Bounding the polynomial approximation errors of frequency response functions

J. Schoukens[1], G. Vandersteen[1], R. Pintelon[1], Z. Emedi[2], Y. Rolain[1]
Vrije Universiteit Brussel, dept. ELEC, Pleinlaan 2, 1050 Brussel, Belgium
Tel. (+32) E-mail: Johan.Schoukens@vub.ac.be
[2]Ecole Polytechnique Fédérale de Lausane

***Abstract*—Frequency response function (FRF) measurements take a central place in the instrumentation and measurement field because many measurement problems boil down to the characterisation of a linear dynamic behaviour. The major problems to be faced are leakage- and noise errors. The local polynomial method (LPM) was recently presented as a superior method to reduce the leakage errors with several orders of magnitude while the noise sensitivity remained the same as that of the classical windowing methods. The kernel idea of the LPM is a local polynomial approximation of the FRF and the leakage errors in a small frequency band around the frequency where the FRF is estimated. Polynomial approximation of FRF's is also present in other measurement and design problems. For that reason it is important to have a good understanding of the factors that influence the polynomial approximation errors. This article presents a full analysis of this problem, and delivers a rule of thumb that can be easily applied in practice to deliver an upper bound on the approximation error of FRF's. It is shown that the approximation error for lowly damped systems is bounded by $(B_{LPM}/B_{3dB})^{R+2}$ with B_{LPM} the local bandwidth of the LPM, R the degree of the local polynomial that is selected to be even (user choices), and B_{3dB} the 3 dB bandwidth of the resonance, which is a system property.**

***Index Terms*—frequency response function, polynomial approximation errors, nonparametric, error analysis,**

I. Introduction

The major challenge for the instrumentation and measurement society is to develop improved and new measurement techniques. Measuring the frequency response function (FRF) to characterise the dynamic behaviour of a system is an important sub-class among these problems. In this paper the focus is on the local polynomial method (LPM) [1], [2], [3] that was recently presented as a superior alternative to the widely spread and popular windowing methods [4], [5], [6] to solve that problem. All nonparametric methods suffer from leakage and noise errors. Leakage errors form a fundamental restriction for the standard methods and are present even in the absence of measurement or process noise. At a cost of an increase of the computation time, the LPM reduces the leakage errors with several orders of magnitude while the disturbing noise sensitivity remains the same as that of the standard procedures. The LPM belongs to a new family of FRF-measurement methods that got recently a lot of attention [7], [8]. Because the continuously increase of the available computer power removes the major drawback of the more calculations demanding LPM and the related methods, the authors believe that this new family of methods will become the standard in many applications where the leakage errors dominate the noise disturbances. For that reason it is extremely important to understand the errors of the LPM.

The basic idea of the LPM is to use local polynomial approximations of the transfer function and the transient behaviour of the system caused by initial condition effects, usually a polynomial of degree two is used. This finite order approximation will create systematic errors, and it is the goal of this paper to provide an upper bound on these errors. This will provide the reader with a better understanding of the underlying error mechanism, and will offer a simple rule of thumb to design the experiment. In [9], a heuristic procedure is proposed to tune one of the parameters of the LPM. Although this method provides good result, it does not give insight in the underlying structure of the problem. That will be provided in this paper and this will allow the user not only to make better use of the method, it will also provide valuable information to design the experiment.

Although this paper focusses on the LPM, the reader should be aware that the results can also be applied to other problems where a polynomial approximation of the frequency response function is needed, like for example control design [10], [11], [12]. In some design techniques, a controller is designed directly from a discrete set of frequency response function measurements that need to be interpolated. An upper bound for the approximation error is needed in these applications. The method discussed in this paper can be tuned to deliver also this information. Starting from the estimated polynomials it is also possible to estimate the maximal gain of the system with a known upper bound for the error which is extremely important in H-infinity control design [13]. However, this work will focus on bounding the errors at the frequency points where the measurements were made.

In Section II a brief introduction to the LPM is given, in Section III the upper bound on the polynomial approximation error for a lightly damped system is obtained, and extended to systems with real poles in Section IV. The theoretic results are verified in Section V, and eventually followed by the conclusions.

This paper extends the results presented in [14]. It does not only show a thorough mathematical analysis, but generalizes the results from a 2nd degree polynomial approximation to an approximation of arbitrary degree. The study is also generalized to include besides lightly damped complex poles also real poles. It turned that the latter extension doubles the requested measurement time which is very important for the practical user.

II. The local polynomial method: a brief introduction

The aim of this section is to show why the polynomial approximation problem becomes important in FRF measurements. First the system and the measurement set-up is described in Section II-A, next the basic idea is discussed very briefly in Section II-B that introduces the polynomial approximation problem, and finally the LPM is formulated as a linear-least-squares problem that is solved frequency per frequency. The latter will set the behaviour of the approximation error. The reader is refered to [1], [3] for more detailed information.

