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Time-Variant Frequency Response Function Measurement of Multivariate Time-Variant Systems Operating in Feedback

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Abstract-The classical time-invariance assumption is often not (exactly) met in real life applications. As a natural extension of the frequency response function (FRF), the time-variant frequency response function (TV-FRF) provides quickly insight into the complex dynamics of time-variant systems. Recently a procedure has been proposed to estimate nonparametrically the TV-FRF from known input, noisy output measurements of time-variant systems operating in open loop. However, in quite some applications feedback is present either due to an explicit control action or due to the interaction between a nonideal actuator and the system under test. The extension of the open loop approach to noisy input, noisy output measurements of timevariant systems operating in closed loop requires the deconvolution of the time-variant impulse response of the cascade of two timevariant systems. In this paper this non-trivial problem is solved for a particular class of time-variant systems. The robustness of the approach w.r.t. the system assumption is demonstrated via simulations and measurements on an electronic circuit.

Index Terms—time-variant frequency response function, nonparametric estimates, time-variant systems, Legendre polynomials, errorsin-variables, feedback

I. INTRODUCTION

Time-variant dynamics are present in quite some engineering applications. Think of, for example, pit corrosion of metals where the time-variation is induced by the changing surface [1]; thermal drift in power electronics [2] and lithium-ion batteries [3] whose characteristics are temperature dependent; mortification in ex vivo experiments [4]; muscle fatigue causing time-dependent relationships between, for instance, the hand-grip force and electromyographic signals [5]; and airplane dynamics which depend on the changing flight speed and height [6]. In all these applications the dynamics evolve in a smooth and non-periodic manner.

The time-variant dynamics can also be smooth and periodic. This is the case in, for example, myocardial electrical impedance measurements where the time-variation is induced by the heart beat [7], [8]; rotating machinery (e.g. helicopters, wind turbines) where the rotor anisotropy causes the dynamics to be periodically time-variant [9], [10]; lung acoustical impedance measurements where the breathing creates the periodic time-variation [11]; and the stability analysis of power systems around a periodic trajectory [12]. Periodically time-variant dynamics are not considered in this paper.

A totally different type of time-variant dynamics are those produced by the switching between a finite number of linear time-invariant systems. Examples of such dynamics can be found in switched power electronics [13], econometrics [14], control applications [15], and more general, hybrid systems (see [16] and the references therein). Non-smooth time-variant dynamics are not considered in this paper.

This paper handles multivariate systems with smooth nonperiodic time-variant dynamics. They are characterised by the timevariant frequency response function (TV-FRF) introduced in [17], [18]. The TV-FRF (see Section II for a precise definition) gives a lot of insight into the dynamic behaviour of a time-variant system and, hence, it can be used for physical interpretation, and for model selection and model validation purposes in the parametric modelling of the time-variant dynamics.

Since the estimated time-variant dynamics are not valid outside the experiment interval, and since some time-variant experiments are not exactly reproducible (e.g. pit corrosion, thermal drift phenomena, fatigue in biomedical experiments), one could wonder whether it is useful at all to model non-repeatable time-variant dynamics. The answer is yes because it allows one to quantify and distinguish the non-repeatability of the experiment from the timevariation and the noise. Hence, given some prior knowledge about the physical model structure, the physical phenomena involved can be quantified from the time-variant transfer function (e.g. the parameters that influence the pit corrosion of metals). If the time-variation is controlled by (a) known external parameter(s) (e.g. flight flutter, extendible robot arm), then the time-variant experiment is repeatable, and the time-variant model (differential, difference or state space equations with time-dependent (matrix) coefficients) is a first step towards the identification of a linear parameter varying (LPV) model with possible dynamic dependencies on the external parameter(s). An LPV model is valid for other trajectories of the external parameter(s) and, hence, can be used for prediction and control (see, for example, [19]).

In [20] a method is presented to estimate nonparametrically the TV-FRF of single-input, single-output systems with smooth nonperiodic time-variant dynamics from measured data. It is assumed (i) that the input is known exactly, and (ii) that the time-variant system operates in open loop. However, in quite some measurement applications a non-ideal actuator (e.g. a voltage source with nonzero output impedance or a current source with finite output impedance) is connected to the system to be measured. This has two important consequences: (i) the true input of the system is unknown, and (ii) a feedback loop is present due to the dynamic interaction between the non-ideal actuator and the system (see [21] for the details). It emphasizes the practical need to generalise the results of [20] to noisy input, noisy output observations of systems operating in feedback.

In this paper the results of [20] are generalised to noisy input, noisy output measurements of multivariate systems operating in closed loop. To handle this problem a reference signal should be known [22], [23], and the impulse response of the cascade of two time-variant systems should be deconvolved, which is a non-trivial task [24] that has not been solved yet. The main contributions of this paper are the development of an estimation algorithm that tackles the deconvolution issue from measured input-output data for a particular class of time-variant systems, and the associated bias/variance analysis.

The outline of the paper is as follows. First, it is shown that estimating nonparametrically the TV-FRF from a known reference, and noisy input, noisy output observations of a time-variant system operating in closed loop, boils down to a deconvolution problem (Section II). Next, the class of time-variant systems for which the theory applies is defined (Section III). Further, the nonparametric estimation procedure is explained and analysed in detail (Section IV). The proposed approach is illustrated with two simulation examples (Section V) and measurements on an electronic circuit (Section VI). Finally, some conclusions are drawn (Section VII).

II. PROBLEM STATEMENT

Consider the generic class of linear time-variant (LTV) systems for which the n_y outputs $y_0(t)$ are related to the n_u inputs $u_0(t)$ by the general convolution integral

$$y_0(t) = \int_{-\infty}^{+\infty} g(t,\tau) u_0(\tau) d\tau$$
(1)

where $g(t, \tau)$ represents the time-variant impulse response as a function of time t when a Dirac impulse input has been applied at time τ (see [17], [18]). For causal systems ($g(t, \tau) = 0$ if $t < \tau$) the upper integration bound in (1) is replaced by t. In [17], [18] the time-variant transfer function corresponding to a causal $g(t, \tau)$ is defined as

$$G(s,t) = \int_0^\infty g(t,t-\tau) e^{-\tau s} d\tau$$
(2)

where the real part of s is chosen large enough to guarantee the existence of the integral. Other definitions exist (see [25], p. 135), however, (2) is the only one which has the following two properties [17], [18]. First, the transient response $y_0(t)$ to an input $u_0(t)$ is calculated as

$$y_0(t) = L^{-1} \{ G(s,t) U_0(s) \}$$
(3)

with $U_0(s)$ the Laplace transform of $u_0(t)$, and L^{-1} {} the inverse Laplace transform operator. Next, the steady state response $y_0(t)$ to a sinewave input $u_0(t) = \sin(\omega_0 t)$ equals

$$y_0(t) = |G(j\omega_0, t)| \sin(\omega_0 t + \angle G(j\omega_0, t))$$
(4)

with $G(j\omega, t)$ the time-variant frequency response function (analytic continuation of (2) evaluated along the $j\omega$ -axis). Note that



Figure 1. Noisy input u(t), noisy output y(t) observations of a linear time-variant plant operating in closed loop. The linear feedback and actuator characteristics might be time-variant. G(s,t) is the time-variant plant transfer function; $u_0(t)$ and $y_0(t)$ are, respectively, the true plant input and output signals; r(t) is the known reference signal; and $m_u(t)$ and $m_y(t)$ are, respectively, the input and output measurement errors.

(3) and (4) are natural extensions of the properties of the transfer function of a linear time-invariant system.

The goal is to estimate nonparametrically the time-variant frequency response function (TV-FRF) $G(j\omega, t)$ (2) from noisy input, noisy output observations (see Fig. 1). In the measurement setup of Fig. 1 the number of actuators equals the number of system inputs. Each of the actuators should be driven by a different reference signal; otherwise the system input $u_0(t)$ is not rich enough to uniquely identify the system dynamics. Therefore, in a natural way the sizes of the reference signal r(t) and the input $u_0(t)$ should be the same.

