A Monotonous Intuitionistic Fuzzy TOPSIS Method under General Linear Orders via Admissible Distance Measures

Xinxing Wu, Zhiyi Zhu, Chuan Chen, Guanrong Chen, *Life Fellow, IEEE,* Peide Liu

***Abstract*—All intuitionistic fuzzy TOPSIS methods contain two key elements: (1) the order structure, which can affect the choices of positive ideal-points and negative ideal-points, and construction of admissible distance/similarity measures; (2) the distance/similarity measure, which is closely related to the values of the relative closeness degrees and determines the accuracy and rationality of decision-making. For the order structure, many efforts are devoted to constructing some score functions, which can strictly distinguish different intuitionistic fuzzy values (IFVs) and preserve the natural partial order for IFVs. This paper proves that such a score function does not exist, namely the application of a single monotonous and continuous function does not distinguish all IFVs. For the distance or similarity measure, some examples are given to show that classical similarity measures based on the normalized Euclidean distance and normalized Minkowski distance do not meet the axiomatic definition of intuitionistic fuzzy similarity measures. Moreover, some illustrative examples are given to show that classical intuitionistic fuzzy TOPSIS methods do not ensure the monotonicity with the natural partial order or linear orders, which may yield some counter-intuitive results. To overcome the limitation of non-monotonicity, we propose a novel intuitionistic fuzzy TOPSIS method, using three new admissible distances with the linear orders measured by a score degree/similarity function and accuracy degree, or two aggregation functions, and prove that the proposed TOPSIS method is monotonous under these three linear orders. This is the first result with a strict mathematical proof on the monotonicity with the linear orders for the intuitionistic fuzzy TOPSIS method. Finally, we show two practical examples and comparative analysis with other decision-making methods to illustrate the efficiency of the developed TOPSIS method.**

***Index Terms*—Intuitionistic fuzzy set, Distance measure, Similarity measure, TOPSIS, Multi-attribute decision making.**

I. INTRODUCTION

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X. Wu is with (1) the School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, China; (2) the Institute for Artificial Intelligence, Southwest Petroleum University, Chengdu, Sichuan 610500, China. (3) the Zhuhai College of Jilin University, Zhuhai, Guangdong 519041, China. e-mail: (wuxinxing5201314@163.com).

Z. Zhu is with the School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, China. e-mail: (zhuzhiyi2019@163.com).

C. Chen is with the School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, China. e-mail: (chenchuan975201314@163.com).

G. Chen is with the Department of Electrical Engineering, City University of Hong Kong, Hong Kong SAR, China. e-mail: (eegchen@cityu.edu.hk).

P. Liu is with a School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan, Shandong 250014, China. e-mail: (peide.liu@gmail.com).

All correspondences should be addressed to X. Wu and P. Liu.

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ZADEH (1965) [1] established the fuzzy set (FS) theory by applying membership degree to measure the importance of a fuzzy element, which generalized the concept of crisp set, characterized by a characteristic function taking value 0 or 1, by taking any value in the unit interval $[0, 1]$. However, due to the limitation of a membership function that only indicates two (supporting and opposing) opposite states of fuzziness, the fuzzy set cannot express the neutral state of "this and also that". According to this, Atanassov (1986) [2] generalized Zadeh's fuzzy set by proposing the concept of intuitionistic fuzzy sets (IFSs) (see also [3]), characterized by a membership function and a non-membership function meeting the condition that the sum of the membership degree and the non-membership degree at every point is less than or equal to 1. Every pair of membership and non-membership degrees for IFSs was called an intuitionistic fuzzy value (IFV) by Xu [4]. Thereafter, various multi-attribute decision making (MADM) methods under the intuitionistic fuzzy framework were developed and widely applied. This paper focuses on the intuitionistic fuzzy TOPSIS method.

Being one of the most well-known MADM methods, TOPSIS was first proposed by Hwang and Yoon (1981) [5]. The main idea of the TOPSIS is that the most desirable alternative should be nearest from the positive ideal-point and meanwhile furthest from the negative ideal-point. Noting that the total order structure '$\leq$' and the absolute distance '$|\cdot|$' on the real line $\mathbb{R}$ are admissible (the bigger the real number, the nearer from the maximum, and the further from the minimum). This can naturally guarantee the establishment of the TOPSIS and further ensure its monotonicity. Due to the complex two-dimensional structure of the space of all IFVs, all existing IF distance measurements for IFVs are not admissible under linear orders on IFVs. Therefore, we [6] introduced the concept of admissible distance measure with the linear order '$\leq_{\rm XY}$' of Xu and Yager [7] and constructed an admissible distance measure ϱ with the linear order '$\leq_{\rm XY}$'.

Through careful analysis of the TOPSIS method, it is not difficult to find that the TOPSIS method contains two key elements: (1) the order structure, which can affect the choices of positive ideal-points and negative ideal-points; (2) the distance/similarity measure, which is closely related to the relative closeness degrees and determines the accuracy and rationality of decision-making. Therefore, essentially speaking, all improvements on IF TOPSIS method provide certain improvements on the order structure and the distance/similarity measures.

To rank IFVs, score function is a useful tool. Xu and Yager [7] introduced the linear order '$\leq_{\text{XY}}$' for IFVs by applying a score function and an accuracy function. According to the TOPSIS idea [5], Zhang and Xu [8] proposed another linear order '$\leq_{\text{ZX}}$' for IFVs by applying a similarity function and an accuracy function. Recently, Xing et al. [9] defined a linear order for IFVs by using a score function expressed by the Euclidean distance from the maximum point $\langle 1, 0\rangle$. Bustince et al. [10] suggested a general construction of linear orders for intervals contained in $[0, 1]$ by means of aggregation functions. Based on this, De Miguel et al. [11], [12] developed a general method for constructing linear orders between pairs of intervals based on aggregation functions, which was successfully applied to construct linear orders for interval-valued IFSs. Wang et al. [13] classified the existing score functions of IFVs into two types, one type consists of score functions without abstention group influence [14], [15], [16], [17], [18], [19], and the other consists of score functions with abstention group influence [20]. Zeng et al. [21] illustrated that these existing score functions, the existing accuracy functions in [22], [23], and the measure methods in [24], [25], [8] for ranking IFVs have a drawback that they may cause some unreasonable raking results or they cannot distinguish some different IFVs. To overcome this drawback, Zeng et al. [21] proposed a new score function $S_{_{CK}}$ for IFVs, which was a monotonically increasing injective with Atanassov's partial order $\subset$ (see [21, Theorems 3.1 and 3.2]). However, we constructed an example to show that this inference in [21] does not hold and proved that such a score function does not exist, i.e., there is no any continuous injective from the space of all IFVs to $\mathbb{R}$ that is increasing with Atanassov's partial order $\subset$. This means that the score function $S_{_{CK}}$ has the same drawback. This trouble mainly arises from the fact that the space of all IFVs has a two-dimensional structure, which is not homeomorphic to any closed interval on the real line $\mathbb{R}$. In fact, we proved that the application of a single monotonous and continuous function does not distinguish all IFVs.

Being a pair of dual concepts, the normalized distance measures and similarity measures have been widely studied. Similarly to the axiomatic definition of similarity measure for fuzzy sets [26], [27], Li and Cheng [28] gave an axiomatic definition of similarity measures for IFSs by using normalization (S1), symmetry (S3), and compatibility with Atanassov's partial order (S4), i.e., the condition $I_1 \subset I_2 \subset I_3$ implies that the similarity measure between I_1 and I_3 is smaller than that between I_1 and I_2 and between I_2 and I_3. Xu [29] introduced some new IF similarity measures and applied them to some practical MADM problems. Xu and Chen [30] presented a comprehensive overview of distance and similarity measures of IFSs and proposed some continuous distance and similarity measures for IFSs using the weighted Hamming distance, weighted Euclidean distance, and weighted Hausdorff distance. Iancu [31] defined some IF similarity measures using Frank t-norms $T_{\gamma}^{\mathbf{F}}$. Szmidt and Kacprzyk [32] pointed out that the third parameter (indeterminacy degree) should be considered when calculating distances for IFSs. Because of the duality between distance and similarity measures, various three-dimensional IF distance and similarity measures including indeterminacy degrees were introduced in [33], [34], [35], [36]. However, Atanassov's partial order only reveals the magnitudes of membership degrees and non-membership degrees between two IFSs, and thus many three-dimensional IF similarity measures considering indeterminacy degrees might not meet the axiomatic condition (S4). In fact, we constructed three examples to show that Euclidean similarity measure in [34], [37], [35], Minkowski similarity measure in [29], [35], [34], [37], and a modified similarity measure in [30] based on the idea of the above TOPSIS does not satisfy the axiomatic condition (S4). In particular, it should be pointed out that some existing IF distance and similarity measures are unreasonable for dealing with some practical decision-making problems. For example, Mitchell [38] showed that Li and Cheng's similarity measure [28] may lead to counter-intuitive situations in some cases. Chen et al. [39] showed some counterexamples to illustrate that the similarity measures in [40], [41], [42], [38], [23], [43] may produce unreasonable results in some cases. As noted above, the application of a single monotonous and continuous function does not distinguish all IFVs. This means that all continuous IF similarity measures are unable to distinguish every pair of different IFSs. For example, when we apply a continuous IF similarity measure for pattern recognition, we always encounter the case that we cannot determine the classification result. Therefore, the comparative analysis on the indistinguishability is meaningless ([44, Tables 2–5], [40, Tables 1–2], [39, Tables 1, 2, 5, 6], [33, Table 2]), because all continuous IF similarity measures will also encounter the same indistinguishability problem. On the other hand, all existing distance and similarity measures are only admissible with Atanassov's partial order, and thus decision-making results obtained by these distance or similarity measures can only guarantee the monotonicity with the partial order. Therefore, to obtain monotonous decision-making methods, we need to develop new admissible distance and similarity measures with linear orders.

During the past decade, various generalized IF TOPSIS methods have been developed. For example, Boran et al. [45] first extended the TOPSIS method to IF group decision-making with IFV weights. Then, some similar IF TOPSIS methods with IFV/linguistic weights were developed [46], [37], [47] and widely applied to practical decision-making problems [46], [48], [49], [50], [51]. Our Examples 6 and 7 below in this paper show that the TOPSIS methods in [45], [48], [37], [47], [50] may yield unreasonable results even when dealing with the simplest decision-making problems. Chen et al. [52] proposed a MADM method with crisp numerical weights based on the TOPSIS method and a new similarity measure for the IF situation. Zeng et al. [21] pointed out that the MADM method in [52] cannot distinguish the alternatives in some special situations. Furthermore, our Example 8 below in this paper demonstrates that the MADM method in [52] is not monotonous with the linear order '$\leq_{\text{XY}}$' of Xu and Yager [7]. Based on a new distance measure, Shen et al. [33] developed an extended IF TOPSIS method and applied it to credit risk evaluation. Here, only some works closely related to this paper are mentioned. For more results on TOPSIS under the IF or interval-valued IF framework, interested readers are referred to [53], [54], [55], [56], [57], [58], [59].

Inspired by the above discussions, this paper establishes a monotonous IF TOPSIS under three popular linear orders, '$\leq_{\text{XY}}$' in [7], '$\leq_{\text{ZX}}$' in [8], and '$\leq_{A,B}$' in ([11], [12], [10]). More precisely, the main contributions of this paper are as follows:

(1) We construct some counterexamples to illustrate that Euclidean similarity measure ([37], [34]), Minkowski similarity measure ([29]), and modified Euclidean similarity measure ([30]) do not satisfy the axiomatic definition of IF similarity measures (see Examples 1–3).

(2) We prove that there is no any continuous and injective function from the space of all IFVs to the real line $\mathbb{R}$ that is increasing with Atanassov's partial order '$\subset$'. This indicates the nonexistence of continuous similarity measure distinguishing between every pair of IFVs. Therefore, the comparative analysis on the indistinguishability for IF similarity measures is meaningless, but unfortunately this is a common problem for all continuous IF similarity measures.

(3) We construct three simple examples to show that some classical IF TOPSIS methods in [37], [52], [48], [45], [47], [50] are not monotonous with Atanassov's partial order '$\subset$' or the linear order '$\leq_{\text{XY}}$' (see Examples 6–8), which may yield counter-intuitive results. To overcome this limitation, by proposing three new admissible distances with the linear order '$\leq_{\text{XY}}$' or '$\leq_{\text{ZX}}$' or '$\leq_{A,B}$', we develop a novel IF TOPSIS method and prove that it is monotonically increasing with these three linear orders.

(4) We provide two practical examples and a comparative analysis with other MADM methods to illustrate the efficiency of our proposed TOPSIS method.

The paper is organized as follows: Section II presents some basic definitions related to the IFSs, including IFSs, orders for IFSs, and IF distance and similarity measure. Section III provides some examples to illustrate that classical similarity measures in [34], [29], [30] based on the normalized Euclidean distance and normalized Minkowski distance do not meet the axiomatic definition of IF similarity measures. Section IV proves that the application of a single monotonous and continuous function does not distinguish all IFVs. Section V applies three examples to demonstrate that the IF TOPSIS methods in [37], [52], [48], [45], [47], [50] are not monotonous with Atanassov's partial order '$\subset$' or the linear order '$\leq_{\text{XY}}$' of Xu and Yager [7]. To overcome this limitation, by constructing three new admissible distances with the linear order '$\leq_{\text{XY}}$', '$\leq_{\text{ZX}}$' or '$\leq_{A,B}$', Section VI develops a new IF TOPSIS method and proves that it is monotonically increasing with these three linear orders. Section VII presents two practical examples to demonstrate the efficiency of the proposed TOPSIS method developed in Section VI. Section VIII concludes the paper with a future research outlook.

