Extension of restricted equivalence functions and similarity measures for type-2 fuzzy sets

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***Abstract*—In this work we generalize the notion of restricted equivalence function for type-2 fuzzy sets, leading to the notion of extended restricted equivalence functions. We also study how under suitable conditions, these new functions recover the standard axioms for restricted equivalence functions in the real setting. Extended restricted equivalence functions allow us to compare any two general type-2 fuzzy sets and to generate a similarity measure for type-2 fuzzy sets. The result of this similarity is a fuzzy set on the same referential set (i.e., domain) as the considered type-2 fuzzy set. The latter is crucial for applications such as explainable AI and decision making, as it enables an intuitive interpretation of the similarity within the domain-specific context of the fuzzy sets. We show how this measure can be used to compare type-2 fuzzy sets with different membership functions in such a way that the uncertainty linked to type-2 fuzzy sets is not lost. This is achieved by generating a fuzzy set rather than a single numerical value. Furthermore, we also show how to obtain a numerical value for discrete referential sets.**

***Index Terms*—Type-2 Fuzzy Sets, Restricted Equivalence Functions, Similarity Measures, Uncertainty, Information Loss, Explainable AI**

I. Introduction

The task of comparing two objects is relevant in virtually every area of study; whether working with images, functions, or algorithms. In the particular case of Fuzzy Sets (FSs), one of the most used techniques to that end is the one based on similarity measures [1].

In the literature we can find several works generalizing the concept of similarity measures for different extensions of FSs [2], such as interval-valued FSs [3], [4], [5] or intuitionistic FSs [6], [7] . In the context of type-2 FSs, there are works dealing only with interval type-2 [8], [9], [10] or with general type-2 FSs [11], [12], [13], [14].

However, a number of fundamental questions remain. Fuzzy sets and their generalizations are characterized for dealing with an additional grade of uncertainty. Considering this fact, two distinct directions can be found in the literature. When generalizing some theoretical notions, is it better to maintain the original format/structure of the result, so that it can be understood in the same way as the previous notion or should the format/structure change to reflect the additional degree of freedom – here, offering the capacity to model the uncertainty? As stated in the literature, there is not a clear answer to this question since "Here naturalness will no longer help us much, for all the evidence suggests that what is natural to one author in the field is not the same as what is natural to another, and readers will surely show at least the same variety."(Bandler and Kohout in [15]).

These two directions can be also found in the context of type-2 similarity measures. In the literature we find both, similarity measures in which the result is a numerical value [4], [12], [16] and similarity measures in which the result is a fuzzy set [17], [18]. We opt for the latter case, coinciding with Bandler and Kohout([15]) which state that "The natural anticipation, it seems to us, is that the fuzziness will not thereby be diminished". The interest of our similarity measure is that it fulfills appropriate properties without losing important uncertainty information. Moreover, if a single representative value is required, the resulting fuzzy set can be defuzzified.

Similarity measures are commonly established in respect to a set of axioms which varies depending on the applications [19], [20], [21], [22]. A commonly used way of building similarity measures in the context of type-1 fuzzy sets and interval-valued fuzzy sets is based on aggregating Restricted Equivalence Functions (REF) [23], [24], [25]. REFs were introduced in the literature as a generalization of equivalence functions introduced by Fodor [26]. They allow the generation of similarity measures of fuzzy sets in terms of simple and well studied functions such as aggregation functions [27], [28], implication operators [29], [30], etc.

As explained in [23], the similarity measures generated through the aggregation of the values of restricted equivalence functions satisfy that:

- the comparison of two sets does not depend on the order in which they are compared, i.e., the similarity measures are symmetric;
- the comparison is the maximum reachable value if and only if we are comparing a set with itself;

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- the comparison of two sets is the minimum reachable value if and only if we are comparing complementary crisp sets;
- the comparison of two sets yields the same value as when comparing their complements.

These properties are especially desirable for image processing problems, see [13], [23], [31], [32], [33].

All the previous considerations have led us to consider the generalization of restricted equivalence functions and similarity measures to the type-2 FSs setting. We call the former Extended Restricted Equivalence Functions (EREF), which are designed to compute a similarity measure between membership functions of type-2 FSs. Moreover, we present the concept of EREF-based Type-2 Similarity Measures, giving a definition and a construction method based on EREFs. The main advantage behind this concept is that the result of the functions is a fuzzy set, i.e., the final result is not a single value but several ones (one per element of the universe of the type-2 FSs). Hence, most of the uncertainty associated with the type-2 FSs is preserved. We show that the aggregation of the considered FSs may be seen as a numerical T2SM. The convenience of this proposal is that it maintains the uncertainty until the last step. Nevertheless, it is important to remark that our definition of similarity is the first one to consider that similarity is given by a fuzzy set over the same referential set (i.e., domain) as the one over which the compared type-2 fuzzy sets are considered.

Observe, in order to compare two objects, a natural way is to compare the different components of those objects. This is the approach followed by our definition of similarity in terms of restricted equivalence functions. We first compute how similar the membership functions for the same element in the referential universe are. This information is later taken into account by considering the type 1 fuzzy set it defines – in which the membership of each element is a measure of how similar the two corresponding membership functions are -.

This paper is organized as follows. In Section II, we recall some preliminary notions. In Section III we discuss the concept of extended restricted equivalence functions and generate a construction method. In Section IV, we introduce and demonstrate some properties of a novel type-2 similarity measure. Finally, in Section V, we present some conclusions and remarks about future works.

II. Background

In this work we denote by X a non-empty universe of discourse that can be either finite or infinite. When the restriction to a finite universe is required, the universe is denoted by U.

A. Type-1 Fuzzy Sets

Definition 2.1: [34] A type-1 Fuzzy Set, or simply, a Fuzzy Set (FS) A is a mapping $A : X \mapsto [0,1]$ where the value $A(x)$ is referred to as the membership degree of the element x to the FS A.

The set of all FSs on X is denoted by $FS(X)$.

We also recall here the definition of aggregation function.

Definition 2.2: Let $m \geq 2$ (where m is the number of sources). An aggregation function is a function $M : [0,1]^m \to [0,1]$ which is increasing in each variable and it satisfies that $M(0,\ldots,0) = 0$ and $M(1,\ldots,1) = 1$.

B. Type-2 fuzzy sets

While the concept of FS has been very useful, establishing discrete degrees of membership is often challenging and counter-intuitive. Trying to solve this difficulty, different generalizations of FSs have been introduced. In this paper, we work with Type-2 FSs (T2FS) as originally introduced by Zadeh in [35].

Definition 2.3: A Type-2 FS (T2FS) $\tilde{A}$ on X is a mapping $\tilde{A} : X \mapsto FS([0,1])$ where the membership degree of an element of the universe $x \in X$, which is denoted by $\tilde{A}_x$, is a FS on the infinite universe $[0,1]$.

It is worth mentioning that two different notations in the context of type-2 fuzzy exist. In this case, for the simplicity of our mathematical developments we have opted for the notation used in [36], [37], [38]. With this notation T2FSs can be expressed mathematically as a mapping $\tilde{A} : X \mapsto \mathcal{F}$, where

$$\mathcal{F} = \{f \mid f : [0,1] \mapsto [0,1]\}.$$

Let $T2FS(X)$ denote the class of all T2FSs in the universe X. In this case, the membership degree of each element $x = x'$ is associated with a function $f : [0,1] \mapsto [0,1]$. In a three-dimensional coordinate framework, note that, with this notation, $\tilde{A}_x(t)$ corresponds to the z-axis, with t in the y-axis, see [39].

