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Optimal Probabilistic Constellation Shaping for Covert Communications

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Abstract-In this paper, we investigate the optimal probabilistic constellation shaping design for covert communication systems from a practical view. Different from conventional covert communications with equiprobable constellations modulation, we propose non-equiprobable constellations modulation schemes to further enhance the covert rate. Specifically, we derive covert rate expressions for practical discrete constellation inputs for the first time. Then, we study the covert rate maximization problem by jointly optimizing the constellation distribution and power allocation. In particular, an approximate gradient descent method is proposed for obtaining the optimal probabilistic constellation shaping. To strike a balance between the computational complexity and the transmission performance, we further develop a framework that maximizes a lower bound on the achievable rate where the optimal probabilistic constellation shaping problem can be solved efficiently using the Frank-Wolfe method. Extensive numerical results show that the optimized probabilistic constellation shaping strategies provide significant gains in the achievable covert rate over the state-of-the-art schemes.

Index Terms—Covert communications, probabilistic constellation shaping, achievable rate.

I. INTRODUCTION

Radio frequency (RF) based wireless communication is inevitably susceptible to eavesdropping due to the broadcast nature of the electromagnetic waves. With the ever-growing Internet of Things (IoT) applications, the security issue has become more and more crucial in the future sixth-generation (6G) wireless networks [1]. For example, in large enterprise buildings, hospitals, factories, communication may be very sensitive to a hostile adversary. The conventional cryptography approaches [2] usually focus on protecting the transmission content or increasing message decoding complexity, while the physical-layer security [3], [4] approaches exploit the intrinsic wireless fading channels properties to minimize the information leakage to the eavesdroppers. In fact, a higher level of security is to hide the existence of the communication, which not only can be applied in all aforementioned application scenarios [5]–[8], but also meets more critical demands from military or security agencies. To address this high level security, covert communications [9], [10], which shields the existence of message transmissions against the detection of a warden, are emerging as a cutting-edge wireless communication security technique, and have recently attracted significant research attention. Note that, covert communication aims to hide the communication behavior from the eavesdropper, while physical layer security tries to reduce the interception information of the eavesdropper.

The basic idea of covert communications is as follows. The legitimate transmitter (Alice) transmits messages to the paired receiver (Bob), while guaranteeing a low detection probability for a Warden (Willie). Although such idea has been realized by spread-spectrum techniques [11] for several decades, the information-theoretic limits of covert communications, which is also referred to as low probability of detection (LPD) communications in some literature, were only recently derived [10], [12]–[14]. In particular, the authors in [12] firstly demonstrated that in additive white Gaussian noise (AWGN) channels, Alice can reliably send at most $\mathcal{O}(\sqrt{n})$ bits to Bob in n channel usages under the covert requirement. Such result is also called the square root law (SRL). Subsequent works have extended this result to various channel models such as binary symmetric channels [15], broadcast communications [16], multiple access channels [17], and interference channels [18].

Although the SRL indicates that the asymptotic achievable rate under the covert requirement approaches zero, many researchers have shown that the SRL limit can be beaten by exploiting additional techniques in the considered covert communication scenario. These methods include: taking advantage of the ignorance of the transmission time at Willie [19]; applying an intelligent reflecting surface [20]-[22]; exploring the molecular absorption or scattering feature of the Terahertz spectrum [23], [24]; cooperative jamming [25] or uninformed jamming [26]–[29]; jointly optimizing the beam training and data transmission for millimeter-wave communication [30]; exploiting the uncertainty noise power or channel state information (CSI) at Willie [31]–[35]; robust beamforming design [36], [37], over a finite number of channel uses [38]; applying a full-duplex transceiver [39]-[42]; intermediate relay [43], [44] or exploring unmanned aerial vehicle (UAV) as mobile relay [45], [46].

To be more specific, it is shown in [19] that Alice can covertly transmit $\mathcal{O}\left(\min\left\{n, \sqrt{n\log T\left(n\right)}\right\}\right)$ bits to Bob. When Willie lacks the knowledge of his noise power, Alice can reliably transmit $\mathcal{O}(n)$ bits [31]–[33]. With the aid of an uninformed jammer [26], Alice can also achieve positive transmission rate. With a finite number of channel uses, a uniformly distributed power allocation scheme was proposed [38] to enhance the covert transmission. In [39], the authors showed that, the effective throughput under delay constraints

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can be improved by adding artificial noise (AN) at the fullduplex receiver. For a one-way relay network, the authors in [43] studied the performance limits of convert communications of an energy harvesting relay. In [44], a multiple-relay network was considered, and both the maximum throughput and the minimum end-to-end delay routing algorithms were developed with multiple Willies. In [36], two probabilistic metrics, called the covert outage probability and the connectivity probability, were analyzed for multi-antenna covert communications with randomly located wardens and interferers. Using Kullback-Leibler divergence, i.e., $D(p_1||p_0)$ or $D(p_0||p_1)$ to measure the covertness, a Gaussian input distribution was shown to be optimal for the covert metric $D(p_1||p_0)$, and not optimal for the covert metric $D(p_0||p_1)$ [9], where p_0 and p_1 represent Willie's received signal distributions when covert communications occur and not occur, respectively.

The aforementioned research advances in covert communications mainly make the assumption of a Gaussian input distribution at the transmitter side, which can hardly be realized in practical communication systems. In fact, the information symbols in practical communication systems are realized in the form of discrete constellation points, i.e., finite alphabet inputs, such as pulse amplitude modulation (PAM) and multiplequadrature amplitude modulation (M-QAM). In [47], both the lower bound of covert transmission probability and throughput maximization have been analyzed with discrete constellation inputs. The discrete constellation points are assumed to be equally likely, which is not optimal for practical covert communications, especially for high-order modulation schemes. So far, the optimal discrete constellation inputs of covert communication are still not well discussed in the literature.

Motivated by the above background, we develop information-theoretic limits of covert communications with probabilistic constellation shaping. First, we derive the achievable rate expression of the system with the discrete constellation input signals, rather than the Gaussian inputs adopted in most of the existing works. Then, we investigate the performance with the optimal input distribution. Our results provide a practical design framework for covert communication systems. The main contributions of this paper are summarized as follows:

- Generally, the inputs of practical communications systems follow a finite-set discrete distribution rather than a Gaussian distribution. To evaluate performance, we derive the achievable rate expressions for an arbitrary discrete distributed input. Comparing to the existing rate expressions with equiprobable discrete constellation points, the derived expressions are more general and practical. Since the derived rate expression is not in closed-form, we further derive both lower and upper bounds. All these results can be used as performance metrics for the considered covert communication system.
- Furthermore, we design optimal discrete constellation inputs to maximize the exact covert rate under the covertness constraints, transmit power limitations, and the signal distribution requirements, which is a challenging problem since neither the exact covert rate nor the covertness constraint has an analytical expression.

To efficiently solve it, we conservatively transform the covertness constraint into its upper bound with closedform expression. Then, we adopt the numerical integration method to approximate the covert rate objective function and its gradient. Afterwards, the optimal probability distributions of the discrete constellation are calculated by the approximate gradient descent method, where the step sizes are calculated by the backtracking line search.

 To reduce the computation complexity of the design problem, we further adopt the derived lower bound as the covert rate performance metric. To overcome the nonconvexity challenge, this problem is iteratively solved by the proposed Frank-Wolfe method.

The rest of this paper is organized as follows. The system model and the derivation of Bob's achievable rate are presented in Section II. The optimal probabilistic constellation shaping design for covert communications is provided in Section III. The probabilistic constellation shaping design and its approximations are presented in Section IV. In Section V, we evaluate the proposed probabilistic constellation shaping design using numerical results. Finally, we conclude the paper in Section VI.

Notations: The vectors and matrices are represented by boldfaced lowercase and uppercase letters, respectively. The notations $(\cdot)^*$, $\mathbb{E}\{\cdot\}$, $\|\cdot\|$, $\operatorname{Tr}(\cdot)$, $\operatorname{Re}(\cdot)$ and $\operatorname{Im}(\cdot)$ represent the conjugate, the expectation, Frobenius norm, trace, the real part and imaginary part of its argument, respectively. And \odot is Hadamard Product, i.e. $A_{m \times n}[a_{ij}] \odot B_{m \times n}[b_{ij}] =$ $C_{m \times n}[a_{ij}b_{ij}]$. The operator $\mathbf{A} \succeq \mathbf{0}$ means \mathbf{A} is positive semidefinite. The notation $\mathcal{CN}(\mu, \sigma^2)$ denotes a complexvalued circularly symmetric Gaussian distribution with mean μ and variance σ^2 .

II. SYSTEM MODEL



Fig. 1: The system model of covert communication.

