Conditional Random-Fuzzy Variables Representing Measurement Results

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I. Introduction

BAYES' THEOREM, or Bayes' rule, is a well-known principle in probability that allows one to merge a subjective degree of belief with the available evidence. It has been often used, in several branches of physics and engineering, to refine the knowledge about a particular phenomenon on the basis of the available experimental evidence [1]. In measurement applications, Bayes' theorem has been widely used to refine the *a priori* knowledge about the measurand, coming either from theoretical assumptions or previously obtained measurement results, on the basis of a new information provided by a measurement result and its associated uncertainty, so that a more accurate *a posteriori* estimate of the measurand is obtained.

Of course, Bayes' theorem is defined within the probability theory and, in its continuous formulation, it requires that both the *a priori* knowledge and the new information are described in terms of probability density functions (pdfs). Starting from these pdfs, the *a posteriori* conditional pdf can be obtained. Therefore, Bayes' theorem can be applied only when measurement results are expressed in terms of probability distributions, as suggested by the standard approach to uncertainty defined by the guide to the expression of uncertainty in measurement (GUM) [2].

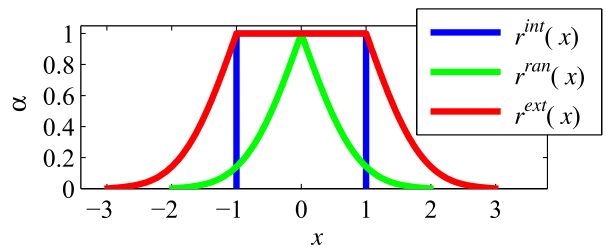

Fig. 1. Example of RFV (red + blue lines) and its PDs r^{int}, r^{ran}, and r^{ext}.

Recently, a more general approach to the expression of measurement uncertainty, based on possibility distributions (PDs), has been proposed [3]–[7]. Very briefly, following this approach, a measurement result can be represented, together with the associated uncertainty, using a random-fuzzy variable (RFV) [6], [7]. As shown in Fig. 1, an RFV is composed by two different PDs: the PD r^{int}, called internal PD, which represents all nonrandom contributions to uncertainty, including the uncompensated systematic ones, and the PD r^{ext}, called external PD, which represents all contributions to uncertainty. This last PD is obtained by combining r^{int} with the random PD r^{ran}, which represents only the random contributions to uncertainty. As shown in [8], r^{int} and r^{ran} can be built starting from the available metrological information.

An in-depth discussion about the RFV approach to uncertainty is out of the scope of this paper, since it has been widely discussed in the available literature to which the reader is addressed for further details. This paper is aimed at extending Bayes' theorem to the possibility domain, extending the already available concepts of conditional possibilities [9]–[11] to the RFVs, so that, starting from an *a priori* knowledge and a new information described in terms of RFVs, the *a posteriori* conditional RFV can be obtained. This paper is organized as follows. After having briefly recalled the definition of conditional pdfs in Section II, Section III proposes a definition of internal conditional PDs and random conditional PDs. Starting from these PDs, the definition of conditional RFVs is discussed in Section IV. Finally, a simple experimental example is considered in Section V to validate the proposed definition of conditional RFVs.

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II. Conditional Probability Distributions

The *a posteriori* conditional pdf $p_{X|Y}(x|y_m)$ represents the knowledge about the possible values of the random variable X, revised according to the knowledge of the actual value y_m assumed by the random variable Y. In mathematical terms, this pdf can be obtained by applying the well-known Bayes' theorem for continuous random variables:

$$p_X(x) \cdot p_{Y|X}(y_m|x) = p_{X,Y}(x, y_m) = p_Y(y_m) \cdot p_{X|Y}(x|y_m) \quad (1)$$

where p_X and p_Y are the marginal pdfs, $p_{Y|X}$ and $p_{X|Y}$ are the conditional pdfs, and $p_{X,Y}$ is the joint pdf. If (1) is written in a different way, the meaning of Bayes' theorem can be more easily perceived:

$$p_{X,Y}(x, y_m) = p_X(x) \cdot p_{Y|X}(y_m|x) \quad (2)$$

$$p_{X|Y}(x|y_m) = p_{X,Y}(x, y_m)/p_Y(y_m). \quad (3)$$

Starting from the *a priori* knowledge about X (p_X) and the relationship between Y and X ($p_{Y|X}$), the joint pdf $p_{X,Y}$ can be evaluated. The *a posteriori* knowledge about X ($p_{X|Y}$) can be obtained dividing the joint pdf by a normalization factor (p_Y). This normalization factor can be obtained from the joint pdf through marginalization:

$$p_Y(y_m) = \int_{-\infty}^{+\infty} p_{X,Y}(x, y_m)dx. \quad (4)$$

Therefore, the *a posteriori* conditional pdf of X can be obtained only if the joint pdf can be evaluated. The joint pdf, in turn, depends on the *a priori* knowledge about X (p_X) and the relationship between Y and X ($p_{Y|X}$).

Bayes' theorem has been employed, in the measurement science, to define a probabilistic model of the measurement process [12]. In this model, the random variables Y and X represent the measured quantity and the measurand, respectively. Moreover, p_X represents the available *a priori* knowledge about the measurand and $p_{Y|X}$ the distribution that describes the measurement process, that is, the process that generates a measured value (y_m) starting from the measurand value (x). Therefore, (2) yields the joint probability distribution and (3) the measurement result $p_{X|Y}$, that is, the *a posteriori* knowledge about measurand X [12].

III. Conditional PDs

To extend the aforementioned concepts to the possibility domain, PDs have to be considered instead of pdfs. The *a posteriori* conditional PD $r_{X|Y}(x|y_m)$ can be defined with a formally similar equation as (1)

$$T[r_X(x), r_{Y|X}(y_m|x)] = r_{X,Y}(x, y_m) = T[r_Y(y_m), r_{X|Y}(x|y_m)] \quad (5)$$

where r_X and r_Y are the marginal PDs, $r_{Y|X}$ and $r_{X|Y}$ are the conditional PDs, $r_{X,Y}$ is the joint PD, and T is a class of algebraical operators called t-norms. t-norms are basically associative functions, satisfying specific mathematical constraints [13], [14]. A detailed survey of t-norms and t-norms parametrized families can be found in [13]–[15] to which the reader is addressed for further details. The main difference between (1) and (5) is related to this class T. The algebraical operator product applied in (1) to combine the pdfs is replaced, in the possibility domain, by a general class of operators, which includes the product as a particular case, thus adding one degree of freedom to (5).