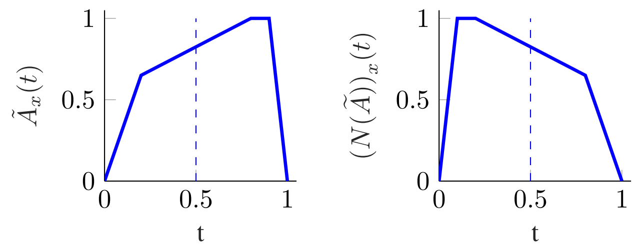
For those accustomed to the notation of Mendel et al. [40], the preceding function is equivalent to the secondary membership degree $(\mu_{\tilde{F}}(x', u))$ for a fixed value $x = x'$ in a vertical slice where u lies in the unit interval $[0,1]$ (adding the value $\mu_{\tilde{F}}(x', u) = 0$ to those $u \notin [\underline{\mu}_{\tilde{F}}(x', u), \overline{\mu}_{\tilde{F}}(x', u)]$). See [14], [41] for more complete information.

The union, intersection and negation operations of T2FSs have been deeply studied, including in [36], [37]. Here, we briefly recapitulate the notion of negation and some important concepts that are crucial for the subsequent parts of the paper.

Definition 2.4: Let X be a universe of discourse and let $\tilde{A} \in T2FS(X)$. The negation of the T2FS $\tilde{A}$, denoted by $N(\widetilde{A})$ is the new T2FS such that for each element x in the universe, the membership degree is given by:

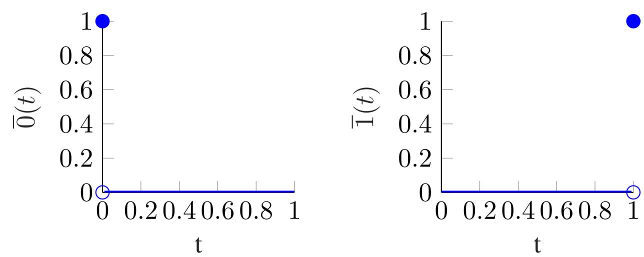
$$(N(\widetilde{A}))_x(t) = \tilde{A}_x(1-t) \text{ for any } t \in [0,1].$$

Observe that this refers to the membership of t on the membership of the element x on $N(\tilde{A})$. Note also that considering $\tilde{A} \in T2FS(X)$, for any $x \in X$, the function representing the membership degree of the negation, $(N(\widetilde{A}))_x \in \mathcal{F}$ is the symmetric function with respect to $t = \frac{1}{2}$ of the function $\tilde{A}_x$ which represents the membership degree of the type-2 FS $\tilde{A}$ (i.e., $(N(\tilde{A}))_x(t) = \tilde{A}_x(1-t)$ for every $t \in [0,1]$, see Fig. 1, where we represent the membership value to the type-2 FS, on the left, and its negation, on the right). Moreover, N is a strong negation, i.e., it is involutive which means that it holds $N(N(\widetilde{A})) = \tilde{A}$ for all $\tilde{A} \in T2FS(X)$.



a) Membership function b) Negation of case a)

Fig. 1. Example of a function degree and its negation.



(a) Function $\overline{0}$ (b) Function $\overline{1}$

Fig. 2. Representation of the functions $\overline{0}$ and $\overline{1}$.

Let X be a universe of discourse and let $\tilde{A} \in T2FS(X)$. Considering $x \in X$, the interpretation of the membership function f_x associated to x is the following. For any $t \in [0,1]$,

- $f_x(t) = 1$ means certainty about the membership of the value t to the type-1 FS associated to x.
- $f_x(t) = 0$ means certainty about the non-membership of the value t to the type-1 FS associated to x.
- $f_x(t) = 0.5$ means uncertainty about the membership of the value t to the type-1 FS associated to x.

An important consequence of the interpretation of T2FSs is the fact that the constant function $\mathbf{0}$ given by $\mathbf{0}(t) = 0$ for all $t \in [0,1]$ means completely certainty that there is no possible value of membership degree (empty set). From here on, all our developments assume that the membership degrees of a T2FS lie in $\mathcal{G} = \mathcal{F} \backslash \{\mathbf{0}\}$ (see also [39]).

Moreover, the functions representing the logical values *False* ($\mathbb{F}$) and *True* ($\mathbb{T}$) in classical logic are the functions $\overline{0}$ and $\overline{1}$ given by

$$\overline{0}(t) = \begin{cases} 1\,, & \text{if } t = 0\,, \\ 0\,, & \text{otherwise}\,; \end{cases} \tag{1}$$

$$\overline{1}(t) = \begin{cases} 1\,, & \text{if } t = 1\,, \\ 0\,, & \text{otherwise} \end{cases} \tag{2}$$

which are displayed in Fig. 2(a)–(b).

Note that, considering the negation introduced in Def. 2.4, it holds that the negation of the membership degree $\overline{0}$ (*False*) is $\overline{1}$ (*True*), while the negation of the membership degree $\overline{1}$ (*True*) is $\overline{0}$ (*False*) as in classical logic.

C. Distance Measures

Quantifying the similarity between two objects is an important field of research. This quantification is often based on distances which capture the proximity of the two considered objects. First of all, we recall the notion of distance between numbers in the unit interval, see, for instance, [42].

Definition 2.5: A distance function on the unit interval is a function $d : [0,1] \times [0,1] \to \mathbb{R}$ that satisfies for any $x, y, z \in [0,1]$ that:

1) $d(x,y) \geq 0$.
2) $d(x,y) = 0$ if and only if $x = y$.
3) $d(x,y) = d(y,x)$.
4) $d(x,z) \leq d(x,y) + d(y,z)$.

Note that, since membership grades of elements in a fuzzy set are numbers between zero and one, it would be possible to measure distances between fuzzy sets by appropriately aggregating distances between the corresponding membership grades.

Example 2.1: Some examples of distances are:

- The Euclidean distance, $d_E(x,y) = |x - y|$;
- The function, $d(x,y) = k|\arctan(x) - \arctan(y)|$, for any $k > 0$.

If we consider a finite referential set $U = \{u_1, \ldots, u_n\}$, if we take as aggregation function the arithmetic mean $M(t_1, \ldots, t_n) = \frac{1}{n}(t_1 + \cdots + t_n)$ and if $A, B \in FS(U)$, then, $M(d_E(A(u_1), B(u_1)), \ldots, d_E(A(u_n), B(u_n)))$ can be seen as a measure of the distance between A and B. Of course, d_E can be replaced by any other distance.

D. Similarity Measures

In this paper, we introduce a similarity measure for type-2 FSs based on the definition of similarity measures for type-1 FSs. Similarity measures are functions which quantify the similarity between two objects. Specifically, in the literature we can find many axiomatizations of similarity measures for FSs [1], [43], [44], [45]. Our definition builds on that by L. Xuecheng [43], although it is not exactly the same since our axioms are more restrictive. In particular, the motivation for Axiom 2 in Def. 2.6 follows from the necessity of setting for which of the considered objects the difference should be the greatest possible one, or equivalently, the similarity should be the smallest possible one. This approach has already been followed in the literature [44], [46], [47], since, in the case of type-1 fuzzy sets and interval-valued fuzzy sets, this property is relevant for some applications such as image processing [47], where it is natural to demand that the value of the similarity measure is 0 when the two sets considered to be the extremal ones are involved.

Definition 2.6: A similarity measure S is a mapping $S : FS(X) \times FS(X) \to [0,1]$ such that:

1) $S(A,B) = S(B,A)$ for every $A, B \in FS(X)$.
2) $S(A,B) = 0$ if and only if $\{A(x), B(x)\} = \{0,1\}$ for all $x \in X$.
3) $S(A,B) = 1$ if and only if $A(x) = B(x)$ for all $x \in X$.
4) For any $A, B, C \in FS(X)$, such that $A \leq B \leq C$, it holds $S(A,C) \leq S(A,B)$ and $S(A,C) \leq S(B,C)$, (where the order relation $A \leq B$ holds if and only if $A(x) \leq B(x)$ for any $x \in X$).