To avoid a bias in the nonparametric estimates, the noisy input, noisy output closed loop problem in Fig. 1 is transformed into two known input, noisy output open loop problems (see Fig. 2) which can be solved using the approach of [20], [26]. However, extracting the plant TV-FRF $G(j\omega, t)$ from the reference to input $G^{ru}(j\omega, t)$ and the reference to output $G^{ry}(j\omega, t)$ TV-FRFs is a non-trivial task. Indeed, unlike for LTI systems, the time-variant transfer function (TV-TF) of a cascade of LTV systems is not equal to the product of the TV-TFs

$$G^{ry}(s,t) \neq G(s,t) G^{ru}(s,t)$$
(5)

The exact relationship between the three TV-TFs is given by [24]

$$G^{ry}(s,t) = L_p^{-1} \{ G(s+p,t) G^{ru}(s,p) \}$$
(6)

with $G^{ru}(s, p)$ the Laplace transform of $G^{ru}(s, t)$ w.r.t. the time t, and $L_p^{-1}\{\}$ the inverse Laplace transform operator acting on the variable p. In Section IV we will unravel G(s, t) from (6) for the class of LTV systems defined in Section III.

Note that the time-variant plant might be unstable (G(s,t)) has poles in the right half plane) so long as it operates in a stabilising feedback loop.

III. CLASS OF LTV SYSTEMS CONSIDERED

As already mentioned in the introduction we consider LTV systems with n_u inputs and n_y outputs whose time-variant dynamics are a smooth function of time. For such systems the $n_y \times n_u$ TV-TF (2) can be expanded in series as



Figure 2. Transformation of the noisy input u(t), noisy output y(t) closed loop problem in Fig. 1 into two known reference r(t), noisy output open loop problems: from known reference r(t) to noisy input u(t), and from known reference r(t) to noisy output y(t). $G^{ru}(s, t)$ and $G^{ry}(s, t)$ are the time-variant transfer functions from, respectively, reference to input and reference to output. The true input $u_0(t)$ and output $y_0(t)$ signals are connected via the time-variant plant transfer function G(s, t).



Figure 3. Direct input-output model of the class of LTV systems considered. $u_0(t), y_0(t)$ and $G_m(s), m = 0, 1, \ldots, N_b$, have the size $n_u \times 1$, $n_y \times 1$ and $n_y \times n_u$ respectively.

$$G(s,t) = \sum_{m=0}^{\infty} G_m(s) f_m(t) \quad \text{for } t \in [0,T]$$
(7)

with $f_m(t)$, m = 0, 1, ..., a complete set of basis functions over [0,T] (e.g. polynomials); T the observation time; and where $G_m(s)$, m = 0, 1, ..., are the $n_y \times n_u$ matrix coefficients of the series expansion. Without any loss in generality we can impose the following constraints on the basis functions

$$f_0(t) = 1$$
 and $\frac{1}{T} \int_0^T f_m(t) dt = 0$ for $m > 0$ (8)

A. Direct Model

Since (7) contains infinitely many transfer functions $G_m(s)$, m = 0, 1, ..., of LTI systems, it is not suitable for nonparametric estimation. Therefore, the class of LTV systems is restricted to those systems for which the series (7) can be reduced to a finite sum

$$G(s,t) = \sum_{m=0}^{N_b} G_m(s) f_m(t) \text{ for } t \in [0,T]$$
 (9)

Assumption 1 (Class of input-output LTV systems): The timevariant transfer function G(s,t) (2) can be written under the form (9) with $f_m(t) = l_m(2t/T - 1)$; and where $l_m(x)$, $x \in [-1,1]$, are Legendre polynomials of degree m.

Note that the basis functions $l_m(2t/T-1)$ satisfy the constraints (8) [27]. Moreover, the Legendre polynomials $l_p(x)$ are orthogonal

over the interval [-1,1] [27] and, therefore, guarantee the good numerical stability of the nonparametric TV-FRF estimates [20], [26].

Under Assumption 1, the response $y_0(t)$ of an LTV system to an input $u_0(t)$ is given by

$$y_0(t) = \sum_{m=0}^{N_b} L_s^{-1} \{ G_m(s) U_0(s) \} f_m(t)$$
 (10)

(proof: combine (3) and (9)). It shows that the response can be written as the sum of the responses of $N_b + 1$ LTI systems $G_m(s)$, $m = 0, 1, \ldots, N_b$, weighted with the basis functions $f_m(t)$. Eq. (10) is called the direct model of the LTV system and Fig. 3 shows the corresponding block diagram.

Note also that (10) resembles the response of a linear parametervarying system for a particular trajectory of the scheduling parameter when it is modelled using a finite set of orthonormal basis functions (see [28], eq. (2)). The major difference between the orthonormal basis function expansion and the direct model (10) is the parametrisation: (i) the $G_m(s)$ dynamics are unknown and should be estimated nonparametrically from the data while the orthonormal basis functions modelling the dynamics are given, and (ii) the basis functions $f_m(t)$ modelling the time-variation are given while the parameter dependent coefficient functions of the orthonormal basis function expansion modelling the time-variation are estimated parametrically from the data.

B. Indirect Model

In [26] it has been shown that the block diagram in Fig. 3 is equivalent to the block diagram in Fig. 4. The latter is called the indirect model of the LTV system and its response is calculated as

$$y_0(t) = \sum_{m=0}^{N_b} L_s^{-1} \{ H_m(s) U_m(s) \}$$
(11)

where $U_m(s)$ is the Laplace transform of $u_m(t) = u_0(t)f_m(t)$. Since the inputs $u_m(t)$ of the LTI systems in Fig. 4 are linearly independent (the basis functions $f_m(t)$ are linearly independent) and since they are known if $u_0(t)$ is known, the transfer function matrices $H_m(s)$ can be estimated using standard LTI techniques (see [26]). The $G_m(s)$ transfer function matrices are then recovered from $H_m(s)$ as

$$G_{m}(s) = H_{m}(s) + \frac{2}{T} (2m+1) \sum_{i=0}^{\left\lfloor \frac{N_{b}-m-1}{2} \right\rfloor} H_{2i+1+m}^{(1)}(s) + \frac{4}{T^{2}} \sum_{i=1}^{\left\lfloor \frac{N_{b}-m}{2} \right\rfloor} \beta_{2i,m} H_{2i+m}^{(2)}(s) + O\left(T^{-3}\right)$$
(12)

with $H_m^{(n)}(s)$ the *n*th order derivative of $H_m(s)$ w.r.t. s, $\lfloor x \rfloor$ the largest integer smaller than or equal to x, and $\beta_{2i,m}$ coefficients defined as



Figure 4. Indirect input-output model of the class of LTV systems considered. $u_m(t)$, $y_0(t)$ and $H_m(s)$, $m = 0, 1, ..., N_b$, have the size $n_u \times 1$, $n_y \times 1$ and $n_y \times n_u$ respectively.

$$\beta_{2i,m} = \gamma_m + \delta_m (i-1) + \mu_m (i-1)^2$$
(13)
$$\gamma_m = 1.5 + 4m + 2m^2 \delta_m = 2.5 + 6m + 2m^2 \mu_m = 1 + 2m$$

(see Theorem 3 of [26]). The bias term $O(T^{-3})$ in (12) depends on the higher order (> 2) derivatives of the *H*-transfer functions (see [26] for the details). These higher order derivatives are not calculated because they are of the same order of magnitude as the numerical approximation errors of $H_m^{(1)}(s)$ and $H_m^{(2)}(s)$ via the central differences (see [29] and [30]).

C. Discussion

One might wonder how general the direct model (9) is. First, the direct model has successfully been applied to real life systems where the finite series condition (9) has not been imposed. See, for example, [7], [26], and [31] for measurements on timevarying electronic circuits; [8] for myocardial electrical impedance measurements; and [32] for measurements on an XY-table. Next, although one cannot show theoretically that the approximation error of (9) is always insignificant, one can in practice increase N_b such that the remaining approximation error is "small enough" or below the noise level. The estimation procedure described in Section IV quantifies the remaining approximation error. This is illustrated in Section V-B by simulations on a time-variant system operating in closed loop and described by time-variant state space equations.

IV. NONPARAMETRIC ESTIMATION OF THE PLANT TV-FRF

This section handles the nonparametric estimation of the $n_y \times n_u$ TV-FRF $G(j\omega, t)$ from noisy input, noisy output observations of a plant operating in open or closed loop (Fig. 1). First, the basic idea of the proposed estimation procedure is explained (Section IV-A). Solving the deconvolution problem (6) is the non-trivial part of this procedure. Therefore, the key steps of the proposed deconvolution algorithm are discussed in detail in Section IV-B. Finally, the stochastic properties (bias, variance) of the nonparametric TV-FRF estimates are analysed (Section IV-C).