Consider a typical covert communication scenario as illustrated in Fig. 1, where Alice and Bob are a legitimate communication pair, and Willie is the eavesdropper. Each one of them is equipped with a single antenna. Let $g_{\rm b} \sim \mathcal{CN}(0, \sigma_1^2)$ and $g_{\rm w} \sim \mathcal{CN}(0, \sigma_2^2)$ denote the Rayleigh flat fading channel from Alice to Bob and Willie, respectively [48], where σ_1^2 and σ_2^2 are the variances of $g_{\rm b}$ and $g_{\rm w}$. Let x [i] denote Alice's transmitted symbol at the *i*-th channel use, where i = 1, ..., N, and N is the total number of channel uses.

A. Achievable Rate of Bob

As it is the case in practice, $x[i] \in \Omega$ follows a discrete constellation distribution instead of Gaussian distribution. Here, Ω denotes a discrete constellation set with K discrete points $\{x_k\}_{1 \le k \le K}$, i.e.,

$$\Omega = \left\{ X \left| \begin{array}{c} \Pr\left(X = x_k\right) = p_k \ge 0, \sum_{k=1}^{K} p_k = 1, \\ \sum_{k=1}^{K} p_k |x_k|^2 \le P_A, x_k \in \mathbb{C}, k = 1, ..., K \end{array} \right\},$$
(1)

where x_k denotes the kth discrete point, p_k denotes the corresponding probability, and P_A denotes the average power.

For the *i*-th channel use, the received signal at Bob $y_{\rm b}[i]$ is given as

$$y_{\rm b}[i] = g_{\rm b}^* x[i] + z_{\rm b}[i],$$
 (2)

where $z_{\rm b}[i] \sim C\mathcal{N}(0, \sigma_{\rm b}^2)$ denotes the received noise at Bob. Since $x[i] \in \Omega$, then the likehood functions of $y_{\rm b}[i]$ is given as

$$p(y_{\rm b}) = \frac{1}{\pi \sigma_{\rm b}^2} \sum_{k=1}^{K} p_k \exp\left(-\frac{|y_{\rm b} - g_{\rm b}^* x_k|^2}{\sigma_{\rm b}^2}\right).$$
(3)

Therefore, given the discrete constellation, the achievable rate of Bob $R_{\rm b}$ is given by

$$R_{\rm b} = I\left(y_{\rm b}\left[i\right]; x\left[i\right]\right) \tag{4a}$$

$$= h\left(y_{\rm b}\left[i\right]\right) - h\left(z_{\rm b}\left[i\right]\right) \tag{4b}$$

$$= -\int_{-\infty} p(y_{\rm b}) \log_2 p(y_{\rm b}) \, dy_{\rm b} - \log_2 \pi e \sigma_b^2 \tag{4c}$$

$$= -\sum_{k=1}^{K} p_{k} \mathbb{E}_{z_{b}} \left\{ \log_{2} \sum_{j=1}^{K} p_{j} \exp\left(-\frac{\left|g_{b}^{*}\left(x_{k}-x_{j}\right)+z_{b}\right|^{2}}{\sigma_{b}^{2}}\right) \right\} - \frac{1}{\ln 2},$$
(4d)

where $h(X) = -\int f(x) \log f(x) dx$ denotes differential entropy, and f(x) represents the probability density function (PDF).

Based on the achievable rate expression in (4), we will further investigate the optimal probability of discrete constellations for covert communications.

B. Hypothesis Testing

According to the received signals, Willie attempts to decide whether Alice covertly transmits information to Bob or not by performing an optimal statistical hypothesis test (such as the Neyman-Pearson test). Specifically, Willies needs to distinguish between two hypotheses: 1) the null hypothesis \mathcal{H}_0 indicating no transmission; 2) the hypothesis \mathcal{H}_1 indicating the transmission. Let $y_w[i]$ denote the received signal at Willie in the *i*-th channel use. Under the two hypotheses, the signal received at Willie is given as

$$\mathcal{H}_0: y_{\mathbf{w}}\left[i\right] = z_{\mathbf{w}}\left[i\right],\tag{5a}$$

$$\mathcal{H}_1: y_{w}[i] = g_{w}^* x[i] + z_{w}[i],$$
 (5b)

where $z_{\rm w}[i] \sim C\mathcal{N}(0, \sigma_{\rm w}^2)$ denotes the received noise at Willie. Let \mathcal{D}_1 and \mathcal{D}_0 , respectively, denote the binary decisions of Willie. Thus, the total detection error probability of Willie is defined as [9], [49], [50]

$$\xi = \Pr\left(\mathcal{D}_1|\mathcal{H}_0\right) + \Pr\left(\mathcal{D}_0|\mathcal{H}_1\right). \tag{6}$$

Note that, $\Pr(\mathcal{D}_1|\mathcal{H}_0)$ denotes the false alarm probability that Willie believes \mathcal{H}_1 when Alice does not transmit, and $\Pr(\mathcal{D}_0|\mathcal{H}_1)$ denotes the missed detection probability that Willie decides \mathcal{H}_0 when Alice transmits. Moreover, let $p_{y,0} = f(y_w | \mathcal{H}_0)$ and $p_{y,1} = f(y_w | \mathcal{H}_1)$ denote the likehood functions of y_w under \mathcal{H}_0 and \mathcal{H}_1 , respectively. According to (5), we have

$$p_{y,0} = \frac{1}{\pi \sigma_{\rm w}^2} \exp\left(-\frac{\left|y_{\rm w}\right|^2}{\sigma_{\rm w}^2}\right),\tag{7a}$$

$$p_{y,1} = \frac{1}{\pi \sigma_{\rm w}^2} \sum_{k=1}^{K} p_k \exp\left(-\frac{|y_{\rm w} - g_{\rm w}^* x_k|^2}{\sigma_{\rm w}^2}\right).$$
(7b)

Let $V_T(p_{y,0} || p_{y,1}) = \frac{1}{2} || p_{y,0} - p_{y,1} ||_1$ denote the total variation distance between $p_{y,0}$ and $p_{y,1}$. According to Theorem 13.1.1 in [50], the optimal detection error probability of Willie is given as

$$\xi^{\text{opt}} = 1 - V_T \left(p_{y,0} \| p_{y,1} \right) = 1 - \frac{1}{2} \| p_{y,0} - p_{y,1} \|_1.$$
 (8)

However, in general, $V_T(p_{y,0} || p_{y,1})$ is difficult to analyze. To address this issue, we apply Pinsker's inequality [49] to obtain an upper bound

$$V_T(p_{y,0} \| p_{y,1}) \le \sqrt{\frac{1}{2} D(p_{y,0} \| p_{y,1})}, \tag{9}$$

$$V_T(p_{y,0} \| p_{y,1}) \le \sqrt{\frac{1}{2}} D(p_{y,1} \| p_{y,0}), \tag{10}$$

where $D(p_{y,0} || p_{y,1}) = \int_y p_{y,0} \log_2 \frac{p_{y,0}}{p_{y,1}} dy_w$ denotes the Kullback-Leibler (KL) divergence from $p_{y,0}$ to $p_{y,1}$, and $D(p_{y,1} || p_{y,0}) = \int_y p_{y,1} \log_2 \frac{p_{y,1}}{p_{y,0}} dy_w$ denotes the KL divergence from $p_{y,1}$ to $p_{y,0}$.

Based on the likehood functions of y_w in (7), $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$ are, respectively, given as

$$D(p_{y,0} || p_{y,1}) = -\frac{1}{\ln 2} - \mathbb{E}_{z_{w}} \left\{ \log_{2} \sum_{k=1}^{K} p_{k} \\ \times \exp\left(-\frac{|z_{w} - g_{w}^{*} x_{k}|^{2}}{\sigma_{w}^{2}}\right) \right\},$$
(11a)
$$D(p_{y,1} || p_{y,0}) = \sum_{l=1}^{K} p_{k} \mathbb{E}_{z_{w}} \left\{ \log_{2} \sum_{l=1}^{K} p_{j} \right\}$$

$$\times \exp\left(-\frac{|g_{\rm w}^*(x_k - x_j) + z_{\rm w}|^2}{\sigma_{\rm w}^2}\right)\right\} + \frac{|g_{\rm w}^2 P_{\rm A}|}{(\ln 2)\,\sigma_{\rm w}^2} + \frac{1}{\ln 2}.$$
(11b)

Covert communication is achieved for a given ε if the detection error probability ξ is no less than $1 - \varepsilon$, i.e.,

$$\xi \ge 1 - \varepsilon, \varepsilon \in [0, 1], \tag{12}$$

where ε is a small number determining the required covertness level.

Therefore, to achieve covert communication with the given ε , i.e., $\xi \ge 1 - \varepsilon$, the KL divergences of the likelihood functions should satisfy one of the following constraints:

$$D(p_{y,0} || p_{y,1}) \le 2\varepsilon^2,$$
(13a)
$$D(p_{y,1} || p_{y,0}) \le 2\varepsilon^2.$$
(13b)

Both of the above constraints can meet the requirements of covert communication, but these two constraints are not exactly the same, as we discuss next.

