (denoted by O) belonging to some one of l classes (denoted by $C_1, C_2, \cdots, C_l$) in an imprecise or uncertain environment. Let the objects have n characteristics. Since the problem discussed by us take on some imprecise or uncertain attributes, it is unsuitable (see Remark 2.1) that we use a crisp $n-$dimensional vector (i.e., a standard $n-$dimensional real number vector) to express the n character values of C_i ($i=1,2,\cdots,l$) or O. Therefore, using the method of statistics, we construct $l+1$ fuzzy $n-$cell numbers to express the n character values of $C_1, C_2, \cdots, C_l$ and O, respectively, and then put forward an algorithmic version of pattern recognition based on the metrics or the difference values defined by us.

Algorithmic version of pattern recognition based on metrics
The first step:

Depending on the practicality, we first find out one domain of the jth character value of C_i for each i ($i=1,2,\cdots,l$) and j ($j=1,2,\cdots,n$), and denote the said domain by D^i_j.

We arbitrarily take m_i samples in C_i ($i=1,2,\cdots,l$), and gain m_i measure values (denoted by $c^i_1, c^i_2 \cdots, c^i_{m_i}$) of the n characters of the m_i samples, denote $c^i_m = (c^i_{m1}, c^i_{m2}, \cdots, c^i_{mn})$ ($m=1,2,\cdots,m_i$), i.e., we gain the following tables:

$$C_1: \begin{matrix} c^1_{11} & c^1_{12} & \cdots & c^1_{1n} \\ c^1_{21} & c^1_{22} & \cdots & c^1_{2n} \\ \vdots & \vdots & & \vdots \\ c^1_{m_11} & c^1_{m_12} & \cdots & c^1_{m_1n} \end{matrix} \quad C_2: \begin{matrix} c^2_{11} & c^2_{12} & \cdots & c^2_{1n} \\ c^2_{21} & c^2_{22} & \cdots & c^2_{2n} \\ \vdots & \vdots & & \vdots \\ c^2_{m_21} & c^2_{m_22} & \cdots & c^2_{m_2n} \end{matrix}$$

$$\cdots \quad C_l: \begin{matrix} c^l_{11} & c^l_{12} & \cdots & c^l_{1n} \\ c^l_{21} & c^l_{22} & \cdots & c^l_{2n} \\ \vdots & \vdots & & \vdots \\ c^l_{m_l1} & c^l_{m_l2} & \cdots & c^l_{m_ln} \end{matrix}$$

For C_i ($i=1,2,\cdots,l$), we directly work out the following

means $\mu^i_j = \dfrac{1}{m_i}\sum_{k=1}^{m_i} c^i_{kj}$ and standard deviations $\sigma^i_j = \sqrt{\dfrac{1}{m_i-1}\sum_{k=1}^{m_i}(c^i_{kj} - \mu^i_j)^2}$ ($j=1,2,\cdots,n$) of the n character values of C_i ($i=1,2,\cdots,l$), respectively. We construct triangular model $1-$dimensional fuzzy numbers u^i_j ($i=1,2,\cdots,l$, $j=1,2,\cdots,n$) as the following:

$$u^i_j(x) = \begin{cases} \dfrac{x-(\mu^i_j - 2\sigma^i_j)}{2\sigma^i_j} & \text{if} \quad x \in [\mu^i_j - 2\sigma^i_j, \mu^i_j] \bigcap D^i_j \\ \dfrac{(\mu^i_j + 2\sigma^i_j) - x}{2\sigma^i_j} & \text{if} \quad x \in (\mu^i_j, \mu^i_j + 2\sigma^i_j] \bigcap D^i_j \\ 0 & \text{if} \quad x \notin [\mu^i_j - 2\sigma^i_j, \mu^i_j + 2\sigma^i_j] \bigcap D^i_j \end{cases}$$

$$(i=1,2,\cdots,l, \quad j=1,2,\cdots,n)$$

or construct Gaussian model one-dimensional fuzzy numbers v^i_j ($i=1,2,\cdots,l$, $j=1,2,\cdots,n$) as the following

$$v^i_j(x) = \begin{cases} \exp(-\dfrac{(x-\mu^i_j)^2}{2\sigma^{i^2}_j}) & \text{if} \quad x \in D^i_j \\ 0 & \text{if} \quad x \notin D^i_j \end{cases}$$

$$(i=1,2,\cdots,l, \quad j=1,2,\cdots,n)$$

We construct fuzzy $n-$cell numbers $u^i = (u^i_1, u^i_2, \cdots, u^i_n)$ ($u^i(x_1, x_2, \cdots, x_n) = \min\{u^i_1(x_1), u^i_2(x_2), \cdots, u^i_n(x_n)\}$) and $v^i = (v^i_1, v^i_2, \cdots, v^i_n)$ ($v^i(x_1, x_2, \cdots, x_n) = \min\{v^i_1(x_1), v^i_2(x_2), \cdots, v^i_n(x_n)\}$), $i=1,2,\cdots,l$, and use u^i or v^i to express the ith class C_i ($i=1,2,\cdots,l$).

The second step:

For the object O to be recognized, taking t samples in O, we can gain t classes of data about the n characters of O as the following:

$$O: \begin{matrix} o_{11} & o_{12} & \cdots & o_{1n} \\ o_{21} & o_{22} & \cdots & o_{2n} \\ \vdots & \vdots & & \vdots \\ o_{t1} & o_{t2} & \cdots & o_{tn} \end{matrix}$$

We work out the following means (denoted by $\bar{o}_1, \bar{o}_2, \cdots, \bar{o}_n$) and standard deviations (denoted by $s_1, s_2, \cdots, s_n$) of the n character values of O:

$$\bar{o}_i = \frac{1}{t}\sum_{k=1}^{t} o_{ki} \quad (i=1,2,\cdots,n)$$

and

$$s_i = \sqrt{\frac{1}{t-1}\sum_{k=1}^{t}(o_{ki} - \bar{o}_i)^2} \quad (i=1,2,\cdots,n)$$