Figure 5. The indirect model from reference r(t) to output $y_0(t)$ (top block diagram) written as the cascade of the indirect models from reference r(t) to input $u_0(t)$ and from input $u_0(t)$ to output $y_0(t)$ (bottom block diagram). r(t), $u_0(t)$, $y_0(t)$, $H_m^{ru}(s)$, $m = 0, 1, \ldots N_{b_u}$, $H_i^{ry}(s)$, $i = 0, 1, \ldots, N_{b_y}$, and $H_n(s)$, $n = 0, 1, \ldots N_b$, have size $n_u \times 1$, $n_u \times 1$, $n_y \times 1$, $n_u \times n_u$, $n_y \times n_u$ and $n_y \times n_u$ respectively.



A. Basic Idea

The proposed estimation algorithm is basically a 2-step procedure. In step 1, following the same lines of [22], [23], the noisy input, noisy output closed loop problem is transformed into a known input, noisy output open loop problem for which a solution exists [26]. Therefore, the TV-FRF $G^{rz}(j\omega,t)$ from reference r(t) to plant input and output simultaneously $z(t) = \begin{bmatrix} y^T(t) & u^T(t) \end{bmatrix}^T$

$$G^{rz}(j\omega,t) = \begin{bmatrix} G^{ry}(j\omega,t) \\ G^{ru}(j\omega,t) \end{bmatrix}$$

is estimated nonparametrically using the known reference and the noisy input, noisy output observations. In step 2, the plant TV-FRF $G(j\omega,t)$ is recovered from $G^{ru}(j\omega,t)$ and $G^{ry}(j\omega,t)$ via (6). This is a deconvolution problem that is handled in the sequel of this section.

To solve the deconvolution problem (6), the following assumption is made.

Assumption 2 (Class of reference to input-output LTV systems): The time-variant transfer functions from reference to input $G_{ru}(s,t)$ and from reference to output $G_{ry}(s,t)$ can be written under the form (9) with $f_m(t) = l_m(2t/T-1)$ and where $l_m(x)$, $x \in [-1,1]$, are Legendre polynomials of degree m. In addition, the reference signal r(t) and the system input $u_0(t)$ have the same size.

Using Assumptions 1 and 2 and the equivalence between the direct (10) and the indirect (11) models, the indirect model from reference r(t) to output $y_0(t)$ (see Fig. 5, top block diagram) can be written as the cascade of the indirect models from reference r(t) to input $u_0(t)$ and from input $u_0(t)$ to output $y_0(t)$ (see Fig. 5, bottom block diagram). To be compatible, the condition $N_{b_y} = N_{b_u} + N_b$ should be satisfied.

To establish the relationship between the transfer functions $H_m^{ry}(s)$, $H_n^{ru}(s)$ and $H_q(s)$ in Fig. 5, the cascade of the indirect models (bottom block diagram) is manipulated to be of the form of the top block diagram. This is done as follows.

First, the input-output indirect model is shifted backward into the reference to input indirect model. The resulting block diagram has $(N_{b_u} + 1)(N_b + 1)$ parallel branches, one of which is shown in Fig. 6.

Next, the output gain $f_n(t)$ of the first LTI system in Fig. 6 is moved to the input using the transformation of Fig. 7. The resulting block diagram is of the form of the top block diagram in Fig. 5 but with input gains $f_m(t)f_n(t)$, $m = 0, 1, \ldots, N_{b_u}$ and $n = 0, 1, \ldots, N_b$.

Further, the products of basis functions $f_m(t)f_n(t)$ are written as a linear combination of the basis functions $f_q(t)$, $q = 0, 1, \ldots, m+$ n. The resulting block diagram resembles the top block diagram in Fig. 5 but has multiple branches with the same input gain $f_q(t)$, $q = 0, 1, \ldots, m + n$.

The transfer functions of the branches with the same input gain are added and set equal to the transfer function of the corresponding branch in the indirect reference to output model (Fig. 5, top block diagram). This results in a linear set of equations in the unknowns $H_q(j\omega)$, $q = 0, 1, \ldots, N_b$, that is solved numerically for each frequency. Using (12), we finally obtain the FRFs $G_q(j\omega)$, $q = 0, 1, \ldots, N_b$, of the direct model.

B. Key Steps of the Deconvolution Algorithm

The first step of the deconvolution procedure consists in shifting the output gain $f_n(t)$ of $H_m^{ru}(s)$ (see Fig. 6) to the input. Since the direct (Eq. (10) and Fig. 3) and indirect (Eq. (11) and Fig. 4) models are only defined for $t \in [0, T]$, the Laplace transform of the windowed signals is used

$$X(s) = L\{x(t)\} = \int_{0}^{T} x(t) e^{-st} dt$$
 (14)

in the lemma describing the shifting of the output gain.

Lemma 1. (Shifting the output gain to the input) The Laplace transforms $U_n(s)$ and $R_m(s)$ of the windowed signals $u_n(t)$ and $r_m(t)$ are related as

$$U_{n}(s) = L\left\{f_{n}(t) L_{s}^{-1}\left\{H_{m}^{ru}(s) R_{m}(s)\right\}\right\} + T_{1}(s)$$
(15)



Figure 7. Shifting the output gain $f_n(t)$ (top block diagram) to the input results in a parallel structure with n+1 branches (bottom block diagram). $f_0(t) = 1$; $\nu = 1$ for n odd and $\nu = 2$ for n even; $k_q^n = -\alpha \zeta_{n-q}$ for q odd, with $\alpha = 2/T$ and $\zeta_i = 2i + 1$; and $k_q^n = \alpha^2 \beta_{q,n-q}$ for q even, with $\beta_{q,n-q}$ defined in (13).

$$U_{n}(s) = H_{m}^{ru}(s) R_{m,n}(s)$$

$$-\alpha \sum_{q=0}^{\left\lceil \frac{n}{2} \right\rceil - 1} \zeta_{n-2q-1} H_{m}^{ru(1)}(s) R_{m,n-(2q+1)}(s)$$

$$+\alpha^{2} \sum_{q=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \beta_{2q,n-2q} H_{m}^{ru(2)}(s) R_{m,n-2q}(s)$$

$$+ T_{2}(s) + O(\alpha^{3})$$
(16)

with $R_{m,v}(s) = L\{r_m(t) f_v(t)\}, \alpha = 2/T, \zeta_i = 2i + 1, \beta_{2i,m}$ defined in (13), and $O(\alpha^3)$ a bias term depending on the higher order derivatives of $H_m^{ru}(s)$. $T_1(s)$ and $T_2(s)$ are smooth functions of s depending on the difference between the initial (t = 0) and final (t = T) conditions of the experiment. At the DFT frequencies $\omega_k = 2\pi k/N$, with N the number of time domain samples and $k = 0, 1, \ldots N/2$, $T_1(j\omega_k)$ and $T_2(j\omega_k)$ are rational functions of $j\omega_k$. [x] is the smallest integer larger than or equal to x.

Proof: See Appendix A.

Note that formula (16) for shifting the output gain to the input resembles formula (48) of [26] for shifting the input gain to the

output: α is replaced by $-\alpha$, and the role of the direct and indirect models is interchanged.

Fig. 7 shows the block diagram corresponding to (15) and (16). It is exact within an $O(T^{-3})$ bias term that depends on the higher order (>2) derivatives of $H_m^{ru}(s)$.

The second step of the deconvolution procedure is the expansion of the product $f_m(t)f_n(t)$, $n = 0, 1, ..., N_b$ and $m = 0, 1, ..., N_{b_u}$, of Legendre polynomials in the Legendre polynomial basis

$$f_m(t) f_n(t) = \sum_{q=0}^{n+m} C_{0[q+1,m+1,n+1]} f_q(t)$$
(17)

where C_0 is a $(N_{b_u} + N_b + 1) \times (N_{b_u} + 1) \times (N_b + 1)$ tensor. Since no analytical expression for the tensor C_0 could be found, the coefficients $C_{0[q+1,m+1,n+1]}$ of the expansion (17) are calculated in Matlab via a linear least squares fit of (17) using 1024 time domain samples uniformly distributed in the interval [0,T]. All coefficients smaller than 10^{-10} are set to zero. Proceeding in this way, the calculated expansion (17) is accurate within Matlab precision. From the calculated expansion it is observed that C_0 is sparse. Namely, if m + n is even (odd), then only the even (odd) degree Legendre polynomials are present in the sum (17). Moreover, the column $C_{0[:,m,n]}$ (":" denotes all entries) contains at most min(n,m) non-zero elements for all values of m.

