Simplified Modeling and Identification of Nonlinear Systems Under Quasi-Sinusoidal Conditions

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Abstract— This paper proposes a simplified Volterra model able to represent the steady-state behavior of nonlinear systems in quasi-sinusoidal conditions. A wide class of nonlinear systems can be modeled using the conventional Volterra approach, but as the order of nonlinearity or the memory length increases, the number of coefficients grows exponentially, thus making the identification of the Volterra model troublesome. By considering a system whose input is a periodic signal containing a main frequency component which is much higher than the others, it is possible to drastically reduce the number of coefficients of its frequency-domain Volterra model without affecting the model accuracy. The proposed technique is particularly suitable to represent the behavior of the electrical devices connected to the ac mains, since they typically operate in quasi-sinusoidal conditions. In particular, its application to voltage and current transducers takes on great importance in the field of instrumen-tation and measurement, since it allows overcoming their usual characterization. Thanks to the proposed model, dynamics and nonlinearities can be considered simultaneously, while avoiding the complexity usually associated with the conventional Volterra approach. For example, the proposed technique is applied to model a Hammerstein system, which is often employed to represent the behavior of electrical devices, and the results are deeply discussed.

Index Terms— Nonlinear systems, nonlinearities, power quality, quasi-sinusoidal conditions, system identification, testing, transducers, Volterra series.

I. Introduction

It is well known that all physical systems are inherently nonlinear to some extent. The study and the identification of nonlinear systems are somewhat complicated and nonintuitive, especially when compared with the straightforwardness of the well-known theory of linear time-invariant (LTI) systems [1]. For this reason, whenever is possible, scientists and engineers try to approximate nonlinear phenomena using linear models [2], [3]. In some cases, the weakness of the nonlinear behavior for a given range of the input signals makes this approximation acceptable with respect to the target accuracy. Unfortunately, in other cases, the aforementioned approach cannot be applied, for example, when the required accuracy cannot be met or the nonlinear effects have to be investigated, and therefore, nonlinear modeling necessarily has to be faced. Knowledge about the system allows one to write a set of equations containing a (limited) set of parameters to be determined that is able to describe its behavior with enough detail. Often, this is not possible due to the lack of information about the physical insight of the system, therefore it has to be treated as a black box, namely, as a mere (nonlinear) relation between the input and the output [4]. Among the several nonparametric approaches to nonlinear system identifi-cation [4]–[6], one of the most widely used is the discrete-time Volterra series [7], named after the mathematician Volterra [8]. A finite-order Volterra system is defined by its kernels, and each of them consists of a set of coefficients whose number rapidly grows with the kernel order and the memory length [9]; therefore, identifying a Volterra model means computing this large number of coefficients.

Although techniques capable of substantially reducing the number of coefficients have been proposed in [10] and [11], in usual applications, only low-order Volterra models are employed; otherwise, this number can still be extremely high and the model identification process critical.

On the other hand, one of the advantages of the Volterra approach lies in the fact that using the multidimensional Fourier transform, it is possible to derive a frequency domain representation [7], [12]. This formulation becomes particularly interesting when just the steady-state response to a multitone excitation has to be computed, since it leads to a consistent reduction in the number of coefficients. Moreover, it should be noticed that some nonlinear systems are fed with an input signal containing a frequency component whose amplitude is much larger than the others. In this paper, this property will be exploited to further reduce the number of coefficients characterizing the nonlinear model. Its identification can be carried out by applying proper quasi-sinusoidal input signals and using the ordinary least squares estimator. The accuracy of the proposed simplified model will be discussed through an example.

Quasi-sinusoidal conditions are typical of the electrical components connected to the ac mains. Therefore, the proposed method is particularly suitable for the modeling of these devices. For example, the approach can be used to characterize the behavior of current and voltage transducers in the presence of harmonic distortion. Together with a medium voltage generator previously developed in [13]–[15], the method allows a model-driven evaluation of the metrological performance of

Manuscript received June 14, 2015;
revised September 18, 2015;
accepted October 21, 2015.
Date of publication February 11, 2016;
date of current version May 10, 2016.
The Associate Editor coordinating the review process was Dr. John Lataire.

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instrument transformers. In other words, it permits overcoming the conventional approach based on ratio and angle errors and on the measurement of the frequency response function, which are not applicable in the presence of distorted input and nonlinear effects.

In Section II, the general theory of the Volterra models is briefly recalled, and in Section III-A, the proposed simplification is presented. Then, Section III-B reports the procedure to identify the coefficients of the model, which is applied to a simple nonlinear system in Section IV in order to discuss its accuracy.