Having selected a particular t-norm T^* from class T, and following the same approach as the one applied to pdfs, (5) can be written as

$$r_{X,Y}(x, y_m) = T^*[r_X(x), r_{Y|X}(y_m|x)] \quad (6)$$

$$r_{X|Y}(x|y_m) = F^*[r_{X,Y}(x, y_m), r_Y(y_m)] \quad (7)$$

where F^* is the inverse algebraical operator of T^*, provided that the employed t-norm T^* can be inverted [16]. Similar to the pdf case, starting from the *a priori* knowledge about X (r_X) and the relationship between Y and X ($r_{Y|X}$), the joint PD $r_{X,Y}$ can be evaluated. The *a posteriori* knowledge about X ($r_{X|Y}$) can be obtained by applying the inverse operator F^* to the joint PD and the normalization factor r_Y. The normalization factor r_Y can be obtained, also in this case, as the marginal distribution of the joint PD that, in the possibility domain, can be defined as

$$r_Y(y_m) = \sup_{x \in [-\infty, +\infty]} r_{X,Y}(x, y_m). \quad (8)$$

Therefore, the *a posteriori* conditional PD of X can be obtained only if the joint PD can be evaluated. The joint PD, in turn, depends on the *a priori* knowledge about X (r_X), the relationship between Y and X ($r_{Y|X}$), and the choice of a specific t-norm T^*.

Let us consider a generic relationship between Y and X, i.e., a generic measurement model:

$$Y = f(X, U). \quad (9)$$

This model is a functional relationship between the inputs of the measurement system X and U, which represent the sets of possible measurand and possible uncertainty values, respectively, and the output of the measurement system Y, which represents the set of the possible measured quantity values.

Also in the PD case, the measurement model leads to the construction of the conditional distribution $r_{Y|X}$. $r_{Y|X}$ is the possibility that Y assumes the fixed value y_m, given that X assumes the fixed value x. When fixed values are assigned to the measurand and the measured quantity, according to the measurement model (9), also uncertainty U assumes a fixed value u. This value can be found as $u = f^{-1}(x, y_m)$, where f^{-1} is the inverse of the measurement model (9). The fact that uncertainty assumes a known value implies the possibility that $Y = y_m$, given that $X = x$, is equal to the possibility that $U = u$. Since this last possibility, namely, the uncertainty distribution r_U, is considered to be known by assumption, $r_{Y|X}$ is also known. In mathematical terms

$$r_{Y|X}(y_m|x) = r_U(u). \quad (10)$$

As far as the choice of the t-norm T^* is concerned, it depends on the nature of the uncertainty contributions represented by the PDs, as stated in [17] and [18]. Since, as briefly recalled in Section I, the internal and the random PDs represent different uncertainty contributions, different t-norms shall be considered for these two PDs. The following two sections will address separately the conditioning of internal and random PDs. For the sake of simplicity, internal PDs are conditioned assuming that no random PDs are present, and vice versa. This assumption will be removed in Section IV.

A. Internal PDs

The internal PDs represent the uncertainty contributions due to all kinds of nonrandom effects, including the uncompensated systematic ones. This has suggested [17], [19] that a minimum specificity principle (or maximum entropy principle) could be followed for the determination of the algebraical operator for their aggregation, i.e., the associated t-norm. The maximum entropy in their aggregation is obtained choosing, as T^*, the min operator [17], [19], that is a particular t-norm ($T_{\min}$) [13], [14]. Therefore, (6) becomes

$$r^{\text{int}}_{X,Y}(x, y_m) = T_{\min}\left[r^{\text{int}}_X(x), r^{\text{int}}_{Y|X}(y_m|x)\right]. \tag{11}$$

The conditional PD $r^{\text{int}}_{X|Y}$ can be obtained by applying, in (7), the inverse algebraical operator $F_{\min}$ [9] of $T_{\min}$:

$$\begin{aligned} r^{\text{int}}_{X|Y}(x|y_m) &= F_{\min}\left[r^{\text{int}}_{X,Y}(x, y_m), r^{\text{int}}_Y(y_m)\right] \\ &= \begin{cases} r^{\text{int}}_{X,Y}(x, y_m) & \text{if } r^{\text{int}}_Y(y_m) > r^{\text{int}}_{X,Y}(x, y_m) \\ \left[r^{\text{int}}_{X,Y}(x, y_m), 1\right] & \text{if } r^{\text{int}}_Y(y_m) = r^{\text{int}}_{X,Y}(x, y_m). \end{cases} \end{aligned} \tag{12}$$

According to (12), $r^{\text{int}}_{X|Y}$ is determined only if the joint PD is dominated by r^{int}_Y or it is equal to one. To avoid this ambiguity, the least specific solution of (12) has to be considered, as proposed in [10]

$$r^{\text{int}}_{X|Y}(x|y_m) = \begin{cases} r^{\text{int}}_{X,Y}(x, y_m) & \text{if } r^{\text{int}}_Y(y_m) > r^{\text{int}}_{X,Y}(x, y_m) \\ 1 & \text{if } r^{\text{int}}_Y(y_m) = r^{\text{int}}_{X,Y}(x, y_m). \end{cases} \tag{13}$$

This solution has the advantage of providing a normalized conditional PD ($\sup_x r^{\text{int}}_{X|Y}(x|y_m) = 1$) for every measured value y_m. Interestingly, if (13) is applied with $r^{\text{int}}_Y(y_m) = 0$, i.e., when the measured value is considered impossible due to an incompatible *a priori* knowledge, a totally uninformative $r^{\text{int}}_{X|Y}(x|y_m) = 1$ for every x value is obtained.[1]