In the third step of the deconvolution procedure the branches with the same input gain are joined and set equal to the corresponding branch of the indirect reference to output model. This results in the following lemma.

Lemma 2. (Joining of branches with the same input gain). Under Assumptions 1 and 2 the dynamics $H_i^{ry}(s)$ of the indirect reference to output model are related to the dynamics of the indirect reference to input $H_m^{ru}(s)$ and input-output $H_n(s)$ models as

$$H_{i}^{ry}(s) = \sum_{n=0}^{N_{b}} \sum_{m=0}^{N_{b_{u}}} C_{0[i+1,m+1,n+1]} H_{n}(s) H_{m}^{ru}(s) + \sum_{n=0}^{N_{b}} \sum_{m=0}^{N_{b_{u}}} C_{1[i+1,m+1,n+1]} H_{n}(s) H_{m}^{ru(1)}(s) + \sum_{n=0}^{N_{b}} \sum_{m=0}^{N_{b_{u}}} C_{2[i+1,m+1,n+1]} H_{n}(s) H_{m}^{ru(2)}(s) + O(\alpha^{3})$$
(18)

for $i = 0, 1, ..., N_{b_y}$ $(N_{b_y} = N_{b_u} + N_b)$, with C_0 the 3-dimensional tensor of the expansion (17) and $\alpha = 2/T$. The 3-dimensional tensors C_1 and C_2 are related to C_0 as

$$C_{1[i,m,n]} = \begin{cases} -\alpha \sum_{q=1}^{\left\lceil \frac{n-1}{2} \right\rceil} \zeta_{n-2q} C_{0[i,m,n-2q+1]} & n > 1\\ 0 & n = 1 \end{cases}$$
(19)

$$C_{2[i,m,n]} = \begin{cases} \alpha^2 \sum_{q=1}^{\lfloor \frac{n-1}{2} \rfloor} \beta_{2q,n-2q-1} C_{0[i,m,n-2q]} & n > 2\\ 0 & n = 1, 2 \end{cases}$$
(20)

with $\zeta_i = 2i + 1$ and $\beta_{2i,m}$ defined in (13).

Proof: Combine Lemma 1 and (17).

In the fourth step of the deconvolution procedure the set of equations (18) is solved for $H_n(j\omega)$. This is the deconvolution of the indirect models.

Theorem 3. (Deconvolution of the indirect models). Under Assumptions 1 and 2 the dynamics $H_n(j\omega)$ of the indirect inputoutput model are the solution of the following linear set of equations

$$(H_i^{ry}(j\omega))^T = \sum_{n=0}^{N_b} C_{[i+1,n+1]}(j\omega) (H_n(j\omega))^T + O(T^{-3})$$
(21)

for $i = 0, 1, \ldots, N_b$, with $C_{[i+1,n+1]}(j\omega) \ n_u \times n_u$ matrices

$$C_{[i+1,n+1]}(j\omega) = \sum_{m=0}^{N_{b_u}} C_{0[i+1,m+1,n+1]} \left(H_m^{ru}(j\omega) \right)^T + \sum_{m=0}^{N_{b_u}} C_{1[i+1,m+1,n+1]} \left(H_m^{ru(1)}(j\omega) \right)^T + \sum_{m=0}^{N_{b_u}} C_{2[i+1,m+1,n+1]} \left(H_m^{ru(2)}(j\omega) \right)^T$$
(22)

and where C_0 , C_1 and C_2 are defined in, respectively, (17), (19) and (20).

Proof: Take for each frequency the matrix transpose of (18) for $i = 0, 1, ..., N_b$ and put them on top of each other. This gives for each frequency $N_b + 1$ matrix equations of size $n_u \times n_y$ in the N_b+1 unknowns $(H_n(j\omega))^T$ of size $n_u \times n_y$. Solving this linear set of equations gives the dynamics $H_n(j\omega)$ within an $O(T^{-3})$ bias error.

In the last step of the deconvolution procedure the indirect inputoutput model is transformed into the direct model via (12).

C. Stochastic Properties

The stochastic properties are analysed under the following assumption on the disturbing input-output noise.

Assumption 3: $m_u(t)$ and $m_y(t)$ in Figure 1 are stationary, (mutually) (un)correlated, filtered white noise disturbances.

In practice the input-output disturbing noise dynamics might be non-stationary. The proposed procedure estimates then an equivalent time-invariant power spectrum of the non-stationary noise signals. Simultaneous nonparametric estimation of the time-variant noise power spectra and the TV-FRF is a very challenging problem that is out of the scope of this paper.

The estimation procedure of Section IV-B has three parameters that should be chosen: the number of time-variant branches N_{b_u} and N_{b_y} ($N_b = N_{b_y} - N_{b_u}$) in the reference to input and reference to output models, and the degree R of the local polynomial approximation used for estimating the indirect model from reference r(t) to plant input and output simultaneously $z(t) = \begin{bmatrix} y^T(t) & u^T(t) \end{bmatrix}^T$ (see [33], [34] for the algorithmic details of the local polynomial method for estimating FRFs). To avoid bias due to a too small number of time-variant branches, N_{b_u} and N_{b_y} are gradually increased until the estimated dynamics

of the time-variant branches $m > N_{b_u}, N_{b_y}$ are equal to their standard deviation. To avoid bias due to a too low degree R of the local polynomial approximation of the FRFs in the indirect model, R is progressively augmented until the standard deviation of the estimated FRFs no longer decreases.

Bias: The bias in the estimated FRFs $\hat{G}_m(j\omega_k)$ of the direct input-output model (10) has five contributions:

- 1) An $O(T^{-(R+1)})$ residual bias term of the local polynomial approximation of the FRFs in the indirect models [33], [34].
- 2) An $O(T^{-3})$ term due to the shifting of the output gain to the input in the deconvolution of the indirect models (see Lemma 1).
- An O(T⁻³) term introduced by the numerical approximation of H^{ru(1)}_m (jω) and H^{ru(2)}_m (jω) in (22) via central differences (see [29] for a uniform frequency grid, and [35] and [30] for a non-uniform frequency grid).
- An O(T⁻³) term introduced by the numerical approximation of H_n⁽¹⁾ (jω) and H_n⁽²⁾ (jω) in (12) via central differences.
 An O(T⁻³) term originating from the transformation of the
- 5) An $O(T^{-3})$ term originating from the transformation of the indirect to the direct model (12).

To verify whether the $O(T^{-2})$ terms in (22) depending on the second order derivatives are significant, the linear set of equations (21) is solved twice, once with the second order derivative terms and once without. The difference between both solutions is then compared to the noise standard deviation of the estimates.

Note that bias contributions no. 2 and 3 originating from the deconvolution of the indirect models are in-existent in the known input, noisy output open loop problem handled in [26]. Note also that the FRFs $G_m(j\omega_k)$ of the direct input-output model (10) can be estimated without the second order derivatives in (12) and (22). The resulting $O(T^{-2})$ bias error can then be predicted via the contributions of these second order derivatives to $\hat{G}_m(j\omega_k)$.

Using (9) it follows that bias results for the FRFs $G_m(j\omega_k)$ are also valid for the estimated TV-FRF $\hat{G}(j\omega_k, t)$.

Variance: Under Assumption 3 the local polynomial method [33], [34] used for estimating the indirect model from reference r(t) to plant input and output simultaneously $z(t) = \begin{bmatrix} y^T(t) & u^T(t) \end{bmatrix}^T$ also quantifies the covariance matrix of the FRF estimates $\hat{H}_m^{ru}(j\omega)$, $m = 0, 1, \ldots, N_{b_u}$, $\hat{H}_i^{ry}(j\omega)$, $i = 0, 1, \ldots, N_{b_y}$ via the squared output residuals of the local polynomial approximation. Linearisation of (21) w.r.t. $H_m^{ru}(j\omega)$ and $H_n(j\omega)$, readily gives the covariance of the FRF estimates $\hat{H}_n(j\omega)$, $n = 0, 1, \ldots N_b$. Next, the covariance of the FRFs $\hat{G}_m(j\omega)$ in the direct input-output model (10) follows immediately from (12). Note that in the covariance calculations only the dominant $O(T^0)$ terms are used: the first sum in the right hand side of (12).