We construct triangular model one-dimensional fuzzy numbers w_i ($i=1,2,\cdots,n$) as the following:

$$w_i(x) = \begin{cases} \dfrac{x-(\overline{o}_i - 2s_i)}{2s_i} & \text{if} \quad x \in [\overline{o}_i - 2s_i, \ \overline{o}_i] \bigcap (\bigcup_{j=1}^l D_i^j) \\[3mm] \dfrac{(\overline{o}_i + 2s_i) - x}{2s_i} & \text{if} \quad x \in (\overline{o}_i, \ \overline{o}_i + 2s_i] \bigcap (\bigcup_{j=1}^l D_i^j) \\[3mm] 0 & \text{if} \quad x \notin [\overline{o}_i - 2s_i, \ \overline{o}_i + 2s_i] \bigcap (\bigcup_{j=1}^l D_i^j) \end{cases}$$

$$(i = 1, 2, \cdots, n)$$

or construct Gaussian model one-dimensional fuzzy numbers w_i' ($i = 1, 2, \cdots, n$) as the following

$$\begin{cases} v_i'(x) = \exp(-\dfrac{(x-\overline{o}_i)^2}{2s_i^2}) & \text{if} \quad x \in \bigcup_{j=1}^l D_i^j \\[3mm] 0 & \text{if} \quad x \notin \bigcup_{j=1}^l D_i^j \end{cases} \quad (i = 1, 2, \cdots, n)$$

We construct fuzzy $n-$ cell numbers $w = (w_1, w_2, \cdots, w_n)$ ($w(x_1, x_2, \cdots, x_n) = \min\{w_1(x_1), w_2(x_2), \cdots, w_n(x_n)\}$) and $w' = (w_1', w_2', \cdots, w_n')$ ($w'(x_1, x_2, \cdots, x_n) = \min\{w_1'(x_1), w_2'(x_2), \cdots, w_n'(x_n)\}$), and use w or w' to express the object O.

The third step:

Taking proper $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$, $i = 1, 2, \cdots, n$, and $p \geq 1$, we compute the metrics

$$\widetilde{D}_{\alpha,p}(w, u^j)$$
$$= \frac{1}{2}(\int_0^1 [r \sum_{i=1}^n \alpha_i \cdot (|\underline{w_i}(r) - \underline{u_i^j}(r)| + |\overline{w_i}(r) - \overline{u_i^j}(r)|)]^p dr)^{1/p}$$

$$(j = 1, 2, \cdots, l)$$

or

$$\widetilde{D}_{\alpha,p}(w', v^j)$$
$$= \frac{1}{2}(\int_0^1 [r \sum_{i=1}^n \alpha_i \cdot (|\underline{w_i'}(r) - \underline{v_i^j}(r)| + |\overline{w_i'}(r) - \overline{v_i^j}(r)|)]^p dr)^{1/p}$$

$$(j = 1, 2, \cdots, l)$$

The fourth step:

We choose u^{j_0} in $u^1, u^2, \cdots, u^l$, or $v^{j_0'}$ in $v^1, v^2, \cdots, v^l$ such that
$$\widetilde{D}_{\alpha,p}(w, u^{j_0}) = \min\{\widetilde{D}_{\alpha,p}(w, u^1), \widetilde{D}_{\alpha,p}(w, u^2), \cdots, \widetilde{D}_{\alpha,p}(w, u^l)\}$$
or
$$\widetilde{D}_{\alpha,p}(w', v^{j_0'}) = \min\{\widetilde{D}_{\alpha,p}(w', v^1), \widetilde{D}_{\alpha,p}(w', v^2), \cdots, \widetilde{D}_{\alpha,p}(w', v^l)\}$$

Then we can consider that object O belongs to the j_0th class C_{j_0}, or belongs to the j_0'th class $C_{j_0'}$.

Remark 5.1. In the third and fourth steps of the above method, we can have the metric $\hat{D}_{\alpha,p}$ replace the metric $\widetilde{D}_{\alpha,p}$, as a result of which, we can also set up a method based on the metric $\hat{D}_{\alpha,p}$.

Algorithmic version of pattern recognition based on difference values

The first step and the second step:

They are respectively same with the first step and the second step of the method of pattern recognition based on the metric, as above.

The third step:

Taking proper $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$, $i = 1, 2, \cdots, n$, and $a = (a_1, a_2) \in (0, +\infty) \times (0, +\infty)$, we compute the difference values

$$\Gamma_{\alpha,a}(w, u^j)$$
$$= \frac{\exp\left(a_2 \sum_{i=1}^n \alpha_i (|\int_{\underline{w_i}(0)}^{w_i^*(1)} w_i(t)dt - \int_{\underline{u_i^j}(0)}^{u_i^j*(1)} u_i^j(t)dt| + |\int_{w_i^*(1)}^{\overline{w_i}(0)} w_i(t)dt - \int_{u_i^j*(1)}^{\overline{u_i^j}(0)} u_i^j(t)dt|)\right)}{\left(\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{w_i}(r) + \underline{u_i^j}(r) + \overline{w_i}(r) + \overline{u_i^j}(r)]dr\right)^{a_1}}$$
$$\cdot \left(\sum_{i=1}^n \alpha_i \int_0^1 r[|(\underline{w_i}(r) - \underline{u_i^j}(r)| + |\overline{w_i}(r) - \overline{u_i^j}(r)|]dr\right)$$

$$(j = 1, 2, \cdots, l)$$

or

$$\Gamma_{\alpha,a}(w', v^j)$$
$$= \frac{\exp\left(a_2 \sum_{i=1}^n \alpha_i (|\int_{\underline{w_i'}(0)}^{w_i'*(1)} w_i'(t)dt - \int_{\underline{v_i^j}(0)}^{v_i^j*(1)} v_i^j(t)dt| + |\int_{w_i'*(1)}^{\overline{w_i'}(0)} w_i'(t)dt - \int_{v_i^j*(1)}^{\overline{v_i^j}(0)} v_i^j(t)dt|)\right)}{\left(\sum_{i=1}^n \alpha_i \int_0^1 r[\underline{w_i'}(r) + \underline{v_i^j}(r) + \overline{w_i'}(r) + \overline{v_i^j}(r)]dr\right)^{a_1}}$$
$$\cdot \left(\sum_{i=1}^n \alpha_i \int_0^1 r[|(\underline{w_i'}(r) - \underline{v_i^j}(r)| + |\overline{w_i'}(r) - \overline{v_i^j}(r)|]dr\right)$$

$$(j = 1, 2, \cdots, l)$$

The fourth step:

We choose u^{j_0} in $u^1, u^2, \cdots, u^l$, or $v^{j_0'}$ in $v^1, v^2, \cdots, v^l$ such that
$$\Gamma_{\alpha,a}(w, u^{j_0}) = \min\{\Gamma_{\alpha,a}(w, u^1), \Gamma_{\alpha,a}(w, u^2), \cdots, \Gamma_{\alpha,a}(w, u^l)\}$$
or
$$\Gamma_{\alpha,a}(w', v^{j_0'}) = \min\{\Gamma_{\alpha,a}(w', v^1), \Gamma_{\alpha,a}(w', v^2), \cdots, \Gamma_{\alpha,a}(w', v^l)\}$$

Then we can consider that object O belongs to the j_0th class C_{j_0}, or belongs to the j_0'th class $C_{j_0'}$.

Remark 5.2. In the third and fourth steps of the method above, we can have the difference value $\Delta_{\alpha,a}$ or $\Lambda_{\alpha,a}$ replace the difference value $\Gamma_{\alpha,a}$, as a result of, we can also set up a method based on the difference value $\Delta_{\alpha,a}$ or $\Lambda_{\alpha,a}$.

In order to be more obvious, we may use the following diagram to illustrate the methods set up by us.

Measure values of the jth character of samples in C_1, $j=1,2,…,n$	……	Measure values of the jth character of samples in C_l, $j=1,2,…,n$	Measure values of the ith character of samples in O, $i=1,2,…,n$
Mean μ_j^1, $j=1,2,…,n$ Standard deviation σ_j^1, $j=1,2,…,n$	……	Mean μ_j^l, $j=1,2,…,n$ Standard deviation σ_j^l, $j=1,2,…,n$	Mean μ_i, $i=1,2,…,n$ Stan-dard deviation σ_i, $i=1,2,…,n$
1-dimensional fuzzy numbers: u_j^1, $j=1,2,…,n$	……	1-dimensional fuzzy numbers: u_j^l, $j=1,2,…,n$	1-dimensional fuzzy numbers: w_i, $i=1,2,…,n$
Fuzzy n-cell number u_1	……	Fuzzy n-cell number u_l	Fuzzy n-cell number w

Difference value $\Gamma_{\alpha,a}(w, u_1)$	……	Difference value $\Gamma_{\alpha,a}(w, u_l)$

The minimal $\Gamma_{\alpha,a}(w, u_{l_0})$

o belongs to the j_0th class C_{j_0}

Example 5.1. Suppose that some terrain consists of five different types of land based cover: C_1: Road; C_2: Farm or Crop; C_3: Korean Pine accounts for the main part; C_4: Boreal and Broad-leaf Mixture Forest; C_5: Birch Forest. For the five types of land cover (C_1, C_2, C_3, C_4, C_5) and by using the four wave bands: MSS-4, MSS-5, MSS-6, MSS-7, we take 10 samples, and acquire the following data:

		MSS - 4	MSS - 5	MSS - 6	MSS - 7
	Sample 1	18.62	20.71	58.20	26.72
	Sample 2	19.76	17.01	51.02	24.32
	Sample 3	18.24	19.46	48.12	26.33
	Sample 4	18.76	15.95	56.35	22.89
C_1:	Sample 5	18.96	18.78	45.32	28.32
	Sample 6	19.90	20.13	50.82	25.05
	Sample 7	19.16	18.58	52.30	22.21
	Sample 8	19.36	17.32	55.02	25.28
	Sample 9	19.36	17.98	48.26	23.86
	Sample 10	18.48	16.46	46.63	27.46

		MSS - 4	MSS - 5	MSS - 6	MSS - 7
	Sample 1	24.51	28.13	52.39	22.43
	Sample 2	19.37	23.41	45.56	23.89
	Sample 3	23.12	24.82	47.80	18.05
	Sample 4	18.93	18.87	43.88	20.43
C_2:	Sample 5	21.36	32.28	53.61	24.95
	Sample 6	26.06	24.15	42.75	19.38
	Sample 7	17.64	17.08	46.73	21.16
	Sample 8	22.23	26.02	49.17	19.04
	Sample 9	20.26	21.56	41.37	24.39
	Sample 10	25.42	30.52	50.46	22.59

		MSS - 4	MSS - 5	MSS - 6	MSS - 7
	Sample 1	15.01	13.30	40.50	19.37
	Sample 2	15.60	12.56	38.81	16.35
	Sample 3	15.82	12.79	37.70	18.16
	Sample 4	14.90	11.70	35.50	14.75
C_3:	Sample 5	16.10	13.80	42.10	20.75
	Sample 6	13.80	11.94	32.10	15.54
	Sample 7	15.90	10.98	30.87	14.29
	Sample 8	16.82	13.67	37.64	18.62
	Sample 9	15.50	12.58	36.10	18.02
	Sample 10	15.38	12.48	34.08	17.45

		MSS - 4	MSS - 5	MSS - 6	MSS - 7
	Sample 1	17.05	13.53	43.32	22.76
	Sample 2	16.09	12.03	38.65	19.47
	Sample 3	15.44	12.87	42.21	21.20
	Sample 4	16.41	12.58	40.87	18.63
C_4:	Sample 5	16.32	13.01	46.56	23.58
	Sample 6	17.21	13.58	45.98	21.33
	Sample 7	15.98	12.14	41.09	20.23
	Sample 8	16.21	12.68	42.51	21.65
	Sample 9	15.09	12.10	38.03	21.01
	Sample 10	16.38	13.31	44.87	22.34

		MSS - 4	MSS - 5	MSS - 6	MSS - 7
	Sample 1	18.12	13.45	45.76	23.64
	Sample 2	16.41	12.62	44.71	23.19
	Sample 3	17.04	12.88	44.95	22.54
	Sample 4	16.58	13.13	43.43	22.98
C_5:	Sample 5	18.21	13.90	46.38	23.41
	Sample 6	17.23	13.73	45.03	23.76
	Sample 7	15.89	12.98	44.32	23.15
	Sample 8	17.02	13.25	45.38	22.61
	Sample 9	15.97	12.70	43.96	23.11
	Sample 10	17.67	13.34	46.09	23.62