II. Preliminaries

A. Intuitionistic fuzzy set (IFS)

Definition 2.1 ([3, Definition 1.1]): Let X be the universe of discourse. An *intuitionistic fuzzy set* (IFS) I in X is defined as an object in the following form:

$$I = \{\langle x; \mu_{_I}(x), \nu_{_I}(x)\rangle \mid x \in X\},$$

where the functions $\mu_{_I} : X \to [0,1]$ and $\nu_{_I} : X \to [0,1]$ define the *degree of membership* and the *degree of non-membership* of an element $x \in X$ to the set I, respectively, and for every $x \in X$, $\mu_{_I}(x) + \nu_{_I}(x) \leq 1$.

Let IFS(X) denote the set of all IFSs in X. For $I \in$ IFS(X), the *indeterminacy degree* $\pi_{_I}(x)$ of an element x belonging to I is defined by $\pi_{_I}(x) = 1 - \mu_{_I}(x) - \nu_{_I}(x)$. In [4], [35], the pair $\langle \mu_{_I}(x), \nu_{_I}(x)\rangle$ is called an *intuitionistic fuzzy value* (IFV) or an *intuitionistic fuzzy number* (IFN). For convenience, use $\alpha = \langle \mu_\alpha, \nu_\alpha\rangle$ to represent an IFV α, which satisfies $\mu_\alpha \in [0,1]$, $\nu_\alpha \in [0,1]$, and $0 \leq \mu_\alpha + \nu_\alpha \leq 1$. Additionally, $s(\alpha) = \mu_\alpha - \nu_\alpha$ and $h(\alpha) = \mu_\alpha + \nu_\alpha$ are called the *score degree* and the *accuracy degree* of α, respectively. Let $\tilde{\mathbb{I}}$ denote the set of all IFVs, i.e., $\tilde{\mathbb{I}} = \{\langle \mu, \nu\rangle \in [0,1]^2 \mid \mu + \nu \leq 1\}$.

Motivated by the basic operations on IFSs, Xu et al. [35], [7] introduced the following basic operational laws for IFVs.

Definition 2.2 ([35, Definition 1.2.2]): Let $\alpha = \langle \mu_\alpha, \nu_\alpha\rangle$, $\beta = \langle \mu_\beta, \nu_\beta\rangle \in \tilde{\mathbb{I}}$. Define

(i) $\alpha^{\complement} = \langle \nu_\alpha, \mu_\alpha\rangle$.
(ii) $\alpha \cap \beta = \langle \min\{\mu_\alpha, \mu_\beta\}, \max\{\nu_\alpha, \nu_\beta\}\rangle$.
(iii) $\alpha \cup \beta = \langle \max\{\mu_\alpha, \mu_\beta\}, \min\{\nu_\alpha, \nu_\beta\}\rangle$.
(iv) $\alpha \oplus \beta = \langle \mu_\alpha + \mu_\beta - \mu_\alpha\mu_\beta, \nu_\alpha\nu_\beta\rangle$.
(v) $\alpha \otimes \beta = \langle \mu_\alpha\mu_\beta, \nu_\alpha + \nu_\beta - \nu_\alpha\nu_\beta\rangle$.
(vi) $\lambda\alpha = \langle 1 - (1-\mu_\alpha)^\lambda, (\nu_\alpha)^\lambda\rangle$, $\lambda > 0$.
(vii) $\alpha^\lambda = \langle (\mu_\alpha)^\lambda, 1 - (1-\nu_\alpha)^\lambda\rangle$, $\lambda > 0$.

B. Orders for IFSs

Atanassov's order '$\subset$' [3], defined by that $\alpha \subset \beta$ if and only if $\alpha \cap \beta = \alpha$, is a partial order on $\tilde{\mathbb{I}}$.

Definition 2.3 ([11, Definition 4.1]): An order $\leq$ on $\tilde{\mathbb{I}}$ is said to be an *IF-admissible order* if it is a linear order and refines Atanassov's order $\subset$.

To compare any two IFVs, Xu and Yager [7] introduced the following linear order '$\leq_{\text{XY}}$' (see also [4, Definition 3.1] and [35, Definition 1.1.3]):

Definition 2.4 ([7, Definition 1]): Let α_1 and α_2 be two IFVs.

- If $s(\alpha_1) < s(\alpha_2)$, then α_1 is smaller than α_2, denoted by $\alpha_1 <_{\text{XY}} \alpha_2$.
- If $s(\alpha_1) = s(\alpha_2)$, then
  - if $h(\alpha_1) = h(\alpha_2)$, then $\alpha_1 = \alpha_2$;
  - if $h(\alpha_1) < h(\alpha_2)$, then $\alpha_1 <_{\text{XY}} \alpha_2$.

If $\alpha_1 <_{\text{XY}} \alpha_2$ or $\alpha_1 = \alpha_2$, then denote it by $\alpha_1 \leq_{\text{XY}} \alpha_2$.

Alongside Xu and Yager's order '$\leq_{\text{XY}}$' in Definition 2.4, Szmidt and Kacprzyk [25] proposed another comparison function, $\rho(\alpha) = \frac{1}{2}(1+\pi(\alpha))(1-\mu(\alpha))$ for IFVs, which is a partial order. However, it sometimes cannot distinguish between two IFVs. Although Xu's method [4] constructs a linear order for ranking any two IFVs, its procedure has the following disadvantages: (1) It may result in that the less we know, the better the IFV, which is not reasonable. (2) It is sensitive to a slight change of the parameters. (3) It is not preserved under multiplication by a scalar, namely, $\alpha \leq_{\text{XY}} \beta$ might not imply $\lambda\alpha <_{\text{XY}} \lambda\beta$, where λ is a scalar (see [60, Example 1]). To overcome such shortcomings of the above two ranking methods, Zhang and Xu [8] improved Szmidt and Kacprzyk's method [25], according to Hwang and Yoon's idea [5] and technique for preference order by similarity to an ideal point. They also introduced a *similarity function* $L(\alpha)$, called the *L-value* in [8], for any IFV $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$, as follows:

$$L(\alpha) = \frac{1-\nu_\alpha}{(1-\mu_\alpha)+(1-\nu_\alpha)} = \frac{1-\nu_\alpha}{1+\pi_\alpha}. \tag{1}$$

In particular, if $\nu_\alpha < 1$, then $L(\alpha) = \frac{1}{\frac{1-\mu_\alpha}{1-\nu_\alpha}+1}$. Furthermore, Zhang and Xu [8] introduced the following order '$\leq_{\text{ZX}}$' for IFVs by applying the similarity function $L(\_)$.

Definition 2.5 ([8]): Let α_1 and α_2 be two IFVs.

- If $L(\alpha_1) < L(\alpha_2)$, then $\alpha_1 <_{\text{ZX}} \alpha_2$;
- If $L(\alpha_1) = L(\alpha_2)$, then
  - if $h(\alpha_1) = h(\alpha_2)$, then $\alpha_1 = \alpha_2$;
  - if $h(\alpha_1) < h(\alpha_2)$, then $\alpha_1 <_{\text{ZX}} \alpha_2$.

If $\alpha_1 <_{\text{ZX}} \alpha_2$ or $\alpha_1 = \alpha_2$, then denote it by $\alpha_1 \leq_{\text{ZX}} \alpha_2$.

Definition 2.6 ([61, Definition 1.1]): An *aggregation function* in $[0,1]^n$ is a function $A : [0,1]^n \to [0,1]$ that

(i) is nondecreasing in each variable;
(ii) fulfills the boundary conditions

$$A(0,\ldots,0) = 0 \text{ and } A(1,\ldots,1) = 1.$$

By an equivalent transformation between intervals and IFVs and [10, Proposition 3.2], the following general construction of linear orders is proposed for IFVs.

Proposition 2.1: Let A, B : $[0,1]^2 \to [0,1]$ be two continuous aggregation functions satisfying that, for (x_1, y_1), $(x_2, y_2) \in [0,1]^2$, the identities $A(x_1, y_1) = A(x_2, y_2)$ and $B(x_1, y_1) = B(x_2, y_2)$ can hold only if $x_1 = x_2$ and $y_1 = y_2$. Define the order $\leq_{A,B}$ on $\tilde{\mathbb{I}}$ as follows: $\langle \mu_1, \nu_1 \rangle \leq_{A,B} \langle \mu_2, \nu_2 \rangle$ if and only if $A(\mu_1, 1-\nu_1) < A(\mu_2, 1-\nu_2)$ or $(A(\mu_1, 1-\nu_1) = A(\mu_2, 1-\nu_2)$ and $B(\mu_1, 1-\nu_1) \leq B(\mu_2, 1-\nu_2))$. Then, $\leq_{A,B}$ is an admissible order on $\tilde{\mathbb{I}}$.

For $\gamma \in [0,1]$, consider an aggregation function K_γ : $[0,1]^2 \to [0,1]$ defined by $K_\gamma(x,y) = x + \gamma(y-x)$. For γ_1, $\gamma_2 \in [0,1]$ with $\gamma_1 \neq \gamma_2$, according to Proposition 2.1, it follows that the order $\leq_{\gamma_1,\gamma_2}$ on $\tilde{\mathbb{I}}$ defined by $\alpha \leq_{\gamma_1,\gamma_2} \beta$ if and only if $K_{\gamma_1}(\mu_\alpha, 1-\nu_\alpha) < K_{\gamma_1}(\mu_\beta, 1-\nu_\beta)$ or $(K_{\gamma_1}(\mu_\alpha, 1-\nu_\alpha) = K_{\gamma_1}(\mu_\beta, 1-\nu_\beta)$ and $K_{\gamma_2}(\mu_\alpha, 1-\nu_\alpha) \leq K_{\gamma_2}(\mu_\beta, 1-\nu_\beta))$ is an admissible order on $\tilde{\mathbb{I}}$.

C. IF distance and similarity measure

Li and Cheng [28] introduced an axiomatic definition of similarity measure for IFSs, which was then improved by Mitchell [62] as follows. More results on the similarity measure can be found in [34], [63].

Definition 2.7 ([62]): Let X be the universe of discourse. A mapping $\mathbf{S}$: $\text{IFS}(X) \times \text{IFS}(X) \to [0,1]$ is called an *admissible similarity measure with the order* $\subset$ on $\text{IFS}(X)$ if it satisfies the following conditions: for any I_1, I_2, $I_3 \in \text{IFS}(X)$,

(S1) $0 \leq \mathbf{S}(I_1, I_2) \leq 1$.
(S2) $\mathbf{S}(I_1, I_2) = 1$ if and only if $I_1 = I_2$.
(S3) $\mathbf{S}(I_1, I_2) = \mathbf{S}(I_2, I_1)$.
(S4) If $I_1 \subset I_2 \subset I_3$, then $\mathbf{S}(I_1, I_3) \leq \mathbf{S}(I_1, I_2)$ and $\mathbf{S}(I_1, I_3) \leq \mathbf{S}(I_2, I_3)$.

Remark 1: The admissible similarity measure with the order $\subset$ was also called *similarity measure* by Hung and Yang [64] and Szmidt [34]. When no ambiguity is possible, we simply call it similarity measure.

Definition 2.8: Let X be the universe of discourse and I_1, $I_2 \in \text{IFS}(X)$. If $\langle \mu_{I_1}(x), \nu_{I_1}(x) \rangle \leq_{\text{XY}} \langle \mu_{I_2}(x), \nu_{I_2}(x) \rangle$ holds for all $x \in X$, then we say that I_1 is *smaller* than or equal to I_2 under the linear order $\leq_{\text{XY}}$, denoted by $I_1 \leq_{\text{XY}} I_2$.

Based on Definition 2.8, we introduce the improved similarity measure definition for IFSs below:

Definition 2.9: Let X be the universe of discourse. A mapping $\mathbf{S}$: $\text{IFS}(X) \times \text{IFS}(X) \to [0,1]$ is called an *admissible similarity measure with the order* $\leq_{XY}$ on $\text{IFS}(X)$ if it satisfies the conditions (S1)–(S3) in Definition 2.7, and the following one (S4′):

(S4′) For any I_1, I_2, $I_3 \in \text{IFS}(X)$, if $I_1 \leq_{\text{XY}} I_2 \leq_{\text{XY}} I_3$, then $\mathbf{S}(I_1, I_3) \leq \mathbf{S}(I_1, I_2)$ and $\mathbf{S}(I_1, I_3) \leq \mathbf{S}(I_2, I_3)$.

Now, we recall some classical distances and similarity measures for IFSs.

The *normalized Hamming distance* in [32] is:

$$d_{\text{Ha}}(I_1, I_2) = \frac{1}{2n}\sum_{j=1}^{n} H_j, \tag{2}$$

where $H_j = |\mu_{I_1}(x_j) - \mu_{I_2}(x_j)| + |\nu_{I_1}(x_j) - \nu_{I_2}(x_j)| + |\pi_{I_1}(x_j) - \pi_{I_2}(x_j)|$.