Finally, the variance of the estimated TV-FRF $\hat{G}(j\omega_k, t)$ is calculated via (9), taking into account the covariance between the estimated FRFs $\hat{G}_m(j\omega_k)$.

Undermodelling: The undermodelling is due to a too small order R of the local polynomial approximation of the FRFs in the indirect models and/or a too small value N_b in the series expansion of the TV-FRF (9). The $O(T^{-(R+1)})$ error of the local polynomial approximation (bias contribution no. 1) and the approximation error of (9) result in local bias errors that depend on the random realisation of the reference signal r(t). Hence, both can be handled

as being random (w.r.t. the random realisation of the input) and are included in the covariance of $\hat{G}_m(j\omega)$ which is based on the squared output residuals of the local polynomial approximation.

V. SIMULATION EXAMPLES

Two simulation examples are discussed in this section. In the first example the system satisfies Assumption 1, which allows us to verify the predicted bias and variance properties (see Section IV-C) of the proposed estimation procedure. In the second example the system is described by time-varying state space equations, for which the finite sum (9) is an approximation. It illustrates that by increasing N_b in (9) the approximation error can be made (arbitrarily) small.

A. Example 1: System Satisfying Assumption 1

1) Description Simulation Example: The configuration of Fig. 1, without the feedback loop and with actuator characteristics equal to $G^{ru}(s,t)$, is used as simulation setup. $G^{ru}(s,t)$ and G(s,t) satisfy Assumption 1 with, respectively, $N_{b_u} = 2$ and $N_b = 3$ time-variant branches. $G_m^{ru}(s)$, m = 0, 1, 2, are second order Chebyshev filters with a passband ripple of 2 dB and cut-off frequencies of 60 Hz, 70 Hz, and 80 Hz respectively. $G_0(s)$ is the cascade of a fourth order Butterworth filter with a passband ripple of 3 dB and a cut-off frequency of 70 Hz, and a second order Chebyshev filter with a passband ripple of 6 dB and a cut-off frequency of 60 Hz; while $G_m(s)$, m = 1, 2, 3, are second order Chebyshev filters with passband ripples and cut-off frequencies of, respectively, 15 dB and 65 Hz, 20 dB and 70 Hz, and 15 dB and 75 Hz. A random phase multisine r(t) is applied to the actuator in Fig. 1

$$r(t) = A \sum_{k=F_1}^{F_2} \sin(2\pi k f_0 t + \phi_k)$$
(23)

with A chosen such that the rms value of r(t) equals 2, ϕ_k independent uniformly $[0, 2\pi)$ distributed random variables, $f_0 = f_s/N$ with $f_s = 1 \text{ kHz}$ the sampling frequency and N the number of samples in one signal period, $F_1 = \lceil 3 \text{ Hz}/f_0 \rceil$, and $F_2 = \lfloor 100 \text{ Hz}/f_0 \rfloor$. This signal excites the frequency band (3 Hz, 100 Hz) and the response to one signal period is calculated.

To check the bias and variance properties of the proposed deconvolution procedure, simulations without and with disturbing noise are made. An order R = 4 and a local bandwidth of 43 excited frequencies are used for the local polynomial approximation in the estimates of the indirect reference to input-output models $H_i^{ry}(j\omega)$ and $H_m^{ru}(j\omega)$.

2) Results Noiseless Data: In the noiseless case three simulations are performed with increasing values of the number of samples per period $N = 128 \times 1024$, 256×1024 , 512×1024 (for a fixed value of f_s , increasing the value of N decreases the speed of the time-variation). For each simulation the direct model is estimated once without $(\hat{G}_m(j\omega))$ and once with $(\hat{G}_m(j\omega))$ the second order derivatives in (13) and (22).

The results are shown in Fig. 8. As predicted by the theory (see Section IV-C), per doubling of N, the bias decreases with about 12 dB (2²) for $\hat{G}_m(j\omega)$ (grey lines), 18 dB (2³) for $\hat{G}_m(j\omega)$ (red lines), and 30 dB (2⁵) for the bias resulting from the local



Figure 8. Estimated dynamics of the direct model for increasing values of N – Example 1: noiseless case. Left column (black lines): true frequency response function $G_m(j\omega)$. Right column: the corresponding relative (w.r.t. the true FRF) bias of the estimates. Grey lines: $|(\hat{G}_m(j\omega) - G_m(j\omega))/G_m(j\omega)|$; red lines: $|(\hat{G}_m(j\omega) - G_m(j\omega))/G_m(j\omega)|$; red lines: $|(\hat{G}_m(j\omega) - G_m(j\omega))/G_m(j\omega)|$; cyan lines: relative bias resulting from the $O(T^{-(R+1)})$ bias of the local polynomial estimate of the indirect model from reference to input-output (R = 4). $\hat{G}_m(j\omega)$ and $\hat{G}_m(j\omega)$ are, respectively, the estimates obtained without and with the second order derivatives in (13) and (22). The bias terms (grey, red and cyan lines) decrease for increasing values of N.

polynomial approximation of order R = 4 in the indirect estimate from reference to input-output (cyan lines, except for the largest value of N in those frequency bands where the bias is close to the Matlab accuracy of -300 dB).

3) Results Noisy Data: A Monte-Carlo simulation of hundred runs is made with $N = 128 \times 1024$. For each run a new random phase realisation of the random phase multisine (23) is generated and the response to one signal period is calculated. Next, zero mean normally distributed noise is added to the true input and output signals. The input noise is white with a standard deviation of 0.003, while the output noise is discrete-time filtered white noise. The output noise filter is a third order discrete-time Chebyshev filter with a passband ripple of 15 dB and a cut-off frequency of 70 Hz, and the driving white noise source of that filter has a standard deviation of 0.015. For each Monte-Carlo run, the direct model and its variance is estimated without the second order derivatives in (13) and (22). Next, the mean value over the hundred estimates of the direct model and its variance is calculated.



Figure 9. Estimated dynamics of the direct model – Example 1: noisy case. Left column (black lines): true frequency response function $G_m(j\omega)$. Right column: the corresponding relative error and relative standard deviations of the mean FRF estimates (relative w.r.t. the true FRF). Grey lines: $|(\hat{G}_m(j\omega) - G_m(j\omega))/G_m(j\omega)|$; green lines: estimated relative standard deviations; red lines: Monte Carlo relative sample standard deviations (coincide with the green lines). $\hat{G}_m(j\omega)$ is the mean FRF estimate obtained without the second order derivatives in (13) and (22).

of the mean FRF estimates (green lines) coincide with the Monte Carlo sample standard deviations of the mean FRF estimates (red lines). Since 37.7% of the differences between the mean FRF estimates and the true values (grey lines) lie outside the corresponding standard deviations (red lines), no bias can be detected (for circular complex normally distributed noise the theoretical fraction outside the noise standard deviation is 36.8%; see [36], p. 48).

B. Example 2: System described by Time-Varying State Space Equations

1) Description Simulation Example: The system is described by the following time-varying state space equations

$$\frac{dx(t)}{u} = A(t)x(t) + B(t)u_0(t)$$
(24)

$$\begin{array}{l} at \\ y_0(t) &= C(t) x(t) \end{array}$$
 (25)

From Fig. 9 it can be seen that the estimated standard deviations w

with

$$A(t) = A_0 + (A_1 - A_0) t/T$$

$$B(t) = B_0 + (B_1 - B_0) t/T$$

$$C(t) = C_0 + (C_1 - C_0) t/T$$

for $t \in [0,T]$. (A_r, B_r, C_r) , r = 0, 1, are the controller canonical forms of a fourth order transfer function

$$\frac{\frac{s^2}{\omega_0^2} + 2\zeta_0 \frac{s}{\omega_0} + 1}{\left(\frac{s^2}{\omega_1^2} + 2\zeta_1 \frac{s}{\omega_1} + 1\right) \left(\frac{s^2}{\omega_2^2} + 2\zeta_2 \frac{s}{\omega_2} + 1\right)}$$
(26)

with $\omega_i = 2\pi f_i$, i = 0, 1 and 2, and where $f_0 = 13$ Hz, $\zeta_0 = 0.05$, $f_1 = 10$ Hz, $\zeta_1 = 0.35$, $f_2 = 20$ Hz, $\zeta_2 = 0.2$ for r = 0, and $f_0 = 16$ Hz, $\zeta_0 = 0.05$, $f_1 = 12$ Hz, $\zeta_1 = 0.30$, $f_2 = 22$ Hz, $\zeta_2 = 0.15$ for r = 1. Equations (24) to (26) mimic a vibrating mechanical structure with time-varying (anti-)resonance frequencies f_i and damping ratios ζ_i .

