Characterization of uninorms with continuous underlying t-norm and t-conorm by their set of discontinuity points

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Abstract

Uninorms with continuous underlying t-norm and t-conorm are discussed and properties of the set of discontinuity points of such a uninorm are shown. This set is proved to be a subset of the graph of a special symmetric, surjective, non-increasing set-valued function. A sufficient condition for a uninorm to have continuous underlying operations is also given. Several examples are included.

Keywords: uninorm, ordinal sum, continuous t-norm, continuous t-conorm, set-valued function

1 Introduction

The (left-continuous) t-norms and their dual t-conorms have an indispensable role in many domains [9, 30, 31]. Generalizations of t-norms and t-conorms that can model bipolar behaviour are uninorms (see [7, 22, 32]). The class of uninorms is widely used both in theory

[18, 28] and in applications [13, 33]. The complete characterization of uninorms with continuous underlying t-norm and t-conorm has been in the center of the interest for a long time, however, only partial results were achieved (see [4, 5, 6, 8, 10, 15, 16, 17, 19, 20, 21, 27, 29]).

In [23] we have introduced ordinal sum of uninorms and in [24] we have characterized uninorms that are ordinal sums of representable uninorms. We would like to characterize all uninorms with continuous underlying functions and obtain a similar representation as in the case of t-norms and t-conorms. In this paper we will show that underlying operations of a uninorm U are continuous if and only if U is continuous on $[0,1]^2 \setminus R$, where R is the graph of a special symmetric, surjective, non-increasing set-valued function and U is in each point $(x,y) \in [0,1]^2$ either left-continuous or right-continuous, or continuous. We will then continue and in [25, 26] we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum of semigroups related to representable uninorms, continuous Archimedean t-norms and t-conorms, internal uninorms and singleton semigroups.

In Section 2 we will recall all necessary basic notions and results. We will characterize uninorms with continuous underlying functions via the properties of their set of discontinuity points (Section 3). We give our conclusions in Section 4.

2 Basic notions and results

Let us now recall all necessary basic notions.

A triangular norm is a function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Note that in this paper we stick to the definition from [11], where a non-decreasing function means an increasing function that need not to be strictly increasing. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a function

$S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa.

Proposition 1 ([11])

Let $t\colon [0,1] \longrightarrow [0,\infty]$ *(*$s\colon [0,1] \longrightarrow [0,\infty]$*) be a continuous strictly decreasing (increasing) function such that* $t(1) = 0$ *(*$s(0) = 0$*). Then the operation* $T\colon [0,1]^2 \longrightarrow [0,1]$ *(* $S\colon [0,1]^2 \longrightarrow [0,1]$*) given by*

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$

$$S(x,y) = s^{-1}(\min(s(1), s(x) + s(y)))$$

is a continuous t-norm (t-conorm). The function t *(*s*) is called an* additive generator *of* T *(*S*).*

An additive generator of an Archimedean continuous t-norm T (t-conorm S) is uniquely determined up to a positive multiplicative constant. Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$ ($S(x,y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator. More details on t-norms and t-conorms can be found in [1, 11].

A uninorm (introduced in [32]) is a function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e \in]0,1[$ (see also

[7]). If we take a uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms and the class of t-conorms. In order to stress that we assume a uninorm with $e \in]0,1[$ we will call such a uninorm *proper*. For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1,0) = 0$ ($U(1,0) = 1$). Due to the associativity, we can uniquely define n-ary form of any uninorm for any $n \in \mathbb{N}$ and therefore in some proofs we will use ternary form instead of binary, where suitable.

For each uninorm U with the neutral element $e \in [0,1]$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U on $[0,1]^2$. Moreover, $\min(x,y) \leq U(x,y) \leq \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$. We will denote the set of all uninorms U such that T_U and S_U are continuous by $\mathcal{U}$.

From any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions.

Proposition 2 (**[14]**)

Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm and $S\colon [0,1]^2 \longrightarrow [0,1]$ a t-conorm and assume $e \in [0,1]$. Then the two functions $U_{\min}, U_{\max}\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$U_{\min}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \textit{otherwise} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \textit{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ and of the *second type by* $\mathcal{U}_{\max}$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [7]).

Proposition 3 ([7])

Let $f\colon [0,1] \longrightarrow [-\infty,\infty]$, $f(0) = -\infty$, $f(1) = \infty$ *be a continuous strictly increasing function. Then a function* $U\colon [0,1]^2 \longrightarrow [0,1]$ *given by*

$$U(x,y) = f^{-1}(f(x)+f(y)),$$

where $f^{-1}\colon [-\infty,\infty] \longrightarrow [0,1]$ *is an inverse function to* f, *with the convention* $\infty + (-\infty) = \infty$ *(or* $\infty + (-\infty) = -\infty$*), is a uninorm, which will be called a* representable *uninorm.*

Note that if we relax the strict monotonicity of the additive generator then the neutral element will be lost and by relaxing the condition $f(0) = -\infty$, $f(1) = \infty$ the associativity will be lost (if $f(0) < 0$ and $f(1) > 0$). In [28] (see also [22]) we can find the following result.

Proposition 4 ([28])

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ *be a uninorm continuous everywhere on the unit square except of the two points* $(0,1)$ *and* $(1,0)$. *Then* U *is representable.*

For our examples we will use the following ordinal sum construction introduced by Clifford.

Theorem 1 ([3])

Let $A \neq \emptyset$ *be a totally ordered set and* $(G_\alpha)_{\alpha\in A}$ *with* $G_\alpha = (X_\alpha, *_\alpha)$ *be a family of semigroups. Assume that for all* $\alpha, \beta \in A$ *with* $\alpha < \beta$ *the sets* X_α *and* X_β *are either disjoint or that* $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, *where* $x_{\alpha,\beta}$ *is both the neutral element of* G_α *and the annihilator of* G_β *and where for each* $\gamma \in A$ *with* $\alpha < \gamma < \beta$ *we have* $X_\gamma = \{x_{\alpha,\beta}\}$. *Put* $X = \bigcup\limits_{\alpha\in A} X_\alpha$ *and define* *the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Therefore in our examples the commutativity and the associativity of the corresponding ordinal sum uninorm will follow from Theorem 1. Monotonicity and the neutral element can be then easily checked by the reader.

Further we will use the following transformation. For any $0 \le a < b \le c < d \le 1$, $v \in [b, c]$, and a uninorm U with the neutral element $e \in [0,1]$ let $f \colon [0,1] \longrightarrow [a, b[\cup \{v\} \cup]c, d]$ be given by

$$f(x) = \begin{cases} (b-a) \cdot \frac{x}{e} + a & \text{if } x \in [0, e[\,, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is linear on $[0, e[$ and on $]e, 1]$ and thus it is a piece-wise linear isomorphism of $[0,1]$ to $([a, b[\cup \{v\} \cup]c, d])$ and a function $U_v^{a,b,c,d} \colon ([a, b[\cup \{v\} \cup]c, d])^2 \longrightarrow ([a, b[\cup \{v\} \cup]c, d])$ given by

$$U_v^{a,b,c,d}(x, y) = f(U(f^{-1}(x), f^{-1}(y))) \tag{2}$$

is an operation on $([a, b[\cup \{v\} \cup]c, d])^2$ which is commutative, associative, non-decreasing in both variables (with respect to the standard order) and v is its neutral element.