By

$$\mu_j^i = \frac{1}{10}\sum_{k=1}^{10} c_{kj}^i \quad (i=1,2,3,4,5,\ j=1,2,3,4)$$

and

$$\sigma_j^i = \sqrt{\frac{1}{10-1}\sum_{k=1}^{10}(c_{kj}^i - \mu_j^i)^2} \quad (i=1,2,3,4,5,\ j=1,2,3,4)$$

we can work out the following means and standard deviations:

$$\mu_1^1 = 19.06,\ \sigma_1^1 = 0.55,\ \mu_2^1 = 18.24,\ \sigma_2^1 = 1.58$$
$$\mu_3^1 = 51.20,\ \sigma_3^1 = 4.28,\ \mu_4^1 = 25.24,\ \sigma_4^1 = 1.98$$
$$\mu_1^2 = 21.89,\ \sigma_1^2 = 2.88,\ \mu_2^2 = 24.68,\ \sigma_2^2 = 4.82$$
$$\mu_3^2 = 47.37,\ \sigma_3^2 = 4.09,\ \mu_4^2 = 21.63,\ \sigma_4^2 = 2.39$$
$$\mu_1^3 = 15.46,\ \sigma_1^3 = 1.22,\ \mu_2^3 = 12.58,\ \sigma_2^3 = 0.88$$
$$\mu_3^3 = 36.54,\ \sigma_3^3 = 3.55,\ \mu_4^3 = 17.33,\ \sigma_4^3 = 2.08$$
$$\mu_1^4 = 16.22,\ \sigma_1^4 = 0.64,\ \mu_2^4 = 12.78,\ \sigma_2^4 = 0.58$$
$$\mu_3^4 = 42.41,\ \sigma_3^4 = 2.87,\ \mu_4^4 = 21.22,\ \sigma_4^4 = 1.50$$
$$\mu_1^5 = 17.01,\ \sigma_1^5 = 0.82,\ \mu_2^5 = 13.20,\ \sigma_2^5 = 0.42$$
$$\mu_3^5 = 45.00,\ \sigma_3^5 = 0.94,\ \mu_4^5 = 23.20,\ \sigma_4^5 = 0.42$$

Taking $D_i^j = (0, +\infty)$ ($i=1,2,3,4$ and $j=1,2,3,4,5$) then according to

$$u_j^i(x) = \begin{cases} \dfrac{x-(\mu_j^i - 2\sigma_j^i)}{2\sigma_j^i} & \text{if}\quad x \in [\mu_j^i - 2\sigma_j^i, \mu_j^i] \cap D_j^i \\[2mm] \dfrac{(\mu_j^i + 2\sigma_j^i)-x}{2\sigma_j^i} & \text{if}\quad x \in (\mu_j^i, \mu_j^i + 2\sigma_j^i] \cap D_j^i \\[2mm] 0 & \text{if}\quad x \notin [\mu_j^i - 2\sigma_j^i, \mu_j^i + 2\sigma_j^i] \cap D_j^i \end{cases}$$

$$(i=1,2,3,4,5,\ j=1,2,3,4)$$

we have

$$u_1^1(x) = \begin{cases} \dfrac{x-17.96}{1.10} & \text{if}\quad x \in [17.96,\ 19.06] \\[2mm] \dfrac{20.16-x}{1.10} & \text{if}\quad x \in (19.06,\ 20.16] \\[2mm] 0 & \text{if}\quad x \notin [17.96,\ 20.16] \end{cases}, u_2^1(x) = \begin{cases} \dfrac{x-15.08}{3.16} & \text{if}\quad x \in [15.08,\ 18.24] \\[2mm] \dfrac{21.40-x}{3.16} & \text{if}\quad x \in (18.24,\ 21.40] \\[2mm] 0 & \text{if}\quad x \notin [15.08,\ 21.40] \end{cases}$$

$$u_3^1(x) = \begin{cases} \dfrac{x-42.64}{8.56} & \text{if}\quad x \in [42.64,\ 51.20] \\[2mm] \dfrac{59.76-x}{8.56} & \text{if}\quad x \in (51.20,\ 59.76] \\[2mm] 0 & \text{if}\quad x \notin [42.64,\ 59.76] \end{cases}, u_4^1(x) = \begin{cases} \dfrac{x-21.28}{3.96} & \text{if}\quad x \in [21.28,\ 25.24] \\[2mm] \dfrac{29.20-x}{3.96} & \text{if}\quad x \in (25.24,\ 29.20] \\[2mm] 0 & \text{if}\quad x \notin [21.28,\ 29.20] \end{cases}$$

$$u_1^2(x) = \begin{cases} \dfrac{x-16.13}{5.76} & \text{if } x \in [16.13,\ 21.89] \\ \dfrac{27.65-x}{5.76} & \text{if } x \in (21.89,\ 27.65] \\ 0 & \text{if } x \notin [16.13,\ 27.65] \end{cases}, \quad u_2^2(x) = \begin{cases} \dfrac{x-15.04}{9.64} & \text{if } x \in [15.04,\ 24.68] \\ \dfrac{34.32-x}{9.64} & \text{if } x \in (24.68,\ 34.32] \\ 0 & \text{if } x \notin [15.04,\ 34.32] \end{cases}$$

$$u_3^2(x) = \begin{cases} \dfrac{x-39.19}{8.18} & \text{if } x \in [39.19,\ 47.31] \\ \dfrac{55.55-x}{8.18} & \text{if } x \in (47.31,\ 55.55] \\ 0 & \text{if } x \notin [39.19,\ 55.55] \end{cases}, \quad u_4^2(x) = \begin{cases} \dfrac{x-16.85}{4.78} & \text{if } x \in [16.85,\ 21.63] \\ \dfrac{26.41-x}{4.78} & \text{if } x \in (21.63,\ 26.41] \\ 0 & \text{if } x \notin [16.85,\ 26.41] \end{cases}$$