II. Volterra Approach

Nonlinear time-invariant (NTI) systems are often modeled by following the Volterra series approach. According to this approach, the input/output relation for continuous single-input and single-output (SISO) homogeneous NTI systems can be expressed as

$$\begin{aligned} y(t) &= \sum_{i=1}^{\infty} y^i(t) \\ y^i(t) &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h^i(\tau_1, \ldots, \tau_i) x(t-\tau_1) \ldots x(t-\tau_i) \\ &\quad \times d\tau_1 \ldots d\tau_i \end{aligned} \tag{1}$$

where x and y are the system input and output, respectively. According to (1), y is the sum of an infinite number of terms y^i. Each y^i is the contribution of a nonlinear homogeneous subsystem of order i, and it can be evaluated by considering the generalized convolution integral of the same order. The core of the generalized convolution integral is the function $h^i(\tau_1, \ldots, \tau_i)$, which is called ith order Volterra kernel. This function can be considered as a higher order impulse response of the system. Since the integrals in (1) are not bounded, from a theoretical point of view, the memory length of each kernel is infinite.

In this paper, the focus is on the modeling of causal and discrete NTI systems. The causality is a common property of all physical systems. On the other hand, the interest in discrete systems is justified by the fact that inputs and outputs are measured by means of analog-to-digital converters (ADCs), and therefore, their values are known only for discrete time instants.

From a practical point of view, when (1) is employed to model NTI systems, only a finite number of y^i contributions can be considered, i.e., order i has to be upper limited to a finite number I. Moreover, since each kernel of order i has to be of finite length, the memory length also has to be upper limited to a finite number K. Starting from these considerations, the following truncated Volterra series is obtained:

$$\begin{aligned} y(n) &= \sum_{i=1}^{I} y^i(n) \\ y^i(n) &= \sum_{k_1=0}^{K-1} \cdots \sum_{k_i=0}^{K-1} h^i(k_1, \ldots, k_i) x(n-k_1) \ldots x(n-k_i). \end{aligned} \tag{2}$$

According to (2), the input/output relation is fully represented by I Volterra kernels of finite length and therefore, by a finite number of coefficients. The total number of coefficients c_{tot} can be found [9] as

$$c_{\text{tot}} = \sum_{i=0}^{I} K^i. \tag{3}$$

A significant drawback of the Volterra representation is highlighted in (3). The total number of coefficients c_{tot} grows rapidly (exponentially) as I and K increase. As a result, a huge number of coefficients has to be handled even for limited values of I and K.

Of course, not all of these coefficients are independent. The number of independent coefficients can be obtained by considering that every Volterra system can be represented by means of infinite kernels, but by a unique set of symmetric kernels h^i_{sym}. A Volterra kernel of order i is said to be symmetric when its value is invariant under permutations of its indices $k_1, \ldots, k_i$. Therefore, when h^i_{sym} is employed, the number of independent coefficients for each order i is given by the i-combination with repetitions of K elements [9], such that the total number of independent coefficient is

$$c_{\text{tot,ind}} = \sum_{i=0}^{I} \frac{(K+i-1)!}{(K-1)!i!}. \tag{4}$$

Even if $c_{\text{tot,ind}}$ is lower than c_{tot}, it may result in an extremely large number. For this reason, the identification of Volterra model (2) is, in most circumstances, very complex, unless low orders and short memory length are considered [10], [16]. This limits the practical application of the approach to mildly nonlinear systems.

III. Proposed Simplified Volterra Model

A. Model Definition

Under suitable assumptions, the Volterra model (2) can be drastically simplified by reducing the number of coefficients. The first one concerns the periodicity of the input signal x. When the input signal is periodic, with fundamental angular frequency ω_0, computing the steady-state response of the system is relatively straightforward by considering the frequency domain expression of (2). Assuming that the highest significant harmonic component in the spectrum of the input signal $X(jn\omega_0)$ is in position $n = N$, a generic harmonic component $Y(jm\omega_0)$ of the output signal can be found as

$$\begin{aligned} Y(m) &= \sum_{i=1}^{I} Y^i(m) \\ Y^i(m) &= \sum_{-N \le n_1, \ldots, n_i \le N} H^i_{\text{sym}}(n_1, \ldots, n_i) X(n_1) \ldots X(n_i) \end{aligned} \tag{5}$$

where

$$n_1 + \cdots + n_i = m. \tag{6}$$

The compact notation $Y(m)$ and $X(n)$ is used in (5) in place of $Y(jm\omega_0)$ and $X(jn\omega_0)$, where m and n are the harmonic orders of the output and input spectral components, respectively. H^i_{sym} corresponds to the i-D Fourier transform of the symmetric Volterra kernel of order i, and it is often called generalized frequency response function (GFRF).

According to (5), the mth harmonic component of the output is the sum of the contributions of I subsystems of order i. For each order i, the subsystem contribution is given by all possible intermodulation products between the input harmonic components $X(n_1)\dots X(n_i)$, such that $n_1 + \cdots + n_i = m$, weighted by the GFRF $H^i_{\text{sym}}(n_1,\dots,n_i)$.

Following (5), the number of coefficients for each order i is again given by the i-combination with repetitions of $M = 2N + 1$ elements, such that the total number of coefficient is again given by (4), where K has now to be replaced with M. This is the main advantage of the frequency domain approach (5). In fact, in the time domain approach (2), the memory length K has to be selected according to the system dynamics, which can also be very slow with respect to the sampling frequency. On the other hand, in the frequency domain approach (5), the number of harmonic components M has to be selected considering the harmonic content of the input signal, which in most practical cases is considerably lower than K. It follows that the number of coefficients needed to compute the steady-state response in the frequency domain is much smaller than that required in the time domain.