Example 1

Assume $U_1 \in \mathcal{U}_{\min}$ and $U_2 \in \mathcal{U}_{\max}$ with respective neutral elements e_1, e_2. Then U_1 is an ordinal sum of semigroups $G_\alpha = ([0, e[\,, T^*_{U_1})$ and $G_\beta = ([e, 1]\,, S^*_{U_1})$ with $\alpha < \beta$, where $T^*_{U_1} =$

$U_1|_{[0,e_1]^2}$ and $S^*_{U_1} = U_1|_{[e_1,1]^2}$. Similarly, U_2 is an ordinal sum of semigroups $G_\alpha = ([0,e], T^*_{U_2})$ and $G_\beta = (]e,1], S^*_{U_2})$ with $\alpha > \beta$. If all underlying operations are continuous then the set of discontinuity points of U_1 is equal to the set $S_1 = \{e\} \times]e,1] \cup]e,1] \times \{e\}$ and the set of discontinuity points of U_2 is equal to the set $S_2 = \{e\} \times [0,e[\cup [0,e[\times \{e\}$. Both uninorms can be seen on Figure 1.

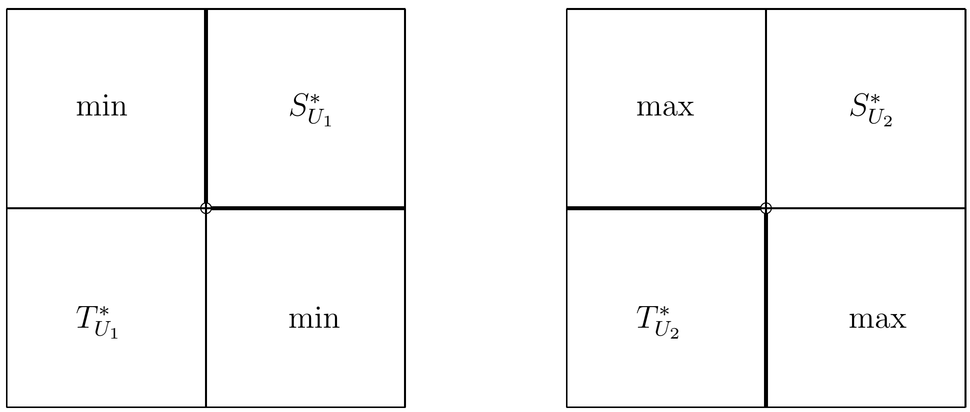

Figure 1: The uninorm U_1 (left) and the uninorm U_2 (right) from Example 1. The bold lines denote the points of discontinuity of U_1 and U_2.

More detailed discussion on the ordinal sum construction for uninorms can be found in [25].

3 Characterization of uninorms $U \in \mathcal{U}$ by means of special set-valued functions

In this section we will show that for a uninorm U we have $U \in \mathcal{U}$ if and only if U is continuous on $[0,1]^2 \setminus R$, where R is the graph of a special symmetric, surjective, non-increasing set-valued function r and U is in each point $(x,y) \in [0,1]^2$ either left-continuous, or right-continuous, or continuous. In the first part we will focus on the necessity part, i.e., we will show that each uninorm $U \in \mathcal{U}$ is continuous on $[0,1]^2 \setminus R$, where R is the graph of some symmetric, surjective, non-increasing set-valued function r (Theorem 2). We will also show that $U \in \mathcal{U}$

implies that U is in each point $(x,y) \in [0,1]^2$ either left-continuous, or right-continuous, or continuous (Theorem 3).

3.1 The necessity part

The following lemmas and propositions are necessary for the proof of Theorem 2 and 3.

Lemma 1 ([24])

Each uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U}$, is continuous in (e,e).

Next we show that for $x, y \in [0,1]$ we have $U(x,y) = \min(x,y)$ or $U(x,y) = \max(x,y)$ if x is an idempotent element of U.

Lemma 2

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm and let $U \in \mathcal{U}$. If $a \in [0,1]$ is an idempotent point of U then U is internal on $\{a\} \times [0,1]$, i.e., $U(a,x) \in \{x,a\}$ for all $x \in [0,1]$.

PROOF: If $a = e$ the result is obvious. Suppose $a < e$ (the case when $a > e$ is analogous). Since T_U is continuous we have $U(a,x) = \min(a,x)$ if $x \in [0,e]$. Suppose that there exists $y \in \,]e,1]$ such that $U(a,y) = c \in \,]a,y[$. Then $U(a,c) = U(a,a,y) = U(a,y) = c$ and if $c \leq e$ then $c = U(a,c) \leq a$ what is a contradiction. Thus $y > c > e$. Then since S_U is continuous there exists a y_1 such that $U(c,y_1) = y$. Then, however,

$$U(a,y) = U(a,c,y_1) = U(c,y_1) = y$$

what is again a contradiction. Thus U is internal on $\{a\} \times [0,1]$. $\square$

For a given uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ and each $x \in [0,1]$ we define a function $u_x\colon [0,1] \longrightarrow [0,1]$ by $u_x(z) = U(x,z)$ for $z \in [0,1]$.

Lemma 3

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$, and assume $x \in [0,1]$. The function u_x is continuous if and only if one of the following conditions:

(i) $u_x(1) < e$,

(ii) $u_x(0) > e$,

(iii) $e \in \mathrm{Ran}(u_x)$

is satisfied.

PROOF: If $e \in \mathrm{Ran}(u_x)$ then there exists a $y \in [0,1]$ such that $U(x,y) = e$. Since U is monotone continuity of u_x is equivalent with the equality $\mathrm{Ran}(u_x) = [a,b]$ for some $a = U(0,x)$ and $b = U(1,x)$. Assume $c \in [0,1]$. Then $U(x,y,c) = c$ and for $z = U(y,c)$ we have $u_x(z) = c$, i.e., $\mathrm{Ran}(u_x) = [0,1]$. If $u_x(1) = v < e$ (the case when $u_x(0) > e$ can be shown similarly) then due to the monotonicity the continuity of u_x is equivalent with the equality $\mathrm{Ran}(u_x) = [0,v]$. Assume $w \in [0,v]$. Since T_U is continuous there exists a $q \in [0,e]$ such that $U(v,q) = w$, i.e., $U(x,1,q) = w$ and then $u_x(U(1,q)) = w$. Therefore $\mathrm{Ran}(u_x) = [0,v]$.

Vice-versa, if u_x is continuous and $u_x(0) \leq e \leq u_x(1)$ then evidently $e \in \mathrm{Ran}(u_x)$.

□

Example 2

For a representable uninorm U the function u_x is continuous for all $x \in \,]0,1[$. If U is conjunctive (disjunctive) then u_0 (u_1) is continuous and u_1 (u_0) is non-continuous in 0 (1). For a uninorm $U \in \mathcal{U}_{\max}$ ($U \in \mathcal{U}_{\min}$) u_x is continuous for all $x \in [e,1]$ ($x \in [0,e]$) and u_x is non-continuous in e for all $x \in [0,e[$ ($x \in \,]e,1]$).

Now we recall a result [12, Proposition 1] which shows a connection between continuity on cuts and joint continuity of a monotone function.

Proposition 5

Let $f(x,y)$ be a real valued function defined on an open set G in the plane. Suppose that $f(x,y)$ is continuous in x and y separately and is monotone in x for each y. Then $f(x,y)$ is (jointly) continuous on the set G.

The following result shows that if $U(a,b) = e$ then U is continuous in the point (a,b). First, however, we introduce two useful lemmas.

Lemma 4

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in [0,1]$. Then if $U(a,b) = e$, for some $a, b \in [0,1]$, there is either $a = b = e$, or a and b are not idempotent elements of U.

PROOF: If a is an idempotent point (similarly for b) then

$$e = U(a,b) = U(a, U(a,b)) = U(a,e) = a,$$

and

$$e = U(a,b) = U(e,b) = b,$$

i.e., $a = b = e$. $\square$

Lemma 5

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in [0,1]$. Then if $U(a,b) = e$, for some $a, b \in [0,1]$, there is either $a = b = e$, or $a < e$, $b > e$, or $a > e$, $b < e$.