There are several construction methods for similarity measures as introduced in Def. 2.6. One approach for finite universes is based on the aggregation of restricted equivalence functions. From the point of view of this work, we are especially interested in aggregation functions which are averaging.

Definition 2.7: An aggregation function $M : [0,1]^m \to [0,1]$ is averaging if, for every $x_1, \ldots, x_m \in [0,1]^m$, the inequalities

$$\min(x_1, \ldots, x_m) \leq M(x_1, \ldots, x_m) \leq \max(x_1, \ldots, x_m)$$

hold.

Some other aggregation functions, such as t-norms or copulas, fail to be averaging. However, there are many well-known functions which are averaging aggregation functions such as the minimum, the maximum or the arithmetic mean [28].

For the next definition we also need to recall that a strong negation is a decreasing continuous function $n : [0,1] \to [0,1]$ such that $n(n(x)) = x$ for every $x \in [0,1]$.

Definition 2.8: [23] A Restricted Equivalence Function (REF) is a function $REF : [0,1]^2 \to [0,1]$ associated with a strong negation n if it satisfies the following:

1) $REF(x,y) = REF(y,x)$ for all $x, y \in [0,1]$;
2) $REF(x,y) = 1$ if and only if $x = y$;
3) $REF(x,y) = 0$ if and only if $\{x,y\} = \{0,1\}$;
4) $REF(x,y) = REF(n(x), n(y))$ for all $x, y \in [0,1]$;
5) For all $x, y, z, t \in [0,1]$, such that $x \leq y \leq z \leq t$, it holds that $REF(y,z) \geq REF(x,t)$.

As we have already said, the relevance of REFs comes from the fact that they can be used to build similarity measures between fuzzy sets, as the next result shows.

Proposition 2.2: [23], [24] Let REF be a restricted equivalence function and let M be an m-ary aggregation function satisfying that $M(x_1, \ldots, x_m) = 0$ if and only if $x_i = 0$ for all $i \in \{1, \ldots, m\}$ and $M(x_1, \ldots, x_m) = 1$ if and only if $x_i = 1$ for all $i \in \{1, \ldots, m\}$. The function $SM : FS(U) \times FS(U) \to [0,1]$, given by

$$SM(A,B) = M_{i=1}^{m}(REF(A(u_i), B(u_i))) \tag{3}$$

is a similarity measure.

For instance, if we take the restricted equivalence function $REF(x,y) = 1 - |x - y|$ and the arithmetic mean, we recover the similarity measure

$$SM(A,B) = \frac{1}{n} \sum_{i=1}^{n} (1 - |A(u_i) - B(u_i)|) \; .$$

III. Extended restricted equivalence functions

The semantics of Restricted Equivalence Functions (REF) are tied to computing the similarity between two membership degrees on a FS. Mathematically, the generalization of Restricted Equivalence Functions for constructing a similarity measure for type-2 FSs should measure how similar two functions $f_1, f_2 \in \mathcal{G}$ are. To do this we need to modify the domain of restricted equivalence functions and replace the values in the unit interval $[0,1]$ with functions in $\mathcal{G}$. We denote these functions by EREFs (*Extended Restricted Equivalence Functions*.)

The special nature of this generalization compels us to extend the domain of REF from a finite cardinality to an infinite one. We do this in order to construct the theoretical development as generally as possible, but the real required cardinal in this case is that of $\mathcal{G}^2$.

For infinite cardinality, it is natural to think about an integral, i.e, formulas similar to

$$EREF(f_1, f_2) = \int d(f_1, f_2),$$

(where d is some distance) dependent on f_1 and f_2 satisfying additional properties.

Axiom 3 of REF adresses cases in which two values are the most dissimilar, i.e., the minimum value given to contrary values. For type-2 FSs, our proposal consists of replacing the values 0, 1 with the functions bottom and top, i.e., with the functions $\overline{0}$ (*False* in classical logic) and $\overline{1}$ (*True* in classical logic) introduced in Eqs. (1) and (2). Hence, the axiom should be:

3') $EREF(f_1, f_2) = 0$ if and only if $\{f_1, f_2\} = \{\overline{0}, \overline{1}\}$.

A first problem of this integral approximation arises due to the integral of two functions which differ in a zero-measure set (in a set of finite points, for example) is equal. A possible solution would be to consider the following modification of axiom 2):

2') $EREF(f_1, f_2) = 1$ if and only if $f_1 = f_2$ $a.e.$,

where $f_1 = f_2$ $a.e.$ means that $f_1 = f_2$ are equal almost everywhere, that is, $f_1(t) = f_2(t)$ for all $t \in [0,1] \backslash S$ being S a set of null measure (i.e., of measure equal to zero).

But even in this case, a second drawback must be taken into account. Considering Axiom 2', since the functions $\overline{0}$ and $\overline{1}$ are the same for every t different from 0 and 1, they only differ in the set $\{0,1\}$, which is a discrete set and hence it is a set of measure zero. According to 2'), this would imply that $EREF(\overline{0}, \overline{1}) = 1$. Moreover, due to Axiom 3' it holds that $EREF(\overline{0}, \overline{1}) = 0$, a contradiction. Our proposal to deal with this problem is based on the cumulative functions defined in [37].

Definition 3.1: Let $f \in \mathcal{G}$. The left- and right-cumulative functions $f^L, f^R : [0,1] \to [0,1]$ are defined for any $t \in [0,1]$ as:

$$f^L(t) = \bigvee_{y \leq t} f(y)$$

and

$$f^R(t) = \bigvee_{y \geq t} f(y).$$

Note that the left cumulative function f^L is an increasing function while f^R is a decreasing one. Moreover, since the constant function $\mathbf{0}$ is not in $\mathcal{G}$, we find that the function $f^L(1) > 0$ and $f^R(0) > 0$. An example of the cumulative functions is depicted in Fig. 3.

Considering the cumulative functions we propose to consider the modification of Axiom 2' as follows:

- $EREF(f_1, f_2) = 1$ if and only if $f_1^L = f_2^L$ *a.e.* and $f_1^R = f_2^R$ *a.e.*

Summarizing our proposal for the definition of EREF is the following.

Definition 3.2: A function $EREF : \mathcal{G}^2 \to [0,1]$ is called an extended restricted equivalence function if it satisfies:

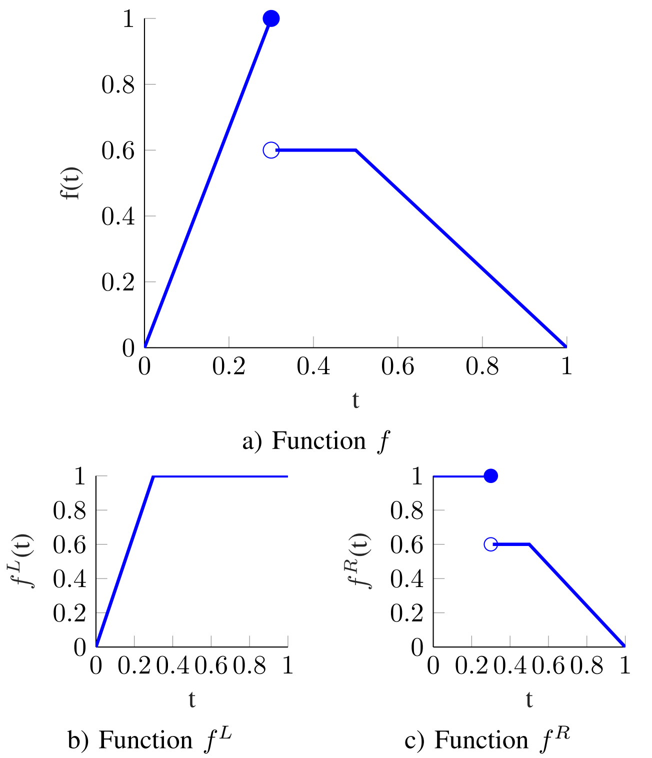


Fig. 3. Representation of a function f and its cumulative functions f^L, f^R: a) function f, b) left-cumulative function and c) right-cumulative function.