A. Set-up

Consider a discrete or continuous time single-input-single-output (SISO) system $G_0\left(q\right)$ or $G_0(s)$ respectively, with q^{-1} the backward shift operator, and s the Laplace variable. Since the focus is on the approximation errors, the impact of disturbing noise is not considered and it is assumed that the input u_0 and the output y_0 are exactly known. For example for a discrete time system, using the standard notation from the field of System Identification [15]:

$$y_0\left(t\right) = G_0\left(q\right) u_0\left(t\right). \tag{1}$$

A similar result applies also for continuous time systems. For a finite record length $t = 0, \ldots, N-1$, as it is in practical applications, this equation has to be extended with the initial conditions (transient) effects of the dynamic plant t_G:

$$y\left(t\right) = G_0\left(q\right) u_0\left(t\right) + t_G\left(t\right) \tag{2}$$

Using the discrete Fourier transform (DFT)

$$X(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t) e^{-j2\pi kt/N}, \tag{3}$$

an exact frequency domain formulation of (2) is obtained [16], [17], [18], [19]:

$$Y_0\left(k\right) = G_0\left(\Omega_k\right) U_0\left(k\right) + T_G\left(\Omega_k\right), \tag{4}$$

where the index k points to the frequency kf_s/N with f_s the sampling frequency, and $\Omega_k = e^{-j2\pi kf_s/N}$. The transfer function G_0, and the transient term T_G are both described by a rational form in the z- or s-domain (for a discrete or continuous time system respectively). Moreover, the denominator is the same for both terms. Although this is very valuable information to build parametric models, it is not used in nonparametric methods. The contributions U_0, Y_0 in (4) are an $O\left(N^0\right)$, the transient term T_G, is an $O\left(N^{-1/2}\right)$ for stationary random inputs. The results are also valid for random phase multisines where the number of excited harmonics grows proportional with N. In these expressions, $O(x)$ stands for ordo(x): a function that goes to zero at least as fast as x. It is most important for the rest of this paper to understand that (4) is an exact relation [16], [17], [18], [19] that is valid for both discrete and continuous time systems. The finite record length requires the use of a transient term in (2), and it turns out that the leakage errors of the DFT are modelled by very similar terms in the frequency domain. Because the terms $t_G\left(t\right), T_G\left(\Omega_k\right)$ are rational forms it are smooth functions of the frequency, all finite order derivatives with respect to Ω exist as long as there are no poles on the imaginary axis (continuous time system), or on the unit circle (discrete time system).

B. The basic idea of the LPM: a polynomial approximation problem

In this section a very brief introduction is given to the polynomial method. A detailed description, together with a full analysis is given in [1], [2], a comparison with the classical spectral windowing methods is found in [3].

The basic idea of the local polynomial method is very simple: the transfer function G_0 and the transient term T_G are smooth functions of the frequency so that they can be approximated in a narrow frequency band around a user specified frequency k by a low order complex polynomial. The complex polynomial parameters are estimated from the experimental data. Next $G_0\left(\Omega_k\right)$, at the central frequency k, is retrieved from this local polynomial model as the measurement of the FRF at that frequency.

C. LPM: a local linear-least-squares estimate

Start from the frequency domain expression (4). Making use of the smoothness of G_0 and T_G, the following polynomial representation holds for the frequency lines $k+r$, with $r = 0, \pm 1, \ldots, \pm n$.

$$\begin{aligned} &G_0(\Omega_{k+r}) = \\ &G_0(\Omega_k) + \textstyle\sum_{s=1}^{R} g_s(k) r^s + O(\left(\frac{r}{N}\right)^{(R+1)}) \end{aligned} \tag{5}$$

$$\begin{aligned} &T_G(\Omega_{k+r}) = \\ &T_G(\Omega_k) + \textstyle\sum_{s=1}^{R} t_s(k) r^s + N^{-\frac{1}{2}} O(\left(\frac{r}{N}\right)^{(R+1)}) \end{aligned} \tag{6}$$

Putting all parameters $G_0(\Omega_k), T_G(\Omega_k)$ and the parameters of the polynomial g_s, t_s, with $s = 1, \ldots, R$ in a column vector θ, and their respective coefficients in a row vector $K(k, r)$ allows (4) to be rewritten (neglecting the remainders) as:

$$Y(k+r) = K(k, r)\theta, \tag{7}$$

Collecting (7) for $r = -n, -n+1, \ldots, 0, \ldots, n$ finally gives

$$Y_{n,k} = K_{n,k}\theta_k, \tag{8}$$

with $Y_{n,k}, K_{n,k}$ the values of $Y(k+r), K(k, r)$, stacked on top of each other for $r = -n, -n+1, \ldots, 0, \ldots, n$. Observe that the matrix $K_{n,k}$ depends upon U_0. Solving this equation in least squares sense eventually provides the polynomial least squares estimate for $\hat{G}_{\text{poly}}(\Omega_k)$. In order to get a full rank matrix $K_{n,k}$, enough spectral lines should be combined: $n \geq R+1$. The smallest approximation error is obtained for $n = R+1$.