PROOF: If $a = e$ then evidently also $b = e$. If $a < e$ then $b \neq e$ and we have

$$e = U(a,b) \leq U(e,b) = b,$$

i.e., $e < b$. Finally, if $a > e$ then $b \neq e$ and we have

$$e = U(a,b) \geq U(e,b) = b,$$

i.e., $e > b$. $\square$

Proposition 6

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ *be a uninorm,* $U \in \mathcal{U}$*. If* $U(a,b) = e$ *for some* $a, b \in [0,1]$*,* $a < e$*, then* U *is continuous on* $[0,1]^2 \setminus ([0,a[\cup]b,1])^2$.

PROOF: Since $a < e$ Lemma 4 implies that a and b are not idempotent elements of U and Lemma 5 implies that $b > e$. From Lemma 3 we know that u_a and u_b are continuous functions. Next we will show that for all $f \in \,]a,b[$ there exists a $v^f \in [0,1]$ such that $U(f, v^f) = e$. Assume $f \in \,]a,e]$ (for $f \in [e,b[$ the proof is analogous). Since T_U is continuous and $U(a,f) \leq a$, $U(f,e) = f$ there exists an $a^f \in [0,e]$ such that $U(f,a^f) = a$. Then

$$e = U(a,b) = U(f, a^f, b)$$

and if $v^f = U(a^f, b)$ then $U(f, v^f) = e$. Summarising, we get that for all $x \in [a,b]$ the function u_x is continuous. Now since a and b are not idempotents we have $U(a,a) = p < a$, $U(b,b) = q > b$ and $U(a,a,b,b) = e$. Thus also all u_x for $x \in [p,q]$ are continuous and then Proposition 5 implies the result. $\square$

Remark 1

If $U \in \mathcal{U}$ then either $U(x,y) = e$ implies $x = y = e$, or there exists an $x \neq e$ such that $U(x,y) = e$ for some $y \in [0,1]$. We will focus on the second case. Then Lemma 5 implies that either $x < e$, $y > e$, or $x > e$, $y < e$. We will suppose that $x < e$ and $y > e$ (as the other case is analogous). Then associativity implies $U(\underbrace{x,\ldots,x}_{n\text{-times}}, \underbrace{y,\ldots,y}_{n\text{-times}}) = e$ for all $n \in \mathbb{N}$ and similarly as in the proof of Proposition 6 we can show that for all $z \in \left[U(\underbrace{x,\ldots,x}_{n\text{-times}}), U(\underbrace{y,\ldots,y}_{n\text{-times}})\right]$ there exists a $q \in [0,1]$ such that $U(z,q) = e$. Further, if $U(b,c) = e$ for some $b, c \in [0,1]$, $b \neq e$, then by Lemma 4 the points b and c are not idempotents. Therefore, in this case,

$U(x,y)=e$ for some $y\in[0,1]$ if and only if $x\in\,]a,d[\,$, where

$$a=\lim_{n\longrightarrow\infty} U(\underbrace{x,\dots,x}_{n\text{-times}})$$

and

$$d=\lim_{n\longrightarrow\infty} U(\underbrace{y,\dots,y}_{n\text{-times}}).$$

Note that a and d are idempotent elements of U which follows from the continuity of the underlying functions of U. Further, the monotonicity of U implies that $a<e<d$. The commutativity of U then implies that if $U(x,y)=e$ for some $x,y\in[0,1]$ then $x,y\in\,]a,d[\,$. Vice versa, for all $x\in\,]a,d[$ there exists a $y\in\,]a,d[$ such that $U(x,y)=e$. Due to Proposition 6 we see that U is continuous on $\{x\}\times[0,1]$ for all $x\in\,]a,d[\,$. If we take the union over all $x\in\,]a,d[$ then the commutativity of U and Proposition 5 implies that U is continuous on $]a,d[\,\times[0,1]\cup[0,1]\times\,]a,d[\,$. In order to include also the case when $U(x,y)=e$ implies $x=y=e$, we can generally say that for an $x\in[0,1]$ there exists a $y\in[0,1]$ such that $U(x,y)=e$ if and only if $x\in\,]a,d[\,\cup\{e\}$. Note that in the case when $U(x,y)=e$ implies $x=y=e$, we take $a=e=d$.

Example 3

Assume two representable uninorms $U_1,U_2\colon[0,1]^2\longrightarrow[0,1]$ with respective neutral elements e_1,e_2. Let U_1^* be a transformation of U_1 to $(\left[0,\frac{1}{3}\right[\cup\{v\}\cup\left]\frac{2}{3},1\right])^2$ given by (2), where $v=\frac{1}{3}$ ($v=\frac{2}{3}$) if U_2 is conjunctive (disjunctive), and let U_2^* be a linear transformation of U_2 to $\left[\frac{1}{3},\frac{2}{3}\right]^2$. Then the ordinal sum of semigroups $G_\alpha=(\left[0,\frac{1}{3}\right[\cup\{v\}\cup\left]\frac{2}{3},1\right],U_1^*)$, $G_\beta=(\left[\frac{1}{3},\frac{2}{3}\right],U_2^*)$, with $\alpha<\beta$, is a semigroup $([0,1],U)$, where U is a uninorm with the neutral element $e=\frac{e_2+1}{3}$. We can find the structure of U on Figure 2. All points of discontinuity of U except $(0,1),(0,1)$ correspond to the transformation of the points $(x,y)\in[0,1]^2$ such that $U_1(x,y)=e_1$. For simplicity, we will assume that $U_1(x,1-x)=e_1=\frac{1}{2}$ for all $x\in\,]0,1[\,$. Moreover, for every $a\in\left]\frac{1}{3},\frac{2}{3}\right[$ there exists a $b\in\left]\frac{1}{3},\frac{2}{3}\right[$ such that $U(a,b)=e$. The previous result then implies that U is continuous in every point from $\left]\frac{1}{3},\frac{2}{3}\right[\times[0,1]$ and from $[0,1]\times\left]\frac{1}{3},\frac{2}{3}\right[$.

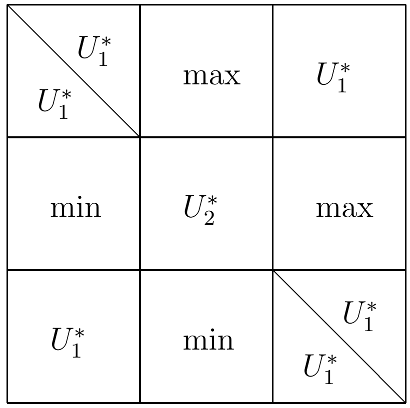

Figure 2: The uninorm U from Example 3. The oblique lines denote the points of discontinuity of U.

In the following results we will continue to investigate properties of the function u_x.

Proposition 7

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then for each $x \in [0,1]$ there is at most one point of discontinuity of u_x. Further, if u_x is non-continuous in $y \in [0,1]$ then $U(x,z) < e$ for all $z < y$ and $U(x,z) > e$ for all $z > y$.

PROOF: If u_x is non-continuous then Lemma 3 implies $e \notin \mathrm{Ran}(u_x)$, $u_x(0) < e$ and $u_x(1) > e$. We will denote

$$f = \sup\{U(x,y) \mid y \in [0,1], U(x,y) \le e\}$$

and

$$g = \inf\{U(x,y) \mid y \in [0,1], U(x,y) \ge e\}.$$

Note that the inequality $u_x(0) < e$ ($u_x(1) > e$) implies that f is the supremum (g is the infimum) of a non-empty set. Fix arbitrary $f_1 < f$. Then there exist an $s > 0$ and y_f such that $f_1 \le f - s \le U(x, y_f) \le f < e$ because f is the supremum. Since $U(U(x,y_f),0) = 0$, $U(U(x,y_f),e) = U(x,y_f)$ and T_U is continuous, there exists an f_3 such that $U(U(x,y_f),f_3) = f_1$. Therefore $U(x, U(y_f, f_3)) = f_1$ and $f_1 \in \mathrm{Ran}(u_x)$.