B. Random PDs

The random PDs represent the uncertainty contributions due to the sole random effects. This has suggested [18]–[20] that a maximum specificity principle (or minimum entropy principle) should be followed in the determination of the algebraical operator for their aggregation, i.e., the associated t-norm. The minimum entropy in their aggregation is obtained choosing, as T^* in (6), a t-norm selected among the Frank parametric family [13], [15], such that (6) becomes

$$\begin{aligned} r^{\text{ran}}_{X,Y}(x, y_m) &= T^F_\gamma\left[r^{\text{ran}}_X(x), r^{\text{ran}}_{Y|X}(y_m|x)\right] \\ &= \log_\gamma\left(1 + \frac{(\gamma^{r^{\text{ran}}_X(x)} - 1)\cdot(\gamma^{r^{\text{ran}}_{Y|X}(y_m|x)} - 1)}{\gamma - 1}\right) \end{aligned} \tag{14}$$

where γ is the parameter value ($\gamma \neq [0, 1, +\infty]$). The optimal $\gamma = 0.05$ value to correctly aggregate random contributions to uncertainty has been proposed in [18] and is here employed.

The conditional PD $r^{\text{ran}}_{X|Y}$ can be obtained by applying, in (7), the inverse algebraical operator F^F_γ of the Frank t-norm family:

$$\begin{aligned} r^{\text{ran}}_{X|Y}(x|y_m) &= F^F_\gamma\left[r^{\text{ran}}_{X,Y}(x, y_m), r^{\text{ran}}_Y(y_m)\right] \\ &= \log_\gamma\left(1 + \frac{\left(\gamma^{r^{\text{ran}}_{X,Y}(x,y_m)} - 1\right)\cdot(\gamma - 1)}{\gamma^{r^{\text{ran}}_Y(y_m)} - 1}\right). \end{aligned} \tag{15}$$

In this case, (15) cannot be applied with $r^{\text{ran}}_Y(y_m) = 0$, i.e., when the measured value is considered impossible according to the *a priori* knowledge. Similar to the case of internal PDs, an uninformative $r^{\text{ran}}_{X|Y}(x|y_m) = 1$ for every x value shall be assumed when $r^{\text{ran}}_Y(y_m) = 0$.

IV. Conditional RFVs

When random and nonrandom effects are simultaneously present in the *a priori* estimate and in the new measurement result, single PDs are no longer adequate to represent the *a posteriori* estimate, and conditional RFVs shall be considered. Conditional RFVs are those distributions expressing the *a posteriori* possibility that the measurand assumes a specific value due to nonrandom contributions only (conditional internal PD) and all contributions (conditional external PD). For the sake of clarity, the *a posteriori* conditional PDs obtained in the presence of both random and nonrandom effects will be labeled as $\hat{r}_{X|Y}$.

The *a posteriori* external PD can be found by combining the *a posteriori* random and internal PDs. The combination can be more easily explained starting from a particular interpretation of the random and internal PDs: the distribution $\hat{r}^{\text{ran}}_{X|Y}(x|y_m)$ represents the *a posteriori* possibility that $X = x$ due to the random contributions only and, assuming that it is unimodal,[2] its peak value identifies the *a posteriori* mode value

$$x^* = \arg\max_x \hat{r}^{\text{ran}}_{X|Y}(x|y_m) \tag{16}$$

Therefore, x^* is the x value for which $\hat{r}^{\text{ran}}_{X|Y}(x|y_m)$ attains its maximum value. On the other hand, the distribution $\hat{r}^{\text{int}}_{X|Y}(x'|y_m)$ represents the *a posteriori* possibility that the mode value x^* of $\hat{r}^{\text{ran}}_{X|Y}$ is shifted, due to the nonrandom contributions, such that the new mode value is x', i.e., that the final *a posteriori* distribution is $\hat{r}^{\text{ran}}_{X|Y}(x - x' + x^*|y_m)$. Starting from these considerations, the *a posteriori* external PD can be found as

$$\hat{r}^{\text{ext}}_{X|Y}(x|y_m) = \sup_{x'}\; T_{\min}\left[\hat{r}^{\text{ran}}_{X|Y}(x - x' + x^*|y_m),\; \hat{r}^{\text{int}}_{X|Y}(x'|y_m)\right] \tag{17}$$

i.e., for each x value, the *a posteriori* external possibility is the sup, on all new mode values x', of $\hat{r}^{\text{ran}}_{X|Y}$ centered on x', limited by the possibility that x' is the new mode value.

[1] This means, in the possibility framework, that no *a posteriori* knowledge about x is provided by the measurand value y_m, so that a PD representing total ignorance is consistently obtained [7].

[2] In the following, only unimodal random distributions are considered, since they represent the most frequent cases in measurement. The extension of this method to multimodal distributions is still an open issue.

For the sake of simplicity, (17) is written with simplified notation as

$$r_{x|y_m} = \sup_{x'} T_{\min}[r_{(x|y_m)|x'},\ r_{x'|y_m}]. \tag{18}$$

According to this formulation, the *a posteriori* external PD $r_{x|y_m}$ is the marginal distribution over x of the joint PD $r_{(x,x')|y_m}$. The minimum specificity principle can be applied also in this case, so that the min t-norm has been selected to aggregate $r_{(x|y_m)|x'}$ and $r_{x'|y_m}$ [14]. In this case, this assumption is justified because the internal and random PDs are originated by independent uncertainty contributions and, therefore, they are independent.

According to (17), the *a posteriori* external PD, and consequently the conditional RFV, can be obtained starting from the *a posteriori* random and internal PDs. As proposed in [21], the simplest, though not always strict way to obtain the *a posteriori* random and internal PDs, in the presence of both random and nonrandom effects, is to assume that random and internal PDs can be conditioned separately, following the equations discussed in the previous section. Therefore

$$\begin{aligned}\hat{r}^{\text{ran}}_{X|Y}(x|y_m) &= r^{\text{ran}}_{X|Y}(x|y_m)\\ \hat{r}^{\text{int}}_{X|Y}(x|y_m) &= r^{\text{int}}_{X|Y}(x|y_m).\end{aligned} \tag{19}$$

Of course, this assumption is valid until the conditioning of the internal PDs is not influenced by the presence of the random PDs, and vice versa, i.e., until there is a predominant uncertainty source between random and nonrandom effects.