Since the true TV-FRF corresponding to (24-26) is unknown, two types of simulations are performed: (i) system (24-26) operating in open loop with $u_0(t) = r(t)$, and (ii) system (24-26)operating in closed loop with

$$u_0(t) = r(t) - y_0(t) \tag{27}$$

The open loop simulations are handled using the estimation procedure of [26] and the resulting TV-FRF serves as reference value for the closed loop simulations. For all simulations r(t) is a random phase multisine excitation (23) with A = 1, $F_1 = 20$, $F_2 = 6000$, and $f_0 = 0.005$ Hz, and the transient response to one signal period ($T = 1/f_0$) under zero initial conditions (x(0) = 0) is calculated using the ODE45 solver of MatlabTM at the sampling period $T_s = 1/f_s$ with $f_s = 300$ Hz, resulting in N = 60000 samples. The relative and absolute tolerance of the solver are set to 10^{-10} and 10^{-15} , respectively. No noise is added to the input-output signals because we want to analyse the approximation error of the finite series expansion (9).

2) Results Identification in Open Loop: Two data sets are generated: an identification data set and a validation data set, which differ in the random phase realisation of the multisine excitation (23). The procedure of [26] is applied to estimate the TV-FRF. First, the indirect input-output model $H_n(j\omega)$ with $N_b = 9$ is estimated using an order R = 8 and a local bandwidth of 199 frequencies for the local polynomial approximation. Next, the estimates $\hat{G}_m(j\omega_k)$, $m = 0, 1, \ldots, N_b$, of the direct model (9) are calculated without the second order derivatives. The latter are used to quantify the $O(T^{-2})$ bias error of $\hat{G}_m(j\omega_k)$. Figure 10 shows the corresponding TV-FRF estimate (28)

$$\hat{G}(j\omega_k, t) = \sum_{m=0}^{N_b} \hat{G}_m(j\omega_k) f_m(t)$$
(28)

The bias error of (28) has two contributions: (i) the undermodelling (R and/or N_b are too small) quantified by the covariance of $\hat{G}_m(j\omega_k)$ (see Section IV-C–Undermodelling) and (ii) the contributions of the second order derivatives in (9) quantified by the



Figure 10. Nonparametric TV-FRF estimate (28) – Example 2: open loop case. Top left: amplitude TV-FRF; top right: phase TV-FRF; bottom left: frequency-time plot amplitude TV-FRF; and bottom right: predicted relative bias error TV-FRF estimate.

bias of $\hat{G}_m(j\omega_k)$. Both contributions can be calculated resulting in the predicted relative bias error shown in the bottom right plot of Figure 10. At the borders the predicted relative bias is -98 dB (almost 5 significant digits are correct) while elsewhere the predicted error is below -124 dB (more than 6 significant digits are correct).

Since the true TV-FRF corresponding to the time-varying state space equations (24–26) is unknown, the predicted bias error in Figure 10 cannot be compared to the true bias. Therefore, we indirectly assess the quality of the nonparametric TV-FRF estimate via the simulation error on both the identification and validation data sets. The output of the time-variant system (24–26) is simulated as

$$\hat{Y}(k) = \sum_{m=0}^{N_b} \text{DFT}\left(\text{IDFT}\left(\hat{G}_m(j\omega_k) U_0(k)\right) f_m(nT_s)\right)$$
(29)

with $G_m(j\omega_k)$ the nonparametric estimate of the direct model (9),

$$X(k) = \text{DFT}(x(nT_s)) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_s) e^{-j2\pi \frac{kn}{N}}$$
(30)

$$x(nT_s) = \text{IDFT}(X(k)) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X(k) e^{j2\pi \frac{kn}{N}}$$
 (31)

and where $U_0(k)$ is the DFT spectrum of the random phase multisine excitation $u_0(t)$ of the identification or validation data set.

Note that Eq. (29) implicitly assumes that the LTI blocks $G_m(s)$ operate in periodic steady state which is not the case in the time domain calculation of the output of the time-varying



Figure 11. Open loop simulation error of the nonparametric TV-FRF estimate – Example 2: open loop identification. Top figures: output DFT spectrum $Y_0(k)$ (black lines), estimated transient term $\hat{T}_Y(j\omega_k)$ (black dashed lines), actual simulation error $\hat{Y}(k) - Y_0(k) - \hat{T}_Y(j\omega_k)$ (grey lines) with $\hat{Y}(k)$ the simulated output (29), and predicted simulation error (red lines). Bottom figure: relative simulation error $(\hat{Y}(k) - Y_0(k) - \hat{T}_Y(j\omega_k))/Y(k)$ on the identification (black line) and validation (grey line) data sets.

state space equations (24). Hence, the simulated output DFT spectrum $\hat{Y}(k)$ (29) and the true output DFT spectrum $Y_0(k) = DFT(y_0(nT_s))$ can only be equal within a transient (leakage) error $T_Y(j\omega_k)$ which is a smooth function of the frequency. Moreover, the transient terms in the identification and validation data sets are different. Therefore, the transient term $T_Y(j\omega_k)$ is removed nonparametrically in the simulation error $\hat{Y}(k) - Y_0(k)$ using the local polynomial method for output observations only (see [33], [34]). Finally, the simulation error $\hat{Y}(k) - Y_0(k)$ is also predicted using the predicted covariance and bias error of $\hat{G}_m(j\omega_k)$ (see Appendix B for the details).

Figure 11 shows the results on the identification and validation data sets. It can be seen that the transient contribution (dashed black lines) is significantly larger than the simulation error (grey lines), and that the actual (grey lines) and predicted (red lines) simulation errors coincide. Note also that the simulation errors of the identification and validation data sets are very similar and almost everywhere at least 120 dB (6 significant digits are correct) below the true output DFT spectrum.

3) Results Identification in Closed Loop: An identification data set of the time-variant system (24–26) operating in feedback (27) is generated as described in Section V-B1. The procedure of Section IV-B is applied to estimate the TV-FRF. First, the indirect reference to input-output models $H_i^{ry}(j\omega)$ and $H_m^{ru}(j\omega)$ with $N_b = 13$ are estimated using an order R = 12 and a local bandwidth of 482 frequencies for the local polynomial approximation. Next, the deconvolution of the indirect models (Theorem 3) is performed without the second order derivatives in (22). Finally, FRFs $\check{G}_m(j\omega_k)$ of the direct model (9) are calculated without the second order derivatives. Note that compared with the open loop identification more time-variant branches are needed for the identification in closed loop. This is due to the fact that the poles



Figure 12. Relative bias $|\check{G}(j\omega_k,t) - \hat{G}(j\omega_k,t)|/\hat{G}(j\omega_k,t)|$ of the nonparametric TV-FRF estimate $\check{G}(j\omega_k,t)$ (32) – Example 2: closed loop identification. $\hat{G}(j\omega_k,t)$ is the nonparametric TV-FRF estimate (28) obtained in open loop.

of $H_i^{ry}(s)$ and $H_m^{ru}(s)$ are closer to the imaginary axis than those of $H_n(s)$.

First, the nonparametric TV-FRF estimate $\check{G}(j\omega_k, t)$ obtained in closed loop

$$\check{G}(j\omega_k, t) = \sum_{p=0}^{N_b} \check{G}_p(j\omega_k) f_p(t)$$
(32)

is compared in Figure 12 to the TV-FRF estimate obtained in open loop. At the borders the relative bias is -88 dB (more than 4 significant digits are correct) while elsewhere the error is below -125 dB (at least 6 significant digits are correct).