Approach (5) can be further simplified by considering that, in this paper, it is applied to NTI systems fed with an input signal containing a frequency component whose amplitude is much larger than the others, i.e., matching the quasi-sinusoidal assumption.[1] This is the case of the electrical devices connected to the mains and, in particular, current and voltage transducers. Let us assume that the fundamental $X(1)$ and its complex conjugate $X(-1)$ are the greatest harmonic components of the input signal. In this case, for a generic order i, all the intermodulation products that consider more than once a harmonic component different from $X(1)$ and $X(-1)$ give a negligible contribution to the output y, considering (5). In other words, the contribution of a specific intermodulation product of order i is significant only if $i - 1$ harmonic components correspond to the fundamental component $X(1)$ or to its complex conjugate $X(-1)$. Following these considerations, the input/output relation (5) up to the ith order can be found by considering i_p times the fundamental positive harmonic component $X(1)$, i_m times the fundamental negative component $X(-1)$, and just once a generic component $X(n)$:

$$
\begin{aligned}
Y(m) &= \sum_{i=1}^{I} Y^i(m) \\
Y^i(m) &= \sum_{-N\le n\le N} H^i_{\text{sym}}\Bigg(\underbrace{1,\dots,1}_{i_p},\underbrace{-1,\dots,-1}_{i_m},n\Bigg) \\
&\quad \times X(1)^{i_p} X(-1)^{i_m} X(n)
\end{aligned}
\tag{7}
$$

[1]At this stage, only a qualitative definition of the considered quasi-sinusoidal assumption can be provided. A quantitative definition will be provided *a posteriori*, at the end of this section.

where due to (6) and considering the ith order

$$
\begin{cases} i_p - i_m + n = m \\ i_p + i_m + 1 = i. \end{cases} \tag{8}
$$

For the sake of simplicity, it is advisable to express H^i_{sym} as a function of numbers i_p and i_m as

$$
H^i_{\text{sym}}\Bigg(\underbrace{1,\dots,1}_{i_p},\underbrace{-1,\dots,-1}_{i_m},n\Bigg) = H^i_{\text{sym}}(i_p, i_m, n). \tag{9}
$$

According to (7), a further reduction in the number of coefficients is obtained, which is now M times the $(i-1)$-combination with repetitions of elements 1 and -1

$$
\begin{aligned}
c'_{\text{tot}} &= \sum_{i=0}^{I} M \frac{(2+(i-1)-1)!}{(2-1)!(i-1)!} = M \sum_{i=0}^{I} i = M\frac{I(I+1)}{2} \\
&= M \cdot L.
\end{aligned}
\tag{10}
$$

Not all of these coefficients are independent due to the complex conjugate symmetry of the two spectra X and Y. Even neglecting these symmetries, by comparing (10) with (4), it can be immediately concluded that under specific assumptions on the spectrum of the input signal, a drastic reduction in the number of coefficients has been obtained.

To further clarify the proposed approach, a specific nonlinearity order I has to be chosen; in this paper, $I = 5$ is considered. It is worth noting that this choice has no theoretical implications about the validity of the proposed method. In fact, all the following equations can be reformulated considering a different order I. Having thus set the nonlinearity order, (7) can be expressed with a compact matrix equation

$$
Y(m) = \mathbf{X}(m)\mathbf{H}(m) \tag{11}
$$

where

$$
\mathbf{X}(m) = \begin{bmatrix}
X(m) \\ \hdashline
X(1)X(m-1) \\
X(-1)X(m+1) \\ \hdashline
X^2(1)X(m-2) \\
X^2(-1)X(m+2) \\
X(1)X(-1)X(m) \\ \hdashline
X^3(1)X(m-3) \\
X^3(-1)X(m+3) \\
X^2(1)X(-1)X(m-1) \\
X(1)X^2(-1)X(m+1) \\ \hdashline
X^4(1)X(m-4) \\
X^4(-1)X(m+4) \\
X^3(1)X(-1)X(m-2) \\
X(1)X^3(-1)X(m+2) \\
X^2(1)X^2(-1)X(m)
\end{bmatrix}^T
$$

$$
\mathbf{H}(m) = \begin{bmatrix} H^1_{\mathrm{sym}}(0,0,m) \\ \hdashline H^2_{\mathrm{sym}}(1,0,m-1) \\ H^2_{\mathrm{sym}}(0,1,m+1) \\ \hdashline H^3_{\mathrm{sym}}(2,0,m-2) \\ H^3_{\mathrm{sym}}(0,2,m+2) \\ H^3_{\mathrm{sym}}(1,1,m) \\ \hdashline H^4_{\mathrm{sym}}(3,0,m-3) \\ H^4_{\mathrm{sym}}(0,3,m+3) \\ H^4_{\mathrm{sym}}(2,1,m-1) \\ H^4_{\mathrm{sym}}(1,2,m+1) \\ \hdashline H^5_{\mathrm{sym}}(4,0,m-4) \\ H^5_{\mathrm{sym}}(0,4,m+4) \\ H^5_{\mathrm{sym}}(3,1,m-2) \\ H^5_{\mathrm{sym}}(1,3,m+2) \\ H^5_{\mathrm{sym}}(2,2,m) \end{bmatrix}. \tag{12}
$$