III. OPTIMAL SIGNALING DESIGN UNDER COVERT **CONSTRAINTS**

In this section, we investigate the design of optimal probability of discrete constellation points for covert transmission with covertness constraints $D(p_{y,0} || p_{y,1}) \leq 2\varepsilon^2$ or $D(p_{u,1} || p_{u,0}) \le 2\varepsilon^2$ [9], [10], [12], [13].

A. Case of $D(p_{y,0} || p_{y,1}) \leq 2\varepsilon^2$

For the probabilistic constellation shaping scheme, we aim to maximize the achievable rate of Bob $R_{\rm b}$ by optimizing the distribution of discrete constellation inputs, while satisfying both the covert transmission constraint and the discrete distribution constraint. Mathematically, the covert discrete constellation input optimization problem can be formulated as follows

$$\max_{\{p_k\}} R_{\rm b} \tag{14a}$$

s.t.
$$D(p_{y,0} || p_{y,1}) \le 2\varepsilon^2$$
, (14b)

$$\Pr\left(X=x_k\right)=p_k\ge 0,\tag{14c}$$

$$\sum_{k=1}^{K} p_k |x_k|^2 \le P_{\mathcal{A}},\tag{14d}$$

$$\sum_{k=1}^{K} p_k = 1, k = 1, ..., K.$$
(14e)

Since the KL-divergence $D(p_{y,0} || p_{y,1})$ in (11b) is not an analytical expression, constraint (14b) is intractable. To circumvent this, we derive an explicit upper bound for the KL-divergence $D(p_{y,0} || p_{y,1})$, which is given by

$$D_{\mathrm{U}}(p_{y,0} \| p_{y,1}) = -\log_2 \sum_{k=1}^{K} p_k \exp\left(-\frac{|g_{\mathrm{w}}^* x_k|^2}{\sigma_{\mathrm{w}}^2}\right). \quad (15)$$

The details of derivations for (15) can be found in Appendix A. Based on (15), problem (14) can be rewritten as

$$\max_{\{p_k\}} R_{\rm b} \tag{16a}$$

s.t.
$$-\log_2 \sum_{k=1}^{K} p_k \exp\left(-\frac{|g_{w}^* x_k|^2}{\sigma_{w}^2}\right) \le 2\varepsilon^2,$$
 (16b)
(14c), (14d), (14e).

In order to put problem (16) in a more concise form, we define the following variables

$$\mathbf{x} \stackrel{\Delta}{=} [x_1, \dots, x_K]^T, \tag{17a}$$

$$\mathbf{p} \stackrel{\Delta}{=} [p_1, \dots, p_K]^T, \tag{17b}$$

$$\mathbf{q} \stackrel{\Delta}{=} \begin{bmatrix} \log_2 \mathbf{p}^T \hat{\mathbf{q}}_1, ..., \log_2 \mathbf{p}^T \hat{\mathbf{q}}_K \end{bmatrix}^T,$$
(17c)

$$\hat{\mathbf{q}}_{l} \stackrel{\Delta}{=} \begin{bmatrix} \exp\left(-\frac{\sigma_{b}^{*}}{\sigma_{b}^{*}}\right) \\ \dots \\ \exp\left(-\frac{|g_{b}^{*}(x_{l}-x_{K})+z_{b}|^{2}}{\sigma_{b}^{*}}\right) \end{bmatrix}, \forall l \in K,$$
(17d)

$$\mathbf{t} \stackrel{\Delta}{=} \left[\exp\left(-\frac{|g_{\mathbf{w}}^* x_1|^2}{\sigma_{\mathbf{w}}^2}\right), ..., \exp\left(-\frac{|g_{\mathbf{w}}^* x_K|^2}{\sigma_{\mathbf{w}}^2}\right) \right]^T, \quad (17e)$$

$$\phi\left(\mathbf{p}\right) \stackrel{\Delta}{=} \mathbb{E}_{z_1} \left\{ \mathbf{p}^T \mathbf{q} \right\}. \quad (17f)$$

$$\phi(\mathbf{p}) \stackrel{\Delta}{=} \mathbb{E}_{z_{\mathrm{b}}} \left\{ \mathbf{p}^{T} \mathbf{q} \right\}.$$
(17f)

Furthermore, the rate of Bob $R_{\rm b}$ and the upper bound of the KL-divergence $D_{\rm U}(p_0||p_1)$ can be, respectively, rewritten as follows

$$R_{\rm b} = -\phi\left(\mathbf{p}\right) - \frac{1}{\ln 2},\tag{18a}$$

$$D_{\mathrm{U}}\left(p_{y,0} \| p_{y,1}\right) = -\log_2 \mathbf{p}^T \mathbf{t}.$$
 (18b)

Since $\frac{1}{\ln 2}$ is constant and maximizing $-\phi(\mathbf{p})$ is equivalent to minimizing $\phi(\mathbf{p})$, problem (16) can be reformulated as

$$\min_{\mathbf{p}} \phi(\mathbf{p}) \tag{19a}$$

s.t.
$$-\log_2 \mathbf{p}^T \mathbf{t} \le 2\varepsilon^2$$
, (19b)

$$\mathbf{p}^T \mathbf{1}_K = 1, \tag{19c}$$

$$\mathbf{p}^T \left(\mathbf{x} \odot \mathbf{x} \right) \le P_{\mathrm{A}},\tag{19d}$$

$$\mathbf{p} \ge \mathbf{0},\tag{19e}$$

where $\mathbf{1}_K$ is a $K \times 1$ vector with all elements equal to 1.

Problem (19) is now a convex problem, and we adopt the gradient projection method to solve it. Specifically, let $\nabla \phi(\mathbf{p})$ denote the gradient of the objective function (19a), which is given by

$$\nabla \phi \left(\mathbf{p} \right) = \mathbb{E}_{z_{\mathrm{b}}} \left\{ \mathbf{q} + \mathbf{Q} \mathbf{p} \right\} = \int_{-\infty}^{\infty} f_{z_{\mathrm{b}}} \left(z_{\mathrm{b}} \right) \left(\mathbf{q} + \mathbf{Q} \mathbf{p} \right) dz_{\mathrm{b}}.$$
(20a)

Here, $f_{z_{\rm b}}(z_{\rm b}) = \frac{1}{\pi \sigma_{\rm b}^2} \exp\left(-\frac{|z_{\rm b}|^2}{\sigma_{\rm b}^2}\right)$ denotes the probability density function of $z_{\rm b}$, $\mathbf{Q} \stackrel{\Delta}{=} [Q_{i,j}]$, where $Q_{i,j} \stackrel{\Delta}{=} \frac{\hat{\mathbf{q}}_j^T \mathbf{e}_i}{\hat{\mathbf{q}}_j^T \mathbf{p} \ln 2}$, and \mathbf{e}_i is the unit vector where the *i*th element is 1 and the other elements are 0.

However, neither the objective function $\phi(\mathbf{p})$ or the gradient $\nabla \phi(\mathbf{p})$ has an analytic expression. To tackle this challenge, we adopt the numerical integration method to approximate $\phi(\mathbf{p})$ and $\nabla \phi(\mathbf{p})$, i.e.,

$$\tilde{\phi}\left(\mathbf{p}\right) = \int_{\substack{\tau_{1} \\ \sigma T_{2}}}^{\tau_{1}} f_{z_{b}}\left(z_{b}\right) \left(\mathbf{p}^{T}\mathbf{q}\right) dz_{b}, \tag{21a}$$

$$\nabla \tilde{\phi} \left(\mathbf{p} \right) = \int_{-\tau_2}^{\tau_2} f_{z_{\rm b}} \left(z_{\rm b} \right) \left(\mathbf{q} + \mathbf{Q} \mathbf{p} \right) dz_{\rm b}, \tag{21b}$$

where $[-\tau_1, \tau_1]$ and $[-\tau_2, \tau_2]$ denote the integration intervals, by defining $\tau_1 > 0, \tau_2 > 0$. Furthermore, $\tilde{\phi}(\mathbf{p})$ and $\nabla \tilde{\phi}(\mathbf{p})$

denote the approximations of the objective function and its gradient, respectively.

Furthermore, let \mathbf{p}_0 denote a feasible starting point, and \mathbf{p}_n denote the *n*th iteration feasible point. With approximate gradient $\nabla \phi(\mathbf{p})$, the gradient descent iteration step is given as

$$\hat{\mathbf{p}}_{n+1} = \mathbf{p}_n - \alpha_n \nabla \phi\left(\mathbf{p}_n\right), \qquad (22)$$

where $\alpha_n \in (0, 1]$ is the stepsize of the *n*th iteration. To choose a proper step size α_n with a sufficient decrease, we adopt the backtracking line search algorithm, given in Algorithm 1.