III. Upper bounding the polynomial approximation error of a FRF

A. Problem formulation

From the previous section, it turns out that the local polynomial approximation plays a central role in the FRF-estimation. The polynomial approximation errors will set directly the errors of the LPM in the noiseless case. These depend upon the polynomial degree R in (5) and (6), and on the selected bandwidth n in (8). For that reason it is important to bound the maximum approximation error. It will be shown that for lightly damped systems the approximation errors can be described by a single invariant of the system/set-up, given by $(B_{LPM}/B_{3dB})^{R+2}$ with B_{LPM} the local bandwidth of the LPM, R the degree of the local polynomial that is selected to be even (user choices), and B_{3dB} the 3 dB bandwidth of the resonance ω_n (where $|G(\omega)|_{dB} \geq G_{\max\, dB} - 3$, see Figure 1), which is a system property. It can be shown that for a second order system $B_{3dB} = 2\varsigma\omega_n$, with ς the damping of the system. The errors for highly damped systems that have no resonance is much smaller than those of the lightly damped systems and are not studied in this paper.

Instead of focusing on the original problem in (4), this section focusses on the underlying and more generic problem of the local approximation of a transfer function in a given frequency band by a polynomial of degree R. In that case put $T_G = 0$, and $U = 1$ in (4) so that $Y_0 = G_0$ (eventually the full problem will be discussed again). Using a similar notation as in (8), it follows that:

$$G_{n,k} = \tilde{K}_{n,k}\tilde{\theta}_k, \tag{9}$$

with $\tilde{\theta}_k^T = [G_0(\Omega_k), g_1, \ldots, g_R]$, and $\tilde{K}_{n,k}$ the corresponding frequency matrix that does no longer depend on U_0.

It is well known that the transfer function of a system can be written as the sum of a set of first order systems with complex or real poles. The complex conjugated poles can be grouped in 2^{nd} order contributions. The local approximation of the transfer function reduces to the approximation of a first order system with a complex pole, if the damping of the poles is assumed to be low enough (e.g. $\varsigma < 0.25$). This is not a hard restriction, because the polynomial approximation errors grow for a decreasing damping, so this is the worst case situation. For that reason is is assumed without great loss of generality that there are only single poles with a low damping. The situation of coinciding poles is excluded from this study. In this section all the calculations are made for simplicity on the continuous time representation of the system, but all results apply without loss of generality also for discrete time systems, as long as the damping is sufficiently small. Eventually, also the approximation error for a first order system is discussed at the end of this section.

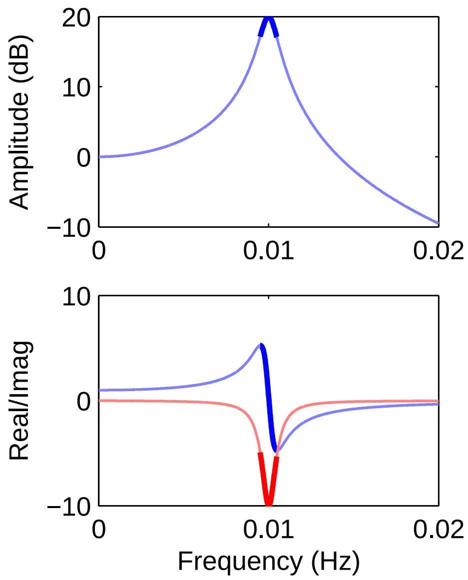

Figure 1. Behaviour of G_0 around its resonance frequency. The 3 dB bandwidth is emphasised in bold. The upper figure shows the amplitude in dB. The lower figure shows the real (blue) and imaginary (red) part of G_0.

B. Normalised second order system

Consider the normalised 2^{nd}order system with resonance frequency ω_n and damping ς :

$$G(s) = \frac{G_{DC}\omega_n^2}{s^2 + 2s\varsigma\omega_n + \omega_n^2} = \frac{b}{s-p} + \frac{\bar{b}}{s-\bar{p}}, \tag{10}$$

with $s = j\omega$ the frequency variable,

$$b = -jG_{DC}\omega_n/(2\sqrt{1-\varsigma^2}) \tag{11}$$

and

$$p = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2}. \tag{12}$$

The over-bar denotes the complex conjugate. The maximum of the transfer function is at $\omega = \omega_n\sqrt{1-2\varsigma^2}$. Around that frequency, the first term

$$\tilde{G}(s) = \frac{b}{s-p} \text{ for } s \simeq j\omega_n \tag{13}$$

in (10) dominates. This term becomes maximum at $\omega = \omega_n\sqrt{1-\varsigma^2}$. This paper focusses on systems with $\varsigma \ll 1$, so that both maxima coincide almost completely with ω_n which is called the resonance frequency $s = j\omega_n$. So the focus can be completely on (13) that can be rewritten as

$$\tilde{G}(s = j\omega) = \frac{jG_{DC}\omega_n/(2\sqrt{1-\varsigma^2})}{jw + \varsigma\omega_n - j\omega_n\sqrt{1-\varsigma^2}}. \tag{14}$$