According to (11), all the output harmonic components $Y(m)$ are generated by independent sets of intermodulation products of the input harmonics $X(n)$. Since the indices of the input harmonics are limited, by assumption, in the range $-N \le n \le N$, the indices of the output harmonics generated according to (11) are limited in the range $-N-4 \le m \le N+4$. However, not all of these harmonic components have to be evaluated. In fact, by exploiting the complex conjugate symmetry of the spectrum Y, it is sufficient to evaluate $Y(m)$ in the subinterval $0 \le m \le N+4$.

In general, for the fifth-order Volterra model (11), the value of a single harmonic component $Y(m)$ is provided by 15 coefficients H_{sym}. However, due to the assumption of the input spectrum $X(n)$ being limited in the range $-N \le n \le N$, vectors $\mathbf{X}(m)$ and $\mathbf{H}(m)$ have no full length for some indices m. For example, let us consider the last harmonic component produced by the considered Volterra system: $m = N+4$. For this component, (11) reduces to

$$
\begin{aligned} Y(N+4) &= \mathbf{X}(N+4)\mathbf{H}(N+4) \\ &= X^4(1)X(N)H^5_{\mathrm{sym}}(4,0,N). \end{aligned} \tag{13}
$$

In other words, the last harmonic component is produced by the single intermodulation product of fifth order which considers four times the fundamental input harmonic $X(1)$ and one time the last input harmonic $X(N)$, and therefore, the single coefficient $H^5_{\mathrm{sym}}(4,0,N)$ is associated with this harmonic component.

Moreover, for some m values, the coefficients H^i_{sym} may be mutually dependent. In fact, considering the invariance of H^i_{sym} to the permutation of the harmonic indices, four constraints can be identified for the fifth-order Volterra model

$$
\begin{aligned} H^3_{\mathrm{sym}}(1,1,1) &= H^3_{\mathrm{sym}}(2,0,-1) \\ H^4_{\mathrm{sym}}(2,1,1) &= H^4_{\mathrm{sym}}(3,0,-1) \\ H^5_{\mathrm{sym}}(3,1,1) &= H^5_{\mathrm{sym}}(4,0,-1) \\ H^5_{\mathrm{sym}}(2,2,1) &= H^5_{\mathrm{sym}}(3,1,-1). \end{aligned} \tag{14}
$$

These constraints can be easily identified since the intermodulation products corresponding to the H^i_{sym} pairs in (14) are equal. To eliminate the dependence, these intermodulation products have to appear once and, therefore the H^i_{sym} coefficients in the right side of (14) are excluded from the model. By excluding these coefficients, the shortened vectors $\mathbf{X}^{\mathrm{red}}(m)$ and $\mathbf{H}^{\mathrm{red}}(m)$ are obtained, and thus, (11) becomes

$$
Y(m) = \mathbf{X}^{\mathrm{red}}(m)\mathbf{H}^{\mathrm{red}}(m). \tag{15}
$$

Other constraints can be identified for the output dc component $Y(0)$. Being y a real output signal, $Y(0)$ has to be a real number. Therefore, the contribution of each order in (11) for $m=0$ has to be real. By imposing this condition and considering the complex conjugate symmetry of the spectrum X, some constraints are obtained for coefficients H^i_{sym} for $m=0$. In order to identify these constraints, $\mathbf{X}(0)$ and $\mathbf{H}(0)$ have to be rewritten as real vectors

$$
Y(0) = \mathbf{X}^0\mathbf{H}^0 \tag{16}
$$

where

$$
\mathbf{X}^0 = \begin{bmatrix} X(0) \\ \hdashline 2X(1)X(-1) \\ \hdashline 2\,\mathrm{Re}(X^2(1)X(-2)) \\ -2\,\mathrm{Im}(X^2(1)X(-2)) \\ X(1)X(-1)X(0) \\ \hdashline 2\,\mathrm{Re}(X^3(1)X(-3)) \\ -2\,\mathrm{Im}(X^3(1)X(-3)) \\ 2X^2(1)X^2(-1) \\ \hdashline 2\,\mathrm{Re}(X^4(1)X(-4)) \\ -2\,\mathrm{Im}(X^4(1)X(-4)) \\ 2\,\mathrm{Re}(X^3(1)X(-1)X(-2)) \\ -2\,\mathrm{Im}(X^3(1)X(-1)X(-2)) \\ X^2(1)X^2(-1)X(0) \end{bmatrix}^T
$$