$$u_1^3(x) = \begin{cases} \dfrac{x-13.02}{2.44} & \text{if } x \in [13.02,\ 15.46] \\ \dfrac{17.90-x}{2.44} & \text{if } x \in (15.46,\ 17.90] \\ 0 & \text{if } x \notin [13.02,\ 17.90] \end{cases}, \quad u_2^3(x) = \begin{cases} \dfrac{x-10.82}{1.76} & \text{if } x \in [10.82,\ 12.58] \\ \dfrac{14.34-x}{1.76} & \text{if } x \in (12.58,\ 14.34] \\ 0 & \text{if } x \notin [10.82,\ 14.34] \end{cases}$$

$$u_3^3(x) = \begin{cases} \dfrac{x-29.44}{7.10} & \text{if } x \in [29.44,\ 36.54] \\ \dfrac{43.64-x}{7.10} & \text{if } x \in (36.54,\ 43.64] \\ 0 & \text{if } x \notin [29.44,\ 43.64] \end{cases}, \quad u_4^3(x) = \begin{cases} \dfrac{x-13.17}{4.16} & \text{if } x \in [13.17,\ 17.33] \\ \dfrac{21.49-x}{4.16} & \text{if } x \in (17.33,\ 21.49] \\ 0 & \text{if } x \notin [13.17,\ 21.49] \end{cases}$$

$$u_1^4(x) = \begin{cases} \dfrac{x-14.94}{1.28} & \text{if } x \in [14.94,\ 16.22] \\ \dfrac{17.50-x}{1.28} & \text{if } x \in (16.22,\ 17.50] \\ 0 & \text{if } x \notin [14.94,\ 17.50] \end{cases}, \quad u_2^4(x) = \begin{cases} \dfrac{x-11.62}{1.16} & \text{if } x \in [11.62,\ 12.78] \\ \dfrac{13.94-x}{1.16} & \text{if } x \in (12.78,\ 13.94] \\ 0 & \text{if } x \notin [11.62,\ 13.94] \end{cases}$$

$$u_3^4(x) = \begin{cases} \dfrac{x-36.67}{5.74} & \text{if } x \in [36.67,\ 42.41] \\ \dfrac{48.15-x}{5.74} & \text{if } x \in (42.41,\ 48.15] \\ 0 & \text{if } x \notin [36.67,\ 48.15] \end{cases}, \quad u_4^4(x) = \begin{cases} \dfrac{x-18.22}{3.00} & \text{if } x \in [18.22,\ 21.22] \\ \dfrac{24.22-x}{3.00} & \text{if } x \in (21.22,\ 24.22] \\ 0 & \text{if } x \notin [18.22,\ 24.22] \end{cases}$$

$$u_1^5(x) = \begin{cases} \dfrac{x-15.37}{1.64} & \text{if } x \in [15.37,\ 17.01] \\ \dfrac{18.65-x}{1.64} & \text{if } x \in (17.01,\ 18.65] \\ 0 & \text{if } x \notin [15.37,\ 18.65] \end{cases}, \quad u_2^5(x) = \begin{cases} \dfrac{x-12.36}{0.84} & \text{if } x \in [12.36,\ 13.20] \\ \dfrac{14.04-x}{0.84} & \text{if } x \in (13.20,\ 14.04] \\ 0 & \text{if } x \notin [12.36,\ 14.04] \end{cases}$$

$$u_3^5(x) = \begin{cases} \dfrac{x-43.02}{1.98} & \text{if } x \in [43.02,\ 45.00] \\ \dfrac{46.98-x}{1.98} & \text{if } x \in (45.00,\ 46.98] \\ 0 & \text{if } x \notin [43.02,\ 46.98] \end{cases}, \quad u_4^5(x) = \begin{cases} \dfrac{x-22.36}{0.84} & \text{if } x \in [22.36,\ 23.20] \\ \dfrac{24.04-x}{0.84} & \text{if } x \in (23.20,\ 24.04] \\ 0 & \text{if } x \notin [22.36,\ 24.04] \end{cases}$$

and for $r \in [0,1]$

$$\underline{u_1^1}(r) = 1.10r + 17.96, \quad \overline{u_1^1}(r) = 20.16 - 1.10r$$

$$\underline{u_2^1}(r) = 3.16r + 15.08, \quad \overline{u_2^1}(r) = 21.40 - 3.16r$$

$$\underline{u_3^1}(r) = 8.56r + 42.64, \quad \overline{u_3^1}(r) = 59.76 - 8.56r$$

$$\underline{u_4^1}(r) = 3.96r + 21.28, \quad \overline{u_4^1}(r) = 29.20 - 3.96r$$

$$\underline{u_1^2}(r) = 5.76r + 16.13, \quad \overline{u_1^2}(r) = 27.65 - 5.76r$$

$$\underline{u_2^2}(r) = 9.64r + 15.04, \quad \overline{u_2^2}(r) = 34.32 - 9.64r$$

$$\underline{u_3^2}(r) = 8.18r + 39.19, \quad \overline{u_3^2}(r) = 55.55 - 8.18r$$

$$\underline{u_4^2}(r) = 4.78r + 16.85, \quad \overline{u_4^2}(r) = 26.41 - 4.78r$$

$$\underline{u_1^3}(r) = 2.44r + 13.02, \quad \overline{u_1^3}(r) = 17.90 - 2.44r$$

$$\underline{u_2^3}(r) = 1.76r + 10.82, \quad \overline{u_2^3}(r) = 14.34 - 1.76r$$

$$\underline{u_3^3}(r) = 7.10r + 29.44, \quad \overline{u_3^3}(r) = 43.64 - 7.10r$$

$$\underline{u_4^3}(r) = 4.16r + 13.17, \quad \overline{u_4^3}(r) = 21.49 - 4.16r$$

$$\underline{u_1^4}(r) = 1.28r + 14.94, \quad \overline{u_1^4}(r) = 17.50 - 1.28r$$

$$\underline{u_2^4}(r) = 1.16r + 11.62, \quad \overline{u_2^4}(r) = 13.94 - 1.16r$$

$$\underline{u_3^4}(r) = 5.74r + 36.67, \quad \overline{u_3^4}(r) = 48.15 - 5.74r$$

$$\underline{u_4^4}(r) = 3.00r + 18.22, \quad \overline{u_4^4}(r) = 24.22 - 3.00r$$

$$\underline{u_1^5}(r) = 1.64r + 15.37, \quad \overline{u_1^5}(r) = 18.65 - 1.64r$$

$$\underline{u_2^5}(r) = 0.84r + 12.36, \quad \overline{u_2^5}(r) = 14.04 - 0.84r$$

$$\underline{u_3^5}(r) = 1.98r + 43.02, \quad \overline{u_3^5}(r) = 46.98 - 1.98r$$

$$\underline{u_4^5}(r) = 0.84r + 22.36, \quad \overline{u_4^5}(r) = 24.04 - 0.84r$$