Approximating $\sqrt{1-\varsigma^2} \approx 1$, results in

$$\tilde{G}(s = j\omega) \approx \frac{jG_{DC}\omega_n/2}{jw + \varsigma\omega_n - j\omega_n}, \tag{15}$$

and eventually, around the resonance frequency $G(j\omega) \approx \tilde{G}(j\omega)$ with

$$\tilde{G}(j\omega) = \frac{\Delta + j}{1+\Delta^2}\frac{G_{DC}}{2\varsigma}, \tag{16}$$

with $\Delta = \frac{\omega-\omega_n}{B_{3dB}/2}$, and $B_{3dB} = 2\varsigma\omega_n$. Observe that the real and imaginary part of $\tilde{G}(\Delta)$ are respectively an odd and even function of Δ. The maximum amplitude of (10) is approximately $\tilde{G}_{max} = G_{DC}/2\varsigma$ so that the previous expression can be also written as:

$$\tilde{G}(s = j\omega) = \frac{\Delta + j}{1+\Delta^2}\tilde{G}_{max}, \tag{17}$$

1) Approximation error of a polynomial least squares fit of degree R: The interest is in $\tilde{G}(s = j\omega) = \tilde{G}(\Delta = 0)$, estimated from a polynomial least squares fit in the frequency band $\tilde{\omega} \in \left[\omega_n - \frac{B_{LPM}}{2}, \omega_n - \frac{B_{LPM}}{2}\right]$, or $\Delta \in \left[-\frac{B_{LPM}}{B_{3dB}}, \frac{B_{LPM}}{B_{3dB}}\right]$. In this section the error of this fit is analysed for $\Delta = 0$.

2) Even and odd part of $\tilde{G}$: The expression in (17) can be written as:

$$\frac{\tilde{G}(\Delta)}{\tilde{G}_{max}} = \frac{\Delta + j}{1+\Delta^2} = \frac{\Delta}{1+\Delta^2} + \frac{j}{1+\Delta^2} = F_o(\Delta) + jF_e(\Delta), \tag{18}$$

with $F_o(\Delta), F_e(\Delta)$, respectively an odd and an even function of Δ. For a symmetric grid around zero, the fit of the odd part will also be an odd function of Δ, and hence it will be equal to zero for $\Delta = 0$. As a consequence, the polynomial approximation error on $F_o(\Delta)$ is equal to zero so that the focus can be completely on the fit of $F_e(\Delta)$.

3) Bounding the error on the polynomial fit of F_e: Since F_e is an even function of Δ, only even terms will appear in the polynomial least squares approximation on a symmetric frequency grid. For that reason R is set to be even, the approximation error will not be reduced by adding an additional odd term. Consider the Taylor expansion of F_e for $|\Delta| < 1$:

$$F_e(\Delta) = \sum_{k=0}^{\infty}(-1)^k\Delta^{2k}, \tag{19}$$

A fit of a polynomial of degree R will capture all the contributions up to degree R of the Taylor expansion. So the error will be dominated by the next contribution of the Taylor approximation which is of degree $R+2$, and the error E_F on the estimate of $F_e(\Delta = 0)$ is then given by

$$E_F = \alpha_{R+2}\Delta_{max}^{R+2} = \alpha_{R+2}\left(\frac{B_{LPM}}{B_{3dB}}\right)^{R+2}. \tag{20}$$

with R even, and α_{R+2} a constant. The corresponding error E_G on $\tilde{G}(\Delta)$ becomes $\tilde{G}_{max}E_F$, or

$$E_G = O(\left(\frac{B_{LPM}}{B_{3dB}}\right)^{R+2}). \tag{21}$$

4) Determining the value of α_{R+2} in (20): In order to determine the constant α_{R+2} the polynomial least squares fitting of $F_e(\Delta)$ is studied in more detail. Consider first the polynomial model $F_e(\Delta, \theta)$ of degree R (assumed to be even) for the function $F_e(\Delta)$:

$$F_e(\Delta, \theta) = \begin{bmatrix} 1 & \Delta & \cdots & \Delta^R \end{bmatrix}\theta \tag{22}$$

with $\theta \in R^{R+1}$ the vector that contains the polynomial coefficients. Similar to (8), this equation can be rewritten at the frequencies Δ_k with $k = -n, \cdots, n$, with Δ_k the k^{th} frequency on an equidistant grid in $\left[-\frac{B_{LPM}}{B_{3dB}}, \frac{B_{LPM}}{B_{3dB}}\right]$:

$$F_e = L\theta \tag{23}$$

with F_e the vector with the stacked values $F_e(\Delta_k)$, with $k = -n, \cdots, n$. Observe that all the entries in L are a priory known:

$$L(i,j) = \Delta_i^{2(j-1)}. \tag{24}$$

The least squares estimates are then given as

$$\hat{\theta} = \arg\min_\theta V, \tag{25}$$

with

$$V = \sum_{k=-n}^{n}\left(F_e(\Delta_k) - F_e(\Delta_k, \theta)\right)^2 \tag{26}$$