1) $EREF(f_1, f_2) = EREF(f_2, f_1)$ for all $f_1, f_2 \in \mathcal{G}$;
2) $EREF(f_1, f_2) = 1$ if and only if $f_1^L = f_2^L$ *a.e.* and $f_1^R = f_2^R$ *a.e.*;
3) $EREF(f_1, f_2) = 0$ if and only if $\{f_1, f_2\} = \{\overline{0}, \overline{1}\}$;
4) $EREF(f_1, f_2) = EREF(N(f_1), N(f_2))$ where $N(f_i)$ $(i = 1, 2)$ is the negation given by $N(f_i)(t) = f_i(1-t)$ for all $t \in [0, 1]$;
5) for any three functions $f_1, f_2, f_3 \in \mathcal{G}$ such that $f_1 \leq f_2 \leq f_3$[1] it holds $EREF(f_1, f_3) \leq EREF(f_1, f_2)$ and $EREF(f_1, f_3) \leq EREF(f_2, f_3)$.

Remark 1: Note that if $f_1 = f_2$, then it holds that $f_1^L = f_2^L$ and $f_1^R = f_2^R$. The converse does not always hold, just consider the function identically equal to 1 and the function which is 1 everywhere except at $t = 0.5$, where it takes the value 0. However, we find a close relation. In [37] it is proved that the union and intersection operations of type-2 FSs generate a lattice only if the considered functions are normal and convex. Namely, in order to generate a lattice the functions must satisfy these additional properties:

- $\bigvee_{t\in[0,1]} f(t) = 1$ (normal functions)
- for any $t \leq y \leq z$, it holds that $f(y) \geq f(t) \wedge f(z)$ (convex functions).

In [37] it is also proved that in the set of normal and convex functions it holds that $(f^L \wedge f^R)(x) = f(x)$. Hence, $f_1^L = f_2^L$ and $f_1^R = f_2^R$ imply that $f_1 = f_2$. This means that the proposed Axiom 2 is the straightforward generalization of Axiom 2 for extended restricted equivalence functions when we consider normal and convex functions.

[1] We consider $f_1 \leq f_2$ if $f_1(t) \leq f_2(t)$ for any $t \in [0, 1]$.

The main reason to introduce left and right cumulative functions is to provide a tool to deal with membership functions as general as possible. Observe that, strictly speaking, if we consider a type-2 fuzzy set A over a referential set X, any function $f : [0, 1] \to [0, 1]$ can be taken as membership function for a given $x \in X$. Hence, it is necessary to provide a mathematical mechanism as general as possible in order to compare any two of these functions, and the use of cumulative functions provides such mechanism.

Theorem 3.1: Let $d : [0, 1]^2 \to [0, 1]$ be a distance on $[0, 1]$ such that

1) $d(x, y) = 1$ if and only if $\{x, y\} = \{0, 1\}$, and
2) for every $x, y, z \in [0, 1]$ such that $x \leq y \leq z$, it holds that $d(x, z) \geq d(x, y)$.

Then, the function

$$EREF(f_1, f_2) = 1 - \frac{1}{2}\int_0^1 d(f_1^L(t), f_2^L(t))dt - \frac{1}{2}\int_0^1 d(f_1^R(t), f_2^R(t))dt$$

is an $EREF$.

Proof: See Appendix. ■

Example 3.2:

1) Considering the Euclidean distance, the function

$$EREF(f_1, f_2) = 1 - \frac{1}{2}\int_0^1 |f_1^L(t) - f_2^L(t)|dt - \frac{1}{2}\int_0^1 |f_1^R(t) - f_2^R(t)|dt\,,$$

is an EREF.

2) Considering the distance function $d(x, y) = \frac{4}{\pi}(|\arctan(x) - \arctan(y)|)$, the function

$$\begin{aligned} EREF(f_1, f_2) &= \\ 1 &- \frac{1}{2}\int_0^1 \frac{4}{\pi}|\arctan(f_1^L(t)) - \arctan(f_2^L(t))|dt \\ &- \frac{1}{2}\int_0^1 \frac{4}{\pi}|\arctan(f_1^R(t)) - \arctan(f_2^R(t))|dt \\ &= 1 - \frac{2}{\pi}\int_0^1 |\arctan(f_1^L(t)) - \arctan(f_2^L(t))|dt \\ &\quad - \frac{2}{\pi}\int_0^1 |\arctan(f_1^R(t)) - \arctan(f_2^R(t))|dt \end{aligned}$$

is an EREF.

IV. Type-2 Similarity Measures

In this section we generalize the definition of a similarity measure, as well as introduce a construction method based on EREFs defined in the preceding section. Like in the definition of similarity measures for Type-1 FSs, a reasonable way to define similarity measures for type-2 FSs is to give a number in the unit interval $[0, 1]$ such that the closer to one, the more similar the T2FSs are. However, that idea implies an immense loss of information. Even worse, type-2 FSs provide a fine-grained model of uncertainty and we believe that to summarize the similarity of two T2FSs in a single value is not an ideal option. Our proposal consists of associating the similarity of any two T2FSs on the universe X with a FS on the universe X, such that the membership degree of each element of the universe is associated with a number which expresses the similarity between the two type-2 memberships at that element.[2] The main advantage of this proposal is that it allows the generalization of the concept of a similarity measure while preserving key uncertainty information inherent in the type-2 FSs which are being compared.

Definition 4.1: A type-2 similarity measure T2SM is a mapping $T2SM : T2FS(X) \times T2FS(X) \longrightarrow FS(X)$ such that, for any element x in the universe of discourse X, the membership degree of the T2SM satisfies that:

1) $T2SM(\tilde{A},\tilde{B})_x = T2SM(\tilde{B},\tilde{A})_x$ for any $\tilde{A},\tilde{B} \in T2FS(X)$.
2) $T2SM(\tilde{A},\tilde{B})_x = 0$ if and only $\{\tilde{A}_x,\tilde{B}_x\} = \{\overline{0},\overline{1}\}$.
3) $T2SM(\tilde{A},\tilde{B})_x = 1$ if and only if $\tilde{A}^L_x = \tilde{B}^L_x$ *a.e.* and $\tilde{A}^R_x = \tilde{B}^R_x$ *a.e* for all $x \in X$.
4) For any $\tilde{A},\tilde{B},\tilde{C} \in T2FS(X)$ such that $\tilde{A} \leq \tilde{B} \leq \tilde{C}$, it holds that $T2SM(\tilde{A},\tilde{C}) \leq T2SM(\tilde{A},\tilde{B})$ and $T2SM(\tilde{A},\tilde{C}) \leq T2SM(\tilde{B},\tilde{C})$, where the order relation $\tilde{A} \leq \tilde{B}$ holds if and only if $\tilde{A}_x(t) \leq \tilde{B}_x(t)$ for all $x \in X$ and for all $t \in [0,1]$.