When random and nonrandom effects lead to comparable uncertainty contributions, random and internal PDs cannot be conditioned separately and a different method shall be devised for the determination of the *a posteriori* random and internal PDs. A more general method can be obtained reinterpreting the initial PDs of the conditioning process, r^{ran}_X and $r^{\text{ran}}_{Y|X}$, as done in (17). Due to r^{int}_X, the mode of r^{ran}_X can be shifted in the new mode values x'. For a given x' value, the distribution $r_{x|x'}$ can be defined as

$$r_{x|x'} = r^{\text{ran}}_X(x - x' + x^*). \tag{20}$$

Moreover, due to $r^{\text{int}}_{Y|X}$, the mode of $r^{\text{ran}}_{Y|X}$ can be shifted in the new mode values x', y'. For given x', y' values, a distribution $r_{(y_m|x)|(x',y')}$ can be defined as

$$r_{(y_m|x)|(x',y')} = r^{\text{ran}}_{Y|X}(y_m - y' + y^*|x - x' + x^*). \tag{21}$$

Therefore, for fixed new mode values x' and y', $r_{x|x'}$ and $r_{(y_m|x)|(x',y')}$ still represent random PDs and, therefore, they can be conditioned independently from the internal PDs. The *a posteriori* random PD, given x' and y', can be obtained by applying the conditioning method for the random part discussed in Section III-B. Applying the conditioning rules, the *a posteriori* random PD, given x' and y', becomes

$$r_{(x|y_m)|(x',y')} = r^{\text{ran}}_{X|Y}(x - x' + x^*|y_m - y' + y^*) \tag{22}$$

i.e., the new mode values x' and y' produce a simple translation of the *a posteriori* random PD. In this respect, the shape of the *a posteriori* random PD is not influenced by the presence of the internal PD, but only the position of its mode value is affected. The *a posteriori* position of the mode, given x' and y', can be defined, similar to (16), as

$$x^*_{\text{pos}} = \arg\max_x r_{(x|y_m)|(x',y')}. \tag{23}$$

Since the shape of $r_{(x|y_m)|(x',y')}$ is not influenced by x', y', the final *a posteriori* random PD $\hat{r}^{\text{ran}}_{X|Y}$ can be defined as $r_{(x|y_m)|(x',y')}$ for arbitrary x', y' values. As proved by (17), a different position of $\hat{r}^{\text{ran}}_{X|Y}$ does not affect the *a posteriori* external PD, and the final *a posteriori* RFV. In other words, the *a posteriori* mode values of $\hat{r}^{\text{ran}}_{X|Y}$ are imposed by the distribution $\hat{r}^{\text{int}}_{X|Y}$. Therefore, it can be assumed as

$$\hat{r}^{\text{ran}}_{X|Y}(x|y_m) \triangleq r_{(x|y_m)|(x'=x^*,y'=y^*)} = r^{\text{ran}}_{X|Y}(x|y_m). \tag{24}$$

Equation (24) states that the conditioning of the random PDs, originated by random contributions only, is never influenced by the presence of the internal PDs, i.e., by the presence of nonrandom contributions to uncertainty.

On the other hand, the *a posteriori* internal PD $\hat{r}^{\text{int}}_{X|Y}$ represents the *a posteriori* distribution of the mode. Therefore, for given x', y' values, this PD is influenced both by the *a posteriori* position of the mode (x^*_{pos}) and the possibility that x^*_{pos} is the actual *a posteriori* mode value. Since x^*_{pos} depends on x', y', this possibility corresponds to the *a posteriori* possibility of x', y' ($r_{(x',y')|y_m}$). Starting from these considerations, the *a posteriori* internal PD can be found as

$$\hat{r}^{\text{int}}_{X|Y}(x|y_m) = \sup_{(x',y')|x=x^*_{\text{pos}}} r_{(x',y')|y_m} \tag{25}$$

i.e., the *a posteriori* internal PD is the sup of the *a posteriori* possibility of x^*_{pos}, on all x', y' values which lead to x^*_{pos}. Equation (25) states that, in general, the conditioning of the internal PDs is always influenced by the presence of the random PDs that affect the *a posteriori* position of the mode and the presence of internal PDs that affect the *a posteriori* possibility of the mode.

The *a posteriori* possibility of x', y' can be defined by applying once again the conditioning process described in Section III-A. The joint PD among y_m, x', and y' can be found, according to (11), as

$$r_{y_m,x',y'} = T_{\min}[r_{y_m|(x',y')},\ r_{(x',y')}]. \tag{26}$$

Since, for a given y', the possibility of the measured value y_m is not affected by the position of the mode value along x, it is possible to write

$$r_{y_m|(x',y')} = r_{y_m|y'}. \tag{27}$$

Similar to (18), once the mode y' of $r_{y_m|y'}$ is assumed, only random contributions affect the possible values taken by y_m about y. Hence

$$r_{y_m|y'} = r^{\text{ran}}_Y(y_m - y^* + y'). \tag{28}$$

On the other hand, similar to (18), the *a priori* possibility of x', y' can be simply obtained as