Since the model (22) is linear-in-the-parameters, the explicit solution of this least squares problem is known and given by:

$$\hat{\theta} = (L^TL)^{-1}L^TF_e, \tag{27}$$

Because only the estimated function at frequency zero is of interest,

$$F_e(\Delta = 0, \theta) \tag{28}$$

and the error should be known for $\Delta = 0$, it is only the coefficient of Δ^0 which is stored at $\theta(1)$ that is of importance. For that reason, the solution is not sensitive to an additional scaling of the frequency variable $\Delta = s\delta$, such that $\delta \in [-1, 1]$. This will simplify the further analysis.

All contributions up to degree R in (19) are covered by the polynomial model of degree R and hence do not contribute to the error E_F. The first term that contributes to the error E_F is the term of degree $R+2$ in $F_e(\Delta)$ (19), keeping in mind the discussion in the first part of this section on even and odd contributions. In the matrix products in (27), sums like

$$\sum_{k=-n}^{n}(\frac{k}{n})^{2r} \tag{29}$$

will appear. These sums can be numerically calculated for the actual grid which will be most accurate. Alternatively integral approximations can be used for these sums, these will be valid for n becoming large while the approximation is poor for n small (see later for more information). Using the integral approximation, it can be shown that for $R = 2$, the error is given by:

$$E_G \simeq \begin{bmatrix} 2.25 & 0 & -3.75 \end{bmatrix}\begin{bmatrix} 1/5 \\ 0 \\ 1/7 \end{bmatrix}\left(\frac{B_{LPM}}{B_{3dB}}\right)^4 G_{max}, \tag{30}$$

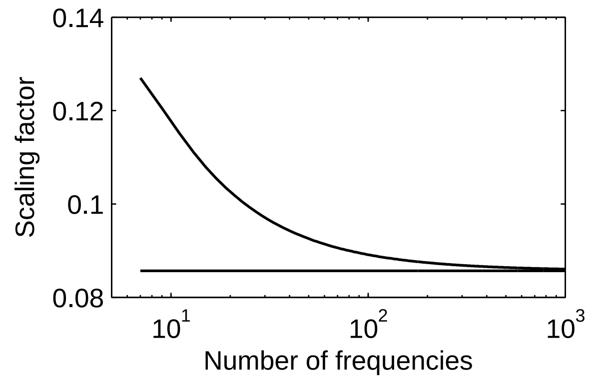

Figure 2. Evolution of the scaling factor as a function of the number of frequencies $2n+1$, for $R=2$. The horizontal line is the value obtained with the integral approximation of the sums.

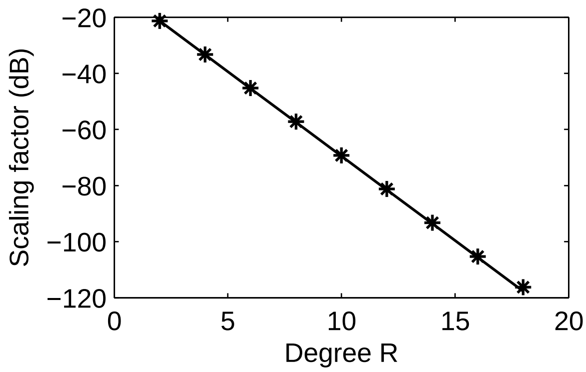

Figure 3. Asymptotic scaling factor ($n = 1000$) as a function of R.

or

$$E_G \simeq 0.0857 \left(\frac{B_{LPM}}{B_{3dB}}\right)^4 G_{max}. \tag{31}$$

It is possible to get more precise estimates for the scale factor in (31) by calculating the coefficient numerically for a given frequency grid. In practice it is advised to use the numerical expression. However, the approximative result (32) gives much more insight in the dependency of the error on the user parameters. In Figure 2, the numerically calculated factor is compared to the result from the integral approximation as a function of the number of frequencies $2n+1$. This learns that the asymptotic result obtained from the integral relations underestimates the error. For that reason it is better to use the numerical expressions to get the best results.