Once again, we have opted for a modification of the axioms of similarity measures in order to preserve key uncertainty information of type-2 FSs. It is worth mentioning many of the expressions considered in the literature, fulfill Axiom 2). For instance, if we consider some of the measures which are discussed in [48]:

- For the Hung and Yang's similarity measure [49], which is based on the use of the Hausdorff distance, it comes out that $d(\tilde{A},\tilde{B}) = 0$ if and only if $\tilde{A} = \overline{0}$ and $\tilde{B} = \overline{1}$ or viceversa.
- Regarding Yang and Lin's similarity measure [12], if we consider two type-2 fuzzy sets over the same referential set X such that, for every $x \in X$, $\tilde{A}_x(t) \cdot \tilde{B}_x(t) = 0$, then the similarity between them is zero.
- Hwang, Yang, Hung and Lee's similarity [50] is not well-defined for general type-2 fuzzy sets, since, if the membership function of all the elements in A is equal to zero a.e, the denominator maybe equal to zero for a general fuzzy measure.
- Finally, both McCulloch, Wagner and Aickelin's [11] and McCulloch and Wagner's [16] similarities fullfill the requested axioms as long as the type-1 similarity used for defining them satisfies an analogous condition; that is, if the similarity between two fuzzy sets A and B is equal to zero if and only if for each element x in the referential set, $\{\tilde{A}_x,\tilde{B}_x\} = \{\overline{0},\overline{1}\}$.

Furthermore, it is worth remarking thatr, in [12], a similarity measure for general type-2 FSs is also introduced. That measure is defined according to the axioms of symmetry (Axiom 1), transitivity (Axiom 4) and minimum and maximum similarity (Axioms 2, 3). The method presented in this paper and that of [12] share Axioms 1 and 4, but they differ in Axioms 2, 3. In [12] they impose:

(S2) $S(D,D^c) = 0$, for every $D \in \mathcal{P}(X)$ (the power set of X).
(S3) $S(E,E) = \max_{A,B \in FS(X)} S(A,B)$, for all $E \in FS(X)$.

Although S2 may look quite similar, indeed the main difference is that we impose that only complementary crisp sets yield the minimum value 0 while Yang and Lin allow minimum value for other sets. Similarly, Axiom 3 imposes maximum similarity measure. In general, our axioms are more restrictive, since they imply a sufficient and necessary condition.

The construction method by means of EREFs still holds, but in this case, without the necessity of an aggregation function.

Proposition 4.1: Let $EREF$ be an extended restricted equivalence function and let U be a finite universe of discourse. We generate a type-2 similarity measure $T2SM$, as follows:

$$T2SM(\tilde{A},\tilde{B}) = \{(u_i, EREF(\tilde{A}_{u_i},\tilde{B}_{u_i}))|u_i \in U\}$$

So, a way to calculate the type-2 similarity measure between two type 2 fuzzy sets $\tilde{A},\tilde{B}$ defined over a referential set X, it is necessary to follow the following steps:

1) Choose an EREF function which fulfills the properties required in Def. 3.2.
2) For each $x \in X$, calculate the value $EREF(\tilde{A}_x,\tilde{B}_x)$.
3) Then, the resulting similarity measure is the fuzzy set over X such that the membership value of the element $x \in X$ is given by $EREF(\tilde{A}_x,\tilde{B}_x)$.

Remark 2: Note that, with our approach, the similarity measure that we obtain is strongly dependent on the EREF that we have chosen. In this sense, our definition has the advantage of providing a whole family of similarity measures rather than a specific expression, enabling (but also requiring) the most appropriate instance to be selected given an application. In this sense, the proposed approach is not dissimilar to requiring practitioners to select amongst the existing set of similarity measures, except that it provides one coherent family of measures within one overall mathematical framework.

A. Numerical type-2 similarity measures

In our axiomatic definition we have opted for a FS instead of a single value as the result of type-2 similarity measures. As explained before, this is partially done with the aim of maintaining the uncertainty of the type-2 fuzzy sets. However, it might be necessary for some applications or comparisons to yield a single representative value. In this context, when the universe of discourse is finite, we propose to fuse the information by means of an aggregation function, retrieving a single value which satisfies suitable properties. Nevertheless, some other defuzzification methods available in the literature may be used (e.g., the centroid).

Definition 4.2: Let $U = \{u_1,\ldots,u_n\}$ be a finite universe of discourse, M be an averaging n-ary aggregation function and $T2SM$ a type-2 similarity measure. We call Numerical Type-2

[2] Note that another option is associating a type-2 FS but we believe that the interpretation of uncertainty in this situation would be intricate.

Similarity Measure ($NT2SM$) with respect to M to the result of fusing all the membership degrees of the $T2SM$, i.e., to the function $NT2SM : T2FS(U) \times T2FS(U) \to [0,1]$ defined by:

$$NT2SM(\tilde{A},\tilde{B}) = M_{i=1}^{n}(T2SM(\tilde{A},\tilde{B})_{u_i}) = M(T2SM_{u_1}, T2SM_{u_2}, \ldots, T2SM_{u_n}).$$

Notice that, as the aggregation function does not have to be necessarily commutative, the previous definition may directly depend on the concrete order of the elements in the set $\{u_1, u_2, \ldots, u_n\}$.

For instance, if we take as $M : [0,1]^n \to [0,1]$ the arithmetic mean, and as EREF the function given in 2) of Example 3.2, we have that, for any $\tilde{A},\tilde{B} \in T2FS(U)$:

$$NT2SM(\tilde{A},\tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{2}{\pi}\int_0^1 |\arctan(\tilde{A}^L_{u_i}(t)) - \arctan(B^L_{u_i}(t))|dt - \frac{2}{\pi}\int_0^1 |\arctan(\tilde{A}^R_{u_i}(t)) - \arctan(B^R_{u_i}(t))|dt\right)$$

If the averaging aggregation function M satisfies that:

- $M(x_1,\ldots,x_n) = 0$ if and only if $x_1 = \ldots = x_n = 0$;
- $M(x_1,\ldots,x_n) = 1$ if and only if $x_1 = \ldots = x_n = 1$;

then the numerical type-2 similarity measure satisfies that

1) $NT2SM(\tilde{A},\tilde{B}) = NT2SM(\tilde{B},\tilde{A})$ for any $\tilde{A},\tilde{B} \in T2FS(X)$.
2) $NT2SM(\tilde{A},\tilde{B}) = 0$ if and only $\{\tilde{A}_x,\tilde{B}_x\} = \{\bar{0},\bar{1}\}$ for all $x \in X$.
3) $NT2SM(\tilde{A},\tilde{B}) = 1$ if and only if $\tilde{A}^L_x = \tilde{B}^L_x$ $a.e.$ and $\tilde{A}^R_x = \tilde{B}^R_x$ $a.e.$ for all $x \in X$.
4) For any $\tilde{A},\tilde{B},\tilde{C} \in T2FS(X)$, such that $\tilde{A} \le \tilde{B} \le \tilde{C}$, it holds that $NT2SM(\tilde{A},\tilde{C}) \le NT2SM(\tilde{A},\tilde{B})$ and $NT2SM(\tilde{A},\tilde{C}) \le NT2SM(\tilde{B},\tilde{C})$.

So, in order to calculate the numerical type-2 similarity measure between two type 2 fuzzy sets $\tilde{A},\tilde{B}$ defined over a finite referential set U, it is necessary to follow the following steps:

1) Choose the similarity function $T2SM$ to be used.
2) For each $u \in U$, calculate the value $T2SM(\tilde{A},\tilde{B})_u$. That is, the membership value of the element $u \in U$ to the fuzzy set $T2SM(\tilde{A},\tilde{B})$.
3) Choose an aggregation function $M : [0,1] \to [0,1]$, where n is the cardinality of the referential set U.
4) Calculate the aggregation by M of all the values $T2SM(\tilde{A},\tilde{B})_u$ with $u \in U$.