Next, the TV-FRF estimate (32) is used to simulate the output of the open loop setup

$$\check{Y}(k) = \sum_{p=0}^{N_b} \text{DFT}\left(\text{IDFT}\left(\check{G}_p(j\omega_k) U_0(k)\right) f_p(t)\right)$$
(33)

where $U_0(k)$ is the input DFT spectrum of the validation data set. The simulation error of (33) is here mainly due to the undermodelling (*R* and/or N_b are too small) quantified by the covariance of $\check{G}_m(j\omega_k)$ and to the contributions of the second order derivatives in (12) and (22) quantified by the bias of $\check{G}_m(j\omega_k)$. Hence, it can be predicted (see Appendix B for the details). Comparing Figure 13 to Figure 11 it can be seen that the actual simulation errors of the TV-FRF estimates (28) and (32) on the open loop validation data set are almost equal. Although the predicted simulation error (red line) is somewhat too conservative, it gives a useful estimate of the order of magnitude of the actual simulation error (grey line). Except at a few frequencies, the relative bias error in Figure 13 is smaller than -120 dB.

VI. MEASUREMENT EXAMPLE

A. Experimental setup

The electronic circuit is a second order time-variant bandpass filter (see Fig. 14) operating in feedback (Fig. 1 with unity feedback loop and linear time-invariant actuator). Three variable resistors controlled by the scheduling voltage p(t) induce the timevariation. The reference signal r(t) in Fig. 1 is a random phase multisine (23) with A chosen such that its rms value equals 48 mV, ϕ_k independent and uniformly $[0, 2\pi)$ distributed random



Figure 13. Open loop simulation error of the nonparametric TV-FRF estimate – Example 2: closed loop identification. Left figure: output DFT spectrum $Y_0(k)$ (black line), estimated transient term $\tilde{T}_Y(j\omega_k)$ (black dashed line), actual simulation error $\check{Y}(k) - Y_0(k) - \check{T}_Y(j\omega_k)$ (grey line) with $\check{Y}(k)$ the simulated output (33), and predicted simulation error (red line). Right figure: relative simulation error $(\check{Y}(k) - Y_0(k) - \check{T}_Y(j\omega_k))/Y(k)$.



Figure 14. Electronic circuit consisting of three high gain operational amplifiers (TL071), two capacitors ($C_1 = C_2 = 10 \text{ nF}$), four resistors ($R_1 = R_2 = 10 \text{ k}\Omega$, $R_3 = 5.31 \text{ k}\Omega$, and $R_4 = 100.8 \text{ k}\Omega$), and three voltage dependent resistors R(p(t)) made with an electro-optical component VTL5C1 (see [21] for the details). The rms values of the measured input u(t) and output y(t) signals are, respectively, 72 mV and 88 mV; and the scheduling voltage p(t) varies between 0.8 V and 1.1 V.

variables, $f_0 = f_s/N$ with $f_s = 625 \text{ kHz}$ the sampling frequency and $N = 128 \times 1024$ the number of samples in one period, $F_1 = \lceil 200 \text{ Hz}/f_0 \rceil = 42$, and $F_2 = \lfloor 40 \text{ kHz}/f_0 \rfloor = 8388$. This signal excites the frequency band (200 Hz, 40 kHz) and the inputoutput responses to one reference signal period are measured for two different random phase realisations. The first data set is used for identification and the second for validation. Fig. 14 also shows the variation of the scheduling voltage p(t) that modifies the timevariant resistors during the experiments. Note that p(t) is not used in the estimation procedure.

Two HP 1445A generator cards (50 Ω output impedance) are used for generating the random phase multisine r(t) and the scheduling voltage p(t). To avoid loading of the electronic circuit, the input-output signals u(t) and y(t) are first buffered (> 5M Ω input impedance, 50 Ω output impedance) before being connected to the HP E1430A data acquisition channels. Both generator and data acquisition cards of the VXI measurement setup are synchronised (coherent sampling at $f_s = 625$ kHz). Fig. 14 shows the measured input-output signals of the identification data set.

B. Results

First, the indirect reference to input-output models $H_i^{ry}(j\omega)$ and $H_m^{ru}(j\omega)$ with $N_b = 13$ are estimated using an order R = 4 and a local bandwidth of 875 frequencies for the local polynomial approximation. Next, the deconvolution of the indirect models (Theorem 3) is performed without the second order derivatives in (22). Finally, FRFs $\check{G}_m(j\omega_k)$ of the direct model (9) are calculated



Figure 15. Nonparametric TV-FRF estimate (32) – Measurement example. Top left: amplitude TV-FRF; top right: phase TV-FRF; bottom left: frequency-time plot amplitude TV-FRF; bottom right: predicted relative bias error TV-FRF estimate in the frequency band [3 kHz, 40 kHz] and the time interval [0.01 s, 0.19 s].



Figure 16. Simulation error of the nonparametric TV-FRF estimate – Measurement example. Left: identification data set; and right: validation data set. Black lines: measured output DFT spectrum Y(k). Grey lines: actual simulation error $\check{Y}(k) - Y(k) - \check{T}_Y(j\omega_k)$, with $\check{T}_Y(j\omega_k)$ the estimated transient term, and $\check{Y}(k)$ the simulated output (33) where $U_0(k)$ is replaced by the measured input DFT spectrum U(k). Red lines: predicted simulation error (eq. (42) where $U_0(k)$ is replaced by U(k)).

without the second order derivatives. The latter are used to quantify the $O(T^{-2})$ bias error contribution of $\check{G}_m(j\omega_k)$ to the TV-FRF and to the mean square error of the simulated output. Figure 15 shows the corresponding TV-FRF estimate (32). It can be seen that the predicted relative bias error of the TV-FRF estimate is almost everywhere smaller than - 40 dB. Note also that the time evolution of the resonance frequency has the same shape as the scheduling voltage p(t) in Fig. 14.

Finally, the nonparametric TV-FRF estimate and the measured input are used to simulate the output of the identification and validation data sets via (33) where the true input DFT spectrum $U_0(k)$ is replaced by the measured spectrum U(k). Via the predicted bias and variance of the $\check{G}_m(j\omega_k)$ estimates, the mean square error of the simulated output is calculated (see Appendix B). The results are shown in Fig. 16. It can be seen that the simulation error (grey lines) is within the predicted bound (red lines) for both the identification and validation data sets, which validates the estimated TV-FRF model (9) with $N_b = 13$ time-variant branches.

VII. CONCLUSIONS

A nonparametric method for estimating the time-variant frequency response function from noisy input-output measurements has been presented. The proposed method accounts for possible linear time-variant interactions between the actuator and the plant and/or the presence of a (time-variant) feedback loop. The key to solve this problem is the deconvolution of the impulse response of the cascade of two linear time-variant systems. Although the deconvolution issue has been solved for a particular class of timevariant systems only, the many simulations performed of which only one example is reported here, and the real measurements suggest – but do not prove – that the proposed estimation procedure is robust w.r.t. the system assumption made. Matlab software is available on demand.

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APPENDIX

A. Proof of Lemma 1

In this appendix (16) is first proven for n = 1, 2, and 3. Next, it is shown by induction that (16) is valid for any $n \in \mathbb{N}$.

To simplify the notations, the interval [0,T] is replaced by [-T/2, T/2]. Proceeding in this way the basis functions $f_n(t)$ are related to the Legendre polynomials $l_n(x)$, $x \in [-1,1]$, as $f_n(t) = l_n(\alpha t)$, with $\alpha = 2/T$ and $t \in [-T/2, T/2]$. Hence, they satisfy the following recurrence formula

$$(n+1)f_{n+1}(t) = (2n+1)\alpha t f_n(t) - n f_{n-1}(t)$$
(34)

with n > 1, $f_0(t) = 1$, and $f_1(t) = \alpha t$ [27]. For notational simplicity, the transient terms $T_1(s)$ and $T_2(s)$ in (15) and (16) will be discarded in the calculations. The equalities in the remainder of the proof should be interpreted as exact within a transient term that is accounted for by $T_1(s)$ and $T_2(s)$.