$$
\mathbf{H}^0 = \begin{bmatrix} H^1_{\mathrm{sym}}(0,0,0) \\ \hdashline H^2_{\mathrm{sym}}(1,0,-1) \\ \hdashline \mathrm{Re}(H^3_{\mathrm{sym}}(2,0,-2)) \\ \mathrm{Im}(H^3_{\mathrm{sym}}(2,0,-2)) \\ H^3_{\mathrm{sym}}(1,1,0) \\ \hdashline \mathrm{Re}(H^4_{\mathrm{sym}}(3,0,-3)) \\ \mathrm{Im}(H^4_{\mathrm{sym}}(3,0,-3)) \\ H^4_{\mathrm{sym}}(2,1,-1) \\ \hdashline \mathrm{Re}(H^5_{\mathrm{sym}}(4,0,-4)) \\ \mathrm{Im}(H^5_{\mathrm{sym}}(4,0,-4)) \\ \mathrm{Re}(H^5_{\mathrm{sym}}(3,1,-2)) \\ \mathrm{Im}(H^5_{\mathrm{sym}}(3,1,-2)) \\ H^5_{\mathrm{sym}}(2,2,0) \end{bmatrix}. \tag{17}
$$

Thus, (15) and (16) represent the proposed approach for the characterization of nonlinear systems with quasi-sinusoidal input. Starting from the definition of this approach, it is now possible to give a more quantitative definition of the considered assumption of quasi-sinusoidal input signal. Considering a given target accuracy in the estimated output spectrum, the input signal is quasi-sinusoidal when the output spectrum predicted by the proposed model differs from that of the complete Volterra model by less than the target accuracy.

B. Model Identification

In order to characterize nonlinear systems by means of (11), $\mathbf{H}^{\text{red}}(m)$ and $\mathbf{H}^0$ have to be identified by inverting (15) and (16). Considering a single input signal x and its response y, these expressions are heavily underdetermined since the (unitary) ranks of $\mathbf{X}^{\text{red}}(m)$ and $\mathbf{X}^0$ are much smaller than the maximum lengths of $\mathbf{H}^{\text{red}}(m)$ and $\mathbf{H}^0$, i.e., the maximum number of coefficients that have to be identified. For the considered fifth-order model, this number L is 15; for a generic order I, $L = I \cdot (I+1)/2$, in accordance with (10).

On the contrary, a proper identification of the coefficients requires that the ranks of $\mathbf{X}^{\text{red}}(m)$ and $\mathbf{X}^0$ are greater than the lengths of $\mathbf{H}^{\text{red}}(m)$ and $\mathbf{H}^0$. This can be obtained by measuring the responses $y_1^{\text{act}}, \ldots, y_L^{\text{act}}$ to a suitable set of at least L input signals $x_1, \ldots, x_L$. Of course, models (15) and (16) need to be rewritten considering the multiple input signals

$$\mathbf{Y}_{\text{id}}(m) = \mathbf{X}_{\text{id}}^{\text{red}}(m)\mathbf{H}^{\text{red}}(m) \tag{18}$$

$$\mathbf{Y}_{\text{id}}(0) = \mathbf{X}_{\text{id}}^{0}\mathbf{H}^{0} \tag{19}$$

where $\mathbf{X}_{\text{id}}^{\text{red}}(m)$, $\mathbf{Y}_{\text{id}}(m)$ and $\mathbf{X}_{\text{id}}^0$, $\mathbf{Y}_{\text{id}}(0)$ are constructed by concatenating the respective vectors obtained for a single signal

$$\mathbf{X}_{\text{id}}^{\text{red}}(m) = \begin{bmatrix} \mathbf{X}_1^{\text{red}}(m) \\ \mathbf{X}_2^{\text{red}}(m) \\ \vdots \\ \mathbf{X}_L^{\text{red}}(m) \end{bmatrix} \quad \mathbf{Y}_{\text{id}}(m) = \begin{bmatrix} Y_1(m) \\ Y_2(m) \\ \vdots \\ Y_L(m) \end{bmatrix} \tag{20}$$

$$\mathbf{X}_{\text{id}}^{0} = \begin{bmatrix} \mathbf{X}_1^{0} \\ \mathbf{X}_2^{0} \\ \vdots \\ \mathbf{X}_L^{0} \end{bmatrix} \quad \mathbf{Y}_{\text{id}}(0) = \begin{bmatrix} Y_1(0) \\ Y_2(0) \\ \vdots \\ Y_L(0) \end{bmatrix}. \tag{21}$$

Supposing that the considered input signals are chosen so that $\mathbf{X}_{\text{id}}^{\text{red}}(m)$ and $\mathbf{X}_{\text{id}}^0$ are, at least, of full rank L, the coefficients $\mathbf{H}^{\text{red}}(m)$ and $\mathbf{H}^0$ can be estimated using the least square approach, i.e., by computing the Moore–Penrose pseudoinverse of $\mathbf{X}_{\text{id}}^{\text{red}}(m)$ and $\mathbf{X}_{\text{id}}^0$. This corresponds to minimize, for the different spectral components m (including $m = 0$), the quantity

$$N^2(m) = \|\mathbf{Y}_{\text{id}}(m) - \mathbf{Y}^{\text{act}}(m)\|^2 \tag{22}$$

where $\mathbf{Y}^{\text{act}}(m)$ is concatenated as $\mathbf{Y}_{\text{id}}(m)$ in (20) and (21), but starting from the spectra $Y_1^{\text{act}}, \ldots, Y_L^{\text{act}}$ of the actual responses $y_1^{\text{act}}, \ldots, y_L^{\text{act}}$, while $\| \cdot \|$ denotes the l^2-norm of the complex vector.