B. The case of Interval type-2 fuzzy sets

Dealing with T2FSs is not an easy task. One of the main challenges is that dealing with such intricate expressions of uncertainty is arduous. Moreover, the definition of well defined operators is quite novel, in particular at the design stage of such FSs. Due to these difficulties most of the applications deal with Interval Type-2 FSs (IT2FS) ([8], [9], [40], [51], [52]). In the literature, there are some controversies on the definition of these sets ([53]). In our context, an IT2FS is a T2FS in which, for any element in the universe of discourse, the associated membership function is given by

$$f_x(t) = \begin{cases} 1\,, & \text{if } t \in [a,b], \\ 0\,, & \text{otherwise} \end{cases}$$

for some $0 \le a \le b \le 1$ which depend on the considered T2FS and x. Specifically, the membership interval of x to the interval valued fuzzy set is precisely $[a,b]$.

The main advantage of these sets is that the considered membership functions are convex and normal [37], and, hence, Axiom 2 of EREF is equivalent to the original one (see Remark 1). Furthermore, although the calculation of the type-2 similarity measure between sets of this type follows the same steps as in the case of general type-2 fuzzy sets, it may be easier in some cases.

For instance, considering $\tilde{A},\tilde{A}' \in IT2FS(X)$, if the type-2 similarity measure is generated by the Euclidean distance as in Example 3.2(1), then

$$T2SM(\tilde{A},\tilde{A}') = \{(x,\mu_x)\}\,,$$

where the value μ_x is

$$\mu_x = 1 - \frac{1}{2}|a-a'| - \frac{1}{2}|b-b'|\,. \tag{4}$$

Note that a, a', b, b' depend on x.

Indeed, we find that $\mu_x = \frac{1}{2}REF(a,a') + \frac{1}{2}REF(b,b')$ where the $REF(x,y) = 1 - |x-y|$.

From here on and for the sake of simplicity, the membership function $\tilde{f}_x$ of an IT2FS, given by

$$f_x(t) = \begin{cases} 1\,, & \text{if } t \in [a,b], \\ 0\,, & \text{otherwise} \end{cases}$$

for some $0 \le a \le b \le 1$, is denoted by the corresponding interval $[a,b]$, i.e., considering $\tilde{A} \in IT2FS(U)$ ($U = \{u_1,\ldots,u_n\}$), the IT2FS will be represented as

$$\tilde{A} = \{(u_1,[a_1,b_1]),\ldots,(u_n,[a_n,b_n])\},$$

where $[a_i,b_i]$ is the interval where the function f_{u_i} takes the value 1.

Example 4.2: Let $U = \{u_1,u_2,u_3\}$ be the universe of $\tilde{A} \in IT2FS(U)$ given by

$$\tilde{A} = \{(u_1,[0.2,0.5]),(u_2,[0.6,0.6]),(u_3,[0.4,0.45])\}$$

(see Fig. 4a))[3] and

$$\tilde{A}' = \{(u_1,[0,0.4]),(u_2,[0.3,0.65]),(u_3,[0.55,0.8])\}$$

(see Fig. 4b)) .

We compute the similarity measure of $\tilde{A}$ and $\tilde{A}'$ using Eq. 4, so the result is $T2SM(\tilde{A},\tilde{A}') = \{(u_1,0.85),(u_2,0.825),(u_3,0.75)\}$.

[3]Note that we have opted for depicting the complete T2FS, including the three elements of the universe (on the x-axis) and on the y-axes a blue vertical line with the values in which the membership degree is 1.

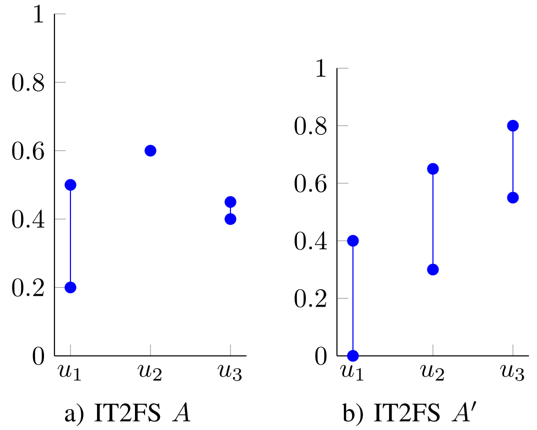


Fig. 4. Interval-type-2 FSs of Example 4.2: a) the set A (left image) and b) set A'.

C. The case of triangular and trapezoidal type-2 fuzzy sets

In this section, we consider general type-2 fuzzy sets in which secondary membership functions are convex and normal $FS([0,1])$. Particularly, we focus on distributions that have been deeply studied in the literature in which the membership functions are triangular or trapezoidal FSs [54], [55], [56], [57], [58], [59]. Triangular membership degrees of T2FSs are mathematically described by:

$$\tilde{f}_x(t) = \begin{cases} \frac{t-a}{b-a}\,, & \text{if } t \in [a,b], \\ 1 - \frac{t-b}{c-b}\,, & \text{if } t \in (b,c], \\ 0\,, & \text{otherwise}, \end{cases} \tag{5}$$

for some $0 \le a < b < c \le 1$.

Similarly, the functions which represent the trapezoidal membership degrees of T2FSs are given by:

$$\tilde{f}_x(t) = \begin{cases} \frac{t-a}{b-a}\,, & \text{if } t \in [a,b], \\ 1\,, & \text{if } t \in (b,c], \\ 1 - \frac{t-c}{d-c}\,, & \text{if } t \in (c,d], \\ 0\,, & \text{otherwise}, \end{cases} \tag{6}$$

for some $0 \le a < b \le c < d \le 1$.

It is worth mentioning that triangular T2FSs can be also thought as a subclass of trapezoidal FSs in which the parameters satisfy that $b = c$.

From here on and for the sake of simplicity, the membership function $\tilde{f}_x$ of a triangular T2FS is denoted by the corresponding triplet (a,b,c) with $a < b < c$, i.e., considering $\tilde{A}$ to be a triangular $T2FS(U)$ on a finite universe $U = \{u_1,\dots,u_n\}$, the triangular T2FS will be represented as

$$\tilde{A} = \{(u_1,(a_1,b_1,c_1)),\dots,(u_n,(a_n,b_n,c_n))\},$$

where (a_i,b_i,c_i) represent the function $\tilde{f}_{u_i}$ (as in Eq. (5)). Similarly,

$$\tilde{A} = \{(u_1,(a_1,b_1,c_1,d_1)),\dots,(u_n,(a_n,b_n,c_n,d_1))\},$$

represents the trapezoidal T2FS where $\tilde{f}_{u_i}$ is expressed as in Eq. (6).

Fig. 5 shows a graphical representation of the triangular membership function $f_x = (0.3, 0.6, 0.7)$ and the trapezoidal membership function $f_x = (0.3, 0.45, 0.8, 0.9)$, respectively.

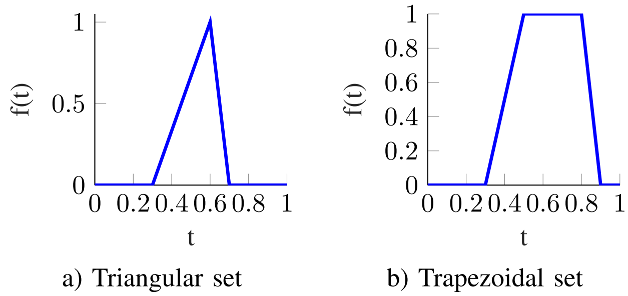


Fig. 5. Functions representing triangular and trapezoidal membership degrees of type-2 FSs: a) triangular membership function, and b) trapezoidal membership function.

Example 4.3: Let be the universe $U = \{u_1,u_2\}$. Let $\tilde{A}$ be the triangular $T2FS(U)$ given by $\tilde{A} = \{(u_1,(0.4,0.6,0.8),(u_2,(0,0.25,0.75)))\}$, (see Fig. 6(a)–(b)) and $\tilde{B}$ be the trapezoidal $T2FS(U)$ given by $\tilde{B} = \{(u_1,(0.4,0.5,0.7,0.8)),(u_2,(0,0.25,0.75,1))\}$ (see Fig. 6(c)–(d)).