Similarly, for each $g_1 > g$ there is $g_1 \in \operatorname{Ran}(u_x)$. Therefore $[0,1] \setminus \operatorname{Ran}(u_x)$ is a connected set. Since u_x is monotone it has only one point of discontinuity. Also, if u_x is non-continuous in $y \in [0,1]$ then $U(x,z) < e$ for all $z < y$ and $U(x,z) > e$ for all $z > y$. □

Proposition 8

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then for all $x \in [0,1]$ the function u_x is either left-continuous or right-continuous.

PROOF: Assume $x \in [0,1]$. From Proposition 7 we know that u_x is non-continuous in at most one point, and thus we will suppose that u_x is non-continuous in the point $p \in [0,1]$. Further, from the proof of the same proposition we know that $[0,1] \setminus \operatorname{Ran}(u_x)$ is a connected set, i.e., an interval I, and $u_x(p)$ is an end point of the interval I. Then it is evident that if $u_x(p) = \inf I$ then u_x is left-continuous and $u_x(p) < e$, and if $u_x(p) = \sup I$ then u_x is right-continuous and $u_x(p) > e$. □

Remark 2

From the proof of Proposition 8 we see that if $u_x(p) < e$ for some $p \in [0,1]$ then u_x is left-continuous on $[0,p]$ and if $u_x(p) > e$ then u_x is right-continuous on $[p,1]$.

Next we will show that the points of discontinuity of u_x are non-increasing with respect to $x \in [0,1]$.

Proposition 9

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Suppose that for $x, x_1 \in [0,1]$, $x_1 < x$, the functions u_x and u_{x_1} are non-continuous in points y and y_1, respectively. Then $y_1 \geq y$.

PROOF: From the proof of Proposition 7 we see that if u_x is non-continuous in y then $U(x,z) < e$ for all $z < y$ and $U(x,z) > e$ for all $z > y$. If u_{x_1} is non-continuous in y_1 then the monotonicity implies $U(x_1,z) < e$ for $z < y$ and thus $y_1 \geq y$. □

Corollary 1

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. If u_{x_1} is non-continuous in y and u_{x_2} is non-continuous in y for some $x_1 < x_2$ then u_x is non-continuous in y for all $x \in [x_1, x_2]$.

PROOF: Assume $x \in \,]x_1, x_2[$. Since u_{x_1} is non-continuous in y we have $U(x_1, z) > e$ for all $z > y$ and the monotonicity gives $U(x, z) > e$ for all $z > y$. Since u_{x_2} is non-continuous in y we have $U(x_2, z) < e$ for all $z < y$ and the monotonicity gives $U(x, z) < e$ for all $z < y$. Thus u_x is either non-continuous in y or $U(x, y) = e$. Assume that $U(x, y) = e$. If $x = y = e$ then $x_1 < e < x_2$ and we get

$$e < U(x_1, x_2) < e$$

what is a contradiction. Therefore by Lemma 4 the points x and y are not idempotent elements of U and $x \neq e$, $y \neq e$.

Suppose that $y > e$. Then $U(x, x, y, y) = e$ with $U(x_1, y, y) > e$ implies $U(x, x) < x_1 < x$ and by Proposition 6 U is continuous on $[0, 1]^2 \setminus ([0, U(x, x)[\cup]U(y, y), 1])^2$. Then, however, u_{x_1} is continuous, what is a contradiction.

In the case when $y < e$ then $U(x, x, y, y) = e$ with $U(x_2, y, y) < e$ implies $x < x_2 < U(x, x)$ and using Proposition 6 again we obtain that u_{x_2} is continuous, what is a contradiction. Therefore in both cases $U(x, y) \neq e$ and thus u_x is non-continuous in y. □

Example 4

Assume a representable uninorm $U_1 \colon [0, 1]^2 \longrightarrow [0, 1]$ with the neutral element e_1 and a uninorm $U_2 \in \mathcal{U}_{\max}$ with the neutral element $e_2 = \frac{1}{2}$. Let U_1^* be a transformation of U_1 to $\left(\left[0, \frac{1}{3}\right[\cup \left[\frac{2}{3}, 1\right]\right)^2$ given by (2), and let U_2^* be a linear transformation of U_2 to $\left[\frac{1}{3}, \frac{2}{3}\right]^2$. Then the ordinal sum of semigroups $G_\alpha = \left(\left[0, \frac{1}{3}\right[\cup \left[\frac{2}{3}, 1\right], U_1^*\right)$, $G_\beta = \left(\left[\frac{1}{3}, \frac{2}{3}\right], U_2^*\right)$, with $\alpha < \beta$, is a semigroup $([0, 1], U)$, where U is a uninorm. We can find the structure of U on Figure 3. Here $u_{\frac{1}{2}}$ is continuous and u_0 (u_1) is continuous if U_1 is conjunctive (disjunctive). In all other cases u_x is non-continuous. Further, $u_{\frac{1}{3}}$ is non-continuous in $e = \frac{1}{2}$ and $u_{\frac{2}{3}}$ is non-continuous in $\frac{1}{3}$.

Now we will show how can be a point of discontinuity of a uninorm U related to the non-continuity of corresponding functions u_x.

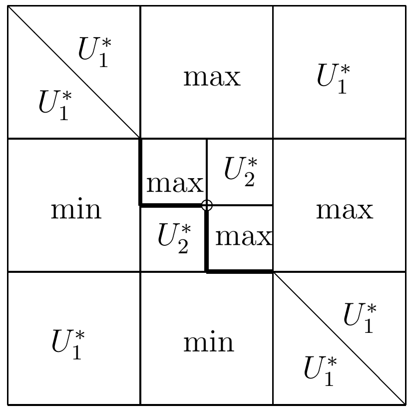

Figure 3: The uninorm U from Example 4. The oblique and bold lines denote the points of discontinuity of U.

Proposition 10

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then U is non-continuous in $(x_0, y_0) \in [0,1]^2$, $(x_0, y_0) \neq (e, e)$, if and only if one of the following is satisfied

(i) u_{x_0} is non-continuous in y_0,

(ii) u_{y_0} is non-continuous in x_0,

(iii) there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that u_z is non-continuous in x_0 and u_v is non-continuous in y_0 either for all $z \in \,]y_0, y_0 + \varepsilon_1]\,$, $v \in \,]x_0, x_0 + \varepsilon_2]\,$, or for all $z \in [y_0 - \varepsilon_1, y_0[\,$, $v \in [x_0 - \varepsilon_2, x_0[\,$.

PROOF: Suppose that U is non-continuous in $(x_0, y_0) \in [0,1]^2$. Then due to Proposition 6 $U(x_0, y_0) \neq e$. Since T_U and S_U are continuous we have $(x_0, y_0) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$. We will assume $(x_0, y_0) \in [0,e] \times [e,1]$ (the other case is analogous). From Proposition 5 it follows that if U is non-continuous in $(x_0, y_0) \in [0,1]^2$ then for all $\delta_1 > 0$ and all $\delta_2 > 0$ there exist $x \in \,]x_0 - \delta_1, x_0 + \delta_1[$ and $y \in \,]y_0 - \delta_2, y_0 + \delta_2[$ such that either u_x is non-continuous in y or u_y is non-continuous in x. Thus U on $[x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2]$ attain values smaller than e and bigger than e as well. Let W be a subset of $[0,1]^2$ such that $(x, y) \in W$ if $U(x_1, y_1) < e$ for all $x_1 < x, y_1 < y$ and $U(x_2, y_2) > e$ for all $x_2 > x, y_2 > y$. Then the set $[x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2] \cap W$ is non-empty for all $\delta_1 > 0$ and all $\delta_2 > 0$. Thus

$(x_0, y_0) \in W$.