$$r_{(x',y')} = r^{\text{int}}_{X,Y}(x', y'). \tag{29}$$

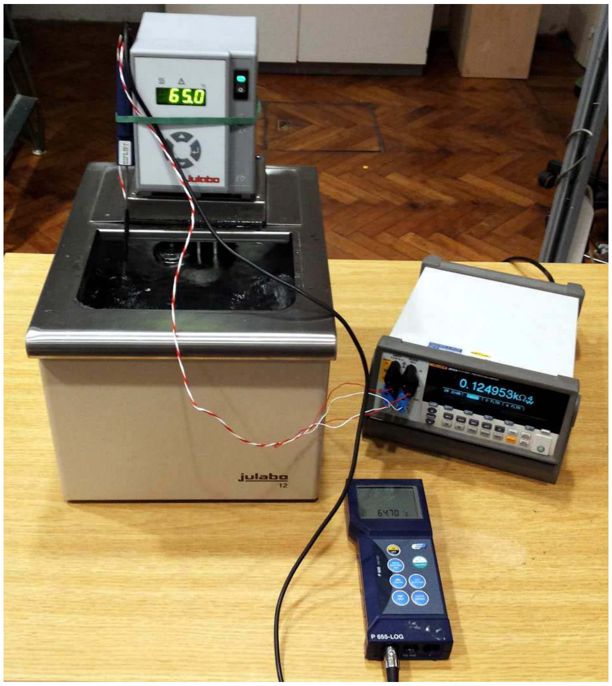

Fig. 2. Picture of the experimental setup.

Therefore, the joint PD $r_{y_m,x',y'}$ in (26) becomes, due to (28) and (29)

$$r_{y_m,x',y'} = T_{\min}\left[r_Y^{\text{ran}}(y_m - y^* + y'),\ r_{X,Y}^{\text{int}}(x', y')\right]. \quad (30)$$

Finally, starting from the marginal distribution of the joint PD $r_{y_m,x',y'}$ that is given by $r_{y_m} = \sup_{x',y'} r_{y_m,x',y'}$, the *a posteriori* possibility of x', y' can be obtained, according to (13), as

$$r_{(x',y')|y_m} = F_{\min}[r_{y_m,x',y'},\ r_{y_m}]. \quad (31)$$

In conclusion, it is worth noting that, despite the complexity of the whole mathematical derivation, the practical implementation of (17), (24), and (25) is relatively simple.

V. Measurement Example

In this section, the proposed method for the definition of the *a posteriori* conditional RFVs is applied to a practical measurement example: a temperature measurement based on a common platinum resistance temperature detector (RTD) (Pt-100). The measured temperature is the temperature of the water contained in a bath tank, which is maintained stable during the measurement process using a Julabo 12-ED heating circulator. A picture of the experimental setup is shown in Fig. 2. The temperature set point of the heating circulator, together with all available information about the accuracy and stability of the temperature control of the circulator, represents the *a priori* knowledge about water temperature. In this respect, the accuracy range of the temperature control is estimated as $\Delta_\theta = \pm 0.01 \cdot \theta_{\text{set}}$ °C, where θ_{set} is the temperature set point, and the standard deviation quantifying the stability is $\sigma_\theta = 0.03$ °C, according to the experimental variability of the temperature measured by the temperature control. Considering Δ_θ as the quantification of nonrandom contributions to uncertainty and σ_θ as the quantification of random contributions, the *a priori* internal and random PDs associated to temperature θ can be built. Fig. 3 shows the *a priori* internal and external PDs r_θ for the set point $\theta_{\text{set}} = 55$ °C.

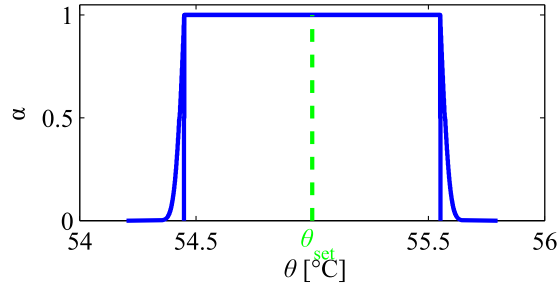

Fig. 3. $\theta_{\text{set}} = 55$ °C. *a priori* PDs r_θ (blue lines).

Water temperature is measured also using the RTD with known manufacturer specifications. For this sensor, the measurement model specifies the functional relationship between temperature θ (the measurand) and resistance R (the measured quantity):

$$R = R_0 \cdot \left[1 + a\theta + b\theta^2\right] \quad (32)$$

where typical values for R_0, a, and b in the temperature range $0\ °\text{C} \le \theta < 850\ °\text{C}$ are $R_0 = 100\ \Omega$, $a = 3.9083 \cdot 10^{-3}\ °\text{C}^{-1}$, and $b = -5.775 \cdot 10^{-7}\ °\text{C}^{-2}$, respectively. For the case $\theta_{\text{set}} = 55$ °C, a value $R_m = 121.16\ \Omega$ has been measured, which corresponds to a temperature value $\theta_m = 54.56$ °C.

Of course, uncertainty U is associated to the resistance measured value, and this should be made explicit in the measurement model (32) to evaluate the conditional PDs correctly. Therefore, the measurement model shall be written in terms of RFVs as

$$\tilde{R} = R_0 \cdot [1 + a\tilde{\theta} + b\tilde{\theta}^2] + \tilde{U} \quad (33)$$

where $\tilde{R}$ is the RFV representing the possible measured value of the RTD resistance, $\tilde{\theta}$ is the RFV representing the possible values of the measurand, and $\tilde{U}$ is the RFV representing the possible values of uncertainty contributions. $\tilde{U}$ can be obtained starting from the specifications of the RTD and of the multimeter used for the resistance measurement.

In particular, it is known that the RTD is in class A ($\Delta_\theta = \pm(0.15 + 0.002 \cdot \theta)$ °C) up to 70 °C and in class B ($\Delta_\theta = \pm(0.3 + 0.005 \cdot \theta)$ °C) for higher temperatures. Moreover, the resistance value of the RTD is measured using a Fluke 8845A digital multimeter in a four-wire configuration. The accuracy range of the multimeter is $\Delta_R = \pm(0.01 + 0.0001 \cdot R_m)\ \Omega$. Starting from the accuracy range Δ_θ of the RTD and Δ_R of the multimeter, the internal PD associated with uncertainty U can be found. To do so, the accuracy range Δ_θ has to be expressed as an interval in the resistance scale exploiting (32), and it has to be added to Δ_R [8].