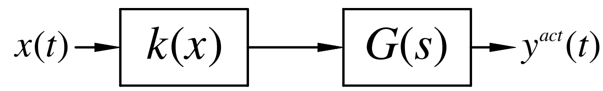

Fig. 1. Block diagram of a Hammerstein system.

TABLE I
COEFFICIENTS OF THE POLYNOMIAL FUNCTION $k(x)$

α_0	α_1	α_2	α_3	α_4	α_5
0	1.3	0.04	-0.07	0.001	0.002

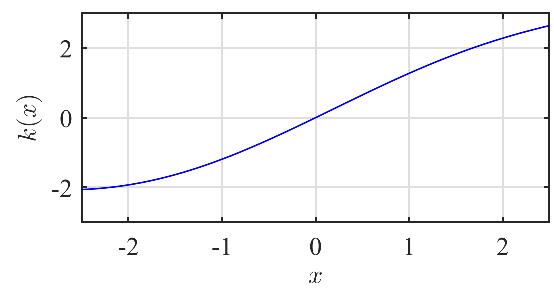

Fig. 2. Polynomial function $k(x)$.

To prove the effectiveness of approach (11) and its identification procedure, in Section IV, the characterization of a simple nonlinear dynamic system is considered.

IV. EXAMPLE

In Section III, a simplified approach allowing the modeling of nonlinear SISO systems driven by quasi-sinusoidal signals has been proposed. In this section, the technique is applied to compute the frequency-domain model of a simple nonlinear dynamic system in these conditions. A Hammerstein model is considered here, since it is often used to represent many nonlinear systems, such as power converters, electrical machines, transducers, and ADCs [17]–[20]. This model is composed by the cascade of a static nonlinearity and of an LTI system [21], as shown in Fig. 1. The proposed approach is here applied to a Hammerstein system whose static nonlinearity is represented by a fifth-order polynomial function k of the input x

$$k(x) = \alpha_5 x^5 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 \tag{23}$$

whose coefficients are reported in Table I. The values of $k(x)$ in the considered input range are plotted in Fig. 2.

Furthermore, let us suppose that the dynamic part of the system is represented by a transfer function $G(s)$ characterized by three poles and a zero

$$G(s) = \frac{g \cdot \left(\frac{s}{\omega_z} + 1\right)}{\left(\frac{s}{\omega_{p1}} + 1\right) \cdot \left[\left(\frac{s}{\omega_{p2}}\right)^2 + \frac{2\xi}{\omega_{p2}} s + 1\right]}. \tag{24}$$

Its parameters can be found in Table II and its magnitude and phase response are shown in Fig. 3.

According to Section III-A, when the considered input signal is quasi-sinusoidal, the Hammerstein model can be represented following the proposed approach (11). It can be proved [7] that when the static nonlinearity of the

TABLE II

PARAMETERS OF THE TRANSFER FUNCTION $G(s)$

ω_{p1} [rad/s]	ω_{p2} [rad/s]	ξ	ω_z [rad/s]	g
50π	600π	0.15	200π	1

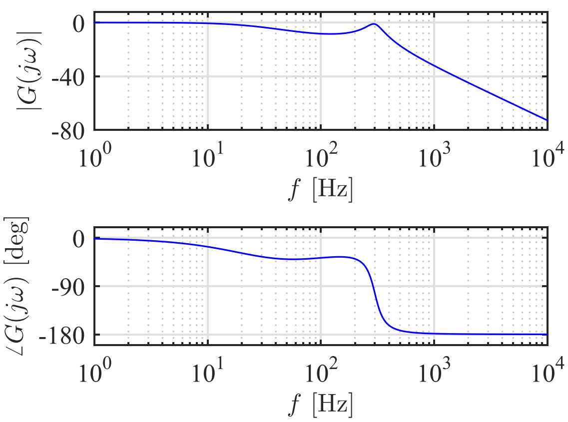

Fig. 3. Magnitude and phase response of $G(s)$.

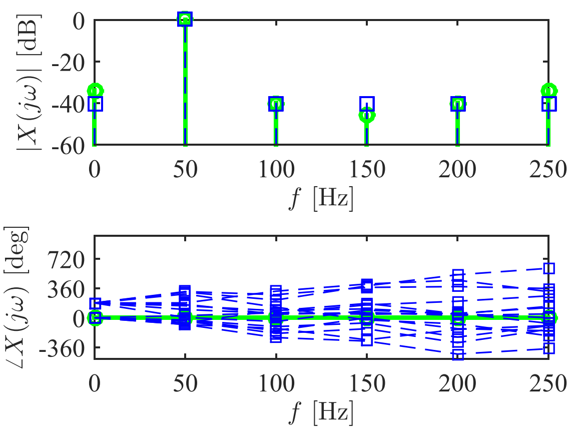

Fig. 4. Magnitude and phase of input identification signals (blue dashed lines) and input test signal (green lines).