Thus the fuzzy $4-$cell numbers $u^i = (u_1^i, u_2^i, u_3^i, u_4^i)$, $i = 1,2,3,4,5$, ie.,

$$u^i(x_1, x_2, x_3, x_4) = \min\{u_1^i(x_1), u_2^i(x_2), u_3^i(x_3), u_4^i(x_4)\}$$
$$(x_1, x_2, x_3, x_4) \in R^4, \ i = 1,2,3,4,5$$

can be used to represent C_i, $i = 1,2,3,4,5$, respectively.

Using the same four wave bands: MSS-4, MSS-5, MSS-6, MSS-7, we now proceed to examine some zone (i.e., object, denoted by O) 12 times, stochastically, at various times or positions, or using various viewers, and obtain the following data:

		MSS−4	MSS−5	MSS−6	MSS−7
	Sample 1	18.31	13.13	45.41	23.52
	Sample 2	17.21	12.71	45.67	23.49
	Sample 3	17.00	12.57	44.82	22.56
	Sample 4	16.32	13.06	43.53	22.77
	Sample 5	18.01	13.91	45.31	23.31
O	Sample 6	16.11	13.59	46.97	23.61
	Sample 7	16.21	12.56	43.96	23.14
	Sample 8	17.13	13.32	46.01	22.84
	Sample 9	16.20	12.34	43.81	23.13
	Sample 10	17.51	13.34	45.89	23.52
	Sample 11	17.02	13.78	45.21	23.18
	Sample 12	18.02	13.25	44.56	23.23

We can work out the following means and standard deviations:

$$\overline{o}_1 = 17.09, \ s_1 = 0.77, \ \overline{o}_2 = 13.17, \ s_2 = 0.50$$
$$\overline{o}_3 = 45.10, \ s_3 = 1.01, \ \overline{o}_4 = 23.19, \ s_4 = 0.33$$

So we can obtain the fuzzy $4-$cell number $o = (o_1, o_2, o_3, o_4)$, i.e.,

$$o(x_1, x_2, x_3, x_4) = \min\{o_1(x_1), o_2(x_2), o_3(x_3), o_4(x_4)\}$$
$$(x_1, x_2, x_3, x_4) \in R^4$$

that can be used to represent O, where,

$$o_1(x) = \begin{cases} \dfrac{x-15.55}{1.54} & \text{if } x \in [15.55,\ 17.09] \\ \dfrac{18.63-x}{1.54} & \text{if } x \in (17.09,\ 18.63] \\ 0 & \text{if } x \notin [15.55,\ 18.63] \end{cases}, \quad o_2(x) = \begin{cases} \dfrac{x-12.17}{1.00} & \text{if } x \in [12.17,\ 13.17] \\ \dfrac{14.17-x}{1.00} & \text{if } x \in (13.17,\ 14.17] \\ 0 & \text{if } x \notin [12.17,\ 14.17] \end{cases}$$

$$o_3(x) = \begin{cases} \dfrac{x-43.08}{2.02} & \text{if } x \in [43.08,\ 45.10] \\ \dfrac{47.12-x}{2.02} & \text{if } x \in (45.10,\ 47.12] \\ 0 & \text{if } x \notin [43.08,\ 47.12] \end{cases}, \quad o_4(x) = \begin{cases} \dfrac{x-22.53}{0.66} & \text{if } x \in [22.53,\ 23.19] \\ \dfrac{23.79-x}{0.66} & \text{if } x \in (23.19,\ 23.79] \\ 0 & \text{if } x \notin [22.53,\ 23.79] \end{cases}$$

and for $r \in [0,1]$.

$$\underline{o_1}(r) = 1.54r + 15.55, \quad \overline{o_1}(r) = 18.63 - 1.54r$$

$$\underline{o_2}(r) = 1.00r + 12.17, \quad \overline{o_2}(r) = 14.17 - 1.00r$$

$$\underline{o_3}(r) = 2.02r + 43.08, \quad \overline{o_3}(r) = 47.12 - 2.02r$$

$$\underline{o_4}(r) = 0.66r + 22.53, \quad \overline{o_4}(r) = 23.79 - 0.66r$$

Taking $\alpha = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $a = (\frac{1}{5}, \frac{1}{5})$, By

$$\Gamma_{\alpha,a}(u,v)$$

$$= \frac{\exp\left(a_2 \sum_{i=1}^{4} \alpha_i \left(\left| \int_{\underline{u_i}(0)}^{u_i^{*(1)}} u_i(t)dt - \int_{\underline{v_i}(0)}^{v_i^{*(1)}} v_i(t)dt \right| + \left| \int_{u_i^{*(1)}}^{\overline{u_i}(0)} u_i(t)dt - \int_{v_i^{*(1)}}^{\overline{v_i}(0)} v_i(t)dt \right| \right)\right)}{\left(\sum_{i=1}^{4} \alpha_i \int_0^1 r[\underline{u_i}(r) + \underline{v_i}(r) + \overline{u_i}(r) + \overline{v_i}(r)]dr \right)^{a_1}}$$

$$\cdot \sum_{i=1}^{4} \alpha_i \int_0^1 r[|\,(\underline{u_i}(r) - \underline{v_i}(r)\,| + |\,\overline{u_i}(r) - \overline{v_i}(r)\,|]dr$$

we can obtain $\Gamma_{\alpha,a}(o,u^1) = 3.36$, $\Gamma_{\alpha,a}(o,u^2) = 10.34$, $\Gamma_{\alpha,a}(o,u^3) = 4.27$, $\Gamma_{\alpha,a}(o,u^4) = 1.11$, $\Gamma_{\alpha,a}(o,u^5) = 0.03$. From $\Gamma_{\alpha,a}(o,u^5) = \min\{\Gamma_{\alpha,a}(o,u^1), \Gamma_{\alpha,a}(o,u^2), \Gamma_{\alpha,a}(o,u^3), \Gamma_{\alpha,a}(o,u^4), \Gamma_{\alpha,a}(o,u^5)\}$, we know that O belongs to C_5, i.e., the zone measured by us is covered by Birch Forest.