Moreover, during the test, the experimental variability of the resistance measured values has been evaluated and found to be $\sigma_R = 7 \cdot 10^{-3}\ \Omega$. Starting from the experimental variability σ_R, the random PD associated with uncertainty U can be found [20].

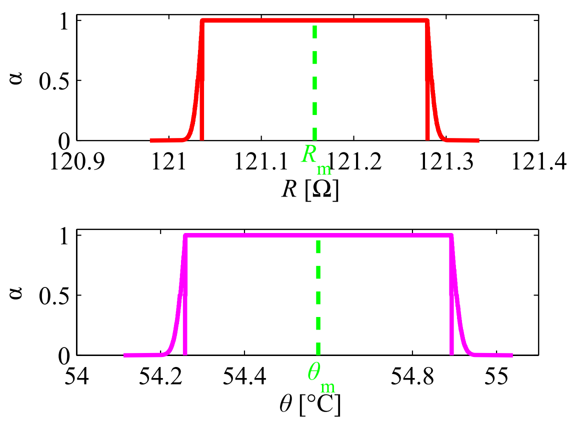

Fig. 4. $\theta_{\text{set}} = 55$ °C. Top plot: PDs $r_{R|\theta_m}$ (red lines). Bottom plot: PDs $r_{\theta|\theta_m}$ (magenta lines).

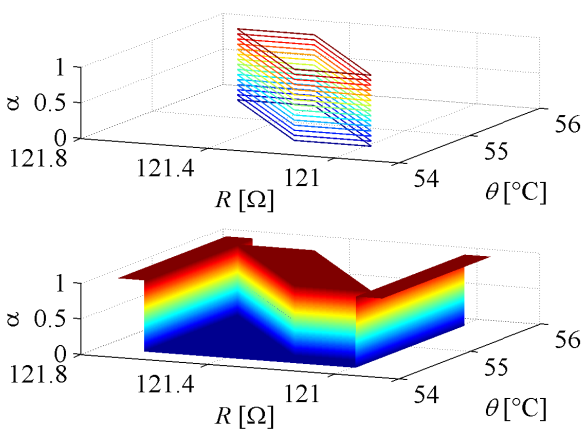

Fig. 5. $\theta_{\text{set}} = 55$ °C. Top plot: joint PD $r^{\text{int}}_{\theta,R}$. Bottom plot: *a posteriori* conditional PD $r^{\text{int}}_{\theta|R}$.

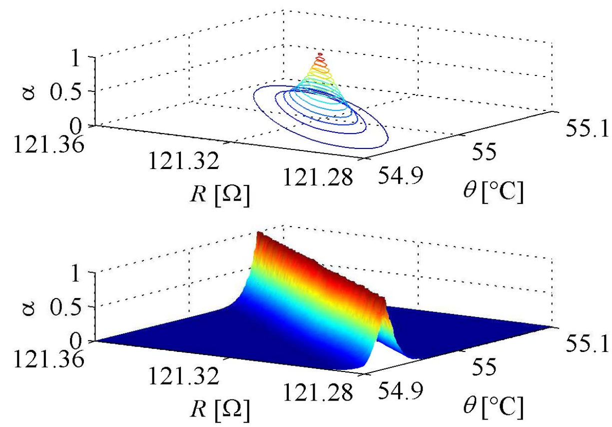

Fig. 6. $\theta_{\text{set}} = 55$ °C. Top plot: joint PD $r^{\text{ran}}_{\theta,R}$. Bottom plot: *a posteriori* conditional PD $r^{\text{ran}}_{\theta|R}$.

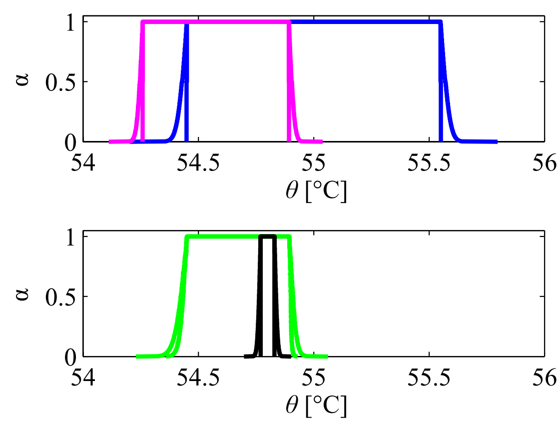

Fig. 7. $\theta_{\text{set}} = 55$ °C. Top plot: *a priori* PDs r_θ (blue lines) and PDs $r_{\theta|\theta_m}$ (magenta lines). Bottom plot: *a posteriori* PDs $\hat{r}_{\theta|R_m}$ (green lines) and PDs $r_{\theta_{\text{ref}}}$ (black lines).

As discussed in Section III, the internal and random PDs $r_{R|\theta}$ can be obtained starting from the internal and random PDs associated with uncertainty, considering the measurement model. The PDs $r_{R|\theta}$ are shown in the upper plot of Fig. 4 for a single value $\theta = \theta_m$. If these PDs are plotted on the θ axis, exploiting again (32), the PDs $r_{\theta|\theta_m}$ shown in the lower plot of Fig. 4 are obtained. These last PDs, which express the estimate of θ only due to the new measurement result, are not necessary for the conditioning process, but are anyway shown here for a better clarification of this process.

Starting from the conditional PD $r^{\text{int}}_{R|\theta}$ and the *a priori* PD r^{int}_{θ}, according to the procedures defined in Section III-A, it is possible to obtain the joint PD $r^{\text{int}}_{\theta,R}$ through (11) and the *a posteriori* conditional PD $r^{\text{int}}_{\theta|R}$ through (13). These distributions are shown in Fig. 5. On the other hand, starting from the conditional PD $r^{\text{ran}}_{R|\theta}$ and the *a priori* PD r^{ran}_{θ}, according to the procedures defined in Section III-B, it is possible to obtain the joint PD $r^{\text{ran}}_{\theta,R}$ through (14) and the *a posteriori* conditional PD $r^{\text{ran}}_{\theta|R}$ through (15). These distributions are shown in Fig. 6.