Hammerstein model consists in a Ith order polynomial function, the model is exactly represented by a Ith order Volterra system. Therefore, an order $I = 5$ is first chosen. Moreover, according to Section III-B, this system can be identified by means of 15 independent signals. Of course, the power spectra of the identification signals should match the quasi-sinusoidal assumption. A possible set of identification signals is shown in Fig. 4 (blue dashed lines). It consists of 15 signals that have been obtained by imposing that all harmonic components have the same magnitude, equal to a fraction (1%) of the fundamental amplitude, and random phases, generated according to a uniform probability density function in the range $[0, 2\pi]$.

Starting from the system responses to the identification signals, the coefficients $\mathbf{H}^{\mathrm{red}}(m)$ and $\mathbf{H}^0$ can be estimated in a least square sense. In order to quantify the goodness of this estimate, the following definitions of residual errors can be considered:

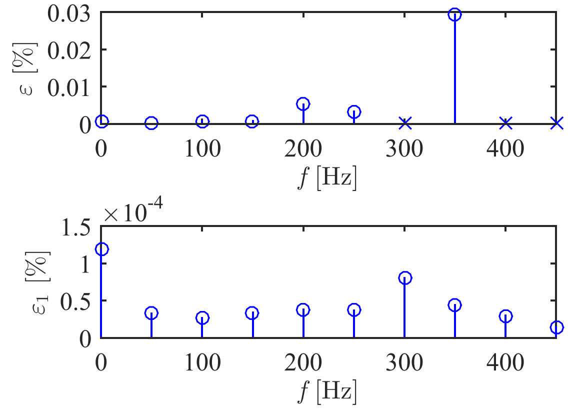

Fig. 5. Residuals ε and ε_1.

$$\varepsilon(m) = \frac{\|\mathbf{Y}_{\mathrm{id}}(m) - \mathbf{Y}^{\mathrm{act}}(m)\|}{\|\mathbf{Y}^{\mathrm{act}}(m)\|} \quad \varepsilon_1(m) = \frac{\|\mathbf{Y}_{\mathrm{id}}(m) - \mathbf{Y}^{\mathrm{act}}(m)\|}{\|\mathbf{Y}^{\mathrm{act}}(1)\|}. \tag{25}$$

That is, ε represents the residuals normalized over the magnitude of each harmonic component and ε_1 represents the residuals normalized over the magnitude of the fundamental harmonic component. These errors are shown in Fig. 5. Of course, ε is not evaluated for negligible harmonic components of the output signal, i.e., with amplitude lower than one thousandth of the fundamental component (-60 dB). These harmonics fall in positions $m = 6, 8, 9$ corresponding to $f = 300, 400, 450$ Hz, having considered a fundamental frequency $f_0 = 50$ Hz. The maximum ε value is 0.03%, obtained for the seventh harmonic. For the same component, ε_1 is significantly lower than ε, meaning that this component has an extremely small amplitude with respect to the fundamental amplitude. The small values of the residuals confirm the validity of the proposed simplified model. In fact, since the considered Hammerstein system is exactly represented by a fifth-order Volterra model, the residuals shown in Fig. 5 are only due to the intrinsic approximation of the proposed approach.

Once the coefficients $\mathbf{H}^{\mathrm{red}}(m)$ and $\mathbf{H}^0$ are identified, one can compute the estimated magnitude and phase response $Y(m)$ to an input test signal. The spectrum of the considered test signal is shown in Fig. 4 (green lines). The magnitudes of the harmonics are slightly different from those in the identification signals, while their phases are set to zero. The estimated output is shown in Fig. 6 (green lines) where it is also compared with the actual magnitude and phase response $Y^{\mathrm{act}}(m)$ (red dashed lines). Fig. 6 shows that the magnitude and phase estimates are very close. In particular, the predicted magnitude is noticeably different from the actual one only for output harmonic component with a negligible amplitude, i.e., lower than -60 dB with respect to the fundamental component.

Starting from $Y(m)$ and $Y^{\mathrm{act}}(m)$, it is possible to compute the total vector error (TVE) as

$$\xi(m) = \frac{|Y(m) - Y^{\mathrm{act}}(m)|}{|Y^{\mathrm{act}}(m)|} \quad \xi_1(m) = \frac{|Y(m) - Y^{\mathrm{act}}(m)|}{|Y^{\mathrm{act}}(1)|} \tag{26}$$

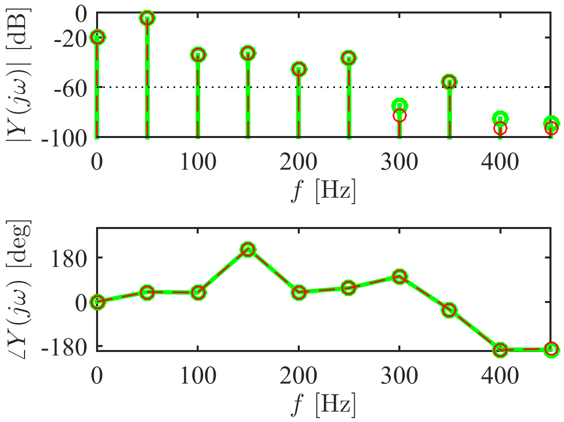

Fig. 6. Magnitude and phase of the estimated $Y(m)$ (green lines) and actual $Y^{\mathrm{act}}(m)$ (red dashed lines).