The *normalized Euclidean distance* in [32] is:

$$d_{\text{Eu}}(I_1, I_2) = \sqrt{\frac{1}{2n}\sum_{j=1}^{n} E_j}, \tag{3}$$

where $E_j = |\mu_{I_1}(x_j) - \mu_{I_2}(x_j)|^2 + |\nu_{I_1}(x_j) - \nu_{I_2}(x_j)|^2 + |\pi_{I_1}(x_j) - \pi_{I_2}(x_j)|^2$.

The *normalized Minkowski distance* in [32], [34] is:

$$d_{\text{M}}^{(\alpha)}(I_1, I_2) = \sqrt[\alpha]{\frac{1}{2n}\sum_{j=1}^{n} M_j}, \tag{4}$$

where $M_j = |\mu_{I_1}(x_j) - \mu_{I_2}(x_j)|^\alpha + |\nu_{I_1}(x_j) - \nu_{I_2}(x_j)|^\alpha + |\pi_{I_1}(x_j) - \pi_{I_2}(x_j)|^\alpha$ and $\alpha \geq 1$.

By using the normalized Hamming distance, Szmidt and Kacprzyk [65] introduced the following similarity measure $\mathbf{S}_{\text{SK}}^1$ for two IFSs I_1 and I_2:

$$\mathbf{S}_{\text{SK}}^1(I_1, I_2) = \frac{d_{\text{Ha}}(I_1, I_2)}{d_{\text{Ha}}(I_1, I_2^{\complement})}, \tag{5}$$

where $I_2^{\complement}$ is the complement of I_2. If we replace the normalized Hamming distance with the normalized Euclidean distance, we can obtain the following "similarity measure" $\mathbf{S}_{\text{SK}}^1$ for two IFSs I_1 and I_2:

$$\mathbf{S}_{\text{SK}}^2(I_1, I_2) = \frac{d_{\text{Eu}}(I_1, I_2)}{d_{\text{Eu}}(I_1, I_2^{\complement})}. \tag{6}$$

Clearly, both $\mathbf{S}_{\text{SK}}^1$ and $\mathbf{S}_{\text{SK}}^2$ are not similarity measures, because their values may exceed 1. Then, the following similarity measures were introduced by Szmidt [34] using the normalized Hamming and Euclidean distances:

$$\mathbf{Sim}^1_{\text{H}}(I_1, I_2) = 1 - d_{\text{Ha}}(I_1, I_2), \tag{7}$$

and

$$\mathbf{Sim}^2_{\text{E}}(I_1, I_2) = 1 - d_{\text{Eu}}(I_1, I_2). \tag{8}$$

Based on the normalized Minkowski distance $d^{(\alpha)}_{\text{M}}$, Xu [29] introduced the following *Minkowski similarity measure* (see also [37], [30]): for $\alpha > 0$,

$$\mathbf{S}^{(\alpha)}_{\text{Xu}}(I_1, I_2) = 1 - d^{(\alpha)}_{\text{M}}(I_1, I_2). \tag{9}$$

Clearly, $\mathbf{S}^{(1)}_{\text{Xu}} = \mathbf{Sim}^1_{\text{H}}$ and $\mathbf{S}^{(2)}_{\text{Xu}} = \mathbf{Sim}^2_{\text{E}}$.

Motivated by the idea of the TOPSIS of Hwang and Yoon [5], Xu and Chen [30] modified Eqs. (5) and (6) as follows:

$$\mathbf{S}^1_{\text{XC}}(I_1, I_2) = \frac{d_{\text{Ha}}(I_1, I_2^{\complement})}{d_{\text{Ha}}(I_1, I_2) + d_{\text{Ha}}(I_1, I_2^{\complement})}, \tag{10}$$

$$\mathbf{S}^2_{\text{XC}}(I_1, I_2) = \frac{d_{\text{Eu}}(I_1, I_2^{\complement})}{d_{\text{Eu}}(I_1, I_2) + d_{\text{Eu}}(I_1, I_2^{\complement})}. \tag{11}$$

III. The drawbacks of some existing similarity measures

This section illustrates that the similarity measures defined by Eqs. (8), (9), and (11) do not meet the property (S4) in the axiomatic definition of intuitionistic fuzzy similarity measures.

Example 1: Let the universe of discourse $X = \{x_1\}$, and $I_1 = \left\{\frac{\langle 0,1\rangle}{x_1}\right\}$, $I_2 = \left\{\frac{\langle 0.1,0\rangle}{x_1}\right\}$, and $I_3 = \left\{\frac{\langle 0.4,0\rangle}{x_1}\right\}$. Clearly, $I_1 \subset I_2 \subset I_3$. By direct calculation, we have

$$\begin{aligned}\mathbf{Sim}^2_{\text{E}}(I_1, I_2) =& 1 - \sqrt{\frac{|0-0.1|^2 + |1-0|^2 + |0-0.9|^2}{2}} \\ =& 1 - \sqrt{0.91},\end{aligned}$$

and

$$\begin{aligned}\mathbf{Sim}^2_{\text{E}}(I_1, I_3) =& 1 - \sqrt{\frac{|0-0.4|^2 + |1-0|^2 + |0-0.6|^2}{2}} \\ =& 1 - \sqrt{0.76},\end{aligned}$$

and thus $\mathbf{Sim}^2_{\text{E}}(I_1, I_2) < \mathbf{Sim}^2_{\text{E}}(I_1, I_3)$. This, together with $I_1 \subset I_2 \subset I_3$, implies that the formula $\mathbf{Sim}^1_{\text{H}}$ defined by Eq. (8) is not a similarity measures on IFSs.

Example 2: Let the universe of discourse $X = \{x_1\}$ and $I_1 = \left\{\frac{\langle 0,1\rangle}{x_1}\right\}$, $I_2 = \left\{\frac{\langle \mu_2,0\rangle}{x_1}\right\}$, and $I_3 = \left\{\frac{\langle \mu_3,0\rangle}{x_1}\right\}$ be three IFSs on X such that $0 < \mu_2 < \mu_3 < 0.5$. Clearly, $I_1 \subset I_2 \subset I_3$. Fix any $\alpha > 1$. By direct calculation, we have

$$\mathbf{S}^{(\alpha)}_{\text{M}}(I_1, I_2) = 1 - \sqrt[\alpha]{\frac{(\mu_2)^\alpha + 1 + (1-\mu_2)^\alpha}{2}},$$

and

$$\mathbf{S}^{(\alpha)}_{\text{M}}(I_1, I_3) = 1 - \sqrt[\alpha]{\frac{(\mu_3)^\alpha + 1 + (1-\mu_3)^\alpha}{2}}.$$

Let $\Gamma(x) = 1 - \sqrt[\alpha]{\frac{x^\alpha + 1 + (1-x)^\alpha}{2}}$ $(x \in (0, 0.5))$. Noting that $\alpha > 1$ and $x \in (0, 0.5)$, by direct calculation, we get $\Gamma'(x) = -\frac{1}{2}\left(\frac{x^\alpha+1+(1-x)^\alpha}{2}\right)^{\frac{1}{\alpha}-1} \cdot (x^{\alpha-1} - (1-x)^{\alpha-1}) > 0$, and thus the function Γ is strictly increasing on $(0, 0.5)$. This, together with $0 < \mu_2 < \mu_3 < 0.5$, implies that $\mathbf{S}^{(\alpha)}_{\text{M}}(I_1, I_2) = \Gamma(\mu_2) < \Gamma(\mu_3) = \mathbf{S}^{(\alpha)}_{\text{M}}(I_1, I_3)$. Therefore, the formula $\mathbf{S}^{(\alpha)}_{\text{M}}$ defined by Eq. (9) is not a similarity measures on IFSs for any $\alpha > 1$.

Example 3: Let the universe of discourse $X = \{x_1\}$, and $I_1 = \left\{\frac{\langle 0,1\rangle}{x_1}\right\}$, $I_2 = \left\{\frac{\langle 0.9,0.01\rangle}{x_1}\right\}$, and $I_3 = \left\{\frac{\langle 0.901,0.007\rangle}{x_1}\right\}$. Clearly, $I_1 \subset I_2 \subset I_3$. By direct calculation, we have

$$\begin{aligned}&\mathbf{S}^2_{\text{XC}}(I_1, I_2)\\ =& \frac{\sqrt{\frac{|0-0.01|^2+|1-0.9|^2+|0-0.09|^2}{2}}}{\sqrt{\frac{|0-0.9|^2+|1-0.01|^2+|0-0.09|^2}{2}} + \sqrt{\frac{|0-0.01|^2+|1-0.9|^2+|0-0.09|^2}{2}}}\\ \approx& 0.09141,\end{aligned}$$

and

$$\begin{aligned}&\mathbf{S}^2_{\text{XC}}(I_1, I_3)\\ =& \frac{\sqrt{\frac{|0.007|^2+|1-0.901|^2+|0.092|^2}{2}}}{\sqrt{\frac{|0.901|^2+|1-0.007|^2+|0.092|^2}{2}} + \sqrt{\frac{|0.007|^2+|1-0.901|^2+|0.092|^2}{2}}}\\ \approx& 0.09148,\end{aligned}$$

and thus $\mathbf{S}^2_{\text{XC}}(I_1, I_2) < \mathbf{S}^2_{\text{XC}}(I_1, I_3)$. This, together with $I_1 \subset I_2 \subset I_3$, implies that the formula $\mathbf{S}^2_{\text{XC}}$ defined by Eq. (11) is not a similarity measures on IFSs.

IV. A remark on score functions for IFVs

Zeng et al. [21] introduced the following *score value* $S_{_{CK}}(\_)$ for IFV $\alpha = \langle \mu_\alpha, \nu_\alpha\rangle$:

$$S_{_{CK}}(\alpha) = (\mu_\alpha - \nu_\alpha) - (1 - \mu_\alpha - \nu_\alpha) \times \frac{\log_2(2 - \mu_\alpha - \nu_\alpha)}{100}. \tag{12}$$

Then, they proved the following basic properties for the score function $S_{_{CK}}(\_)$.

Theorem 4.1 ([21, Theorem 3.1]): Assume that α and β are two IFVs. If $\alpha \neq \beta$, then $S_{_{CK}}(\alpha) \neq S_{_{CK}}(\beta)$.

Theorem 4.2 ([21, Theorem 3.2]): Assume that α and β are two IFVs. If $\alpha \supset \beta$, then $S_{_{CK}}(\alpha) > S_{_{CK}}(\beta)$.

However, the following example shows that Theorem 4.1 does not hold.

Example 4: Choose $\alpha = \langle 0, 0\rangle$ and $\beta = \langle \frac{99}{200}, \frac{101}{200}\rangle$. By direct calculation, we have $S_{_{CK}}(\alpha) = (0-0) - (1-0-0) \times \frac{\log_2 2}{100} = -\frac{1}{100}$ and $S_{_{CK}}(\beta) = (\frac{99}{200} - \frac{101}{200}) - (1 - \frac{99}{200} - \frac{101}{200}) \times \frac{\log_2(2-1)}{100} = -\frac{1}{100}$. This implies that Theorem 4.1 does not hold since $\alpha \neq \beta$.

In fact, we can prove that there is no any continuous function from $\tilde{\mathbb{I}}$ to $\mathbb{R}$ simultaneously meeting the conditions in Theorems 4.1 and 4.2, which indicates that the two-dimensional structure of IFVs is too complex to distinguish all IFVs with only a single monotonous and continuous function, where the monotonicity is under Atanassov's order '$\subset$', and the continuity is under the topology of subset of $\mathbb{R}^2$.

Theorem 4.3: There is no any continuous function $f : \tilde{\mathbb{I}} \to \mathbb{R}$ satisfying the following two conditions:

(1) f is injective, i.e., for any α, $\beta \in \tilde{\mathbb{I}}$ with $\alpha \neq \beta$, $f(\alpha) \neq f(\beta)$;

(2) f is increasing under the partial order $\subset$, i.e., for any α, $\beta \in \tilde{\mathbb{I}}$ with $\alpha \subset \beta$, $f(\alpha) \leq f(\beta)$.

Proof: Suppose on the contrary that there exists a continuous function $f : \tilde{\mathbb{I}} \to \mathbb{R}$ simultaneously satisfying the conditions (1) and (2).

(i) Let $\varphi(\nu) = f(\langle 0.25, \nu\rangle)$ ($\nu \in [0.25, 0.75]$). Clearly, φ is continuous since f is continuous on $\tilde{\mathbb{I}}$. For any $0.25 \le \nu_1 \le \nu_2 \le 0.75$, by $\langle 0.25, \nu_1\rangle \supset \langle 0.25, \nu_2\rangle$ and condition (2), one has $\varphi(\nu_1) = f(\langle 0.25, \nu_1\rangle) \ge f(\langle 0.25, \nu_2\rangle) = \varphi(\nu_2)$. This, together with condition (1), implies that $\varphi(\_)$ is strictly decreasing on $[0.25, 0.75]$. Thus, $\varphi((0.25, 0.75]) = [\varphi(0.75), \varphi(0.25)) = [f(\langle 0.25, 0.75\rangle), f(\langle 0.25, 0.25\rangle))$ by the intermediate value theorem.

(ii) Let $\psi(\nu) = f(\langle \mu, 0.25\rangle)$ ($\mu \in [0, 0.25]$). Clearly, ψ is continuous since f is continuous on $\tilde{\mathbb{I}}$. For any $0 \le \mu_1 \le \mu_2 \le 0.25$, by $\langle \mu_1, 0.25\rangle \subset \langle \mu_2, 0.25\rangle$ and condition (2), one has $\psi(\mu_1) = f(\langle \mu_1, 0.25\rangle) \le f(\langle \mu_2, 0.25\rangle) = \psi(\mu_2)$. This, together with condition (1), implies that $\psi(\_)$ is strictly increasing on $[0, 0.25]$. Thus, $\psi([0, 0.25)) = [\psi(0), \psi(0.25)) = [f(\langle 0, 0.25\rangle), f(\langle 0.25, 0.25\rangle))$ by the intermediate value theorem.