Using the numerical method, it is also possible to evaluate the expression for higher values of R. The results are shown in Figure 3 which reveals a linear relation between the logarithmic error and the degree R. Using a linear regression on the logarithmic error the following relation is retrieved:

$$\begin{aligned} E_G &\simeq \frac{0.3428}{2^R}\left(\frac{B_{LPM}}{B_{3dB}}\right)^{R+2} G_{max} & (32)\\ &= 1.3712\left(\frac{B_{LPM}}{2B_{3dB}}\right)^{R+2} G_{max}. & (33)\end{aligned}$$

with R even and

$$|\frac{B_{LPM}}{2B_{3dB}}| < 1, \tag{34}$$

so that the Taylor expressions that were used to obtain the error bound converge.

5) Discussion: A full characterization of the approximation error is given for a second order system with a sufficiently low damping. It is important here to observe that the invariants that are determining the maximum approximation error are: the degree R of the polynomial, the ratio B_{LPM}/B_{3dB}, and $G_{\max}$. In Section V it will be shown that with these 3 parameters it is possible to cover indeed the approximation error for different resonance frequencies ω_n, damping ς, and polynomial degrees R, as long as the damping $\varsigma < 0.25$.

IV. NORMALISED FIRST ORDER SYSTEM

In this section the results that were obtained for second order systems is transferd to first order systems. This can be done because a second order system can be considered as the sum of two first order systems with a pair of complex conjugated poles (10). The final analysis was done on one of these first order systems $\tilde{G}(s) = \frac{b}{s-p}$ in (13), the other one was neglected because it did not significantly contribute to the error. For that reason it is clear that all the results apply also to a first order system, by replacing a complex pole p by a real pole, resulting in a system with a resonance frequency at $\omega = 0$. The 3 dB bandwidth is then defined around zero, including as well the positive as the negative frequencies. From here on everything remains the same and all the results of the second order system can be carried over to the first order system. However, the reader should be aware that this change will have a significant impact on the required experiment time. By selecting the local bandwidth of the LPM symmetrically around $\omega = 0$, the Fourier coefficients of the positive and negative frequencies are each others complex conjugate and hence carry only half of the information with respect to the general situation of a second order system with a resonance frequency that is different from zero. For that reason the width of the local window should be doubled to $[-2n \cdots 2n]$ for the first order system, which results also in a doubled measurement time if the error level should remain the same.

Define the normalized first order system:

$$G_1(s) = \frac{G_{max}\omega_{3dB}}{s + \omega_{3dB}}. \tag{35}$$

Observe that the maximum amplitude of a first order system appears at $j\omega = 0$. Equation (35) can be put under the same format as that used in (17) by observing that in this case

$$\omega_n = 0 \tag{36}$$

and hence

$$B_{3dB} = \omega_{3dB} - (-\omega_{3dB}) = 2\omega_{3dB}. \tag{37}$$

The scaled frequency becomes

$$\Delta = \frac{\omega - \omega_n}{B_{3dB}/2} = \frac{\omega}{\omega_{3dB}}. \tag{38}$$

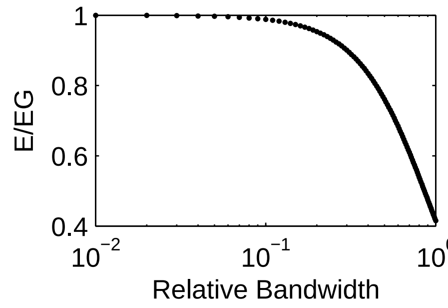

Figure 4. Evaluation of the upper bound on the normalized problem (16) as a function of the relative bandwidth B_{LPM}/B_{3dB}. The ratio of the observed maximum error over the theoretical error is shown.

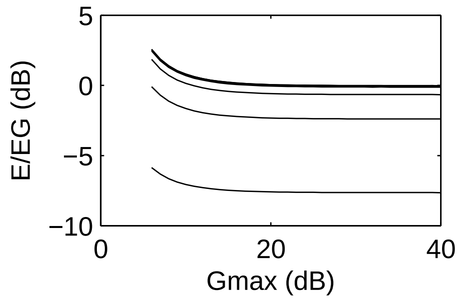

Figure 5. Evolution of the upper bound on the original problem (10) as a function of the maximal amplitude of the system, calculated for relative bandwidths 0.01, 0.025, 0.05, 0.1. 0.25, 0.5, and 1. The ratio drops for increasing relative bandwidths B_{LPM}/B_{3dB}.

Putting these results in (35),

$$G_1(s) = \frac{G_{max}\omega_{3dB}}{s+\omega_{3dB}} = \frac{1}{j\Delta+1}G_{max} = \frac{1-j\Delta}{1+\Delta^2}G_{max}. \tag{39}$$

This is completely similar to the expression equation (17) from which the analysis for second order systems started. This shows that all results can be carried over to first order systems using the normalizations and definitions of this section.

V. Verification of the upper bound

In this section the error bound (21) is verified. First the results on the normalised problem (16) is checked for varying values of B_{LPM}/B_{3dB}, next the original problem (10) is picked up again to verify the results for varying damping ς, and resonance frequencies ω_n.