Then, we project $\hat{\mathbf{p}}_{n+1}$ in the feasible region of problem (19). Specifically, when $\hat{\mathbf{p}}_{n+1}$ satisfies constraints (19b)-(19e), $\mathbf{p}_{n+1} = \hat{\mathbf{p}}_{n+1}$ is in the feasible region. Otherwise, we need to find the closest point \mathbf{p}_{n+1} in the feasible region as the projection of $\hat{\mathbf{p}}_{n+1}$.

Mathematically, the projection operation of $\hat{\mathbf{p}}_{n+1}$ can be formulated as follows

$$\min_{\mathbf{p}_{n+1}} \|\mathbf{p}_{n+1} - \hat{\mathbf{p}}_{n+1}\|^2$$
(23a)

s.t.
$$-\log_2 \mathbf{p}_{n+1}^T \mathbf{t} \le 2\varepsilon^2$$
, (23b)

$$\mathbf{p}_{n+1}^T \mathbf{1}_K = 1, \tag{23c}$$

$$\mathbf{p}_{n+1}^T \left(\mathbf{x} \odot \mathbf{x} \right) \le P_{\mathrm{A}},\tag{23d}$$

$$\mathbf{p}_{n+1} \ge \mathbf{0}.\tag{23e}$$

Algorithm 1 : Backtracking Line Search for Stepsize α_n .

1: Input: \mathbf{p}_n , $\tilde{\phi}(\mathbf{p}_n)$, $\nabla \tilde{\phi}(\mathbf{p}_n)$ and $\bar{\alpha} > 0$, $\rho, c \in (0, 1)$; 2: Update $\hat{\mathbf{p}}_{n+1} = \mathbf{p}_n - \alpha_n \nabla \phi(\mathbf{p}_n);$ 3: if $\hat{\mathbf{p}}_{n+1}$ satisfies constraints (19b)-(19e) 4: $\mathbf{p}_{n+1} = \mathbf{\hat{p}}_{n+1};$ 5: **else** Solve problem (23) over $\hat{\mathbf{p}}_{n+1}$ to obtain \mathbf{p}_{n+1} ; 6: 7: end; While $\tilde{\phi}(\mathbf{p}_{n+1}) \leq \tilde{\phi}(\mathbf{p}_n) + c\bar{\alpha}\nabla\tilde{\phi}(\mathbf{p}_n)^T (\mathbf{p}_{n+1} - \mathbf{p}_n);$ 8: 9: $\bar{\alpha} \leftarrow \rho \bar{\alpha};$ 10: Update $\hat{\mathbf{p}}_{n+1} = \mathbf{p}_n - \alpha_n \nabla \phi(\mathbf{p}_n);$ **if** $\hat{\mathbf{p}}_{n+1}$ satisfies constraints (19b)-(19e) 11: 12: $\mathbf{p}_{n+1} = \mathbf{\hat{p}}_{n+1};$ else 13: 14: solve problem (23) over $\hat{\mathbf{p}}_{n+1}$ to obtain \mathbf{p}_{n+1} ; 15: end; 16: end; 17: return $\alpha_n = \bar{\alpha}$.

Therefore, we propose the approximate gradient descent projection method to efficiently solve problem (19), which is summarized in Algorithm 2.

B. Case of $D(p_{u,1} || p_{u,0}) \leq 2\varepsilon^2$

In this subsection, we further consider the other covert constraint $D(p_{u,1} || p_{u,0}) \leq 2\varepsilon^2$, and the corresponding covert

Algorithm 2 Inexact Gradient Descent Projection Method.

1: Input: choose $K \ge 2$ and choose a random starting point \mathbf{p}_0 which satisfies constraints (19b)-(19e), set c_2 as the stopping parameter and n = 0;

2: Repeat

Let $n \leftarrow n+1$; 3: Update $\tilde{\phi}(\mathbf{p}_{n-1}) = \int_{-\tau_1}^{\tau_1} f_{z_{\rm b}}(z_{\rm b}) \left(\mathbf{p}_{n-1}^T\mathbf{q}\right) dz_{\rm b};$ Update $\nabla \tilde{\phi}(\mathbf{p}_{n-1}) = \int_{-\tau_2}^{\tau_2} f_{z_{\rm b}}(z_{\rm b}) \left(\mathbf{q} + \mathbf{Q}\mathbf{p}_{n-1}\right) dz_{\rm b};$ 4: 5: Compute stepsize α_{n-1} by Algorithm 1; 6: Update $\hat{\mathbf{p}}_n = \mathbf{p}_{n-1} - \alpha_{n-1} \nabla \phi(\mathbf{p}_{n-1});$ 7: if $\hat{\mathbf{p}}_n$ satisfies constraints (19b)-(19e) 8: 9: $\mathbf{p}_n = \hat{\mathbf{p}}_n;$ 10: else solve problem (23) over $\hat{\mathbf{p}}_n$ to obtain \mathbf{p}_n ; 11: 12: end;

Until IIn 12.

13: Until
$$\|\mathbf{p}_n - \mathbf{p}_{n-1}\| \le c_2;$$

14: Output $\mathbf{P}^{\text{opt}} = \mathbf{p}_n$.

rate optimization problem can be formulated as

$$\max_{\{p_k\}} R_{\rm b} \tag{24a}$$

s.t.
$$D(p_{y,1} || p_{y,0}) \le 2\varepsilon^2$$
, (24b)

$$\Pr\left(X=x_k\right)=p_k\ge 0,\tag{24c}$$

$$\sum_{k=1}^{K} p_k |x_k|^2 \le P_{\mathcal{A}},\tag{24d}$$

$$\sum_{k=1}^{K} p_k = 1, k = 1, \dots, K.$$
 (24e)

To handle intractable constraint (24b), we first derive an upper bound on $D(p_{y,1} || p_{y,0})$ denoted by $D_U(p_{y,1} || p_{y,0})$, which is given by

$$D_{\mathrm{U}}(p_{y,1} \| p_{y,0}) = \sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\mathrm{w}}^*(x_k - x_j)|^2}{2\sigma_{\mathrm{w}}^2}\right) + \frac{1}{\ln 2} + \frac{|g_{\mathrm{w}}^2 P_{\mathrm{A}}|}{(\ln 2) \sigma_{\mathrm{w}}^2} - 1.$$
(25)

The details of derivations for (25) are given in Appendix B. Then, problem (24) can be reformulated as

$$\max_{\{p_k\}} R_{\rm b} \tag{26a}$$

s.t.
$$D_{\mathrm{U}}(p_{y,1} \| p_{y,0}) \le 2\varepsilon^2$$
, (26b)
(24c), (24d), (24e).

By defining the following equations to simplify $D_{\rm U}(p_{y,1} \| p_{y,0})$

$$\mathbf{s}_{\mathbf{w},k} \stackrel{\Delta}{=} \begin{bmatrix} \exp\left(-\frac{|g_{\mathbf{w}}^{*}(x_{k}-x_{1})|^{2}}{2\sigma_{\mathbf{w}}^{2}}\right), ..., \exp\left(-\frac{|g_{\mathbf{w}}^{*}(x_{k}-x_{K})|^{2}}{2\sigma_{\mathbf{w}}^{2}}\right) \end{bmatrix}^{T}$$
(27a)
(27a)

$$\mathbf{v}_{w}\left(\mathbf{p}\right) \stackrel{\Delta}{=} \left[\log_{2}\mathbf{p}^{T}\mathbf{s}_{w,1}, ..., \log_{2}\mathbf{p}^{T}\mathbf{s}_{w,K}\right]^{T},$$
(27b)

we obtain

$$D_{\mathrm{U}}(p_{y,1} \| p_{y,0}) = \mathbf{p}^{T} \mathbf{v}_{\mathrm{w}}(\mathbf{p}) + \frac{1}{\ln 2} + \frac{\left| g_{\mathrm{w}}^{2} P_{x} \right|}{(\ln 2) \sigma_{\mathrm{w}}^{2}} - 1.$$
(28)