Remark 5.3. Although, only for the example, using mean vectors $(\mu_1^i, \mu_2^i, \mu_3^i, \mu_4^i)$, $i = 1,2,3,4,5$ and $(\overline{o}_1, \overline{o}_2, \overline{o}_4, \overline{o}_4)$ to represent respectively C_1, C_2, C_3, C_4 and O, we perhaps also judge that O belongs to C_5 by the usual Euclidean metrics, we still emphasize that using fuzzy n-cell numbers to deal with imprecise or uncertain quantities is better than using crisp n-dimensional vectors. The following example (to simplify and shorten the problem, we only consider 1-dimensional case) will show this.

Example 5.2. The following two classes of ferrous quantities (kilogram per hundred kilogram) of minerals come respectively two different mine areas (denoted by A and B).
A: 10.20, 11.76, 8.31, 9.02, 9.63, 8.33, 11.36, 12.30, 12.03, 7.98
B: 52.33, 79.34, 34.51, 62.34, 82.26, 28.36, 17.37, 25.32, 10.11, 8.34

Suppose that one group (denoted by C) of minerals comes from the one of A and B. The problem to be solved is to identify if C comes from A or B. We take samples, and acquire the following data for C:
C: 18.20, 20.31, 76.02, 9.36, 28.56, 23.32, 15.36, 20.51, 13.27, 20.32

We can work out: $\mu_A = 10.09$, $\sigma_A = 1.67$, $\mu_B = 40.03$, $\sigma_B = 27.43$, $\mu_C = 24.52$ and $\sigma_C = 18.86$.

If we use crisp means to represent A, B and C, then we have $A = 10.09$, $B = 40.03$, $C = 24.52$ and $d(C,A) = 14.43 < 15.51 = d(C,B)$. If we regard $d(C,A) < d(C,B)$ as evidence, we can draw a conclusion that C comes from A. However, the conclusion does not accord with fact. We should note that

although $d(C,A) < d(C,B)$, the difference of $d(C,A)$ and $d(C,B)$ is small. Furthermore, from the statistical data, we can see that the ferrous quantities of minerals coming from A are more coincident, but B and C are not. It is almost impossible that some minerals in C come from A, such as, minerals with ferrous quantities 76.02 and 28.56. So we may judge that C comes from B.

If we use fuzzy 1-cell numbers to represent A, B and C, then we have

$$A(x) = \begin{cases} \dfrac{x-6.75}{3.34} & \text{if } x \in [6.75,\ 10.09] \\ \dfrac{13.43-x}{3.34} & \text{if } x \in (10.09,\ 13.43] \\ 0 & \text{if } x \notin [6.75,\ 13.43] \end{cases}, \quad B(x) = \begin{cases} \dfrac{x+14.83}{54.86} & \text{if } x \in [0,\ 40.03] \\ \dfrac{94.89-x}{54.86} & \text{if } x \in (40.03,\ 94.89] \\ 0 & \text{if } x \notin [0,\ 94.89] \end{cases}$$

$$C(x) = \begin{cases} \dfrac{x+13.20}{37.72} & \text{if } x \in [0,\ 24.52] \\ \dfrac{62.24-x}{37.72} & \text{if } x \in (24.52,\ 62.24] \\ 0 & \text{if } x \notin [0,\ 62.24] \end{cases}$$

$$\underline{A}(r) = 3.34r + 6.75, \quad \underline{B}(r) = \begin{cases} 54.86r - 14.83 & \text{if } r \in (0.27,1] \\ 0 & \text{if } r \in [0,0.27] \end{cases},$$

$$\underline{C}(r) = \begin{cases} 37.72r - 13.20 & \text{if } r \in (0.35,1] \\ 0 & \text{if } r \in [0,0.35] \end{cases}, \quad \overline{A}(r) = 13.43 - 3.34r,$$

$\overline{B}(r) = 94.89 - 54.86r$, $\overline{C}(r) = 62.24 - 37.72r$, so $\Gamma_{(0.1,\,0.1)}(C,A)$

$$= e^{3.27}\frac{14.01}{3.487} = 106.39 > 21.77 = e^{1.178}\frac{15.38}{4.019} = \Gamma_{(0.1,\,0.1)}(C,B).$$

Thus we can affirm that C comes from B.

VI. Conclusion

In this paper, we suggest using fuzzy n-cell numbers to represent imprecise or uncertain multichannel digital signals, and put forward a method (see the first or second step of the algorithmic version in Section 5, or see Example 2.1) of constructing such fuzzy n-cell numbers. Although the metrics D and D_L have been studied formerly in [14,15], in view of the roughness of D and D_L, we define two new metrics on fuzzy n-cell number space in order that they can better characterize the degree of the difference of two objects in some imprecise or uncertain environment, and we study their properties (Section 3). In some applications, metrics are unsuitable for use in finding the difference of two fuzzy n-cell numbers, so we introduce the concepts of difference values $\Delta_{\alpha,a}$, $\Lambda_{\alpha,a}$ and $\Gamma_{\alpha,a}$, study their properties, and show the rationality for their use in characterizing the degree of the difference of two fuzzy n-cell numbers by Remarks and examples (see Section 4). Finally, in Section 5, we put forward an algorithmic version of pattern recognition in an imprecise or uncertain environment based on the metrics and difference values defined by us, and give examples to show the application (see Example 5.1) and rationality (see Example 5.2) of the methods.