A. Study of the normalised problem

This section studies the observed error for the normalised problem for $R = 2$ and a varying choice of B_{LPM}/B_{3dB}. Twenty one frequencies are equidistantly distributed in the interval [-1 1] to calculate the polynomial fit (27). The numerically obtained value for $\alpha_4 = 0.1018$ to be compared to the integral expression in (31) that was 0.0857. In Figure 4 the ratio of the actual observed error and the error predicted from the theory is shown. It can be seen that for $B_{LPM}/B_{3dB} < 0.2$, a very good agreement between the theory and the actual observations is found. For larger values, the theoretic bound is too conservative, and hence gives still a safe bound. The deviations are due to the fact that higher order terms should be included in (20) for larger values of B_{LPM}/B_{3dB}. Since the terms in the Taylor series in (19) have alternating signs, the next term that should be included will lower the theoretical error which explains the too conservative estimate for higher values of B_{LPM}/B_{3dB}. For values close to 1 the Taylor series is no longer converging. However, the reader should not conclude from that observation that the polynomial approximation does not exist.

B. Study of the original problem

Next the observed and the theoretical error is compared again, but now on the original second order system (10). The ratio B_{LPM}/B_{3dB} is varied between 0.01 and 1, the damping is varied so that G_{max} varied from 6 to 40 dB (with $G_{DC} = 1$). It can agin be concluded from Figure 5 that the bound is precise. Two deviations from the simplified theoretic bound can be seen:

- For high damping values (ς large, leading to small values of G_{max}) the theoretical bound is too small. This error is due to the fact that B_{3dB} that is estimated from the original second order system G is larger than the value that is obtained for the system $\tilde{G}$ (transfer function of a single pole) and should be used in the bound $O\left((B_{LPM}/B_{3dB})^{R+2}\right)$. Since this value is not readily available without making a parametric estimate, it is replaced it by the over estimated value leading to an under estimate of the error.

- It can also be sees that for larger bandwidths the theoretical bound is again too conservative. This was discussed in the previous section.

C. Solving the LPM problem

In this section the original problem is reconsidered. In the previous sections only the plant $G(\omega)$ was considered, here both the plant $G(\omega)$ and the transient $T_G(\omega)$ are taken into account. By adding the transient term, it is possible to remove the leakage errors in FRF-measurements. The previous results for $G(\omega)$ can also be applied to the transient term $T_G(\omega)$ because it is also a rational form with the same poles as the plant. For that reason the previous results remain valid.

When carrying over the results from Sections V-A and V-B, it should be kept in mind that all these results relied heavily on the symmetry/anti-symmetry of the real/imaginary part. In equation (8) this symmetry can be lost if the spectrum of the excitation signal is not symmetric around the centre frequency. In that case the convergence will be reduced with one order to $O\left((B_{LPM}/B_{3dB})^{R+1}\right)$. In practice it turns out that for the LPM the gain that is obtained by moving from R_{even} to $R_{even}+1$ is much smaller than that by moving from R_{even} to $R_{even}+2$. This is still due to the previous explained mechanism. As a general conclusion it can be stated that the local bandwidth B_{LPM} that is used should be (significantly) smaller than B_{3dB}.

VI. Minimum required measurement time

From the previous sections it turned out that the local bandwidth B_{LPM} is restricted by the $2B_{3dB}$. This sets immediately an under limit on the acceptable measurement time since B_{3dB} is directly linked to the measurement time:

$$\tau = 1/(\varsigma\omega_n) = 2/B_{3dB}. \tag{40}$$

Since at least $2R + 3$ frequencies are needed in the local interval B_{LPM}, and since B_{LPM} should be chosen to be smaller than $2B_{3dB}$, it follows that the measurement time T_{meas} (which is the inverse of the frequency resolution in Hz) should be larger than

$$T_{meas} > 2\pi\frac{2R+2}{2B_{3dB}} = (R+1)\,\pi\tau \tag{41}$$

in order to be in the good operational conditions to use the LPM around the resonance frequency of the system. For example, for $R = 2$, the strict minimum will be $T_{meas} = 10\tau$ (corresponding to having 7 frequency points in B_{3dB}). Although this might look to be a very long measurement time with respect to the time constant of the system, it is shown in [20], using the Fisher information, that for a flat and equidistant multisine at least 5 frequencies should be excited within the 3dB bandwidth of a second order system to avoid a significant increase of the variance on the estimated transfer function with respect to the optimal input excitation [20]. This leads to a minimum measurement time of $4\pi\tau = 13\tau$ which is in the same order of magnitude as the result in (41).