Summing (i) and (ii), one can easily verify that $\Lambda = \varphi((0.25, 0.75]) \cap \psi([0, 0.25)) = [\max\{f(\langle 0, 0.25\rangle), f(\langle 0.25, 0.75\rangle)\}, f(\langle 0.25, 0.25\rangle))$ is a non-degenerate interval, i.e.,

$$\max\{f(\langle 0, 0.25\rangle), f(\langle 0.25, 0.75\rangle)\} < f(\langle 0.25, 0.25\rangle),$$

implying that, for any $\xi \in \Lambda$, there exist $\nu \in (0.25, 0.75]$ and $\mu \in [0, 0.25)$ such that $\varphi(\nu) = \xi$ and $\psi(\mu) = \xi$, and thus there exist $0 < \mu < 0.25 < \nu \le 0.75$ such that $\varphi(\nu) = f(\langle 0.25, \nu\rangle) = \xi = f(\langle \mu, 0.25\rangle) = \psi(\mu)$. This contradicts with condition (1). ■

Shen et al. [33] pointed out that many existing distance measures cannot determine the classification results for some pattern recognition problems (see [33, Table 2]), i.e., their dual similarity measures cannot distinguish between some pair of IFVs. To overcome this drawback, they proposed a new distance measure d_{Sh} as follows: for $\alpha = \langle \mu_\alpha, \nu_\alpha\rangle$, $\beta = \langle \mu_\beta, \nu_\beta\rangle \in \tilde{\mathbb{I}}$,

$$d_{\mathrm{Sh}}(\alpha, \beta) = \sqrt{\frac{(\tilde{\mu}_\alpha - \tilde{\mu}_\beta)^2 + (\tilde{\nu}_\alpha - \tilde{\nu}_\beta)^2}{2}},$$

where $\tilde{\mu}_\alpha = \mu_\alpha(1 + \frac{2}{3}\pi_\alpha(1 + \pi_\alpha))$, $\tilde{\nu}_\alpha = \nu_\alpha(1 + \frac{2}{3}\pi_\alpha(1 + \pi_\alpha))$, $\tilde{\mu}_\beta = \mu_\beta(1 + \frac{2}{3}\pi_\beta(1 + \pi_\beta))$, and $\tilde{\nu}_\beta = \nu_\beta(1 + \frac{2}{3}\pi_\beta(1 + \pi_\beta))$.

Fix $\beta \in \tilde{\mathbb{I}}$ and define $\mathcal{G}(\alpha) = 1 - d_{\mathrm{Sh}}(\alpha, \beta)$ for $\alpha = \langle \mu_\alpha, \nu_\alpha\rangle \in \tilde{\mathbb{I}}$. From [33, Theorem 1], it follows that, for α_1 $\alpha_2 \in \tilde{\mathbb{I}}$ with $\alpha_1 \subset \alpha_2$, one has $\mathcal{G}(\alpha_1) \le \mathcal{G}(\alpha_2)$, i.e., the function $\mathcal{G}$ satisfies the condition (2) of Theorem 4.3. This, together with Theorem 4.3 and the continuity of $\mathcal{G}$, implies that $\mathcal{G}$ is not injective, and thus there exist two different IFVs α_1 and $\alpha_2 \in \tilde{\mathbb{I}}$ such that $\mathcal{G}(\alpha_1) = \mathcal{G}(\alpha_2)$, implying that $d_{\mathrm{Sh}}(\alpha_1, \beta) = d_{\mathrm{Sh}}(\alpha_2, \beta)$. Therefore, the distance measure d_{Sh} has the same drawback (see Example 5). In fact, by Theorem 4.3, we conclude that there is no any continuous distance measure that can overcome the above drawback. Therefore, the comparative analysis in [44, Tables 2–5], [40, Tables 1–2], [39, Tables 1, 2, 5, 6], and [33, Table 2] on the indistinguishability is meaningless.

Example 5: Let $\beta = \langle 0, 0\rangle$ and $\alpha = \langle x, y\rangle \in \tilde{\mathbb{I}}$. Then,

$$d_{\mathrm{Sh}}(\alpha, \beta) = \sqrt{\frac{[1 + \frac{2}{3}(1 - x - y)(2 - x - y)]^2 \times (x^2 + y^2)}{2}} = 0.5,$$

i.e.,

$$\left[1 + \frac{2}{3}(1 - x - y)(2 - x - y)\right]^2 \times (x^2 + y^2) = 0.5. \quad (13)$$

Clearly, Eq. (13) has infinitely many solutions. This means that there exist infinitely many IFVs, whose distances from β are all equal to 0.5.

V. A MONOTONOUS IF TOPSIS METHOD WITH THE LINEAR ORDERS $\le_{\mathrm{XY}}$ AND $\le_{\mathrm{ZX}}$

Suppose that there are n alternatives A_i ($i = 1, 2, \ldots, n$) evaluated with respect to m attributes $\mathscr{O}_j$ ($j = 1, 2, \ldots, m$). The sets of the alternatives and attributes are denoted by $A = \{A_1, A_2, \ldots, A_n\}$ and $\mathscr{O} = \{\mathscr{O}_1, \mathscr{O}_2, \ldots, \mathscr{O}_m\}$, respectively. The rating (or evaluation) of each alternative $A_i \in A$ ($i = 1, 2, \ldots, n$) on each attribute $\mathscr{O}_j$ ($j = 1, 2, \ldots, m$) is expressed with an IFS $r_{ij} = \left\{\frac{\langle \mu_{ij}, \nu_{ij}\rangle}{(A_i, \mathscr{O}_j)}\right\}$, denoted by $r_{ij} = \langle \mu_{ij}, \nu_{ij}\rangle$ for short, where $\mu_{ij} \in [0, 1]$ and $\nu_{ij} \in [0, 1]$ are respectively the satisfaction (or membership) degree and dissatisfaction (or non-membership) degree of the alternative $A_i \in A$ on the attribute $\mathscr{O}_j$ satisfying the condition $0 \le \mu_{ij} + \nu_{ij} \le 1$. A multi-attribute decision-making (MADM) problem with IFSs is expressed in matrix form shown in Table I.

TABLE I
IF DECISION MATRIX $R = (r_{ij})_{n \times m}$

	$\mathscr{O}_1$	$\mathscr{O}_2$	$\ldots$	$\mathscr{O}_m$
A_1	$\langle \mu_{11}, \nu_{11}\rangle$	$\langle \mu_{12}, \nu_{12}\rangle$	$\ldots$	$\langle \mu_{1m}, \nu_{1m}\rangle$
A_2	$\langle \mu_{21}, \nu_{21}\rangle$	$\langle \mu_{22}, \nu_{22}\rangle$	$\ldots$	$\langle \mu_{2m}, \nu_{2m}\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	$\langle \mu_{n1}, \nu_{n1}\rangle$	$\langle \mu_{n2}, \nu_{n2}\rangle$	$\ldots$	$\langle \mu_{nm}, \nu_{nm}\rangle$

To follow the common sense, a good method for MADM must guarantee the monotonicity, i.e., the higher the score of each attribute of the alternative is, the higher the ranking is. Meanwhile, decision-making results obtained by this good method must be consistent with our intuitive judgment, when dealing with the simplest problems, for which the decision making results can be obtained by direct observation and comparison (see Examples 6 and 7). In analyzing the TOPSIS method with crisp score values in $[0, 1]$, we can find that the unit interval $[0, 1]$ has excellent algebraic and topological structures as follows: (1) The unit interval $[0, 1]$ has a natural linear order structure $\le$ and a natural metric structure $|\cdot|$; (2) This natural metric structure $|\cdot|$ can ensure that the smaller the value in $[0, 1]$ is, the farther away from 1, and the closer away from 0; (3) The order topology induced by the linear order $\le$ is consistent with the topology induced by the metric $|\cdot|$. These good structures of $[0, 1]$ can ensure that the TOPSIS method of Hwang and Yoon [5] is monotonous. Recently, we [6] proved that the space $\tilde{\mathbb{I}}$ of all IFVs with the order topology induced by the linear order $<_{\text{XY}}$, defined in Definition 2.4, is not metrizable, i.e., there is no such good distance for $\tilde{\mathbb{I}}$ with the linear order $<_{\text{XY}}$. Nevertheless, we still construct an admissible distance with the order $\leq_{\text{XY}}$ in [6]. In the following, we will show that this distance is very important for our proposed monotonous IF TOPSIS method.

A. *Limitation in TOPSIS method of Li [37]*

First, we recall a fundamental IF TOPSIS method from [37] and use two examples show that the TOPSIS method in [37] does not have the basic monotonicity with the partial order $\subset$, which may lead the decision-making result to be unreasonable and inconsistent with the actual situation, when dealing with some even simplest decision-making problems.

The main process of IF TOPSIS method in [37, Section 3.3] is summarized as follows:

Step 1: Determine the alternatives $A = \{A_1, A_2, \ldots, A_n\}$ and attributes $\mathscr{O} = \{\mathscr{O}_1, \mathscr{O}_2, \ldots, \mathscr{O}_m\}$, respectively;

Step 2: Construct the IF decision matrix $R = (r_{ij} = \langle \mu_{ij}, \nu_{ij}\rangle)_{m\times n}$, as shown in Table I;

Step 3: Determine the weights of the attributes expressed with the IF weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^\top$, where $\omega_j = \langle \rho_j, \vartheta_j\rangle \in \tilde{\mathbb{I}}$;

Step 4: Compute the weighted IF decision matrix $\overline{R} = (\overline{r}_{ij} = \omega_j \otimes r_{ij})$ by Definition 2.2 (v), i.e., $\overline{r}_{ij} = \langle \rho_j\mu_{ij}, \vartheta_j + \nu_{ij} - \vartheta_j\nu_{ij}\rangle$;

Step 5: Determine the IF positive ideal-point $\mathbf{A}^+ = (\langle \mu_1^+, \nu_1^+\rangle, \langle \mu_2^+, \nu_2^+\rangle, \ldots, \langle \mu_m^+, \nu_m^+\rangle)^\top$ and the IF negative ideal-point $\mathbf{A}^- = (\langle \mu_1^-, \nu_1^-\rangle, \langle \mu_2^-, \nu_2^-\rangle, \ldots, \langle \mu_m^-, \nu_m^-\rangle)^\top$ as follows:

$$\mu_j^+ = \max_{1\leq i\leq n}\{\bar{\mu}_{ij}\}, \quad \nu_j^+ = \min_{1\leq i\leq n}\{\bar{\nu}_{ij}\},$$
$$\mu_j^- = \min_{1\leq i\leq n}\{\bar{\mu}_{ij}\}, \quad \nu_j^- = \max_{1\leq i\leq n}\{\bar{\nu}_{ij}\};$$

Step 6: Compute the Euclidean distances $d_{\text{Eu}}(A_i, \mathbf{A}^+)$ and $d_{\text{Eu}}(A_i, \mathbf{A}^-)$ of the alternatives A_i $(i = 1, 2, \ldots, n)$ from $\mathbf{A}^+$ and $\mathbf{A}^-$ by using formula (3);

Step 7: Calculate the relative closeness degrees $\mathscr{C}_i$ of the alternatives A_i $(i = 1, 2, \ldots, n)$ to the IF positive ideal-point $\mathbf{A}^+$ by the following formula:

$$\mathscr{C}_i = \frac{d_{\text{Eu}}(A_i, \mathbf{A}^-)}{d_{\text{Eu}}(A_i, \mathbf{A}^+) + d_{\text{Eu}}(A_i, \mathbf{A}^-)};$$

Step 8: Rank the alternatives A_i $(i = 1, 2, \ldots, n)$ according to the nonincreasing order of the relative closeness degrees $\mathscr{C}_i$ and select the most desirable alternative.

Example 6: Suppose that there exist 4 alternatives A_1, A_2, A_3, A_4 evaluated with respect to 2 benefit attributes $\mathscr{O}_1$, $\mathscr{O}_2$. The sets of the alternatives and attributes are denoted by $\{A_1, A_2, A_3, A_4\}$ and $\{\mathscr{O}_1, \mathscr{O}_2\}$, respectively. Assume that the IF weight vector of $\mathscr{O}_1$ and $\mathscr{O}_2$ is $\omega = (\omega_1, \omega_2)^\top = (\langle 1, 0\rangle, \langle 1, 0\rangle)^\top$. The IF decision-making matrix is expressed as shown in Table II.

TABLE II
IF DECISION MATRIX $R = (r_{ij})_{4\times 2}$

	$\mathscr{O}_1$	$\mathscr{O}_2$
A_1	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
A_2	$\langle 0.9, 0.01\rangle$	$\langle 0.9, 0.01\rangle$
A_3	$\langle 0.901, 0.007\rangle$	$\langle 0.901, 0.007\rangle$
A_4	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$

If we use the above TOPSIS method [37, Section 3.3], by direct calculation, it can be verified that the weighted IF decision matrix is given as shown in Table III.