If u_{x_0} is continuous in y_0 then there exists an $\varepsilon_1 > 0$ such that either $u_{x_0}(z) < e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ or $u_{x_0}(z) > e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$. Similarly, if u_{y_0} is continuous in x_0 then there exists an $\varepsilon_2 > 0$ such that either $u_{y_0}(v) < e$ for all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$ or $u_{y_0}(v) > e$ for all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. Since we cannot have both $U(x_0, y_0) < e$ and $U(x_0, y_0) > e$ we have either $u_{y_0}(v) < e$ and $u_{x_0}(z) < e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$, or $u_{y_0}(v) > e$ and $u_{x_0}(z) > e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. As these two cases are analogous we will assume

$$u_{y_0}(v) < e \text{ and } u_{x_0}(z) < e \text{ for all } z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1] \text{ and all } v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2].$$

Then $U(x_0, y) < e$ for $y \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and $U(x, y_0) < e$ for $x \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. However, since $(x_0, y_0) \in W$, $U(f, g) > e$ for all $f > x_0, g > y_0$. Thus u_z is non-continuous in x_0 and u_v is non-continuous in y_0 for all $z \in]y_0, y_0 + \varepsilon_1]$, $v \in]x_0, x_0 + \varepsilon_2]$.

Vice versa, if u_{x_0} is non-continuous in y_0, or if u_{y_0} is non-continuous in x_0, then evidently U is non-continuous in (x_0, y_0). Suppose that there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that u_z is non-continuous in x_0 and u_v is non-continuous in y_0 either for all $z \in]y_0, y_0 + \varepsilon_1]$, $v \in]x_0, x_0 + \varepsilon_2]$, or for all $z \in [y_0 - \varepsilon_1, y_0[$, $v \in [x_0 - \varepsilon_2, x_0[$. Then $(x_0, y_0) \in W$ and either $U(x_0, y_0) = e$, or U is non-continuous in (x_0, y_0). However, if $U(x_0, y_0) = e$ then since $(x_0, y_0) \neq (e, e)$ Lemma 4 implies that x_0 and y_0 are not idempotents and Proposition 6 implies that U is continuous on $[0, 1]^2 \setminus ([0, U(x_0, x_0)[\cup]U(y_0, y_0), 1])^2$ if $x_0 < e < y_0$ and on $[0, 1]^2 \setminus ([0, U(y_0, y_0)[\cup]U(x_0, x_0), 1])^2$ if $x_0 > e > y_0$. In both cases we obtain a contradiction with the non-continuity of u_z and u_v. Therefore $U(x_0, y_0) \neq e$ and thus U is non-continuous in (x_0, y_0). □

Example 5

Assume two t-norms $T_1, T_2 \colon [0, 1]^2 \longrightarrow [0, 1]$, such that T_2 has no zero divisors, and a t-conorm $S\colon [0,1]^2 \longrightarrow [0,1]$. Let T_1^* (T_2^*) be a linear transformation of T_1 (T_2) to $\left[0,\frac{1}{3}\right]^2$ ($\left[\frac{1}{3},\frac{2}{3}\right]^2$), and let S_2^* be a linear transformation of S_2 to $\left[\frac{2}{3},1\right]^2$. Then the ordinal sum of semigroups $G_\alpha = (\left[0,\frac{1}{3}\right], T_1^*)$, $G_\beta = (\left[\frac{1}{3},\frac{2}{3}\right], T_2^*)$, $G_\gamma = (\left]\frac{2}{3},1\right], S_2^*)$, with $\alpha < \gamma < \beta$, is a semigroup $([0,1], U)$, where U is a uninorm (see Figure 4). If we define an operation $V\colon [0,1]^2 \longrightarrow [0,1]$ by

$$V = \begin{cases} \min(x,y) & \text{if } x = \frac{1}{3}, y \in \left[\frac{2}{3},1\right], \\ \min(x,y) & \text{if } y = \frac{1}{3}, x \in \left[\frac{2}{3},1\right], \\ U(x,y) & \text{otherwise,} \end{cases}$$

then V is also a uninorm. Here V is non-continuous in the point $(\frac{1}{3},\frac{2}{3})$, however, both $v_{\frac{1}{3}}$ and $v_{\frac{2}{3}}$ are continuous. Note that $([0,1], V)$ is an ordinal sum of semigroups G_α, G_γ and $G_{\beta^*} = (\left]\frac{1}{3},\frac{2}{3}\right], T_2^*)$, $G_\delta = (\{\frac{1}{3}\}, T_2^*)$, where $\alpha < \delta < \gamma < \beta^*$.

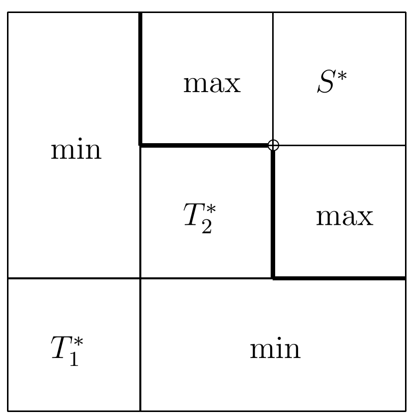

Figure 4: The uninorm U from Example 5. The bold lines denote the points of discontinuity of U.

The following two results show that the set of discontinuity points of a uninorm $U \in \mathcal{U}$ from the set $[0,e] \times [e,1]$ ($[e,1] \times [0,e]$) is connected.

Proposition 11

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Let u_{x_1} be non-continuous in y_1 and u_{x_2} be non-continuous in y_2 for $x_1 < x_2 \le e$ ($e \le x_1 < x_2$). Then for all $y \in [y_2, y_1]$ either there exists $x^* \in [x_1, x_2]$ such that u_{x^*} is non-continuous in y or there is an interval $[c,d]$, where

$y \in [c,d] \subset [0,1]$, and $p \in [x_1, x_2]$ such that u_z is non-continuous in p for all $z \in [c,d]$.

PROOF: If u_{x_1} is non-continuous in y_1 and u_{x_2} is non-continuous in y_2 for $x_1 < x_2 \leq e$ (the case when $e \leq x_1 < x_2$ is analogous) then $U(x_2, z) < e$ for all $z < y_2$ and $U(x_1, z) > e$ for all $z > y_1$ and the monotonicity implies that for all $x \in [x_1, x_2]$ the function u_x is non-continuous in some point $z \in [y_2, y_1]$. Note that $e \notin \mathrm{Ran}(u_x)$ since otherwise by Proposition 6 either u_{x_1} or u_{x_2} would be continuous. Assume the function $g\colon [x_1, x_2] \longrightarrow [y_2, y_1]$ which assigns to $v \in [x_1, x_2]$ a point $w \in [y_2, y_1]$ such that u_v is non-continuous in w. Then by Proposition 9 the function g is non-increasing. If $q \in [y_2, y_1] \setminus \mathrm{Ran}(g)$ then by the monotonicity there exists a $p \in [x_1, x_2]$ such that $g(d) > q$ if $d < p$ and $g(d) < q$ if $d > p$. Further, since g is monotone there exists an interval $[c,d]$, such that $q \in [c,d] \subset [y_2, y_1] \setminus \mathrm{Ran}(g)$. Then for $z \in [c,d]$ we have $U(z,v) < e$ for all $v < p$ and $U(z,v) > e$ for all $v > p$ thus u_z has a point of discontinuity in p. □

Lemma 6

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Let u_x be non-continuous in y_1 and u_{y_2} be non-continuous in x for some $y_1 \neq y_2$. Then for all $y \in]y_1, y_2]$ ($y \in [y_2, y_1[$) the function u_y is non-continuous in x.

PROOF: We will assume $y_1 < y_2$ (the case when $y_1 > y_2$ is analogous). Then $U(x,y) > e$ for all $y > y_1$ and $U(z,y) \leq U(z,y_2) < e$ for all $z < x$, $y \leq y_2$. Then since $U(x,y) \neq e$ the function u_y is non-continuous in x. □

In the following result we show that the set of discontinuity points of a uninorm $U \in \mathcal{U}$ has a non-empty intersection with the border of the unit square.