In this way, the nonrandom and random contributions are initially conditioned separately. However, as discussed in Section IV, to find the final *a posteriori* PDs $\hat{r}_{\theta|R_m}$ composing the *a posteriori* RFV, the influence of the random PDs on the conditioning of the internal PDs has to be considered. In this respect, the *a posteriori* PD $\hat{r}^{\text{int}}_{\theta|R_m}$ can be found by applying (25), and the *a posteriori* PD $\hat{r}^{\text{ran}}_{\theta|R_m}$ can be found by applying (24).

The resulting *a posteriori* PDs, composing the final *a posteriori* RFV, are shown in the lower plot of Fig. 7 (green lines). The *a posteriori* RFV is compared with the *a priori* RFV (blue lines) and with the RFV given by the measurement result (magenta lines) in the upper plot. It can be seen that the measured value refines the *a priori* knowledge about temperature reducing both the nonrandom and random contributions in the *a posteriori* estimate of θ, i.e., excluding those temperature values that *a posteriori* are no longer possible.

The measurement result is therefore represented by the whole green RFV in the lower plot of Fig. 7, which provides all possible intervals, at all possible confidence levels $(1 - \alpha)$, where the temperature value is supposed to lie. If a single interval is desired, as in an usual measurement report, a single α-cut of this RFV can be provided instead

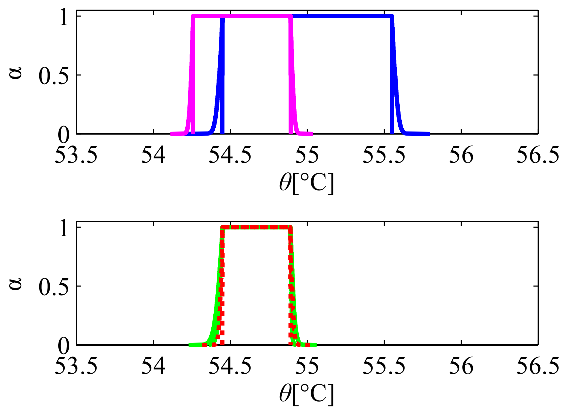

Fig. 8. $\theta_{\text{set}} = 55$ °C. Top plot: *a priori* PDs r_θ (blue lines) and PDs $r_{\theta|\theta_m}$ (magenta lines). Bottom plot: *a posteriori* PDs $\hat{r}_{\theta|R_m}$ (green lines) and $r_{\theta|R_m}$ (red-dashed lines).

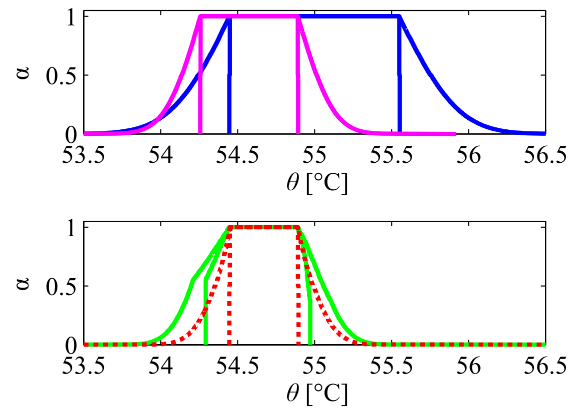

Fig. 9. $\theta_{\text{set}} = 55$ °C. All σ values have been multiplied by 10. Top plot: *a priori* PDs r_θ (blue lines) and PDs $r_{\theta|\theta_m}$ (magenta lines). Bottom plot: *a posteriori* PDs $\hat{r}_{\theta|R_m}$ (green lines) and $r_{\theta|R_m}$ (red-dashed lines).

of the whole variable. For instance, 95% confidence interval is the α-cut for $\alpha = 0.05$ and, in this case, would provide $\theta = (54.7 \pm 0.29)$ °C.

The *a posteriori* RFV is also compared with an RFV (black lines) provided by a Dostmann P 655-LOG high precision temperature probe, featuring an accuracy range $\Delta_{\theta_{\text{ref}}} = \pm 0.03$ °C and showing an experimental variability with $\sigma_{\theta_{\text{ref}}} = 0.01$ °C. According to the comparison, as expected, the *a posteriori* RFV is a better estimate of water temperature than the *a priori* RFV and the RFV representing the measurement result.

Fig. 7 shows also that, due to the presence of the random PDs, the resulting *a posteriori* internal PD has not rectangular shape. Of course, since in this case, the random PDs are much narrower than the internal PDs, their effect of modifying the shape of the *a posteriori* internal PD is almost negligible. Therefore, in this case, the *a posteriori* internal and random PDs can be also determined using the simplified approach (19), whose assumption is that random and internal PDs can be conditioned separately. The obtained *a posteriori* PDs $r_{\theta|R_m}$ are shown in the lower plot of Fig. 8, where they are also compared with the PDs $\hat{r}_{\theta|R_m}$. It can be observed that $r_{\theta|R_m}$ and $\hat{r}_{\theta|R_m}$ are almost overlapping. However, if the same comparison is performed starting, for instance, from experimental σ values multiplied by 10 with respect to the original σ values, as shown in Fig. 9, the difference between the simplified and the general approaches for the determination of the *a posteriori* conditional RFV becomes evident: when the internal and random PDs show similar magnitudes, the simplified approach, already proposed in [21], may lead to an underestimate of uncertainty in the *a posteriori* conditional RFV.