Unfortunately, $D_{\rm U}(p_{y,1} || p_{y,0})$ is non-convex in **p**, and the covert constraint is also non-convex. To handle this issue, we apply the first order Taylor expansion to $D_{\rm U}(p_{y,1} || p_{y,0})$. Specifically, the derivative of $D_{\rm U}(p_{y,1} || p_{y,0})$ at $\bar{\mathbf{p}}_n$ is given by

$$\nabla D_{\mathrm{U}}\left(p_{y,1} \| p_{y,0}\right) |_{\mathbf{p} = \bar{\mathbf{p}}_{n}} = \mathbf{v}_{\mathrm{w}}\left(\bar{\mathbf{p}}_{n}\right) + \nabla \mathbf{v}_{\mathrm{w}}\bar{\mathbf{p}}_{n}, \qquad (29)$$

where $\nabla \mathbf{v}_{w} = \left[\frac{\mathbf{s}_{w,1}}{\mathbf{\bar{p}}_{n}^{T}\mathbf{s}_{w,1}}, ..., \frac{\mathbf{s}_{w,K}}{\mathbf{\bar{p}}_{n}^{T}\mathbf{s}_{w,K}}\right]_{K \times K}$. Then, the first order Taylor expansion of $\mathbf{p}^{T}\mathbf{v}_{w}(\mathbf{p})$ is given as follows

$$L\left(\mathbf{p}\right) \approx \bar{\mathbf{p}}_{n}^{T} \mathbf{v}_{w}\left(\bar{\mathbf{p}}_{n}\right) + \left(\mathbf{v}_{w}\left(\bar{\mathbf{p}}_{n}\right) + \nabla \mathbf{v}_{w} \bar{\mathbf{p}}_{n}\right)^{T} \left(\mathbf{p} - \bar{\mathbf{p}}_{n}\right).$$
(30)

Then, constraint (26b) can be recast to a convex form as

$$L(\mathbf{p}) + \frac{1}{\ln 2} + \frac{|g_{w}^{2}P_{A}|}{(\ln 2)\sigma_{w}^{2}} - 1 \le 2\varepsilon^{2}.$$
 (31)

Thus, problem (24) can be reformulated as

$$\min_{\mathbf{p}} \phi(\mathbf{p}) \tag{32a}$$

s.t.
$$L(\mathbf{p}) + \frac{1}{\ln 2} + \frac{|g_{w}^{2}P_{A}|}{(\ln 2)\sigma_{w}^{2}} - 1 \le 2\varepsilon^{2},$$
 (32b)

$$\mathbf{p}^T \mathbf{1}_K = 1, \tag{32c}$$

$$\mathbf{p}^T \left(\mathbf{x} \odot \mathbf{x} \right) \le P_{\mathrm{A}},\tag{32d}$$

$$\mathbf{p} \ge 0, \tag{32e}$$

which is convex.

Similarly, problem (32) can be efficiently solved by the inexact gradient descent projection method. The details are omitted due to space limitation.

IV. SIGNALING DESIGN WITH APPROXIMATE COVERT RATE EXPRESSION

Due to the expectation operation, the achievable rate in (4) does not have a closed-form expression, and can only be computed numerically using the approximate gradient descent method at the expense of high computational complexity. To strike a balance between complexity and performance, we further derive analytical upper bound and lower bound on the achievable rate in (4).

Lemma 1: An upper bound $R_{\rm b}^{\rm U}$ on the rate $R_{\rm b}$ is given by

$$R_{\rm b}^{\rm U} = -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^* (x_k - x_j)|^2}{\sigma_{\rm b}^2}\right), \quad (33)$$

while a lower bound $R_{\rm b}^{\rm L}$ on the rate $R_{\rm b}$ is given as

$$R_{\rm b}^{\rm L} = -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^* (x_k - x_j)|^2}{2\sigma_{\rm b}^2}\right) - \frac{1}{\ln 2} + 1.$$
(34)

Please find the derivation in Appendices C and D.

In this section, we adopt the upper bound and lower bound on the achievable rate $R_{\rm b}$ in our following analysis.

A. Maximizing $R_{\rm b}^U$

In this subsection, we consider the upper bound on the achievable rate for Bob $R_{\rm b}^{\rm U}$ as the objective function to find the optimal probability of discrete constellation points set. Specifically, we study beamforming design with the objective of maximizing $R_{\rm b}^{\rm U}$, subject to the covert transmission constraint, and the discrete constellation set.

1) $D_{\rm U}(p_{y,0} || p_{y,1}) \le 2\varepsilon^2$: Finding the optimal probability of discrete constellation set can be equivalently written as the following optimization problem

$$\max_{\{p_k\}} R_{\rm b}^{\rm U} \tag{35a}$$

s.t.
$$D_{\mathrm{U}}(p_{y,0} \| p_{y,1}) \le 2\varepsilon^2$$
, (35b)

$$\Pr\left(X=x_k\right)=p_k\ge 0,\tag{35c}$$

$$\sum_{k=1}^{K} p_k |x_k|^2 \le P_{\mathcal{A}},\tag{35d}$$

$$\sum_{k=1}^{K} p_k = 1, k = 1, ..., K.$$
 (35e)

In order to solve problem (35), we first define the following variables

$$\mathbf{r}_{k} \stackrel{\Delta}{=} \begin{bmatrix} \exp\left(-\frac{|g_{\mathrm{b}}^{*}(x_{k}-x_{1})|^{2}}{\sigma_{\mathrm{b}}^{2}}\right), ..., \exp\left(-\frac{|g_{\mathrm{b}}^{*}(x_{k}-x_{K})|^{2}}{\sigma_{\mathrm{b}}^{2}}\right) \end{bmatrix}^{T},$$
(36a)

 $\mathbf{u}(\mathbf{p}) \stackrel{\simeq}{=} \left[\log_2 \mathbf{p}^T \mathbf{r}_1, ..., \log_2 \mathbf{p}^T \mathbf{r}_K \right]^{\mathsf{T}}.$ (36b)

In this case, we can obtain

$$R_{\rm b}^{\rm U} = -\mathbf{p}^T \mathbf{u} \left(\mathbf{p} \right). \tag{37}$$

Therefore, problem (35) can be reformulated as follows

$$\min_{\mathbf{p}} f_{\mathrm{U}}(\mathbf{p}) \tag{38a}$$

s.t.
$$-\log_2 \mathbf{p}^T \mathbf{t} \le 2\varepsilon^2$$
, (38b)

$$\mathbf{p}^T \mathbf{1}_K = 1, \tag{38c}$$

$$\mathbf{p}^{T}\left(\mathbf{x}\odot\mathbf{x}\right)\leq P_{\mathrm{A}},\tag{38d}$$

$$\mathbf{p} \ge \mathbf{0},\tag{38e}$$

where $f_{\rm U}(\mathbf{p}) = \mathbf{p}^T \mathbf{u}(\mathbf{p})$. Since the Frank-Wolf method is an algorithm for solving linearly-constrained problems, it makes a linear approximation of the objective function, obtains the feasible descending direction by solving the linear programming, and conducts a one-dimensional search in the feasible region along this direction. Therefore, we will apply the Frank-Wolf method to solve the optimization problem.

We use Taylor's expansion to make a linear approximation of the objective function $f_{\rm U}(\mathbf{p})$. The first order Taylor expansion at \mathbf{p}_n is as follows

$$f_{\mathrm{U}}(\mathbf{p}) \approx \mathbf{p}_{n}^{T} \mathbf{u}(\mathbf{p}_{n}) + \nabla f_{\mathrm{U}}(\mathbf{p}_{i})^{T} (\mathbf{p} - \mathbf{p}_{n}), \qquad (39a)$$
$$\nabla f_{\mathrm{U}}(\mathbf{p}_{n}) = \mathbf{u}(\mathbf{p}_{n}) + \nabla \mathbf{u}\mathbf{p}_{n}, \qquad (39b)$$

where $\nabla \mathbf{u} = \left[\frac{\mathbf{r}_1}{\mathbf{p}_n^T \mathbf{r}_1}, ..., \frac{\mathbf{r}_K}{\mathbf{p}_n^T \mathbf{r}_K}\right]_{K \times K}$, and \mathbf{p}_n denotes the current iteration point. Then, we reformulate the optimization

problem of (38) as follows

$$\min_{\mathbf{p}} \nabla f_{\mathrm{U}}(\mathbf{p}_n)^T \mathbf{p}$$
(40)
s.t. (38b), (38c), (38d), (38e).

By applying the Frank-Wolf method, the detailed procedures for solving (40) are summarized in Algorithm 3. Note that λ_n is the stepsize of the *n*th iteration and d_n denotes the feasible descending direction of the nth iteration.

Algorithm 3 : Solving (40) by Frank-Wolf method.