TABLE III
WEIGHTED IF DECISION MATRIX $\overline{R} = (\omega_j \otimes r_{ij})_{4\times 2}$

	$\mathscr{O}_1$	$\mathscr{O}_2$
A_1	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
A_2	$\langle 0.9, 0.01\rangle$	$\langle 0.9, 0.01\rangle$
A_3	$\langle 0.901, 0.007\rangle$	$\langle 0.901, 0.007\rangle$
A_4	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$

The IF positive ideal-point $\mathbf{A}^+$ and the IF negative ideal-point $\mathbf{A}^-$ are obtained as follows:

$$\mathbf{A}^+ = (\langle 1, 0\rangle, \langle 1, 0\rangle) \text{ and } \mathbf{A}^- = (\langle 0, 1\rangle, \langle 0, 1\rangle),$$

respectively. According to the Euclidean distance of the alternatives A_1, A_2, A_3, and A_4 from $\mathbf{A}^+$ and $\mathbf{A}^-$ obtained by [37, Eqs. (3.27) and (3.28)], the relative closeness degrees $\mathscr{C}_j$ of the alternatives A_1, A_2, A_3, and A_4 to the IF positive ideal-point can be calculated as follows:

$$\mathscr{C}_1 = \frac{d_{\text{Eu}}(A_1, \mathbf{A}^-)}{d_{\text{Eu}}(A_1, \mathbf{A}^+) + d_{\text{Eu}}(A_1, \mathbf{A}^-)} = 0,$$

$$\mathscr{C}_2 = \frac{d_{\text{Eu}}(A_2, \mathbf{A}^-)}{d_{\text{Eu}}(A_2, \mathbf{A}^+) + d_{\text{Eu}}(A_2, \mathbf{A}^-)} = 0.9085917,$$

$$\mathscr{C}_3 = \frac{d_{\text{Eu}}(A_3, \mathbf{A}^-)}{d_{\text{Eu}}(A_3, \mathbf{A}^+) + d_{\text{Eu}}(A_3, \mathbf{A}^-)} = 0.9085194,$$

and

$$\mathscr{C}_4 = \frac{d_{\text{Eu}}(A_4, \mathbf{A}^-)}{d_{\text{Eu}}(A_4, \mathbf{A}^+) + d_{\text{Eu}}(A_4, \mathbf{A}^-)} = 1,$$

respectively. Therefore, the ranking order of A_1, A_2, A_3, and A_4 is: $A_4 \succ A_2 \succ A_3 \succ A_1$. However, $A_4 \succ A_3 \succ A_2 \succ A_1$ by a direct observation with $\langle 1, 0\rangle \supset \langle 0.901, 0.007\rangle \supset \langle 0.9, 0.01\rangle \supset \langle 0, 1\rangle$. This means that the ranking order obtained by the IF TOPSIS method in [37] is not consistent with the real situation.

The following example demonstrates that the above IF TOPSIS may yield some unreasonable decision-making results, even if we restrict the normalized weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^\top$ in Step 3 to be positive real numbers, i.e., $\omega_j \in (0, 1]$ and $\sum_{j=1}^m \omega_i = 1$.

Example 7: Suppose that there exist 4 alternatives A_1, A_2, A_3, A_4 evaluated with respect to 2 benefit attributes $\mathscr{O}_1$, $\mathscr{O}_2$. The sets of the alternatives and attributes are denoted by $\{A_1, A_2, A_3, A_4\}$ and $\{\mathscr{O}_1, \mathscr{O}_2\}$, respectively. Assume that the weight vector of $\mathscr{O}_1$ and $\mathscr{O}_2$ is $\omega = (\omega_1, \omega_2)^\top = (0.5, 0.5)^\top$.The IF decision-making matrix is expressed as shown in Table IV.

TABLE IV
IF DECISION-MAKING MATRIX $R = (r_{ij})_{4\times 2}$

	$\mathscr{O}_1$	$\mathscr{O}_2$
A_1	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
A_2	$\langle 0.99, 0.0001\rangle$	$\langle 0.99, 0.0001\rangle$
A_3	$\langle 0.990199, 0.49 \times 10^{-4}\rangle$	$\langle 0.990199, 0.49 \times 10^{-4}\rangle$
A_4	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$

By direct calculation, it can be verified that the weighted IF decision matrix is given as shown in Table V.

TABLE V
WEIGHTED IF DECISION MATRIX $\overline{R} = (\omega_j \cdot r_{ij})_{4\times 2}$

	$\mathscr{O}_1$	$\mathscr{O}_2$
A_1	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
A_2	$\langle 0.9, 0.01\rangle$	$\langle 0.9, 0.01\rangle$
A_3	$\langle 0.901, 0.007\rangle$	$\langle 0.901, 0.007\rangle$
A_4	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$

If we use the TOPSIS method in [37], by Example 6, we know that the ranking order of A_1, A_2, A_3, and A_4 is: $A_4 \succ A_2 \succ A_3 \succ A_1$. This is also an unreasonable decision-making result.

Remark 2: (1) Examples 6 and 7 show that, for some weight vector with either IFVs or real numbers, even for some simple decision-making problems, the TOPSIS method in [37] may lead to some unreasonable decision-making results.

(2) Careful readers can verify that by applying the TOPSIS methods in [45], [48], [47], [50] to Examples 6, the same result can be obtained. This means that the TOPSIS methods in [45], [48], [47], [50] may produce unreasonable results when dealing with the simplest decision-making problems.

B. Limitation in TOPSIS method of Chen et al. [52]

The above two examples show that the TOPSIS method in [37] is not monotonous with Atanassov's partial order $\subset$. Recently, Chen et al. [52] developed a monotonous TOPSIS method with the partial order $\subset$ based on a new similarity measure. However, the following example shows that the TOPSIS methods in [52] is not monotonous with the linear order $\leq_{\text{XY}}$.

The main process of IF TOPSIS method in [52] is summarized as follows:

Step 1: Determine the alternatives $A = \{A_1, A_2, \dots, A_n\}$ and attributes $\mathscr{O} = \{\mathscr{O}_1, \mathscr{O}_2, \dots, \mathscr{O}_m\}$, respectively, and construct the IF decision matrix $R = (r_{ij} = \langle \mu_{ij}, \nu_{ij}\rangle)_{m\times n}$, as shown in Table I;

Step 2: Determine the IF positive ideal-point $\mathbf{A}^+ = (\langle \mu_1^+, \nu_1^+\rangle, \langle \mu_2^+, \nu_2^+\rangle, \dots, \langle \mu_m^+, \nu_m^+\rangle)^\top$ and the IF negative ideal-point $\mathbf{A}^- = (\langle \mu_1^-, \nu_1^-\rangle, \langle \mu_2^-, \nu_2^-\rangle, \dots, \langle \mu_m^-, \nu_m^-\rangle)^\top$ as follows:

$$\langle \mu_j^+, \nu_j^+\rangle = \begin{cases} \langle \max\limits_{1\leq i\leq n}\{\mu_{ij}\}, \min\limits_{1\leq i\leq n}\{\nu_{ij}\}\rangle, & \mathscr{O}_j \in \mathscr{O}^+, \\ \langle \min\limits_{1\leq i\leq n}\{\mu_{ij}\}, \max\limits_{1\leq i\leq n}\{\nu_{ij}\}\rangle, & \mathscr{O}_j \in \mathscr{O}^-, \end{cases}$$

and

$$\langle \mu_j^-, \nu_j^-\rangle = \begin{cases} \langle \min\limits_{1\leq i\leq n}\{\mu_{ij}\}, \max\limits_{1\leq i\leq n}\{\nu_{ij}\}\rangle, & \mathscr{O}_j \in \mathscr{O}^+, \\ \langle \max\limits_{1\leq i\leq n}\{\mu_{ij}\}, \min\limits_{1\leq i\leq n}\{\nu_{ij}\}\rangle, & \mathscr{O}_j \in \mathscr{O}^-, \end{cases}$$

where $\mathscr{O}^+$ is the set of benefit attributes and $\mathscr{O}^-$ is the set of cost attributes;

Step 3: Compute the degree of indeterminacy $\pi_j^+ = 1 - \mu_j^+ - \nu_j^+$ of the positive ideal-point $\langle \mu_j^+, \nu_j^+\rangle$ for each attribute $\mathscr{O}_j$ ($j = 1, 2, \dots, n$);

Step 4: Compute the degree of indeterminacy $\pi_j^- = 1 - \mu_j^- - \nu_j^-$ of the negative ideal-point $\langle \mu_j^-, \nu_j^-\rangle$ for each attribute $\mathscr{O}_j$ ($j = 1, 2, \dots, n$);

Step 5: Compute the degree of similarity g_{ij}^+ between the evaluating IFV r_{ij} of the alternative A_i with respect to the attribute $\mathscr{O}_j$ and the positive ideal-point $\langle \mu_j^+, \nu_j^+\rangle$ of the attribute $\mathscr{O}_j$ to construct the positive similarity matrix $G^+ = (g_{ij}^+)_{m\times n}$, where $g_{ij}^+ = 1 - \frac{|2(\mu_j^+ - \mu_{ij}) - (\nu_j^+ - \nu_{ij})|}{3} \times (1 - \frac{\pi_j^+ + \pi_{ij}}{2}) - \frac{|2(\nu_j^+ - \nu_{ij}) - (\mu_j^+ - \mu_{ij})|}{3} \times \frac{\pi_j^+ + \pi_{ij}}{2}$;

Step 6: Compute the degree of similarity g_{ij}^- between the evaluating IFV r_{ij} of the alternative A_i with respect to the attribute $\mathscr{O}_j$ and the negative ideal-point $\langle \mu_j^-, \nu_j^-\rangle$ of the attribute $\mathscr{O}_j$ to construct the negative similarity matrix $G^- = (g_{ij}^-)_{m\times n}$, where $g_{ij}^- = 1 - \frac{|2(\mu_j^- - \mu_{ij}) - (\nu_j^- - \nu_{ij})|}{3} \times (1 - \frac{\pi_j^- + \pi_{ij}}{2}) - \frac{|2(\nu_j^- - \nu_{ij}) - (\mu_j^- - \mu_{ij})|}{3} \times \frac{\pi_j^- + \pi_{ij}}{2}$;

Step 7: Compute the weighted positive score $S_i^+ = \sum_{j=1}^m \omega_j g_{ij}^+$ and the weighted negative score $S_i^- = \sum_{j=1}^m \omega_j g_{ij}^-$ of each alternative A_i ($i = 1, 2, \dots, n$), where ω_j is the weight of criterion $\mathscr{O}_j$ such that $\omega_j \in (0, 1]$ and $\sum_{j=1}^m \omega_j = 1$;

Step 8: Compute the relative degree of closeness $T(A_i) = \frac{S_i^+}{S_i^+ + S_i^-}$ of each alternative A_i. The larger the value of $T(A_i)$, the better the preference order of alternative A_i. Then, rank the alternatives A_i ($i = 1, 2, \dots, n$) according to the nonincreasing order of the relative closeness degrees $T(A_1)$, $T(A_2)$, $\dots$, $T(A_n)$.

Example 8: Suppose that there exist 4 alternatives A_1, A_2, A_3, A_4 evaluated with respect to 2 benefit attributes $\mathscr{O}_1$, $\mathscr{O}_2$. The sets of the alternatives and attributes are denoted by $\{A_1, A_2, A_3, A_4\}$ and $\{\mathscr{O}_1, \mathscr{O}_2\}$, respectively. Assume that the weight vector of $\mathscr{O}_1$ and $\mathscr{O}_2$ is $\omega = (0.5, 0.5)^\top$. The IF decision-making matrix is expressed as shown in Table VI.

TABLE VI
IF DECISION MATRIX R

	$\mathscr{O}_1$	$\mathscr{O}_2$
A_1	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
A_2	$\langle 0.3, 0\rangle$	$\langle 0.3, 0\rangle$
A_3	$\langle 0.64, 0.36\rangle$	$\langle 0.64, 0.36\rangle$
A_4	$\langle 1, 0\rangle$	$\langle 1, 0\rangle$

The IF positive ideal-point $\mathbf{A}^+$ and the IF negative ideal-point $\mathbf{A}^-$ are obtained as follows: $\mathbf{A}^+ = (\langle 1, 0\rangle, \langle 1, 0\rangle)$ and $\mathbf{A}^- = (\langle 0, 1\rangle, \langle 0, 1\rangle)$, respectively. By using the TOPSIS method in [52], we obtain the positive similarity matrix G^+ and the negative similarity matrix G^- as follows:

$$G^+ = (g_{ij}^+)_{4\times 2} = \begin{bmatrix} 0 & 0 \\ 0.615 & 0.615 \\ 0.64 & 0.64 \\ 1 & 1 \end{bmatrix},$$

and

$$G^- = (g_{ij}^-)_{4\times 2} = \begin{bmatrix} 1 & 1 \\ 0.385 & 0.385 \\ 0.36 & 0.36 \\ 0 & 0 \end{bmatrix}.$$

Then, the weighted positive scores $S_i^+ = \omega_1 g_{i1}^+ + \omega_2 g_{i2}^+$ ($i = 1, 2, 3, 4$) and the weighted negative scores $S_i^- = \omega_1 g_{i1}^- + \omega_2 g_{i2}^-$ ($i = 1, 2, 3, 4$) of the alternatives A_1, A_2, A_3, and A_4 can be calculated as follows:

$$S_1^+ = 0,\ S_2^+ = 0.615,\ S_3^+ = 0.64,\ S_4^+ = 1,$$

and

$$S_1^- = 1,\ S_2^+ = 0.385,\ S_3^+ = 0.36,\ S_4^- = 0.$$

Therefore, the relative degree of closeness $T(A_i) = \frac{S_i^+}{S_i^+ + S_i^-}$ ($i = 1, 2, 3, 4$) of the alternatives A_1, A_2, A_3, and A_4 are given as follows:

$$T(A_1) = 0,\ T(A_2) = 0.615,\ T(A_3) = 0.64,\ T(A_4) = 1,$$

and thus the ranking order of A_1, A_2, A_3, and A_4 is: $A_4 \succ A_3 \succ A_2 \succ A_1$. However, it can be verified that $A_4 \succ A_2 \succ A_3 \succ A_1$ by a direct observation with $\langle 1, 0\rangle \geq_{\text{XY}} \langle 0.3, 0\rangle \geq_{\text{XY}} \langle 0.64, 0.36\rangle \geq_{\text{XY}} \langle 0, 1\rangle$.