Lemma 7

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Assume $x < e$ ($x > e$) such that u_x is continuous on $[0,1]$ and let u_y be non-continuous in x. Then for all $q \in [y,1]$ ($q \in [0,y]$) the function u_q is non-continuous in x.

PROOF: We will assume $x < e$ (the case for $x > e$ is analogous). If $U(x,z) = e$ for some $z \in [0,1]$ then by Lemma 4 the points x, z are not idempotents and Proposition 6 implies that U is continuous on $[0,1]^2 \setminus ([0,a[\cup]b,1])^2$ for some $a < x$ and $b > z$. Therefore for all $y \in [0,1]$ the function u_y is continuous in x. Since $x < e$ by Lemma 3 we have $u_x(1) < e$, i.e., $U(x,z) < e$ for all $z \in [0,1]$. If u_y is non-continuous in x then $U(p,y) > e$ for all $p > x$ and $U(p,y) < e$ for all $p < x$. Assume any $q \in [y,1]$. Then $U(p,q) \leq U(x,q) < e$ if $p < x$ and $U(p,q) \geq U(p,y) > e$ if $p > x$, i.e.,u_q is non-continuous in x. □

Next we define a set-valued function.

Definition 1

A mapping $p \colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called a set-valued function on $[0,1]$ if to every $x \in [0,1]$ it assigns a subset of $[0,1]$, i.e., $p(x) \subseteq [0,1]$. Assuming the standard order on $[0,1]$, a set-valued function p is called

(i) *non-increasing* if for all $x_1, x_2 \in [0,1]$, $x_1 < x_2$, we have $y_1 \geq y_2$ for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$ and thus the cardinality $\mathrm{Card}(p(x_1) \cap p(x_2)) \leq 1$,

(ii) *symmetric* if $y \in p(x)$ if and only if $x \in p(y)$.

The graph of a set-valued function p will be denoted by $G(p)$, i.e., $(x,y) \in G(p)$ if and only if $y \in p(x)$.

The following is evident.

Lemma 8

A symmetric set-valued function $p \colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is surjective, i.e., for all $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y \in p(x)$, if and only if we have $p(x) \neq \emptyset$ for all $x \in [0,1]$.

The graph of a symmetric, surjective, non-increasing set-valued function $p \colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is a connected line (i.e., a connected set with no interior) containing points $(0,1)$ and $(1,0)$. Indeed, the monotonicity of such a set-valued function ensures that the graph of p has no interior. Further, since p is surjective, monotone and symmetric the graph of p contains points $(0,1)$ and $(1,0)$. If $G(p)$ is not a connected set then either $p(x)$ is not a connected set for some $x \in [0,1]$, which, however, due to the monotonicity implies that p is not surjective, or due to the monotonicity there exists an $x \in [0,1]$ such that either

$$\inf(\bigcup_{q<x} p(q)) > \sup(p(x)),$$

or

$$\sup(\bigcup_{q>x} p(q)) < \inf(p(x)),$$

which, however, due to the symmetry implies that p is not surjective.

The previous results can be summarized in the following theorem. First, however, we introduce one remark.

Remark 3

For any uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U}$ denote $A = \inf\{x \mid U(x,0) > 0\}$, $B = \sup\{x \mid U(x,1) < 1\}$ and let $a, d \in [0,1]$ be such that $U(x,y) = e$ for some $y \in [0,1]$ if and only if $x \in]a,d[\cup \{e\}$ (see Remark 1). If U is conjunctive, i.e., $U(0,1) = 0$, then A is the infimum of an empty set on $[0,1]$, i.e., $A = 1$. If U is disjunctive, i.e., $U(0,1) = 1$, then B is the supremum of an empty set on $[0,1]$, i.e., $B = 0$. Therefore we have either $A = 1, B \neq 0$, or $A \neq 1, B = 0$, or $A = 1, B = 0$. Further, $U(x,0) \leq e$ for some $x \in [0,1]$ implies

$$0 = U(e,0) \geq U(x,0,0) = U(x,0)$$

and thus for all $x \in [0,1]$ either $U(x,0) = 0$ or $U(x,0) > e$. Therefore U is non-continuous in $(0,A)$ if $A \neq 1$. Similarly, $U(x,1) \geq e$ for some $x \in [0,1]$ implies

$$1 = U(e,1) \leq U(x,1,1) = U(x,1)$$

and thus for all $x \in [0,1]$ either $U(x,1) = 1$ or $U(x,1) < e$. Therefore U is non-continuous in $(B,1)$ if $B \neq 0$. Finally, if $A = 1, B = 0$ then $U(x,0) = 0$ for all $x < 1$ and $U(x,1) = 1$

for all $x > 0$ and therefore U is non-continuous in $(0,1)$.

Due to Remark 1 either $a = d = e$, or U is continuous on $]a,d[\times [0,1] \cup [0,1] \times]a,d[$ and therefore we have $0 \le B \le a \le e \le d \le A \le 1$.

Theorem 2

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then there exists a symmetric, surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus R$, where $R = G(r)$. Note that U need not to be non-continuous in all points from R.

PROOF: We will define the set $R^* = \{(x,y) \in [0,1]^2 \mid U \text{ is non-continuous in } (x,y)\}$. Then due to the commutativity of U the set R^* is symmetric, i.e., $(x,y) \in R^*$ if and only if $(y,x) \in R^*$. If we define a set-valued function $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ by

$$r(x) = \begin{cases} \{1\} & \text{if } x \in]0,B[\,, \\ \{0\} & \text{if } x \in]A,1[\,, \\ [0,B] & \text{if } x = 1, \\ [A,1] & \text{if } x = 0, \\ \{y \mid U(x,y) = e\} & \text{if } x \in]a,d[\cup \{e\}, \\ \{y \mid (x,y) \in R^*\} & \text{otherwise} \end{cases} \tag{3}$$

then r is a symmetric set-valued function (see Remark 3). Since u_x is continuous if and only if $x \in [0,B[\cup]a,d[\cup \{e\} \cup]A,1]$ (which follows from Lemma 3 and Proposition 6) Lemma 8 implies that r is surjective. Moreover, it is evident that if U is non-continuous in (x_0,y_0) then $x_0 \in r(y_0)$.

We will further define the set

$$P = \{(x,y) \in [0,1]^2 \mid u_x \text{ is continuous in } y \text{ and } u_y \text{ is continuous in } x\}.$$

Assume $x_1 < x_2$ and $y_1 \in r(x_1)$, $y_2 \in r(x_2)$.

Case 1: If $(x_1, y_1), (x_2, y_2) \in R \setminus P$ then Proposition 9 implies $y_1 \geq y_2$.

Case 2: Assume $(x_1, y_1) \in P \cap R$, $(x_2, y_2) \in R \setminus P$. Then Proposition 10 implies that either $(x_3, y_1) \in R \setminus P$ for some $x_3 \in [0, 1]$, $x_1 < x_3 < x_2$, or $(x_1, y_3) \in R \setminus P$ for some y_3, $y_3 < y_1$. Now since $(x_2, y_2) \in R \setminus P$ the case when $(x_3, y_1) \in R \setminus P$ implies by Proposition 9 $y_1 \geq y_2$. In the case when $(x_1, y_3) \in R \setminus P$ we have $y_1 > y_3 \geq y_2$.

Case 3: Assume $(x_2, y_2) \in P \cap R$ and $(x_1, y_1) \in R \setminus P$. This case can be shown similarly as the Case 2.