- 1: Initialization: Choose a feasible starting point \mathbf{p}_0 , set $\delta >$ 0 as the stopping parameter, let n = 0;
- 2: While $\left\| \nabla f(\mathbf{p}_n)^T \mathbf{d}_n \right\| \le \delta$; 3: Solve the linear programming problems (40) and obtain optimal solution $\bar{\mathbf{p}}_n$;
- Construct the feasible descending direction $\mathbf{d}_n = \bar{\mathbf{p}}_n \mathbf{p}_n$ 4: \mathbf{p}_n ;
- Obtain optimal solution $\lambda_n = \operatorname*{arg\,min}_{0 \le \lambda \le 1} f(\mathbf{p}_n + \lambda \mathbf{d}_n);$ 5: Let $\mathbf{p}_{n+1} = \mathbf{p}_n + \lambda_n \mathbf{d}_n, n \leftarrow n + 1;$ 6:
- 7: end
- 8: Output \mathbf{p}_n

2) $D_{\mathrm{U}}\left(p_{y,1} \, \| p_{y,0}\right) \leq 2 \varepsilon^2$: Furthermore, we consider the optimal probabilistic constellation shaping for covert communications with covert constraint $D_{\rm U}(p_{u,1} || p_{u,0}) \leq 2\varepsilon^2$, such that

$$\max_{\{p_k\}} R_{\rm b}^{\rm U} \tag{41a}$$

s.t.
$$D_{\mathrm{U}}(p_{y,1} \| p_{y,0}) \le 2\varepsilon^2$$
, (41b)

which is non-convex.

Similar to problem (26), by replacing constraint (41b) by constraint (31), we can obtain the optimization problem

$$\min_{\mathbf{p}} \nabla f_U(\mathbf{p}_n)^T \mathbf{p} \tag{42a}$$

s.t.
$$L(\mathbf{p}) + \frac{1}{\ln 2} + \frac{|g_{w}^{2}P_{A}|}{(\ln 2)\sigma_{w}^{2}} - 1 \le 2\varepsilon^{2},$$
 (42b)
(38b), (38c), (38d), (38e).

Then, we apply the Frank-Wolf method to solve problem (35), and the details are omitted since the corresponding algorithm is similar to Algorithm 3.

B. Maximizing $R_{\rm b}^L$

In this subsection, we further study the lower bound beamforming design for covert communication by maximizing the lower bound $R_{\rm b}^{\rm L}$, while satisfying the covert transmission requirement and the discrete constellation set with K.

1) $D_{\rm U}(p_{y,0} \| p_{y,1}) \leq 2\varepsilon^2$: Under the covert constraint $D_{\rm U}(p_{y,0} \| p_{y,1}) \leq 2\varepsilon^2$, the optimal probabilistic constellation shaping for covert communications is formulated as follows

$$\max_{\{p_k\}} R_{\rm b}^{\rm L} \tag{43a}$$

s.t.
$$D_{\mathrm{U}}(p_{y,0} \| p_{y,1}) \le 2\varepsilon^2,$$
 (43b)

$$\Pr\left(X=x_k\right)=p_k\ge 0,\tag{43c}$$

$$\sum_{k=1}^{K} p_k |x_k|^2 \le P_{\mathcal{A}},\tag{43d}$$

$$\sum_{k=1}^{K} p_k = 1, k = 1, ..., K.$$
(43e)

To solve the problem, we define the following equations

$$\mathbf{s}_{\mathrm{b},k} \stackrel{\Delta}{=} \left[\exp\left(-\frac{|g_{\mathrm{b}}^*(x_k - x_1)|^2}{2\sigma_{\mathrm{b}}^2}\right), ..., \exp\left(-\frac{|g_{\mathrm{b}}^*(x_k - x_K)|^2}{2\sigma_{\mathrm{b}}^2}\right) \right]^T,$$
(44a)

$$\mathbf{v}_{\mathrm{b}}\left(\mathbf{p}\right) \stackrel{\Delta}{=} \left[\log_{2}\mathbf{p}^{T}\mathbf{s}_{\mathrm{b},1}, ..., \log_{2}\mathbf{p}^{T}\mathbf{s}_{\mathrm{b},K}\right]^{T}, \tag{44b}$$

and then the lower bound $R_{\rm b}^{\rm L}$ can be transformed to

$$R_{\rm b}^{\rm L} = -\mathbf{p}^T \mathbf{v}_{\rm b}\left(\mathbf{p}\right) - \frac{1}{\ln 2} + 1.$$
(45)

Thus, the covert optimization problem is recast as follows

$$\max_{\mathbf{p}} - \mathbf{p}^T \mathbf{v}_{\mathrm{b}} \left(\mathbf{p} \right) \tag{46a}$$

s.t.
$$-\log_2 \mathbf{p}^T \mathbf{t} \le 2\varepsilon^2$$
, (46b)

$$\mathbf{p}^T \mathbf{1}_K = 1, \tag{46c}$$

$$\mathbf{p}^T \left(\mathbf{x} \odot \mathbf{x} \right) \le P_{\mathrm{A}},\tag{46d}$$

$$\mathbf{p} \ge \mathbf{0},\tag{46e}$$

Similar to problem (35), we can apply the Frank-Wolf method to solve problem (43), and the details are omitted.

2) $D_{\mathrm{U}}(p_{y,1} \| p_{y,0}) \leq 2\varepsilon^2$: With the covert constraint $D_{\mathrm{U}}(p_{y,1} \| p_{y,0}) \leq 2\varepsilon^2$, the optimal probabilistic constellation shaping for covert communications is given as

$$\max_{\{n_{\rm b}\}} R_{\rm b}^{\rm L} \tag{47a}$$

s.t.
$$D_{\rm U}(p_{y,1} || p_{y,0}) \le 2\varepsilon^2$$
, (47b)
(43c), (43d), (43e).

By replacing the constraint (47b) with (31), we obtain the optimization problem

$$\max_{\mathbf{p}} - \mathbf{p}^T \mathbf{v}_{\mathrm{b}}(\mathbf{p}) \tag{48a}$$

s.t.
$$L(\mathbf{p}) + \frac{1}{\ln 2} + \frac{|g_{w}^{2}P_{A}|}{(\ln 2)\sigma_{w}^{2}} - 1 \le 2\varepsilon^{2},$$
 (48b)
(46c), (46d), (46e).

Similar to problem (41), we can apply a similar method to solve problem (47).

V. NUMERICAL RESULTS

In this section, we present and discuss numerical results to assess the performance of the proposed probabilistic constellation shaping designs. In our simulations, the discrete constellation input is QAM modulation, the total transmit power of Alice is $P_{\rm A} = 10$ W, the noise variance of Willie



Fig. 2: The empirical CDF of a) $D(p_{y,0} || p_{y,1})$ and (b) $D(p_{y,1} || p_{y,0})$ with the covertness threshold $2\varepsilon^2 = 0.02$ for the proposed optimal probabilistic constellation shaping design and the equiprobable design.

is $\sigma_{\rm w}^2 = 1$ W, and the noise variance of Bob is $\sigma_{\rm b}^2 = \frac{P_{\rm A}}{10^{\rm SNR/10}}$. Moreover, we assume that all channels experience Rayleigh flat fading, and $\sigma_1 = \sigma_2 = 1$ [48].

We first compare the proposed optimal probabilistic constellation shaping design with the equiprobable design, starting from the empirical CDF of KL divergence and the rate comparison.

Fig. 2 shows the empirical CDF of the achieved $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$, respectively, for both the proposed optimal probabilistic constellation shaping design and the equiprobable design with SNR = 10dB and K = 8, where the covertness threshold is $2\varepsilon^2 = 0.02$, i.e., $D(p_{y,0} || p_{y,1}) \leq 0.02$ and $D(p_{y,1} || p_{y,0}) \leq 0.02$. As observed from Fig. 2, the proposed optimal probabilistic constellation



Fig. 3: The covert rate of a) $D(p_{y,0} || p_{y,1})$ and (b) $D(p_{y,1} || p_{y,0})$ for the proposed optimal probabilistic constellation shaping design and the equiprobable design.

shaping design satisfies the covertness constraint. On the other hand, the equiprobable design cannot satisfy the covertness constraints.

Fig. 3 (a) and (b) show the achievable covert rate of Bob $R_{\rm b}$ for the proposed optimal probabilistic constellation shaping design and the equiprobable design with $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$, respectively. We observe that the optimal probabilistic constellation shaping design is superior when the signal-to-noise ratio is low. Therefore, in practical applications, the proposed optimal probabilistic constellation shaping design is advantageous in medium and low SNR, and the equiprobable design is suitable for high SNR.

Next, we evaluate the performance of the proposed optimal probabilistic constellation shaping design.

Fig. 4 shows the optimal probability distribution of input



Fig. 4: The optimal probability of discrete constellation points with (a) $D(p_{y,0} || p_{y,1})$ and (b) $D(p_{y,1} || p_{y,0})$ for the proposed optimal probabilistic constellation shaping design.

 $\{p_{i,j}\}$ with SNR = 12dB of the proposed optimal probabilistic constellation shaping design, for $D(p_{y,0} || p_{y,1})$ in Fig. 4 (a), and for $D(p_{y,1} || p_{y,0})$ in Fig. 4 (b). As it can be seen from Fig. 4, for the proposed optimal probabilistic constellation shaping design, the optimal probability distribution is not equiprobable, and the symmetrical points have equal probabilities. Specifically, when the number of discrete constellation points is sixteen, the probability of each constellation point for $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$ is given in Table I. From the table, we can clearly see that when the coordinates of the constellation points are symmetrical, their probabilities are the same, and vice versa.