Summing up Examples 6–8, an interesting question is *whether there exists an IF TOPSIS method that is monotonous with the linear order $\leq_{\text{XY}}$ or $\leq_{\text{ZX}}$?* In the following section, we will establish an IF TOPSIS method that is monotonous with the linear order $\leq_{\text{XY}}$ or $\leq_{\text{ZX}}$.

VI. A MONOTONOUS IF TOPSIS METHOD

The fundamental cause for counterintuitive decision-making results in Examples 6–8 lies in the structure of metrics for IFVs. In [6], we defined a metric ϱ in $\tilde{\mathbb{I}}$ as follows: for α, $\beta \in \tilde{\mathbb{I}}$,

$$\varrho(\alpha, \beta) = \begin{cases} \frac{1}{3}(1 + |s(\alpha) - s(\beta)|), & s(\alpha) \neq s(\beta), \\ \frac{1}{3}(|h(\alpha) - h(\beta)|), & s(\alpha) = s(\beta), \end{cases}$$

where $s(\alpha)$ and $h(\alpha)$ are the score degree and the accuracy degree of α, respectively. Furthermore, we [6] proved the following basic properties of ϱ.

Theorem 6.1 ([6]):

(1) $\varrho(\alpha, \beta) \in [0, 1]$ and $\varrho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.
(2) $\varrho(\alpha, \beta) = 1$ if and only if ($\alpha = \langle 0, 1\rangle$ and $\beta = \langle 1, 0\rangle$) or ($\alpha = \langle 1, 0\rangle$ and $\beta = \langle 0, 1\rangle$).
(3) $\varrho(\alpha, \beta) = \varrho(\beta, \alpha)$.
(4) For any α, β, $\gamma \in \tilde{\mathbb{I}}$, $\varrho(\alpha, \beta) + \varrho(\beta, \gamma) \geq \varrho(\alpha, \gamma)$.
(5) For any α, β, $\gamma \in \tilde{\mathbb{I}}$, if $\alpha \leq_{\text{XY}} \beta \leq_{\text{XY}} \gamma$, then $\varrho(\alpha, \beta) \leq \varrho(\alpha, \gamma)$ and $\varrho(\beta, \gamma) \leq \varrho(\alpha, \gamma)$.

Based on the similarity function $L(\alpha)$, similarly to the metric ϱ, define the parametric metrics $\varrho^{(\lambda)}$ and $\tilde{\varrho}^{(\lambda)}$ in $\tilde{\mathbb{I}}$ as follows: for α, $\beta \in \tilde{\mathbb{I}}$,

$$\begin{aligned} &\varrho^{(\lambda)}(\alpha, \beta) \\ &= \begin{cases} \frac{1}{1+2\lambda}(1 + \lambda \cdot |s(\alpha) - s(\beta)|), & s(\alpha) \neq s(\beta), \\ \frac{1}{1+2\lambda}(|h(\alpha) - h(\beta)|), & s(\alpha) = s(\beta), \end{cases} \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\tilde{\varrho}^{(\lambda)}(\alpha, \beta) \\ &= \begin{cases} \frac{1}{1+\lambda}(1 + \lambda \cdot |L(\alpha) - L(\beta)|), & L(\alpha) \neq L(\beta), \\ \frac{1}{1+\lambda}(|h(\alpha) - h(\beta)|), & L(\alpha) = L(\beta), \end{cases} \end{aligned} \tag{15}$$

where $\lambda \geq 1$ is a parameter, and $L(\alpha)$ and $h(\alpha)$ are the similarity function and the accuracy degree of α, respectively.

Let A and B be two aggregation functions satisfying the condition in Proposition 2.1. Based on Proposition 2.1, define another parametric metric $\varrho_{A,B}^{(\lambda)}$ in $\tilde{\mathbb{I}}$ as follows: for α, $\beta \in \tilde{\mathbb{I}}$,

$$\begin{aligned} &\varrho_{A,B}^{(\lambda)}(\alpha, \beta) \\ &= \begin{cases} \frac{1}{2}(1 + |\overline{A}(\alpha) - \overline{A}(\beta)|), & \overline{A}(\alpha) \neq \overline{A}(\beta), \\ \frac{1}{2}(|\overline{B}(\alpha) - \overline{B}(\beta)|), & \overline{A}(\alpha) = \overline{A}(\beta), \end{cases} \end{aligned} \tag{16}$$

where $\lambda \geq 1$ is a parameter, $\overline{A}(\alpha) = A(\mu_\alpha, 1 - \nu_\alpha)$, and $\overline{B}(\alpha) = B(\mu_\alpha, 1 - \nu_\alpha)$. In particular, by taking $A = K_{\gamma_1}$ and $B = K_{\gamma_2}$ with $\gamma_1 \neq \gamma_2$ and direct calculation, we have

$$\begin{aligned} &\varrho_{K_{\gamma_1}, K_{\gamma_2}}^{(\lambda)}(\alpha, \beta) \\ &= \begin{cases} \frac{1}{1+\lambda}(1 + \lambda \cdot E_{\gamma_1}(\alpha, \beta)), & E_{\gamma_1}(\alpha, \beta) \neq 0, \\ \frac{1}{1+\lambda} E_{\gamma_2}(\alpha, \beta), & E_{\gamma_1}(\alpha, \beta) = 0, \end{cases} \end{aligned} \tag{17}$$

where $E_\gamma(\alpha, \beta) = |\overline{K}_\gamma(\alpha) - \overline{K}_\gamma(\beta)| = |(1 - \gamma)(\mu_\alpha - \mu_\beta) - \gamma(\nu_\alpha - \nu_\beta)|$.

Remark 3: The parameter γ_1 in Eq. (17) can be regarded as the preference for decision-makers to choose the membership and non-membership:

(1) If $\gamma_1 > 0.5$, then the decision-makers prefer non-membership to membership, i.e., the decision-makers are pessimistic.

(2) If $\gamma_1 < 0.5$, then the decision-makers prefer membership to non-membership, i.e., the decision-makers are optimistic.

(3) If $\gamma_1 = 0.5$, then the decision-makers have no preference for membership and non-membership, i.e., the decision-makers are neutral.

Similarly to the proof of Theorem 6.1 in [6], we can prove that the metrics $\varrho^{(\lambda)}$, $\tilde{\varrho}^{(\lambda)}$, and $\varrho_{A,B}^{(\lambda)}$ have the following basic properties.

Theorem 6.2: Let $\lambda \geq 1$ and $\rho \in \{\varrho^{(\lambda)}, \tilde{\varrho}^{(\lambda)}, \varrho_{A,B}^{(\lambda)}\}$. Then,

(1) $\rho(\alpha, \beta) \in [0, 1]$ and $\rho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.
(2) $\rho(\alpha, \beta) = 1$ if and only if ($\alpha = \langle 0, 1\rangle$ and $\beta = \langle 1, 0\rangle$) or ($\alpha = \langle 1, 0\rangle$ and $\beta = \langle 0, 1\rangle$).
(3) $\rho(\alpha, \beta) = \varrho(\beta, \alpha)$.
(4) For any α, β, $\gamma \in \tilde{\mathbb{I}}$, $\rho(\alpha, \beta) + \rho(\beta, \gamma) \geq \rho(\alpha, \gamma)$.
(5) For any α, β, $\gamma \in \tilde{\mathbb{I}}$, if $\alpha \leq \beta \leq \gamma$, then $\rho(\alpha, \beta) \leq \rho(\alpha, \gamma)$ and $\rho(\beta, \gamma) \leq \rho(\alpha, \gamma)$.

For the MADM problem with IFSs, by using the three metrics defined by Eqs. (14)–(16), we propose a new IF TOPSIS method as follows:

Step 1: (Construct the decision matrix) Supposing that the decision-maker gave the rating (or evaluation) of each alternative $A_i \in A$ ($i = 1, 2, \ldots, n$) on each attribute $\mathscr{O}_j$ ($j = 1, 2, \ldots, m$) in the form of IFNs $r_{ij} = \langle \mu_{ij}, \nu_{ij}\rangle$, construct an IF decision matrix $R = (r_{ij})_{n\times m}$ as shown in Table I.

Step 2: (Normalize the decision matrix) Transform the IF decision matrix $R = (r_{ij})_{n\times m}$ to the normalized IF decision matrix $\overline{R} = (\bar{r}_{ij})_{n\times m} = (\langle \bar{\mu}_{ij}, \bar{\nu}_{ij}\rangle)_{n\times m}$ as follows:

$$\bar{r}_{ij} = \begin{cases} r_{ij}, & \text{for benefit attribute } \mathscr{O}_j, \\ r_{ij}^{\complement}, & \text{for cost attribute } \mathscr{O}_j, \end{cases}$$

where $r_{ij}^{\complement}$ is the complement of r_{ij}.

Step 3: (Determine the positive and negative ideal-points) Determine the IF positive ideal-point $\mathbf{A}^+ = (\langle \mu_1^+, \nu_1^+\rangle, \langle \mu_2^+, \nu_2^+\rangle, \dots, \langle \mu_m^+, \nu_m^+\rangle)^\top$ and IF negative ideal-point $\mathbf{A}^- = (\langle \mu_1^-, \nu_1^-\rangle, \langle \mu_2^-, \nu_2^-\rangle, \dots, \langle \mu_m^-, \nu_m^-\rangle)^\top$ as follows:

$$\mu_j^+ = \max_{1\le i\le n} \{\bar{\mu}_{ij}\}, \quad \nu_j^+ = \min_{1\le i\le n} \{\bar{\nu}_{ij}\},$$
$$\mu_j^- = \min_{1\le i\le n} \{\bar{\mu}_{ij}\}, \quad \nu_j^- = \max_{1\le i\le n} \{\bar{\nu}_{ij}\}.$$

Step 4: (Compute the weighted similarity measures) Choose $\lambda \ge 1$ and compute the weighted similarity measures between the alternatives A_i $(i = 1, 2, \dots, n)$ and the IF positive ideal-point $\mathbf{A}^+$ and between the alternatives A_i $(i = 1, 2, \dots, n)$ and the IF negative ideal-point $\mathbf{A}^-$ by using the following formulas:

$$\mathbf{S}(A_i, \mathbf{A}^+) = 1 - \sum_{j=1}^{m} \omega_j \cdot \varrho^{(\lambda)}(\langle \bar{\mu}_{ij}, \bar{\nu}_{ij}\rangle, \langle \mu_j^+, \nu_j^+\rangle), \quad (18)$$

$$(\text{resp., } \mathbf{S}(A_i, \mathbf{A}^+) = 1 - \sum_{j=1}^{m} \omega_j \cdot \tilde{\varrho}^{(\lambda)}(\langle \bar{\mu}_{ij}, \bar{\nu}_{ij}\rangle, \langle \mu_j^+, \nu_j^+\rangle), \quad (19)$$

$$\mathbf{S}(A_i, \mathbf{A}^+) = 1 - \sum_{j=1}^{m} \omega_j \cdot \varrho_{A,B}^{(\lambda)}(\langle \bar{\mu}_{ij}, \bar{\nu}_{ij}\rangle, \langle \mu_j^+, \nu_j^+\rangle)), \quad (20)$$

and

$$\mathbf{S}(A_i, \mathbf{A}^-) = 1 - \sum_{j=1}^{m} \omega_j \cdot \varrho^{(\lambda)}(\langle \bar{\mu}_{ij}, \bar{\nu}_{ij}\rangle, \langle \mu_j^-, \nu_j^-\rangle), \quad (21)$$

$$(\text{resp., } \mathbf{S}(A_i, \mathbf{A}^-) = 1 - \sum_{j=1}^{m} \omega_j \cdot \tilde{\varrho}^{(\lambda)}(\langle \bar{\mu}_{ij}, \bar{\nu}_{ij}\rangle, \langle \mu_j^-, \nu_j^-\rangle), \quad (22)$$

$$\mathbf{S}(A_i, \mathbf{A}^-) = 1 - \sum_{j=1}^{m} \omega_j \cdot \varrho_{A,B}^{(\lambda)}(\langle \bar{\mu}_{ij}, \bar{\nu}_{ij}\rangle, \langle \mu_j^-, \nu_j^-\rangle)). \quad (23)$$

By Theorems 6.1 and 6.2, it is easy to see that the similarity measures obtained by Eqs. (18)–(20) are admissible similarity measures with the orders $\le_{\text{XY}}$, $\le_{\text{ZX}}$, and $\le_{A,B}$, respectively.