Case 4: Assume $(x_1, y_1), (x_2, y_2) \in P \cap R$. Then Proposition 10 implies that either $(x_4, y_2) \in R \setminus P$ for some $x_4 \in [0, 1]$, $x_3 < x_4 < x_2$ $(x_1 < x_4 < x_2)$, or $(x_2, y_4) \in R \setminus P$ for some y_4, $y_4 > y_2$. If $(x_3, y_1) \in R \setminus P$ and $(x_4, y_2) \in R \setminus P$ we have $y_1 \geq y_2$. If $(x_3, y_1) \in R \setminus P$ and $(x_2, y_4) \in R \setminus P$ we have $y_1 \geq y_4 > y_2$. If $(x_1, y_3) \in R \setminus P$ and $(x_4, y_2) \in R \setminus P$ we have $y_1 > y_3 \geq y_2$. If $(x_1, y_3) \in R \setminus P$ and $(x_2, y_4) \in R \setminus P$ we have $y_1 > y_3 \geq y_4 > y_2$.

Therefore in all cases $y_1 \geq y_2$ and thus we have shown that r is non-increasing on $[B, a] \cup [d, A]$. Since r is evidently non-increasing also on $[0, B[\cup]a, d[\cup \{e\} \cup]A, 1]$ we see that r is non-increasing.

$\square$

Remark 4

U need not to be non-continuous in all points of R. From the previous proof we see that U is continuous in all points from $\{x\} \times [0, 1]$ for all $x \in [0, B[\cup]a, d[\cup \{e\} \cup]A, 1]$. The symmetric non-increasing set-valued function from the previous theorem need not to be unique. The differences can appear on $]a, d[^2$. However, if we require additionally that $U(x, y) = e$ implies $(x, y) \in R$ for all $(x, y) \in [0, 1]^2$, such a set-valued function is uniquely given and we will call such a set-valued function the *characterizing* set-valued function of a uninorm U for $U \in \mathcal{U}$.

Example 6

Assume a representable uninorm $U_1\colon [0, 1]^2 \longrightarrow [0, 1]$ and a continuous t-norm $T\colon [0, 1]^2 \longrightarrow$

$[0,1]$ and a continuous t-conorm $S\colon [0,1]^2 \longrightarrow [0,1]$. For simplicity we will assume that $\frac{1}{2}$ is the neutral element of U_1 and that $U_1(x, 1-x) = \frac{1}{2}$ for all $x \in \,]0,1[$. Let U_1^* be a linear transformation of U_1 to $\left[\frac{1}{3}, \frac{2}{3}\right]^2$, let T^* be a linear transformation of T to $\left[0, \frac{1}{3}\right]^2$ and let S^* be a linear transformation of S to $\left[\frac{2}{3}, 1\right]^2$. Then the ordinal sum of semigroups $G_\alpha = (\left[0, \frac{1}{3}\right], T^*)$, $G_\beta = (\left[\frac{1}{3}, \frac{2}{3}\right], U_1^*)$, $G_\gamma = (\left[\frac{2}{3}, 1\right], S^*)$, with $\gamma < \alpha < \beta$, is a semigroup $([0,1], U)$, where U is a uninorm, $U \in \mathcal{U}$. On Figure 5 we can see the characterizing set-valued function r of U as well as its set of discontinuity points.

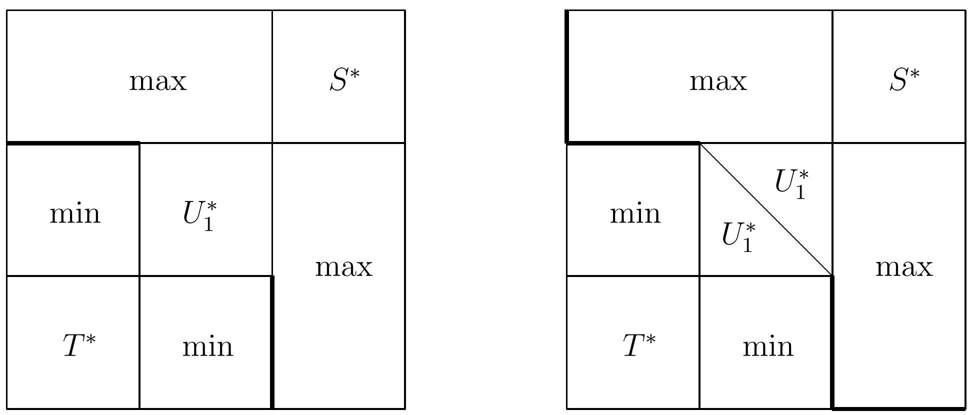

Figure 5: The uninorm U from Example 6. Left: the bold lines denote the points of discontinuity of U. Right: the oblique and bold lines denote the characterizing set-valued function of U.

Remark 5

It is easy to see that for $U \in \mathcal{U}$ its characterizing set-valued function r divides the uninorm U into two parts: U on points below the characterizing set-valued function attains values smaller than e, and U on points above the characterizing set-valued function attains values bigger than e.

Proposition 12

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then in each point $(x_0, y_0) \in [0,1]^2$ the uninorm U is either left-continuous or right continuous.

PROOF: From Proposition 8 we know that for all $x \in [0,1]$ the function u_x is either left-continuous or right continuous. If (x_0, y_0) is the point of continuity of U the claim is trivial.

Thus suppose that (x_0, y_0) belongs to the graph of the characterizing set-valued function r of U. If $U(x_0, y_0) = e$ then by Proposition 6 the uninorm U is continuous in (x_0, y_0) and thus either $U(x_0, y_0) < e$ or $U(x_0, y_0) > e$. If $U(x_0, y_0) < e$ then for all $x \le x_0$, $y \le y_0$ also $U(x, y) < e$ and thus u_x is left-continuous in y and u_y is left-continuous in x (see Remark 2). Now for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|U(x_0 - \delta_1, y_0) - U(x_0, y_0)| < \frac{\varepsilon}{2}$. Since also $u_{x_0-\delta_1}$ is left-continuous in y_0 there exists $\delta_2 > 0$ such that $|U(x_0-\delta_1, y_0-\delta_2)-U(x_0-\delta_1, y_0)| < \frac{\varepsilon}{2}$. The monotonicity of U then implies that

$$0 \le U(x_0, y_0) - U(x_0 - \delta_1, y_0 - \delta_2) =$$

$$U(x_0, y_0) - U(x_0 - \delta_1, y_0) + U(x_0 - \delta_1, y_0) - U(x_0 - \delta_1, y_0 - \delta_2) < \varepsilon.$$

Taking $\delta = \min(\delta_1, \delta_2)$, by the monotonicity of U we have shown that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in [x_0 - \delta, x_0]$ and $y \in [y_0 - \delta, y_0]$ we have $|U(x, y)-U(x_0, y_0)| < \varepsilon$, i.e., that U is left-continuous in (x_0, y_0). Similarly, if $U(x_0, y_0) > e$ then U is right-continuous in (x_0, y_0). □

The previous proposition and the construction of the characterizing set-valued function r of a uninorm U implies the following.

Corollary 2

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then there exists a symmetric, surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus R$, where $R = G(r)$ and if $U(x, y) = e$ then $(x, y) \in R$. Moreover, in each point $(x, y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous.

3.2 The sufficiency part

In this part we will show that if for a uninorm U there exists a symmetric, surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus R$, where $R = G(r)$, and $U(x, y) = e$ implies $(x, y) \in R$, then $U \in \mathcal{U}$ if and only if in each point

$(x, y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous.

We will denote the set of all uninorms $U\colon [0,1]^2 \longrightarrow [0,1]$ such that U is continuous on $[0,1]^2 \setminus R$, where $R = G(r)$ and r is a symmetric, surjective, non-increasing set-valued function such that $U(x, y) = e$ implies $(x, y) \in R$, by $\mathcal{UR}$. First we will show that there exists a uninorm $U \in \mathcal{UR}$ such that $U \notin \mathcal{U}$.