Fig. 5 considers the proposed optimal probabilistic constellation shaping design and depicts, the achievable covert rate of Bob $R_{\rm b}$ versus SNR with different number of constellation



Fig. 5: The achievable covert rate of Bob versus SNR with different number of points K for $D(p_{y,0} || p_{y,1})$.

points K = 2, 4, 8, 16 for $D(p_{y,0} || p_{y,1})$. It can be seen from Fig. 5 that when SNR increases, the covert rate of Bob $R_{\rm b}$ increases. In addition, we observe that larger number of points K results in higher covert rate of Bob $R_{\rm b}$, especially for high SNRs. Thus, as the modulation order increases, the rate of covert communication increases.

Fig. 6 shows the covert rate versus ε for the proposed optimal probabilistic constellation shaping design for $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$, where K = 8, SNR = 6dB. It can be observed from the figure that as ε increases, the covert constraint becomes loose, resulting in a covert rate increase. The rate of the covert constraint $D(p_{y,1} || p_{y,0}) \le 2\varepsilon^2$ is higher than that of the case $D(p_{y,1} || p_{y,0}) \le 2\varepsilon^2$. This is because $D(p_{y,1} || p_{y,0})$ is less than $D(p_{y,1} || p_{y,0})$ for the same probability distribution, and thus $D(p_{y,1} || p_{y,0}) \le 2\varepsilon^2$ is more stringent than $D(p_{y,0} || p_{y,1}) \le 2\varepsilon^2$.

Finally, we compare the performance and complexity of three objective functions.

Fig. 7 depicts the achievable rate of Bob $R_{\rm b}$ with the proposed optimal probabilistic constellation shaping design, as well as the objective functions $R_{\rm b}^{\rm L}$ and $R_{\rm b}^{\rm U}$ versus the SNR for the case of $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$, respectively. It can be observed that the mutual information of Bob $R_{\rm b}$ increases as the SNR increases, while $R_{\rm b}$ of the proposed optimal probabilistic constellation shaping design is between the objective functions $R_{\rm b}^{\rm L}$ and $R_{\rm b}^{\rm U}$, and the objective function $R_{\rm b}^{\rm L}$ is higher than the objective function $R_{\rm b}^{\rm L}$.

From Fig. 7 we observe that the proposed optimal probabilistic constellation shaping design is between the objective functions $R_{\rm b}^{\rm L}$ and $R_{\rm b}^{\rm U}$. Here, we compare the computational complexity of the three designs by computational time in Table II, and all simulations of the three methods are performed using MATLAB 2016b with 2.30GHz, 2.29GHz dual CPUs and a 128GB RAM, where K = 8. Specifically, Table III shows that the computational time of objective functions $R_{\rm b}$,

$\{p_{i,j}\} \qquad \operatorname{Re}\left\{x_{i,j}\right\}$	$D\left(p_{y,0} \parallel p_{y,1} ight)$				$D\left(p_{y,1} \ p_{y,0}\right)$			
$\operatorname{Im}\left\{x_{i,j}\right\}$	-3	-1	1	3	-3	-1	1	3
-3	0.0484	0.0524	0.0524	0.0484	0.0454	0.0502	0.0502	0.0454
-1	0.0524	0.0968	0.0968	0.0524	0.0502	0.1041	0.1041	0.0502
1	0.0524	0.0968	0.0968	0.0524	0.0502	0.1041	0.1041	0.0502
3	0.0484	0.0552	0.0552	0.0484	0.0454	0.0502	0.0502	0.0454

TABLE I: The optimal probability distribution of the optimal probabilistic constellation shaping design for $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$ with SNR = 12dB and K = 16



Fig. 6: The covert rate versus ε for the proposed optimal probabilistic constellation shaping design with the cases of $D(p_{y,0} || p_{y,1})$ and $D(p_{y,1} || p_{y,0})$.

 $R_{\rm b}^{\rm L}$ and $R_{\rm b}^{\rm U}$ for the covert constraint condition $D(p_{y,0} || p_{y,1})$ is 55.04, 2.845 and 3.012 seconds, respectively. The computational times of the latter two cases is approximately 95 percent shorter than that of the probabilistic constellation shaping design. Under the covert constraint condition $D(p_{y,0} || p_{y,1})$ the computational time of objective functions $R_{\rm b}$, $R_{\rm b}^{\rm L}$ and $R_{\rm b}^{\rm U}$ is 107.77, 3.219 and 3.438 seconds, respectively. The computational times of the latter two cases is approximately improved by 97 percent compared to that of the probabilistic constellation shaping design. Moreover, the computational time of the design for $D(p_{y,0} || p_{y,1})$ is less than that of $D(p_{y,1} || p_{y,0})$.

TABLE II: Computational time comparison among the objective functions $R_{\rm b}$, $R_{\rm b}^{\rm L}$ and $R_{\rm b}^{\rm U}$

Time/second Objective Constraint	$R_{ m b}$	$R_{\rm b}^{\rm L}$	$R_{\rm b}^{\rm U}$
$D\left(p_{y,0} \ p_{y,1}\right)$	55.04	2.845	3.012
$D\left(p_{y,1} \ p_{y,0}\right)$	107.77	3.219	3.438



Fig. 7: The achievable rate of Bob versus SNR for (a) $D(p_{y,0} || p_{y,1})$ and (b) $D(p_{y,1} || p_{y,0})$ with the covertness threshold $2\varepsilon^2 = 0.1$.

VI. CONCLUSIONS

In this paper, we propose an optimal probabilistic constellation shaping design for covert communications, where Alice covertly sends a message to Bob while avoiding being discovered by Willie. We derive the achievable rate expressions of the covert communications system, and we study the covert rate maximization problem via optimizing the constellation distribution. In addition, to strike a balance between the computational complexity and the transmission performance, we further develop a framework that maximizes the upper and lower bounds of the achievable rate. Numerical results quantify the gains of the proposed beamformers design over state-ofthe-art schemes in terms of the achievable covert rate.

APPENDIX A DERIVATION OF THE FORMULATION (15)

The upper bound $D_{\rm U}(p_{y,0} || p_{y,1})$ on the KL divergence $D(p_{y,0} || p_{y,1})$ is derived as follows

$$D(p_{y,0} || p_{y,1}) \leq -\frac{1}{\ln 2} - \log_2 \sum_{k=1}^{K} p_k \exp\left(-\mathbb{E}_{z_{w}}\left\{\frac{|z_{w} - g_{w}^* x_k|^2}{\sigma_{w}^2}\right\}\right)$$
(49a)
$$= -\frac{1}{\ln 2} - \log_2 \sum_{k=1}^{K} p_k \exp\left(-\frac{\mathbb{E}_{z_{w}}\left\{z_{w}^2\right\} + |g_{w}^* x_k|^2}{\sigma_{w}^2}\right)$$
(49b)

$$= -\frac{1}{\ln 2} - \log_2 \sum_{k=1}^{K} p_k \exp\left(-1 - \frac{|g_{w}^* x_k|^2}{\sigma_{w}^2}\right)$$
(49c)
$$= -\frac{1}{\ln 2} - \left(-\frac{1}{\ln 2} + \log_2 \sum_{k=1}^{K} p_k \exp\left(-\frac{|g_{w}^* x_k|^2}{\sigma_{w}^2}\right)\right)$$
(49d)

$$= -\log_2 \sum_{k=1}^{K} p_k \exp\left(-\frac{|g_{\mathbf{w}}^* x_k|^2}{\sigma_{\mathbf{w}}^2}\right),\tag{49e}$$

where inequality (49a) holds due to Jensen's Inequality.