Step 5: (Compute the relative closeness degrees) Calculate the relative closeness degrees $\mathscr{C}_i$ of the alternatives A_i $(i = 1, 2, \dots, n)$ to the IF positive ideal-point $\mathbf{A}^+$ by using the following formula:

$$\mathscr{C}_i = \frac{\mathbf{S}(A_i, \mathbf{A}^+)}{\mathbf{S}(A_i, \mathbf{A}^+) + \mathbf{S}(A_i, \mathbf{A}^-)}. \quad (24)$$

Step 6: (Rank the alternatives) Rank the alternatives A_i $(i = 1, 2, \dots, n)$ according to the nonincreasing order of the relative closeness degrees $\mathscr{C}_i$ and select the most desirable alternative.

Remark 4: (1) By Theorems 6.1 and 6.2, it is easy to see that $\mathbf{S}(A_i, \mathbf{A}^+) + \mathbf{S}(A_i, \mathbf{A}^-)$ in Eq. (24) is always nonzero. This overcomes the limitation that many TOPSIS method may lead to the situation that the denominator is equal to 0 when computing the relative closeness degrees.

(2) For practical MADM problems, in order to eliminate the effect of the constant term 1 in formulas (14)–(17) as much as possible, the parameter λ should be chosen as large as possible.

Theorem 6.3 (Monotonicity): Using Eqs. (18) and (21), the above proposed method is increasing with the linear order $\le_{\text{XY}}$, i.e., for the MADM problem expressed in Table I, if there exist $1 \le i_1, i_2 \le n$ such that $\bar{r}_{i_1 j} \le_{\text{XY}} \bar{r}_{i_2 j}$ holds for all $1 \le j \le m$, then $\mathscr{C}_{i_1} \le \mathscr{C}_{i_2}$, i.e., A_{i_2} is better than A_{i_1} ranked by the proposed method. In particular, the proposed method is increasing with Atanassov's order '$\subset$'.

Proof: Fix $\lambda \ge 1$. Let $\mathbf{A}^+ = (\langle \mu_1^+, \nu_1^+\rangle, \langle \mu_2^+, \nu_2^+\rangle, \dots, \langle \mu_m^+, \nu_m^+\rangle)^\top$ and $\mathbf{A}^- = (\langle \mu_1^-, \nu_1^-\rangle, \langle \mu_2^-, \nu_2^-\rangle, \dots, \langle \mu_m^-, \nu_m^-\rangle)^\top$ be the IF positive ideal-point and the IF negative ideal-point obtained by Step 3, respectively. Clearly, $\langle \mu_j^-, \nu_j^-\rangle \le_{\text{XY}} \bar{r}_{i_1 j} \le_{\text{XY}} \bar{r}_{i_2 j} \le_{\text{XY}} \langle \mu_j^+, \nu_j^+\rangle$. By Theorem 6.2, we have

$$\varrho^{(\lambda)}(\langle \mu_j^-, \nu_j^-\rangle, \bar{r}_{i_1 j}) \le \varrho^{(\lambda)}(\langle \mu_j^-, \nu_j^-\rangle, \bar{r}_{i_2 j}),$$

and

$$\varrho^{(\lambda)}(\langle \mu_j^+, \nu_j^+\rangle, \bar{r}_{i_2 j}) \le \varrho^{(\lambda)}(\langle \mu_j^+, \nu_j^+\rangle, \bar{r}_{i_1 j}),$$

and thus,

$$\mathbf{S}(A_{i_1}, \mathbf{A}^-) \ge \mathbf{S}(A_{i_2}, \mathbf{A}^-),$$

and

$$\mathbf{S}(A_{i_1}, \mathbf{A}^+) \le \mathbf{S}(A_{i_2}, \mathbf{A}^+) \text{ by Eqs. (18) and (21).}$$

This, together with Eq. (24), implies that

(1) if $\mathbf{S}(A_{i_1}, \mathbf{A}^+) = 0$, then

$$\mathscr{C}_{i_1} = \frac{\mathbf{S}(A_{i_1}, \mathbf{A}^+)}{\mathbf{S}(A_{i_1}, \mathbf{A}^+) + \mathbf{S}(A_{i_1}, \mathbf{A}^-)} = 0 \le \mathscr{C}_{i_2};$$

(2) if $\mathbf{S}(A_{i_1}, \mathbf{A}^+) > 0$, then $\mathbf{S}(A_{i_2}, \mathbf{A}^+) \ge \mathbf{S}(A_{i_1}, \mathbf{A}^+) > 0$, and thus

$$\mathscr{C}_{i_1} = \frac{\mathbf{S}(A_{i_1}, \mathbf{A}^+)}{\mathbf{S}(A_{i_1}, \mathbf{A}^+) + \mathbf{S}(A_{i_1}, \mathbf{A}^-)} = \frac{1}{1 + \frac{\mathbf{S}(A_{i_1}, \mathbf{A}^-)}{\mathbf{S}(A_{i_1}, \mathbf{A}^+)}}$$
$$\le \frac{1}{1 + \frac{\mathbf{S}(A_{i_2}, \mathbf{A}^-)}{\mathbf{S}(A_{i_2}, \mathbf{A}^+)}} = \frac{\mathbf{S}(A_{i_2}, \mathbf{A}^+)}{\mathbf{S}(A_{i_2}, \mathbf{A}^+) + \mathbf{S}(A_{i_2}, \mathbf{A}^-)} = \mathscr{C}_{i_2}.$$

Therefore, $\mathscr{C}_{i_1} \le \mathscr{C}_{i_2}$. ∎

By Theorem 6.2, similarly to the proof of Theorem 6.3, it is not difficult to check that the following result holds.

Theorem 6.4 (Monotonicity): Using Eqs. (19) and (22) (resp., Eqs. (20) and (23)), the above proposed method is increasing with the linear order $\le_{\text{ZX}}$ (resp., $\le_{A,B}$).

Example 9 (Continuation of Example 7): Consider the MADM problem described in Example 7. If the proposed TOPSIS method in this section is used based on Eq. (14) with $\lambda = 1$, by direct calculation, it can be verified that $\mathscr{C}_1 = 0$, $\mathscr{C}_2 = 0.99495$, $\mathscr{C}_3 = 0.995075$, $\mathscr{C}_4 = 1$. Thus, the ranking order of the alternatives A_1, A_2, A_3, and A_4 is: $A_4 \succ A_3 \succ A_2 \succ A_1$, which is consistent with the result obtained by directly observing in Example 7.

Example 10 (Continuation of Example 8): Consider the MADM problem described in Example 8. If the proposed TOPSIS method in this section is used based on Eq. (14) with $\lambda = 1$, by direct calculation, it can be verified that $\mathscr{C}_1 = 0$, $\mathscr{C}_2 = 0.65$, $\mathscr{C}_3 = 0.64$, $\mathscr{C}_4 = 1$. Thus, the ranking order of the alternatives A_1, A_2, A_3, and A_4 is: $A_4 \succ A_2 \succ A_3 \succ A_1$, which is consistent with the result obtained by directly observing in Example 8.

Remark 5: Observing from Examples 9 and 10, it can be seen that the proposed TOPSIS method can effectively overcome the limitations of the TOPSIS methods in [37], [52], which is consistent with the result proved in Theorem 6.3. Furthermore, this shows that the proposed TOPSIS method is superior to those in [37], [52].

VII. Illustrative Examples

This section provides two practical examples to illustrate the efficiency of the above proposed TOPSIS method. One is an IF MADM problem on the choice of suppliers in the supply chain management (see Example 11). The ranking order obtained by the proposed TOPSIS method is slightly different from the results obtained by those TOPSIS methods in [46], [49], [51], [33]. However, the most desirable alternatives are consistent. The other is an IF MADM problem on the choice of project managers (see Example 12). The ranking order, obtained by the proposed TOPSIS method under the case that the decision-maker is neutral or pessimistic, is consistent with those results obtained by the TOPSIS methods in [52], [66], [46], [49], [51].

Example 11 ([52, Example 5.1]): Assume that there are five alternatives A_1, A_2, A_3, A_4, and A_5 of suppliers and four attributes $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$, and $\mathscr{O}_4$ to assess these five alternatives, so as to choose the best supplier among these five alternatives in the supply chain management, where $\mathscr{O}_1$ is the "Product Quality", $\mathscr{O}_2$ is the "Service", $\mathscr{O}_3$ is the "Delivery", $\mathscr{O}_4$ is the "Sustainability" and $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$, and $\mathscr{O}_4$ are benefit attributes, with weight vector $\omega = (0.25, 0.4, 0.2, 0.15)^\top$.

Step 1: (Construct the decision matrix) The decision matrix $R = (r_{ij})_{5\times 4}$ given by the decision maker is listed in Table VII.

TABLE VII
The decision matrix R

	$\mathscr{O}_1$	$\mathscr{O}_2$	$\mathscr{O}_3$	$\mathscr{O}_4$
A_1	$\langle 0.6, 0.3\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.2, 0.5\rangle$	$\langle 0.1, 0.6\rangle$
A_2	$\langle 0.8, 0.2\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.3, 0.4\rangle$
A_3	$\langle 0.6, 0.3\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.5, 0.2\rangle$
A_4	$\langle 0.9, 0.1\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.2, 0.3\rangle$	$\langle 0.1, 0.5\rangle$
A_5	$\langle 0.7, 0.1\rangle$	$\langle 0.3, 0.2\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.4, 0.2\rangle$

Step 2: (Normalize the decision matrix) Since $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$, and $\mathscr{O}_4$ are all benefit attributes, we have $\overline{R} = (\overline{r}_{ij})_{5\times 4} = R$.

Step 3: (Determine the positive and negative ideal-points) The IF positive ideal-point is

$$\mathbf{A}^+ = (\langle 0.9, 0.1\rangle, \langle 0.8, 0.1\rangle, \langle 0.6, 0.1\rangle, \langle 0.5, 0.2\rangle)^\top,$$

and IF negative ideal-point is

$$\mathbf{A}^- = (\langle 0.6, 0.3\rangle, \langle 0.3, 0.3\rangle, \langle 0.2, 0.5\rangle, \langle 0.1, 0.6\rangle)^\top.$$

Steps 4 and 5: (Compute the relative closeness degrees) Choose $\lambda = 100$ and calculate the relative closeness degrees $\mathscr{C}_i$ of the alternatives A_i ($i = 1, 2, 3, 4, 5$) to the IF positive ideal-point $\mathbf{A}^+$ by Eqs. (18), (21), and (24): $\mathscr{C}_1 = 0.4321$, $\mathscr{C}_2 = 0.5709$, $\mathscr{C}_3 = 0.4750$, $\mathscr{C}_4 = 0.4876$, $\mathscr{C}_5 = 0.5053$.

Step 6: (Rank the alternative) Because $\mathscr{C}_2 > \mathscr{C}_5 > \mathscr{C}_4 > \mathscr{C}_3 > \mathscr{C}_1$, the ranking order of the alternatives A_i ($i = 1, 2, 3, 4, 5$) is: $A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$.

Repeating Steps 1–3, by applying Eqs. (19) and (22), we obtain the following result:

Step 4 and 5: (Compute the relative closeness degrees) Choose $\lambda = 100$ and calculate the relative closeness degrees $\mathscr{C}_i$ of the alternatives A_i ($i = 1, 2, 3, 4, 5$) to the IF positive ideal-point $\mathbf{A}^+$ by Eqs. (19), (22), and (24): $\mathscr{C}_1 = 0.4371$, $\mathscr{C}_2 = 0.5618$, $\mathscr{C}_3 = 0.4694$, $\mathscr{C}_4 = 0.4922$, $\mathscr{C}_5 = 0.4925$.

Step 6: (Rank the alternatives) Because $\mathscr{C}_2 > \mathscr{C}_5 > \mathscr{C}_4 > \mathscr{C}_3 > \mathscr{C}_1$, the ranking order of the alternatives A_i ($i = 1, 2, 3, 4, 5$) is: $A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$.

Comparative analysis

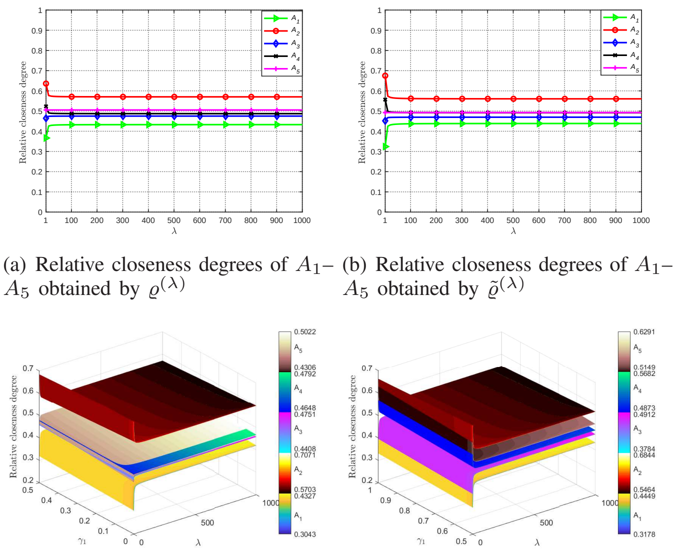
From Table VIII, which shows a comparison of the ranking orders of the alternatives in Example 11 for different MADM methods, it can be observed that (1) our results are exactly the same, which are consistent with the result $A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$ in [52], [66], [21]; (2) the result obtained by Xu's IFWA operator in [4] is different from the results obtained by all other methods; (3) since the TOPSIS method of Büyüközkan and Güleryüz in [49] is based on the normalized Euclidean distance defined by Eq. (3), which does not satisfy the axiomatic definition of IF distance measure (see Example 1), the ranking result $A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$ may be unreasonable; (4) the best choice is always A_2.