Example 7

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be given by

$$U(x, y) = \begin{cases} 0 & \text{if } \max(x, y) < e, \\ x & \text{if } y = e, \\ y & \text{if } x = e, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Then Proposition 2 implies that $U \in \mathcal{U}_{\max}$ is a uninorm, where the underlying t-norm is the drastic product and the underlying t-conorm is the maximum. This uninorm is non-continuous in points from $\{e\} \times [0, e] \cup [0, e] \times \{e\}$. Thus the corresponding set-valued function is given by (see Figure 6)

$$r(x) = \begin{cases} [e, 1] & \text{if } x = 0, \\ e & \text{if } x \in \left]0, e\right[, \\ [0, e] & \text{if } x = e, \\ 0 & \text{otherwise.} \end{cases}$$

Since $U(x, y) = e$ implies $x = y = e$ we see that U is continuous on $[0,1]^2 \setminus R$, where $R = G(r)$ and r is a symmetric, surjective, non-increasing set-valued function such that $U(x, y) = e$ implies $(x, y) \in R$. However, the drastic product t-norm is not continuous and thus $U \notin \mathcal{U}$.

Assume $U \in \mathcal{UR}$. Then for the corresponding characterizing set-valued function r we

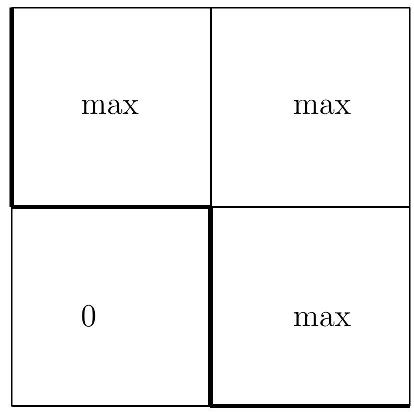

Figure 6: The uninorm U from Example 7. The bold lines denote the characterizing set-valued function r of U.

have $(e,e) \in G(r)$. Denote

$$D = \{e\} \times [0,1] \cup [0,1] \times \{e\}.$$

We have two possibilities: either $G(r) \cap D = \{(e,e)\}$, or $\mathrm{Card}(G(r) \cap D) > 1$. First we will assume the case when $G(r) \cap D = \{(e,e)\}$. Then T_U (S_U) is continuous in all points from $[0,e]^2$ $([e,1]^2)$ except possibly the point (e,e) and we have the following result.

Lemma 9

Let $T\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-norm which is continuous on* $[0,1]^2 \setminus \{(1,1)\}$*. Then* T *is continuous on* $[0,1]^2$.

PROOF: Assume that T is not continuous in $(1,1)$. Then there exist two sequences $\{a_n\}_{n\in\mathbb{N}}$, $a_n \in \,]0,1[$ and $\{b_n\}_{n\in\mathbb{N}}$, $b_n \in \,]0,1[$ such that $\lim\limits_{n\longrightarrow\infty} a_n = \lim\limits_{n\longrightarrow\infty} b_n = 1$ and $\lim\limits_{n\longrightarrow\infty} T(a_n,b_n) < 1$. Since $T(a_n,b_n) \geq T(\min(a_n,b_n), \min(a_n,b_n))$ we see that there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$, $c_n \in \,]0,1[\,$, $\lim\limits_{n\longrightarrow\infty} c_n = 1$ such that $\lim\limits_{n\longrightarrow\infty} T(c_n,c_n) = 1-\delta < 1$, for some $\delta > 0$. Since T is a t-norm we have $T(1-\frac{\delta}{2},1) = 1-\frac{\delta}{2}$ and necessarily $T(1-\frac{\delta}{2},1-\frac{\delta}{2}) \leq 1-\delta$. Since T is continuous on $[0,1]^2 \setminus \{(1,1)\}$ there exists an $\varepsilon > 0$ such that $T(1-\frac{\delta}{2},1-\varepsilon) = 1-\frac{2\delta}{3}$ and the monotonicity of T implies $\varepsilon < \frac{\delta}{2}$. Thus

$$1 - \frac{2\delta}{3} = T(1 - \frac{\delta}{2}, 1 - \varepsilon) \leq T(1 - \varepsilon, 1 - \varepsilon) \leq 1 - \delta,$$

what is a contradiction. $\square$

By duality between t-norms and t-conorms we get the following.

Lemma 10

Let $S\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-conorm which is continuous on* $[0,1]^2 \setminus \{(0,0)\}$. *Then* S *is continuous on* $[0,1]^2$.

From the two previous results we see that if $U \in \mathcal{UR}$ and $G(r) \cap D = \{(e,e)\}$ then $U \in \mathcal{U}$.

Further we will suppose that $\mathrm{Card}(G(r) \cap D) > 1$. Then we obtain the following result.

Lemma 11

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ *be a uninorm,* $U \in \mathcal{UR}$, $U \notin \mathcal{U}$. *Then there exists a point* $(x,y) \in [0,1]^2$ *such that* U *is neither left-continuous, nor right-continuous in* (x,y).

PROOF: Since $U \notin \mathcal{U}$ Lemmas 9 and 10 imply that $\mathrm{Card}(G(r) \cap D) > 1$. Then there exists an $x_1 \in [0,1]$, $x_1 \neq e$ such that $(x_1, e) \in G(r)$. We will suppose that $x_1 < e$ (the case when $x_1 > e$ is analogous). Let

$$x_0 = \inf\{x \in [0,e] \mid (x,e) \in G(r)\}.$$

Then the monotonicity of r implies that S_U is continuous and $]x_0, e] \times \{e\} \subset G(r)$. Moreover, $U(x,y) = e$ implies $x = y = e$ for all $x, y \in [0,1]$. Since U is continuous on $]x_0, e] \times]e, 1] \cup]e, 1] \times]x_0, e]$ we see that $U(x,y) > e$ for all $x \in]x_0, e]$, $y \in]e, 1]$. On the other hand, the neutral element e and the monotonicity of U implies $U(x,y) \in [x,y]$ for all $x \in]x_0, e]$, $y \in]e, 1]$. Thus for all $x \in]x_0, e[$ we have $\lim\limits_{s \longrightarrow e^+} U(x,s) = e$. Therefore on $]x_0, e[$ the uninorm U is not right-continuous. Since $U \notin \mathcal{U}$ and T_U is continuous on $[0,1[^2$ we see that U is not left-continuous in some point (x,e) for $x \in [x_0, e]$. Now similarly as in Lemma 9 we can show that U is not left-continuous in some point (x, e) for $x \in [x_0, e[$. Finally, the neutral element and the monotonicity of U imply that U is not left-continuous in some point (x, e) for $x \in]x_0, e[$. Summarising, there exists a point $(x, y) \in [0, 1]^2$ such that U is neither left-continuous, nor right-continuous in (x, y). □

All previous results can be compiled into the following theorem.

Theorem 3

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{UR}$. Then $U \in \mathcal{U}$ if and only if in each point $(x, y) \in [0, 1]^2$ the uninorm U is either left-continuous or right-continuous.

Corollary 3

Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm. Then $U \in \mathcal{U}$ if and only if $U \in \mathcal{UR}$ and in each point $(x, y) \in [0, 1]^2$ the uninorm U is either left-continuous or right-continuous.

4 Conclusions

We have shown that a uninorm with continuous underlying t-norm and t-conorm is continuous on $[0, 1]^2 \setminus R$, where R is the graph of some symmetric, surjective, non-increasing set-valued function. On the other hand, we have shown also a sufficient condition for a uninorm to have continuous underlying operations. In the follow up papers [25, 26] we will employ these results and using the characterizing set-valued function of a uninorm we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms, internal uninorms and singleton semigroups. Thus these three papers together offer a complete characterization of uninorms from $\mathcal{U}$, i.e., of uninorms with continuous underlying t-norm and t-conorm. The applications of these results are expected in all domains where uninorms are used.

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