It is worth noting that, despite the shape of the internal PD of the conditional RFV can be affected by the original random PDs, it still represents the effects of the nonrandom contributions only. As graphically shown by the green and red-dashed internal PDs in the lower plot of Fig. 9, the proposed mathematical method may provide an internal PD (green line) larger than the narrower one (red-dashed line) obtained when random and nonrandom PDs are processed independently. However, the conditional internal PD is always limited [by definition (25)] by the max between the internal

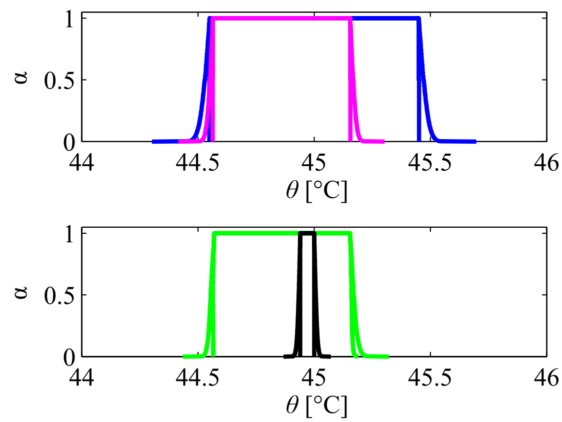

Fig. 10. $\theta_{\text{set}} = 45$ °C. Top plot: *a priori* PDs r_θ (blue lines) and PDs $r_{\theta|\theta_m}$ (magenta lines). Bottom plot: *a posteriori* PDs $\hat{r}_{\theta|R_m}$ (green lines) and PDs $r_{\theta_{\text{ref}}}$ (black lines).

a priori (blue line in Fig. 9) and measured (magenta line in Fig. 9) PDs. This confirms that the resulting conditional internal PD preserves its nature of representing nonrandom effects only. Even if it is not strictly necessary to represent the *a posteriori* measurement result, which is globally represented by the external PD, it is nevertheless necessary if this result has to be employed in a subsequent conditioning process.

The *a posteriori* conditional RFV has been determined for different temperature set points of the heating circulator. As an example, Fig. 10 shows the resulting RFVs for $\theta_{\text{set}} = 45$ °C. This figure simply shows that, when the conditioning process is applied starting from a noninformative *a priori* RFV, the *a posteriori* RFV does not provide any additional information with respect to the RFV given by the measurement result.

Figs. 11 and 12 show the resulting RFVs for $\theta_{\text{set}} = 75$ °C. In Fig. 11, it has been erroneously assumed that for $\theta_{\text{set}} = 75$ °C, the employed RTD is in class A. The figure shows that, starting from erroneous information about temperature, the conditioning process may lead again to an underestimation of uncertainty in the *a posteriori* RFV. The nonrandom contributions are excessively compensated in the *a posteriori* RFV which turns out to be incompatible with the reference

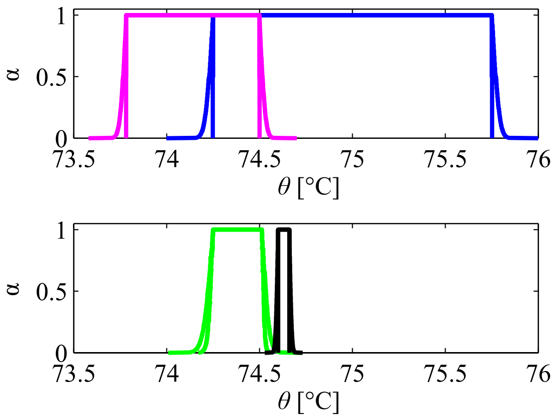

Fig. 11. $\theta_{\text{set}} = 75$ °C. Pt-100 is assumed in class A. Top plot: *a priori* PDs r_θ (blue lines) and PDs $r_{\theta|\theta_m}$ (magenta lines). Bottom plot: *a posteriori* PDs $\hat{r}_{\theta|R_m}$ (green lines) and PDs $r_{\theta_{\text{ref}}}$ (black lines).

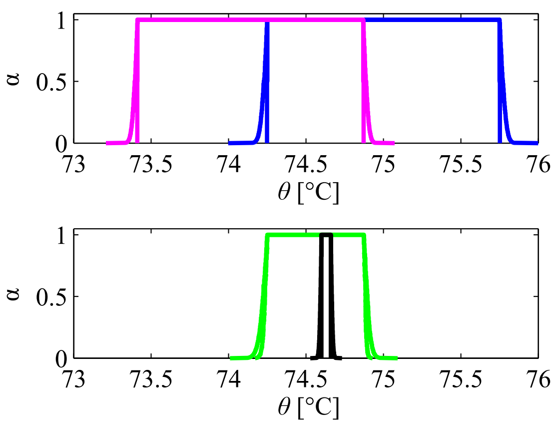

Fig. 12. $\theta_{\text{set}} = 75$ °C. Pt-100 is assumed in class B. Top plot: *a priori* PDs r_θ (blue lines) and PDs $r_{\theta|\theta_m}$ (magenta lines). Bottom plot: *a posteriori* PDs $\hat{r}_{\theta|R_m}$ (green lines) and PDs $r_{\theta_{\text{ref}}}$ (black lines).

temperature PD $r_{\theta_{\text{ref}}}$. On the other hand, assuming correctly that for $\theta_{\text{set}} = 75$ °C, the employed RTD is in class B as in Fig. 12, the conditioning process leads to an effective *a posteriori* temperature estimate.

VI. Conclusion

This paper has presented an extension to the possibility domain of the well-known Bayes' theorem, so that, starting from the RFV expressing the *a priori* knowledge and the RFV associated with a new measurement result, the *a posteriori* conditional RFV can be obtained. In particular, different conditional PDs have been obtained for the internal and random PDs composing an RFV, under the assumption that only nonrandom and random effects, respectively, are present. After removing this assumption, it has been shown that, in general, internal and random PDs cannot be conditioned separately, since the *a posteriori* internal PD is influenced also by the presence of random effects.

The proposed method for the determination of the *a posteriori* conditional RFV has been also applied to a simple experimental example to prove its usefulness and effectiveness. The results have shown that, starting from reliable *a priori* knowledge about the measurand, it is possible to obtain a more accurate *a posteriori* estimate of the measurand. They have also shown the effect of the random PDs on the shape of the *a posteriori* internal PD: the more comparable the random PDs with respect to the internal PDs, the more severe the effect. Moreover, the results have confirmed that the simplified approach for the determination of the *a posteriori* conditional RFV is a valid approach when nonrandom contributions to uncertainty are predominant with respect to random contributions, and vice versa.

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