The left and right cumulative functions of the membership function of u_1 are given by:

$$\tilde{a}^L_{u_1}(t) = \begin{cases} 0\,, & \text{if } t < 0.4, \\ \frac{t-0.4}{0.2} = 5t-2\,, & \text{if } t \in [0.4,0.6], \\ 1\,, & \text{if } t > 0.6\,, \end{cases} \tag{7}$$

and

$$\tilde{a}^R_{u_1}(t) = \begin{cases} 1\,, & \text{if } t < 0.6, \\ 1 - \frac{t-0.6}{0.2} = -5t+4\,, & \text{if } t \in [0.6,0.8], \\ 0\,, & \text{if } t > 0.8\,. \end{cases} \tag{8}$$

Similarly, the left and right cumulative functions of the membership function of u_1 in $\tilde{B}$ are given by:

$$\tilde{b}^L_{u_1}(t) = \begin{cases} 0\,, & \text{if } t < 0.4, \\ \frac{t-0.4}{0.1} = 10t-4\,, & \text{if } t \in [0.4,0.5], \\ 1\,, & \text{if } t > 0.5\,, \end{cases} \tag{9}$$

and

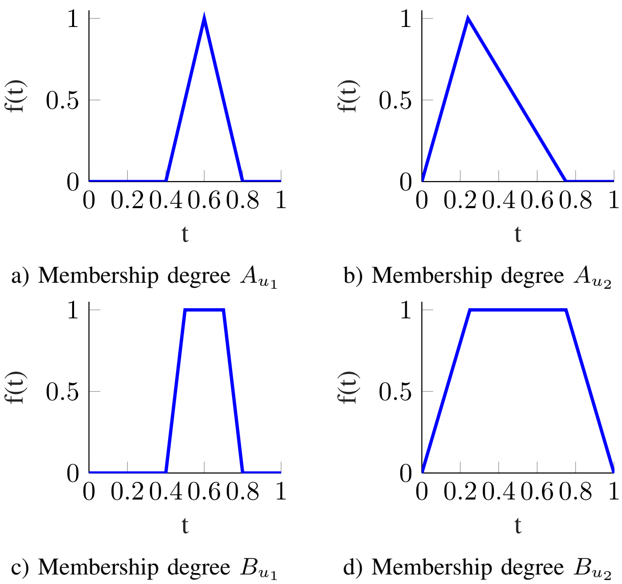
$$\tilde{b}^R_{u_1}(t) = \begin{cases} 1\,, & \text{if } t < 0.7, \\ 1 - \frac{t-0.7}{0.1} = -10t+8\,, & \text{if } t \in [0.7,0.8], \\ 0\,, & \text{if } t > 0.8\,. \end{cases} \tag{10}$$

We compute the similarity measure of $\tilde{A}$ and $\tilde{A}'$, which is for u_1:

$$EREF(\tilde{A}_{u_1},\tilde{B}_{u_1}) = 1 - \frac{1}{2}\int_0^1 d(\tilde{a}^L_{u_1}(t),\tilde{b}^L_{u_1}(t))dt$$
$$-\frac{1}{2}\int_0^1 d(\tilde{a}^R_{u_1}(t),\tilde{b}^R_{u_1}(t))dt = 1 - \frac{1}{2}0.05 - \frac{1}{2}0.05 = 0.95\,.$$

Making a similar calculation for the membership functions of the element u_2, we have a type-2 similarity measure $T2SM(\tilde{A},\tilde{B}) = \{(u_1,0.95),(u_2,0.8125)\}$.

Note that, since triangular and trapezoidal type-2 FSs are quite similar in this example, the similarity measure between them is quite high in both elements of the universe. Moreover,



a) Membership degree A_{u_1} b) Membership degree A_{u_2}

c) Membership degree B_{u_1} d) Membership degree B_{u_2}

Fig. 6. Triangular and trapezoidal type-2 FSs of Example 4.3: a) the membership function $\tilde{A}_{u_1}$, b) the membership function $\tilde{A}_{u_2}$, c) the membership function $\tilde{B}_{u_1}$ and d) the membership function $\tilde{B}_{u_2}$.

it holds that the membership function $\tilde{A}_{u_1}$ and $\tilde{B}_{u_1}$ are more similar than $\tilde{A}_{u_2}$ and $\tilde{B}_{u_2}$. Moreover, if it is necessary a single representative value for the similarity, we can fuse both values with some aggregation function, such as the arithmetic mean and then $NT2SM(\tilde{A},\tilde{B}) = 0.88125$.

It is also worth to mention that, if we consider the expression for similarity given in [18], which is done in terms of the α-plane or zSlice representation, considering for α the values $\{0.25, 0.5, 0.75, 1\}$, we get that the similarity between $\tilde{A}$ and $\tilde{B}$ of Example 4.3 is given by the type-1 fuzzy set:

$$\{(0.25, 0.778), (0.5, 0.7560), (0.75, 0.7426), (1, 0.7255)\}\ .$$

Note that this set is defined over the referential $\{0.25, 0.5, 0.75, 1\}$ so it is not possible to directly compare it to the result $T2SM(\tilde{A},\tilde{B})$ obtained in our example. However, if we calculate the centroid of this set to produce a single numerical, representative value (as proposed in [18]), we get a value of 0.7420, which, again, is a high similarity although lower than the numerical value $NT2SM(\tilde{A},\tilde{B})$ obtained in our case. Nevertheless, we maintain that a direct comparison of the similarity scores for such methods,- operating on different referential sets-, is not meaningful. In this particular case, our proposed method is designed to evaluate the similarity of the sets primarily in respect to the primary degree of membership, while [18] focuses on the similarity of the secondary memberships.

V. Conclusions and Future Work

This work has two main contributions. First of all, we have introduced the family of extended restricted equivalence functions, a generalization of the notion of restricted equivalence functions for type-2 FSs. The value of this definition lies that:

A) it supports the direct operation with general type-2 FSs, i.e, is not restricted to (but is also compatible with) interval type-2 FSs;

B) although some modifications on the axioms of restricted equivalence functions must be considered, the standard axioms are recovered when considering secondary membership degrees that are normal and convex functions as in [36], [37].

The second contribution has been the definition of similarity measures for type-2 FSs constructed by means of EREFs. This concept has been introduced to compare two type-2 FSs with a distance measure while preserving key uncertainty information inherent in the type-2 FSs. Finally, we have applied these similarity measures to different examples of type-2 FSs, focusing on interval, triangular and trapezoidal type-2 FSs. The benefit of focusing on these commonly used sets is that their secondary membership functions are convex and normal, substantially simplifying the formulas and computation.

We expect to expand the present work through two different lines of research. First, we will conduct an independent theoretical study to analyze the differences in the modified axioms of similarity measures when general (instead of normal and convex) type-2 FSs are considered, i.e., deepening the study of the behaviour of similarity measures dealing with general type-2 FSs. Secondly, we expect to introduce and generalize inference methods for dealing with general type-2 FSs. We believe that the proposed similarity measures can be used for measuring the alikeness of general type-2 FSs leading to improvements in some applications, as well as to compare antecedents and consequents of fuzzy rules defined in terms of type-2 fuzzy sets. Our final goal consists of finding a real problem in which these inference methods can be applied. Taking into account that similarity functions built in terms of REF functions have shown themselves very useful in grayscale image processing algorithms, we also think that our developments can be useful for colour image processing, when we assume that the colours are defined in terms of linguistic labels.

Finally, it is worth mentioning that in this work we have not included a comparison to other similarity measures that can be found in the literature, since this is the first time a similarity measure between type-2 fuzzy sets has been defined as a fuzzy set defined over the same referential set as the one of the compared sets. As part of a future publication we will provide a detailed set of empirical analyses which illustrate the difference in output and general behavior between the proposed and traditional similarity measures.