VII. Conclusions

In this paper the polynomial approximation error of a 2nd order system is analysed. This is a generic problem that appears in many modelling and measurement techniques. A theoretical bound on the approximation error was derived that results in a set of normalised numbers (order and relative bandwidth of the fit). The study provides also a lot of insight in the behaviour of the approximation errors. Using these results, it is for example possible to better understand the errors of the local polynomial method that can be used to measure the frequency response function of a dynamical system. This allows the user not only to tune better the design of the experiment, it also allows the reader to make a better choice for the user parameters in the local polynomial method. The results of the study are also translated into very practical advices for the reader to select the minimum required measurement time that is needed to get good measurments.

VIII. Acknowledgement

This work is sponsored by the Research Council of the Vrije Universiteit Brussel, the Research Foundation Flanders (FWO-Vlaanderen), the Flemish Government (Methusalem Fund METH1), and the Belgian Federal Government (Interuniversity Attraction Poles programme IAP VI/4 DYSCO).

References

[1] R. Pintelon, J. Schoukens, G. Vandersteen, and K. Barbé, Estimation of nonparametric noise and FRF models for multivariable systems - part I: theory. *Mechanical Systems and Signal Processing*, vol. 24, no. 3, pp. 573-595, 2010.

[2] R. Pintelon, J. Schoukens, and G. Vandersteen, Estimation of nonparametric noise and FRF models for multivariable systems— Part II: applications and extensions. *Mechanical Systems and Signal Processing*, Vol. 24, pp. 596-616, 2010.

[3] J. Schoukens, G. Vandersteen, K. Barbe, and R. Pintelon, Nonparametric Preprocessing in System Identification: a Powerful Tool. *European Journal of Control*, vol. 15, 260-274, 2009.

[4] J. S. Bendat, and A. G. Piersol, *Engineering Applications of Correlations and Spectral Analysis.* New York (USA):Wiley, 1980.

[5] P. E. Wellstead, Non-parametric methods of system identification, *Automatica*, vol. 17, no. 1, pp. 55-69, 1981.

[6] A.V. Oppenheim, A.S. Willsky, and S. Hamid. *Signals and Systems* (2nd Edition), Prentice Hall, 1996.

[7] T. McKelvey, G. Guillaume. Non-Parametric Frequency Response Estimation Using a Local Rational Model. *Sysid 2012, 16th IFAC Symposium on System Identification*, Brussels, paper WeA02.3, 2012.

[8] P. Hägg, and H. Hjalmarsson. Non-Parametric Frequency Function Estimation Using Transient Impulse Response Modelling. *Sysid 2012, 16th IFAC Symposium on System Identification*, Brussels, 2012.

[9] P. Thummala, J. Schoukens, "Estimation of the FRF through the improved local bandwidth selection in the local polynomial method," *IEEE Trans. Instrum. Meas.*, Accepted for publication, doi: 10.1109/TIM.2012.2196393.

[10] G. Galdos, A. Karimi, and R. Longchamp, Robust controller design by convex optimization based on finite frequency samples of spectral models. *49th IEEE conference on Decision and Control*, Atlanta (USA), December 15-17, 2010, pp. 4317-4322.

[11] A. Karimi and G. Galdos, Fixed-order Hinf controller design for nonparametric models by conves optimization, *Automatica*, vol. 46, pp. 1388-1394, 2010.

[12] T. Oomen, C.R. Rojas, H. Hjalmarsson, B. Wahlberg, Analysing iterations in identification with application to nonparametric Hinf-norm estimation. *18th IFAC World congress*, Milano (Italy), August 28 - September 2, 2011, pp. 9972-9977.

[13] S. Skogestad, and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design* (2nd ed.), Wiley, 2005.

[14] J. Schoukens, G. Vandersteen, R. Pintelon, Z. Emedi, Y. Rolain. Study of the maximal interpolation errors of the local polynomial method for frequency response function measurements. IEEE Instrumentation and Measurement conference, Graz (Austria), May 13-16, 2012.

[15] L. Ljung L. System Identification: *Theory for the User (second edition).* Prentice Hall, Upper Saddle River, New Jersey, 1999.

[16] R. Pintelon, J. Schoukens, and G. Vandersteen, Frequency domain system identification using arbitrary signals. *IEEE Trans. Automatic Control*, vol. 42, pp. 1717-1720, 1997.

[17] R. Pintelon and J. Schoukens, *System Identification. A Frequency Domain Approach.* IEEE-press, Piscataway, 2001.

[18] T. McKelvey, Frequency domain identification methods. *Circuits Systems Signal Processing*, vol. 21, pp. 39-55, 2002.

[19] J.C. Agüero, J.I. Yuz, G.C. Goodwin, and R.A. Delgado, On the equivalence of time and frequency domain maximum likelihood estimation. *Automatica*, vol. 46, pp. 260-270, 2010.

[20] E. Geerardyn, Y. Rolain, and J. Schoukens. Quasi-logarithmic Multisine Excitations for Broad Frequency Band Measurements. *I2MTC 2012, IEEE International Instrumentation and Measurement Technology Conference*, Graz, Austria, May 13-16, 2012, pp.737 - 741.