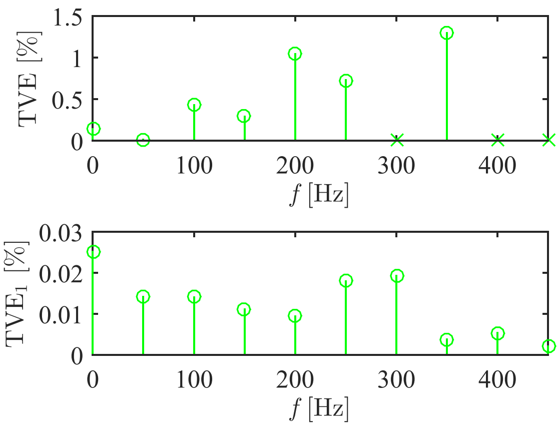

Fig. 7. TVE and TVE_1.

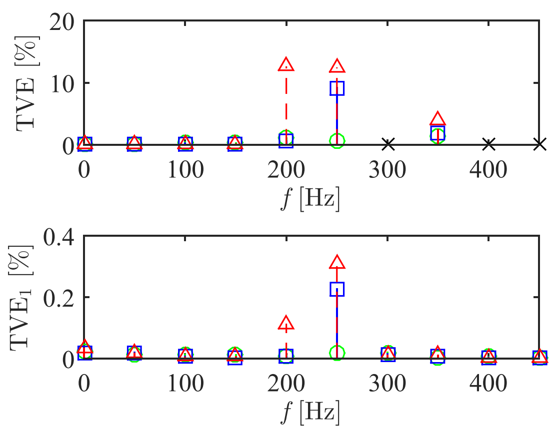

Fig. 8. TVE and TVE_1 in the $Y(j\omega)$ estimate for $I = 5$ (green circles), $I = 4$ (blue squares), and $I = 3$ (red triangles).

where $|\cdot|$ denotes the magnitude, ξ represents the TVE normalized over the magnitude of each harmonic component, and ξ_1 represents the TVE normalized over the magnitude of the fundamental harmonic component. These errors are shown in Fig. 7. Once again, it is proved that approach (11) provides an effective nonlinear model of the considered Hammerstein system.

Finally, the model robustness to the considered order I is discussed, since, in the practical applications, this order is often lower than that of the physical system. Fig. 8 shows the resulting ξ and ξ_1 values when the fifth-order Hammerstein system is undermodeled, namely, modeled by considering an order $I = 4$ and $I = 3$. For $I = 4$ (blue squares), the estimates of the harmonic components in positions $m = 5, 7$ ($f = 250, 350$ Hz) get worse since some relevant intermodulation products are neglected in this model. For example, for the fifth harmonic, ξ increases from 0.7% to 9.2%, while ξ_1 rises from 0.02% to 0.23%. On the other hand, the estimates of the other components are not affected. For $I = 3$ (red triangles), also the estimate of the fourth harmonic gets worse, since ξ and ξ_1 become 12.6% and 0.11%, respectively (1% and 0.01% using the fifth-order model). Therefore, for this component, the contribution of the fourth-order intermodulation product of the input fundamental harmonic component is significant.

According to these results, the model remains stable also in case of severe undermodeling. In these conditions, the TVE values can noticeably increase for some harmonic components. Of course, these values are acceptable (or not) depending on the desired target accuracy. In other words, the final user of the simplified model shall find a compromise between the increase in the complexity of the model (its number of coefficients is approximately proportional to the square of the order, in accordance with (10)) and the loss of accuracy due to undermodeling.

V. Conclusion

In this paper, a simplified Volterra-based steady-state model, specifically devised for nonlinear systems operating under quasi-sinusoidal conditions, has been presented. In general, the main drawback of Volterra models is related to their intrinsic complexity, which, from a practical point of view, restricts the model applicability to weakly nonlinear systems or with small memory length. In this respect, the key feature of the proposed approach is the drastic reduction in the number of coefficients, which has been achieved by exploiting a peculiarity of the input signal, thus making it applicable to a wider class of nonlinear systems.

The proposed technique has been applied to the modeling of a Hammerstein system capable of representing the behavior of a wide variety of nonlinear devices. According to the simulations, the model provides an accurate estimate of the output harmonic components by considering all nonnegligible intermodulation products of the input harmonic components.

A possible application of this method is the modeling and testing of voltage and current transducers employed in the mains power grid, since they typically operate in quasi-sinusoidal conditions. In this case, the proposed technique allows overcoming the conventional characterization of these devices, which is not suited to deal with distorted input and nonlinear phenomena. It is worth reminding that a more accurate testing of these transducers is extremely important in power quality applications, where the measurement of the harmonic components may also involve economic issues.

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