APPENDIX B Derivation of the formulation (25)

The upper bound $D_{U}(p_{y,1} || p_{y,0})$ of the KL divergence $D(p_{y,1} || p_{y,0})$ is given as

$$D(p_{y,1} || p_{y,0}) \leq \sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \\ \times \mathbb{E}_{z_{w}} \left\{ \exp\left(-\frac{|g_{w}^{*}(x_k - x_j) + z_{w}|^2}{\sigma_{w}^2}\right) \right\} + \frac{1}{\ln 2} + \frac{|g_{w}^{2}P_{A}|}{(\ln 2)\sigma_{w}^2}$$
(50a)
$$= \sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \mathbb{E}_{z_{w,R}} \left\{ \exp\left(-\frac{(c_R + z_{w,R})^2}{\sigma_{w}^2}\right) \right\} \\ \times \mathbb{E}_{z_{w,I}} \left\{ \exp\left(-\frac{(c_I + z_{w,I})^2}{\sigma_{w}^2}\right) \right\} + \frac{1}{\ln 2} + \frac{|g_{w}^{2}P_{A}|}{(\ln 2)\sigma_{w}^2}$$
(50b)

$$=\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \left[\int_{-\infty}^{\infty} \frac{dz_{\mathrm{w},R}}{\sqrt{\pi}\sigma_{\mathrm{w}}} \exp\left(-\frac{(c_R + z_{\mathrm{w},R})^2}{\sigma_{\mathrm{w}}^2}\right) + \frac{z_{\mathrm{w},R}^2}{\sigma_{\mathrm{w}}^2} \right) \right] \times \left[\int_{-\infty}^{\infty} \frac{dz_{\mathrm{w},I}}{\sqrt{\pi}\sigma_{\mathrm{w}}} \exp\left(-\frac{(c_R + z_{\mathrm{w},I})^2 + z_{\mathrm{w},I}^2}{\sigma_{\mathrm{w}}^2}\right) \right] + \frac{1}{\ln 2} + \frac{|g_{\mathrm{w}}^2 P_{\mathrm{A}}|}{(\ln 2)\,\sigma_{\mathrm{w}}^2}$$
(50c)

$$= \sum_{k=1}^{K} p_{k} \log_{2} \sum_{j=1}^{K} p_{j} \left[\frac{1}{\sqrt{\pi} \sigma_{w}} \frac{1}{2} \sqrt{\frac{\sigma_{w}^{2} \pi}{2}} \exp\left(-\frac{c_{R}^{2}}{2 \sigma_{w}^{2}}\right) 2 \right] \\ \times \left[\frac{1}{\sqrt{\pi} \sigma_{w}} \frac{1}{2} \sqrt{\frac{\sigma_{w}^{2} \pi}{2}} \exp\left(-\frac{c_{I}^{2}}{2 \sigma_{w}^{2}}\right) 2 \right] + \frac{1}{\ln 2} + \frac{|g_{w}^{2} P_{A}|}{(\ln 2) \sigma_{w}^{2}}$$
(50d)

$$=\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \frac{1}{2} \exp\left(-\frac{c_R^2 + c_I^2}{2\sigma_{\rm w}^2}\right) + \frac{1}{\ln 2} + \frac{\left|g_{\rm w}^2 P_{\rm A}\right|}{(\ln 2)\,\sigma_{\rm w}^2}$$
(50e)

$$=\sum_{k=1}^{K} p_k \log_2 \frac{1}{2} + \sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm w}^* (x_k - x_j)|^2}{2\sigma_{\rm w}^2}\right) + \frac{1}{|z_k|^2} + \frac{|g_{\rm w}^2 P_{\rm A}|}{(|z_k|^2) - z^2}$$
(50f)

$$\ln 2 \quad (\ln 2) \sigma_{\rm w}^{2} = \sum_{k=1}^{K} p_{k} \log_{2} \sum_{j=1}^{K} p_{j} \exp\left(-\frac{|g_{\rm w}^{*}(x_{k}-x_{j})|^{2}}{2\sigma_{\rm w}^{2}}\right) + \frac{1}{\ln 2} + \frac{|g_{\rm w}^{2}P_{\rm A}|}{(\ln 2) \sigma_{\rm w}^{2}} - 1,$$
(50g)

where inequality (50a) is true due to Jensen's Inequality. Equality (50b) holds because of the definitions $z_{w,R} \stackrel{\Delta}{=} \operatorname{Re} \{z_w\}$ and $z_{w,I} \stackrel{\Delta}{=} \operatorname{Im} \{z_w\}, z_{w,R}, z_{w,I} \sim \mathcal{N}\left(0, \frac{1}{2}\sigma_w^2\right)$, and $c_R \stackrel{\Delta}{=} \operatorname{Re}\left(g_w^*\left(x_k - x_j\right)\right), c_I \stackrel{\Delta}{=} \operatorname{Im}\left(g_w^*\left(x_k - x_j\right)\right)$.

APPENDIX C Derivation of the formulation (33)

The upper bound $R_{\rm b}^{\rm U}$ of the covert rate $R_{\rm b}$ is derived as follows

$$R_{\rm b} \leq -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j$$

$$\times \exp\left(-\mathbb{E}_{z_{\rm b}} \left\{ \frac{|g_{\rm b}^*(x_k - x_j) + z_{\rm b}|^2}{\sigma_b^2} \right\} \right) - \frac{1}{\ln 2} \qquad (51a)$$

$$= -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^*(x_k - x_j)|^2}{\sigma_b^2} - \frac{\mathbb{E}_{z_{\rm b}} \left\{|z_{\rm b}|^2\right\}}{\sigma_b^2}\right)$$

$$- \frac{1}{\ln 2} \qquad (51b)$$

$$= -\frac{1}{\ln 2} - \sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^*(x_k - x_j)|^2}{\sigma_b^2} - 1\right)$$

(51c)

$$= -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^*(x_k - x_j)|^2}{\sigma_{\rm b}^2}\right) \exp\left(-1\right) \\ -\frac{1}{\ln 2}$$
(51d)

$$= -\sum_{k=1}^{K} p_k \left(\log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^*(x_k - x_j)|^2}{\sigma_{\rm b}^2}\right) - \frac{1}{\ln 2} \right) - \frac{1}{\ln 2}$$
(51e)

$$= -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^*(x_k - x_j)|^2}{\sigma_{\rm b}^2}\right), \quad (51f)$$

where inequality (51a) is true due to Jensen's Inequality.

APPENDIX D DERIVATION OF THE FORMULATION (34)

The lower bound $R_{\rm b}^{\rm L}$ on the covert rate $R_{\rm b}$ is given as

$$R_{\rm b} \ge -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \\ \times \mathbb{E}_{z_{\rm b}} \left\{ \exp\left(-\frac{(g_{\rm b}^*(x_k - x_j) + z_{\rm b})^2}{\sigma_{\rm b}^2}\right) \right\} - \frac{1}{\ln 2}$$
(52a)

$$= -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \left[\int_{-\infty}^{\infty} \frac{dz_{\mathrm{b},R}}{\sqrt{\pi}\sigma_\mathrm{b}} \exp\left(-\frac{(a_R + z_{\mathrm{b},R})^2 + z_{\mathrm{b},R}^2}{\sigma_\mathrm{b}^2} \right] \right]$$

$$\times \left[\int_{-\infty}^{\infty} \frac{dz_{\mathrm{b},I}}{\sqrt{\pi}\sigma_{\mathrm{b}}} \exp\left(-\frac{(a_{R}+z_{\mathrm{b},I})^{2}+z_{\mathrm{b},I}^{2}}{\sigma_{\mathrm{b}}^{2}}\right) \right] - \frac{1}{\ln 2}$$
(52b)

$$= -\sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \left[\frac{1}{\sqrt{\pi}\sigma_{\rm b}} \frac{1}{2} \sqrt{\frac{\sigma_{\rm b}^2 \pi}{2}} \exp\left(-\frac{a_R^2}{2\sigma_{\rm b}^2}\right) 2 \right]$$

$$\times \left[\frac{1}{\sqrt{\pi}\sigma_{\rm b}} \frac{1}{2} \sqrt{\frac{\sigma_{\rm b}^2 \pi}{2}} \exp\left(-\frac{a_I^2}{2\sigma_{\rm b}^2}\right) 2 \right] - \frac{1}{\ln 2}$$
(52c)

$$= -\frac{1}{\ln 2} - \sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \frac{1}{2} \exp\left(-\frac{a_R^2 + a_I^2}{2\sigma_b^2}\right)$$
(52d)

$$= -\frac{1}{\ln 2} + 1 - \sum_{k=1}^{K} p_k \log_2 \sum_{j=1}^{K} p_j \exp\left(-\frac{|g_{\rm b}^*(x_k - x_j)|^2}{2\sigma_{\rm b}^2}\right),$$
(52e)

where inequality (52a) is true due to Jensen's inequality, equality (52b) holds because of the definitions $z_{\mathrm{b},R} \stackrel{\Delta}{=} \operatorname{Re} \{z_{\mathrm{b}}\},$ $z_{\mathrm{b},I} \stackrel{\Delta}{=} \operatorname{Im} \{z_{\mathrm{b}}\}, z_{\mathrm{b},R}, z_{\mathrm{b},I} \sim \mathcal{N}\left(0, \frac{1}{2}\sigma_{\mathrm{b}}^{2}\right), \text{ and } a_{R} \stackrel{\Delta}{=} \operatorname{Re}\left(g_{\mathrm{b}}^{*}\left(x_{k}-x_{j}\right)\right), a_{I} \stackrel{\Delta}{=} \operatorname{Im}\left(g_{\mathrm{b}}^{*}\left(x_{k}-x_{j}\right)\right).$

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