REFERENCES

[1] D.Butnariu, "Methods of solving optimization problems and linear equations in the space of fuzzy vectors," *Libertas Math*, vol. 17 , pp. 1–7, 1997.

[2] S.S.L.Chang, and L.A.Zadeh, "On fuzzy mappings and control," *IEEE Trans. Syst. Man Cybernet*, vol. 2, pp. 30–34, 1972.

[3] P. Diamond, and P. Kloeden, *Metric Space of Fuzzy Sets*. World Scientific, Singapore, 1994.

[4] D. Dubois, and H. Prade, "Operations on fuzzy numbers," *Internat. J. Systems Sci.,* vol. 9, pp. 613–626, 1978.

[5] D. Dubois, and H. Prade, "Towards fuzzy differential calculus—Part I, II, III," *Fuzzy Sets and Systems,* vol. 8, pp. 1–17, 105–116, 225–234, 1982.

[6] R.Goetschel, W.Voxman, "Elementary calculus," *Fuzzy Sets and Systems*, vol. 18, pp. 31–43, 1986.

[7] O.Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, pp. 301–317, 1987.

[8] A.Kaufmann, M.M.Gupta, *Introduction to Fuzzy Arithmetic-Theory and Applications*. Van Nostrand, Reinhold, NY, 1985.

[9] M.Miyakawa, K.Nakamura, J.Ramik, I.G.Rosenberg, "Joint canonical fuzzy numbers," *Fuzzy Sets and Systems*, vol. 53, pp. 39–49, 1993.

[10] K.Nakamura, "Canonical fuzzy number of dimension two and fuzzy utility difference for understanding preferential judgements," *Inform. Sci.*, vol. 50, pp. 1–22, 1990.

[11] M.L.Puri, D.A.Ralescu, "Differentials for fuzzy functions," *J.Math. Anal.Appl.*, vol. 91, pp. 552–558, 1983.

[12] J.Ramik, K.Nakamura, "Canonical fuzzy numbers of dimension two," *Fuzzy Sets and Systems*, vol. 54, 167–181, 1993.

[13] S.Theodoridis, K.Koutroumbas, *Pattern Recognition (Third Edition)*. Elsevier, USA, 2006.

[14] G.Wang, Y.Li, C.Wen, "On fuzzy $n-$ cell numbers and $n-$ dimension fuzzy vectors," *Fuzzy Sets and Systems*, vol. 158, pp. 71–84, 2007.

[15] G.Wang, C.Wu, "Fuzzy $n-$ cell numbers and the differential of fuzzy $n-$ cell number value mappings," *Fuzzy Sets and Systems*, vol. 130, pp. 367–381, 2002.

[16] G.Wang, C.Wu, "The integral over a directed line segment of fuzzy mapping and its applications," *International Journal of Uncertainty, Fuzzyness and Knowledge-Based Systems*, vol. 12, pp. 543–556, 2004.

[17] G.Wang, C.Zhao, "The measurability and (K)-integrability about fuzzy n-cell number value mappings," *The Journal of Fuzzy Mathematics*, vol. 13, pp. 117–128, 2005.

[18] C.Wu, G.Wang, "convergence of sequences of fuzzy numbers and fixed point theorems for inceasing fuzzy mappings and application," *Fuzzy Sets and Systems*, vol. 130, pp. 383–390, 2002.

[19] T. Wang, Y. Chen and S. Tong, "Fuzzy reasoning models and algorithms on type-2 fuzzy sets", *Int. J. of Innovative Computing Information and Control*, vol.4, no.10, pp. 2451-2460, 2008.

[20] M. Shimakawa, "A proposal of extension fuzzy reasoning method", *Int. J. of Innovative Computing Information and Control*, vol.4, no.10, pp. 2603-2615, 2008.

[21] Kimiaki Shinkai, Decision Analysis of Fuzzy Partition Tree Applying Fuzzy Theory, *Int. J. of Innovative Computing Information and Control*, vol.4, no.10, pp.2581-2594, 2008.

Guixiang Wang received the BSc degree in Mathematics from Hebei Normal University, China in 1982; the MSc degree in Mathematics from Hebei University, China in 1989; and the PhD degree in Mathematics from Harbin Institute of Technology, China in 2002. He worked as an associate professor in Agricultural University of Hebei, China (1992-1999). He joined in Hangzhou Dianzi University, China in 2002 as a professor. He was a post doctoral fellow in Harbin Engineering University, China (2003-2005), and a visiting senior fellow in the University of Glamorgan, UK (2007-2008). Dr Wang's recent research interests include fuzzy set theory and application, nonlinear systems, signal and information processing, data fusion, pattern recognition.
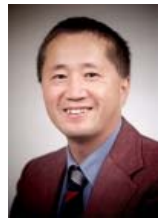
Peng Shi received the BSc degree in Mathematics from Harbin Institute of Technology, China in 1982; the ME degree in control theory from Harbin Engineering University, China in 1985; the PhD degree in electrical engineering from the University of Newcastle, Australia in 1994; and the PhD degree in mathematics from the University of South Australia in 1998. He was awarded the degree of Doctor of Science (Higher Doctorate) by the University of Glamorgan, UK, in 2006.
Dr Shi worked as a lecturer in Heilongjiang University, China (1985-1989), in the University of South Australia (1997-1999) and a senior scientist in Defence Science and Technology Organisation, Department of Defence, Australia (1999-2005). He joined in the University of Glamorgan, UK, in 2004 as a professor. Dr Shi's research interests include robust control and filtering, fault detection techniques, Markov decision processes, and optimization techniques. He has published a number of papers in these areas. In addition, Dr Shi is a co-author of the two research monographs, Fuzzy Control and Filtering Design for Uncertain Fuzzy Systems (Berlin, Springer, 2006), and Methodologies for Control of Jump Time-Delay Systems (Boston, Kluwer, 2003). Currently, Dr Shi serves as Editor-in-Chief of Int. J. of Innovative Computing, Information and Control, and Associate Editor for a number of other journals, such as IEEE Trans on Automatic Control, IEEE Trans on Systems, Man and Cybernetics, Part B, and IEEE Trans on Fuzzy Systems. Dr Shi is a Fellow of Institute of Mathematics and its Applications (UK), and a Senior Member of IEEE.