Appendix

In this Appendix, we include the proof of Theorem 3.1.

Proof: Note that since the codomain of the distance d is restricted to $[0, 1]$, the function $EREF$ is well defined. Let us show that it satisfies properties $1) - 5)$.

1) $EREF(f_1, f_2) = EREF(f_2, f_1)$ is trivially satisfied, due to the commutativity of d.

2)

$$EREF(f_1, f_2) = 1 - \frac{1}{2}\int_0^1 d(f_1^L(t), f_2^L(t))dt - \frac{1}{2}\int_0^1 d(f_1^R(t), f_2^R(t))dt = 1$$

if and only if

$$\int_0^1 d(f_1^L(t), f_2^L(t))dt = 0 \quad \text{and} \tag{11}$$

$$\int_0^1 d(f_1^R(t), f_2^R(t))dt = 0\,. \tag{12}$$

Since the distance d satisfies that $d(x,y) \geq 0$ for all $x, y \in [0,1]$ and $d(x,y) = 0$ if and only if $x = y$, we find that (11) and (12) hold if and only if

$$d(f_1^L(t), f_2^L(t)) = 0 \ a.e. \text{ in } [0,1] \ \text{ and}$$
$$d(f_1^R(t), f_2^R(t)) = 0 \ a.e. \text{ in } [0,1]$$

which is equivalent to

$$f_1^L(t) = f_2^L(t) \ a.e. \text{ in } [0,1] \ \text{ and}$$
$$f_1^R(t) = f_2^R(t) \ a.e. \text{ in } [0,1].$$

3) Similarly to $\mathbf{0}$, which denotes the constant function $\mathbf{0}(t) = 0$ for all $t \in [0,1]$, let $\mathbf{1}$ denote the constant function 1.

$$EREF(f_1, f_2) = 1 - \frac{1}{2}\int_0^1 d(f_1^L(t), f_2^L(t))dt - \frac{1}{2}\int_0^1 d(f_1^R(t), f_2^R(t))dt = 0$$

if and only if

$$\int_0^1 d(f_1^L(t), f_2^L(t))dt = 1 \quad \text{and} \tag{13}$$

$$\int_0^1 d(f_1^R(t), f_2^R(t))dt = 1. \tag{14}$$

Since the distance d satisfies that $d(x,y) \leq 1$ for all $x, y \in [0,1]$ and $d(x,y) = 1$ if and only if $\{x,y\} = \{0,1\}$, we find that (13) and (14) hold if and only if

$$d(f_1^L(t), f_2^L(t)) = 1 \ a.e. \text{ in } [0,1] \ \text{ and}$$
$$d(f_1^R(t), f_2^R(t)) = 1 \ a.e. \text{ in } [0,1]$$

which is equivalent to

$$\{f_L^1(t), f_2^L(t)\} = \{0,1\} \ a.e. \text{ in } [0,1] \ \text{ and}$$
$$\{f_1^R(t), f_2^R(t)\} = \{0,1\} \ a.e. \text{ in } [0,1].$$

Let us show that this holds if and only if one of the left cumulative functions is the constant null function $\mathbf{0}$ *a.e.* in $[0,1]$, and the other one is the constant one function $\mathbf{1}$ *a.e.* in $[0,1]$.

Without loss of generality, suppose $f_1^L = \mathbf{0}$ *a.e.*. Since $\mathbf{0} \notin \mathcal{G}$ and due to the increasingness of the left cumulative function, the function f_1 is given as follows:

$$f_1(t) = \begin{cases} 0\,, & \text{if } t \neq 1\,, \\ a\,, & \text{if } t = 1 \end{cases}$$

where $a > 0$. Furthermore, we know that $\{f_1^R(t), f_2^R(t)\} = \{\mathbf{0}, \mathbf{1}\}$ *a.e.* in $[0,1]$. Since the right cumulative function is a decreasing function and $f_1^R(1) = a > 0$, we find that $f_1^R = \mathbf{1}$ *a.e.* in $[0,1]$. However, this only holds if $a = 1$, i.e., $f_1 = \overline{1}$.

Finally, if $f_1^R = \mathbf{1}$ *a.e.* then $f_2^R = \mathbf{0}$ *a.e.*. Since $\mathbf{0} \notin \mathcal{G}$ and due to the decreasingness of the right cumulative function,

$$f_2(t) = \begin{cases} 0\,, & \text{if } t \neq 0\,, \\ b\,, & \text{if } t = 0 \end{cases}$$

where $b > 0$. Besides, since $f_2^L(t) = \mathbf{1}$ *a.e.* in $[0,1]$ it holds that $b = 1$ and hence, the function $f_2 = \overline{0}$.

4) First of all, let us show that $(N(f))^L(t) = f^R(1-t)$ for any $t \in [0,1]$.

$$(N(f))^L(t) = \bigvee_{y \leq t} (N(f))(y) = \bigvee_{y \leq t} f(1-y) = \bigvee_{1-z \leq t} f(z) = \bigvee_{1-t \leq z} f(z) = f^R(1-t).$$

Similarly, it can be seen that $(N(f))^R(t) = f^L(1-t)$ for any $t \in [0,1]$.

Hence,

$$\begin{aligned} &EREF(N(f_1), N(f_2)) = \\ &\quad 1 - \frac{1}{2}\int_0^1 d((N(f_1))^L(t), (N(f_2))^L(t))dt \\ &\quad - \frac{1}{2}\int_0^1 d((N(f_1))^R(t), (N(f_2))^R(t))dt \\ &= 1 - \frac{1}{2}\int_0^1 d(f_1^R(1-t), f_2^R(1-t))dt \\ &\quad - \frac{1}{2}\int_0^1 d(f_1^L(1-t), f_2^L(1-t)))dt \end{aligned}$$

Changing the variable $1 - t = y$, it holds that

$$\begin{aligned} EREF(N(f_1), N(f_2)) &= 1 - \frac{1}{2}\int_1^0 d(f_1^R(y), f_2^R(y))(-dy) \\ &- \frac{1}{2}\int_1^0 d(f_1^L(y), (f_2^L(y))(-dy) = \\ &1 - \frac{1}{2}\int_0^1 d(f_1^R(y), f_2^R(y))dy \\ &- \frac{1}{2}\int_0^1 d(f_1^L(y), (f_2^L(y))dy = EREF(f_1, f_2)\,. \end{aligned}$$

5) Let $f_1, f_2, f_3 \in \mathcal{G}$ such that $f_1 \leq f_2 \leq f_3$. One easily verifies that, since $f_1(t) \leq f_2(t) \leq f_3(t)$ for all $t \in [0,1]$, it holds that

$$f_1^L(t) \leq f_2^L(t) \leq f_3^L(t)$$
$$f_1^R(t) \leq f_2^R(t) \leq f_3^R(t)\,.$$

Hence,

$$d(f_3^L(t), f_1^L(t)) \geq d(f_3^L(t), f_2^L(t))$$
$$d(f_3^R(t), f_1^R(t)) \geq d(f_3^R(t), f_2^R(t))\,.$$

Consequently, it holds that

$$\begin{aligned}EREF(f_1,f_3) &= 1-\frac{1}{2}\int_0^1 d(f_1^L(t),f_3^L(t))dt\\ &\quad -\frac{1}{2}\int_0^1 d(f_1^R(t),f_3^R(t))dt\\ &\leq 1-\frac{1}{2}\int_0^1 d(f_2^L(t),f_3^L(t))dt\\ &\quad -\frac{1}{2}\int_0^1 d(f_2^R(t),f_3^R(t))dt = EREF(f_2,f_3).\end{aligned}$$

■

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