TABLE VIII
A comparison of the ranking orders of the alternatives in Example 11 for different MADM methods

Methods	Ranking orders
Chen et al.'s TOPSIS method in [52]	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
Wang and Wei's TOPSIS method in [66]	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
Altan Koyuncu et al.'s TOPSIS method in [46]	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
Büyüközkan and Güleryüz's TOPSIS method in [49]	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
Zhang et al.'s TOPSIS method in [51]	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
Zeng et al.'s VIKOR method in [21]	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
Shen et al.'s TOPSIS method in [33]	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
Xu's IFWA operator method in [4]	$A_2 \succ A_4 \succ A_5 \succ A_3 \succ A_1$
Our TOPSIS based on $\varrho^{(100)}$	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
Our TOPSIS based on $\tilde{\varrho}^{(100)}$	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
Our TOPSIS based on $\varrho^{(100)}_{K_{0.2},K_{0.4}}$	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
Our TOPSIS based on $\varrho^{(100)}_{K_{0.5},K_{0.4}}$	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
Our TOPSIS based on $\varrho^{(100)}_{K_{0.6},K_{0.4}}$	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$

To illustrate the detailed influence of the parameters λ and γ_1 on the decision-making results in Example 11 by using metrics $\varrho^{(\lambda)}$, $\tilde{\varrho}^{(\lambda)}$, and $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$, the relative closeness degrees $\mathscr{C}_i$ of each alternative A_i obtained by $\varrho^{(\lambda)}$, $\tilde{\varrho}^{(\lambda)}$, and $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$ are shown in Figs. 1 (a), (b), and (c)–(d), respectively. As can be seen from Fig. 1, the ranking orders of the alternatives using different values of parameters λ and γ_1 remain the same and are stabilized , when the parameter λ is large enough, and thus the preferences of decision makers do not affect the ranking results in this example. This indicates that our method is effective and stable.

In summary, our proposed TOPSIS method has the following advantages:



(a) Relative closeness degrees of A_1–A_5 obtained by $\varrho^{(\lambda)}$

(b) Relative closeness degrees of A_1–A_5 obtained by $\tilde{\varrho}^{(\lambda)}$

(c) Relative closeness degrees of A_1–A_5 obtained by $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$ $(0 \leq \gamma_1 < 0.5,\ \gamma_2 = 1)$

(d) Relative closeness degrees of A_1–A_5 obtained by $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$ $(0.5 < \gamma_1 < 1,\ \gamma_2 = 1)$

Fig. 1. Relative closeness degrees of A_1–A_5 in Example 11

(1) It is monotonous under the linear order '$\leq_{\text{XY}}$' or '$\leq_{\text{ZX}}$' or '$\leq_{A,B}$'. This can overcome the limitation of non-monotonicity for some classical IF TOPSIS methods in [48], [45], [52], [37], [47], [50]. In addition, other IF TOPSIS methods can at most guarantee the monotonicity under Atanassov's partial order $\subset$.

(2) Based on the admissible distances with linear orders, our method is more in line with the essential features of the original TOPSIS introduced by Hwang and Yoon [5].

(3) As can be seen from Example 11, the ranking orders of the alternatives using different MADM methods are slightly different. However, the best choice is the same. Moreover, our preference order is stable when the parameter λ is large enough. This indicates that our method is effective and stable.

(4) Compared to the TOPSIS methods in [33], [52], our method requires less computation and fewer steps.

Example 12 ([21, Example 5.2]): Assume that there is a committee of a company, which decides to choose a project manager from five alternatives A_1, A_2, A_3, A_4, and A_5 with four attributes $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$, and $\mathscr{O}_4$, where $\mathscr{O}_1$ is "Self-Confidence", $\mathscr{O}_2$ is "Personality", $\mathscr{O}_3$ is "Past Experience", $\mathscr{O}_4$ is the "Proficiency in Project Management" and $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$, and $\mathscr{O}_4$ are all benefit attributes, with weight vector $\omega = (0.1, 0.2, 0.3, 0.4)^{\top}$.

Assume the decision matrix $R = (r_{ij})_{5\times 4}$ given by the committee is as listed in Table IX.

TABLE IX
THE DECISION MATRIX R

	$\mathscr{O}_1$	$\mathscr{O}_2$	$\mathscr{O}_3$	$\mathscr{O}_4$
A_1	$\langle 0.4, 0.5\rangle$	$\langle 0.3, 0.6\rangle$	$\langle 0.4, 0.4\rangle$	$\langle 0.5, 0.3\rangle$
A_2	$\langle 0.4, 0.4\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.3, 0.4\rangle$
A_3	$\langle 0.4, 0.6\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.4, 0.6\rangle$
A_4	$\langle 0.3, 0.4\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.1, 0.9\rangle$	$\langle 0.4, 0.4\rangle$
A_5	$\langle 0.5, 0.4\rangle$	$\langle 0.3, 0.6\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.47, 0.5\rangle$

Step 1: (Normalize the decision matrix) Since $\mathscr{O}_1$, $\mathscr{O}_2$, $\mathscr{O}_3$, and $\mathscr{O}_4$ are all benefit attributes, we have $\overline{R} = (\overline{r}_{ij})_{5\times 4} = R$.

Step 2: (Determine the positive and negative ideal-points) The IF positive ideal-point is

$$\mathbf{A}^{+} = (\langle 0.5, 0.4\rangle, \langle 0.5, 0.4\rangle, \langle 0.4, 0.4\rangle, \langle 0.5, 0.3\rangle)^{\top},$$

and IF negative ideal-point is

$$\mathbf{A}^{-} = (\langle 0.3, 0.6\rangle, \langle 0.2, 0.6\rangle, \langle 0.1, 0.9\rangle, \langle 0.3, 0.6\rangle)^{\top}.$$

Step 3: (Compute the relative closeness degrees) Choose $\lambda = 100$ and calculate the relative closeness degrees $\mathscr{C}_i$ of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ to the IF positive ideal-point $\mathbf{A}^{+}$ by Eqs. (18), (21), and (24): $\mathscr{C}_1 = 0.5565$, $\mathscr{C}_2 = 0.5295$, $\mathscr{C}_3 = 0.5058$, $\mathscr{C}_4 = 0.4555$, $\mathscr{C}_5 = 0.5171$.

Step 4: (Rank the alternatives) Because $\mathscr{C}_1 > \mathscr{C}_2 > \mathscr{C}_5 > \mathscr{C}_3 > \mathscr{C}_4$, the ranking order of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ is: $A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$.

Repeating Steps 1–2, by applying Eqs. (19) and (22), we obtain the following result:

Step 3: (Compute the relative closeness degrees) Calculate the relative closeness degrees $\mathscr{C}_i$ of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ to the IF positive ideal-point $\mathbf{A}^{+}$ by Eqs. (19), (22), and (24): $\mathscr{C}_1 = 0.5531$, $\mathscr{C}_2 = 0.5329$, $\mathscr{C}_3 = 0.5042$, $\mathscr{C}_4 = 0.4583$, $\mathscr{C}_5 = 0.5198$.

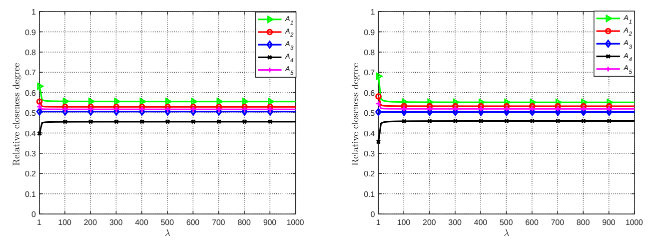
Step 4: (Rank the alternative) Because $\mathscr{C}_1 > \mathscr{C}_2 > \mathscr{C}_5 > \mathscr{C}_3 > \mathscr{C}_4$, the ranking order of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ is: $A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$.

From Table X, which shows a comparison of the ranking orders of the alternatives in Example 12 for different MADM methods, it can be observed that our results based on the metrics $\varrho^{(100)}$, $\tilde{\varrho}^{(100)}$, $\varrho^{(100)}_{K_{0.5},K_{0.4}}$, and $\varrho^{(100)}_{K_{0.6},K_{0.4}}$ are consistent with the ranking orders obtained by the MADM methods in [52], [66], [46], [49], [51], [21].

TABLE X
A COMPARISON OF THE RANKING ORDERS OF THE ALTERNATIVES IN EXAMPLE 12 FOR DIFFERENT MADM METHODS

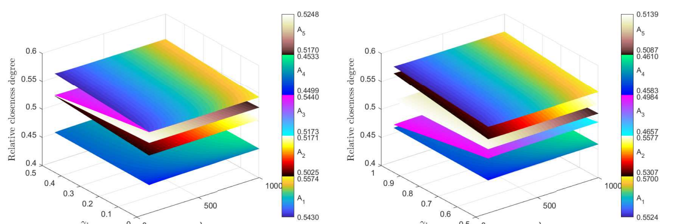
Methods	Ranking orders
Chen et al.'s TOPSIS method in [52]	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Wang and Wei's TOPSIS method in [66]	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Altan Koyuncu et al.'s TOPSIS method in [46]	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Büyüközkan and Güleryüz's TOPSIS method in [49]	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Zhang et al.'s TOPSIS method in [51]	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Zeng et al.'s VIKOR method in [21]	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Our TOPSIS based on $\varrho^{(100)}$	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Our TOPSIS based on $\tilde{\varrho}^{(100)}$	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Our TOPSIS based on $\varrho^{(100)}_{K_{0.2},K_{0.4}}$	$A_1 \succ A_3 \succ A_5 \succ A_2 \succ A_4$
Our TOPSIS based on $\varrho^{(100)}_{K_{0.5},K_{0.4}}$	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$
Our TOPSIS based on $\varrho^{(100)}_{K_{0.6},K_{0.4}}$	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$

To illustrate the detailed influence of the parameters λ and γ_1 on the decision-making results in Example 12 by using the metrics $\varrho^{(\lambda)}$, $\tilde{\varrho}^{(\lambda)}$, and $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$, the relative closeness degrees $\mathscr{C}_i$ of each alternative A_i obtained by $\varrho^{(\lambda)}$, $\tilde{\varrho}^{(\lambda)}$, and $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$ are shown in Fig. 2 (a), (b), and (c)–(d), respectively. Compared to Example 11, the preferences of decision makers greatly affect the ranking results in this example: (1) For $0 \le \gamma_1 < 0.5$, the ranking order is: $A_1 \succ A_3 \succ A_5 \succ A_2 \succ A_4$. (2) For $0.5 \le \gamma_1 < 1$, the ranking order is: $A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$.



(a) Relative closeness degrees of A_1–A_5 obtained by $\varrho^{(\lambda)}$

(b) Relative closeness degrees of A_1–A_5 obtained by $\tilde{\varrho}^{(\lambda)}$



(c) Relative closeness degrees of A_1–A_5 obtained by $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$ ($0 \le \gamma_1 < 0.5$, $\gamma_2 = 1$)

(d) Relative closeness degrees of A_1–A_5 obtained by $\varrho^{(\lambda)}_{K_{\gamma_1},K_{\gamma_2}}$ ($0.5 < \gamma_1 < 1$, $\gamma_2 = 1$)

Fig. 2. Relative closeness degrees of A_1–A_5 in Example 12

VIII. Conclusions

This paper is devoted to establishing a monotonous IF TOPSIS method with three typical linear orders, '$\le_{\rm XY}$' in [7], '$\le_{\rm ZX}$' in [8], and '$\le_{A,B}$' in [11], [12]. Noting that the TOPSIS method is closely related to the order structure and the metric/similarity measure, we first discuss some examples to show that some classical similarity measures in [37], [34], [29], [30], including Euclidean similarity measure, Minkowski similarity measure, and modified Euclidean similarity measure, do not satisfy the axiomatic definition of IF similarity measures. Then, we prove the nonexistence of a continuous function that can distinguish IFV by a real number and is increasing with Atanassov's order '$\subset$'. As a direct corollary, we prove that there is no any continuous similarity measure that can distinguish between each pair of IFVs. Moreover, we show some illustrative examples to demonstrate that some classical IF TOPSIS methods in [48], [45], [52], [37], [47], [50] are not monotonous with Atanassov's partial order '$\subset$' or the linear order '$\le_{\rm XY}$', which may yield counter-intuitive results. To overcome this limitation, by using three new parametric admissible distances with the linear order '$\le_{\rm XY}$' or '$\le_{\rm ZX}$' or '$\le_{A,B}$', we develop a novel IF TOPSIS method and prove that it is monotonically increasing with these two linear orders. Finally, we show two practical examples with comparative analysis to other MADM methods to illustrate the efficiency of our TOPSIS method.

Because the proposed TOPSIS method depends on the choice of the linear orders, choosing an appropriate order for a given problem is very important for practical decision-making. Meanwhile, because the construction method of admissible distances with linear orders presented in this paper is relatively rough, which fails to capture all properties of the the corresponding linear order, this may cause inaccurate decision-making results in some cases. In the future, therefore, we will further study the general construction of linear orders and admissible distance/similarity measures for IFVs, which will be useful for building more effective IF TOPSIS methods.

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