Since the interval [0,T] is replaced by [-T/2,T/2], the onesided Laplace transform (14) is replaced by the two-sided Laplace transform [37] of the windowed signals

$$Y_0(s) = L\{y_0(t)\} = \int_{-T/2}^{+T/2} y_0(t) e^{-st} dt$$
(35)

A key property of the Laplace transform (35) used in the proof is that $L\{t^r x(t)\} = (-1)^r X^{(r)}(s)$ [37]. Note that (15) remains valid for the two-sided Laplace transform (35) (proof: see Appendix A of [38]).

Base case: (16) is valid for n = 1, 2, and 3

Consider the top block diagram of Fig. 7 with n = 1, and where the output of the dynamic block is denoted as u(t). Using $f_1(t) = \alpha t$, $u_1(t) = f_1(t)u(t)$, $U(s) = H_m^{ru}(s)R_m(s)$, and $L\{tx(t)\} = -X^{(1)}(s)$, we find

$$U_{1}(s) = -\alpha U^{(1)}(s)$$

= $H_{m}^{ru}(s)(-\alpha R_{m}^{(1)}(s)) - \alpha H_{m}^{ru(1)}(s)R_{m}(s)$
= $H_{m}^{ru}(s)L\{f_{1}(t)r_{m}(t)\} - \alpha H_{m}^{ru(1)}(s)R_{m}(s)$
= $H_{m}^{ru}(s)R_{m,1}(s) - \alpha H_{m}^{ru(1)}(s)R_{m}(s)$

which equals (16) for n = 1 and $O(\alpha^3) = 0$.

Consider now the block diagram in Fig. 7 with n = 2. Using $f_2(t) = (3\alpha^2 t^2 - 1)/2$, $u_2(t) = f_2(t)u(t)$, $U(s) = H_m^{ru}(s)R_m(s)$, and $L\{t^r x(t)\} = (-1)^r X^{(r)}(s)$ for r = 1, 2, we find via similar calculations as for n = 1

$$U_{2}(s) = \frac{3}{2}\alpha^{2}U^{(2)}(s) - \frac{1}{2}U(s)$$

= $H_{m}^{ru}(s)R_{m,2}(s) - 3\alpha H_{m}^{ru(1)}(s)R_{m,1}(s)$
+ $\frac{3}{2}\alpha^{2}H_{m}^{ru(2)}(s)R_{m}(s)$

which proves (16) for n = 2 and $O(\alpha^3) = 0$.

Repeating the calculations for n = 3 in Fig. 7 with $f_3(t) = (5\alpha^3 t^3 - 3\alpha t)/2$ gives after some calculations

$$U_{3}(s) = -\frac{5}{3}\alpha^{3}U^{(3)}(s) + \frac{3}{2}\alpha U^{(1)}(s)$$

= $H_{m}^{ru}(s)R_{m,3}(s) - 5\alpha H_{m}^{ru(1)}(s)R_{m,2}(s)$
 $- \alpha H_{m}^{ru(1)}(s)R_{m}(s) + \frac{15}{2}\alpha^{2}H_{m}^{ru(2)}(s)R_{m,1}(s)$
 $- \frac{5}{2}\alpha^{3}H_{m}^{ru(3)}(s)R_{m}(s)$

which demonstrates (16) for n = 3 and $O(\alpha^3) = -\frac{5}{2}\alpha^3 H_m^{ru(3)}(s)R_m(s)$.

Inductive step: (16) is valid for any natural number n

Assuming that (16) is valid for all natural numbers smaller than or equal to n, we will prove that (16) holds for n + 1. Combining (15), where n is replaced by n + 1, with (34) using $L\{tx(t)\} = -X^{(1)}(s)$, gives

$$U_{n+1}(s) = -\frac{2n+1}{n+1}\alpha U_n^{(1)}(s) - \frac{n}{n+1}U_{n-1}(s)$$
(36)

By the induction hypothesis (16) holds for $U_n(s)$ and $U_{n-1}(s)$. Using (16), the term $-\alpha U_n^{(1)}(s)$ in (36) can be calculated. This involves derivatives of the form $-\alpha R_{m,p}^{(1)}(s)$ which, using (34), can be simplified as

$$-\alpha R_{m,p}^{(1)}(s) = L\{\alpha tr_m(t)f_p(t)\}\$$

= $\frac{p+1}{2p+1}L\{r_m(t)f_{p+1}(t)\} + \frac{p}{2p+1}L\{r_m(t)f_{p-1}(t)\}\$
= $\frac{p+1}{2p+1}R_{m,p+1}(s) + \frac{p}{2p+1}R_{m,p-1}(s)$ (37)

Substituting in (36) the expressions for $U_n^{(1)}(s)$ (derivative of (16), where the terms $-\alpha R_{m,p}^{(1)}(s)$ are replaced by (37)) and $U_{n-1}(s)$ ((16), where *n* is replaced by n-1), gives after some calculations Eq. (16), where *n* is replaced by n+1.

B. Prediction of the Simulation Error

Using (30) and (31), the simulated output (29) can be rewritten as

$$\hat{Y}(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N_b} \sum_{l=0}^{N-1} \hat{G}_m(j\omega_l) U_0(l) F_m(k-l)$$
(38)

where $F_m(k) = \text{DFT}(f_m(nT_s))$. In the sequel the mean squared error of $\hat{Y}_{[r]}(k)$, the *r*th entry of $\hat{Y}(k)$, is calculated under the following assumptions on the true input $u_0(t)$.

Assumption 4 (true input): The true input DFT spectrum $U_0(k)$ (i) is uncorrelated over the frequency, (ii) has a diagonal power spectrum ($\mathbb{E}\{U_0(k)U_0^H(k)\}$ is diagonal), and (iii) is independently distributed of $\hat{G}_m(j\omega_l)$.

Assumption 4(i) is exact for random phase multisine excitations (23) and is asymptotically (for $N \to \infty$) true for filtered white noise. Assumption 4(ii) requires that the n_u input signals are uncorrelated, and Assumption 4(iii) is fulfilled for the validation data set.

The mean squared error (MSE) of $\hat{Y}_{[r]}(k)$ is given by

$$MSE(\hat{Y}_{[r]}(k)) = \mathbb{E}\left\{ \left| \hat{Y}_{[r]}(k) - Y_{0[r]}(k) \right|^{2} \right\}$$
(39)

$$\hat{Y}(k) - Y_0(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N_b} \sum_{l=0}^{N-1} \Delta \hat{G}_m(j\omega_l) U_0(l) F_m(k-l)$$
(40)

with $Y_0(k)$ the true output DFT spectrum, $\Delta \hat{G}_m(j\omega_l)$ the sum of the noise and bias contributions of the estimate $\hat{G}_m(j\omega_l)$, and where the expected value is taken w.r.t. $\hat{G}_m(j\omega_l)$ and $U_0(k)$. Elaboration of (39) requires the diagonal elements of the following matrix

$$C^{mp} = \mathbb{E} \left\{ \Delta \hat{G}_m(j\omega_l) U_0(l) U_0^H(l') \Delta \hat{G}_p^H(j\omega_{l'}) \right\}$$
$$= \mathbb{E} \left\{ \Delta \hat{G}_m(j\omega_l) \mathbb{E} \left\{ U_0(l) U_0^H(l) \right\} \Delta \hat{G}_p^H(j\omega_l) \right\}$$
(41)

where the second equality uses Assumptions 4(i, iii). Note that (41) is only approximately true for the identification data set $(\hat{G}_m(j\omega_k)$ depends on $U_0(k)$). Combining (39) and (41) taking into account Assumption 4(ii) finally gives

$$MSE(\hat{Y}_{[r]}(k)) = \frac{1}{N} \sum_{m,p=0}^{N_b} \sum_{l=0}^{N-1} C_{[r,r]}^{mp}(l) Q_{mp}(k-l)$$
$$= \frac{1}{\sqrt{N}} \sum_{m,p=0}^{N_b} DFT \left(IDFT \left(C_{[r,r]}^{mp}(k) \right) IDFT \left(Q_{mp}(k) \right) \right)$$
(42)

with

$$C_{[r,r]}^{mp} = \sum_{v=1}^{n_u} \text{MSE}\left(\hat{G}_{m[r,v]}\left(j\omega_l\right), \hat{G}_{p[r,v]}\left(j\omega_l\right)\right) \mathbb{E}\left\{\left|U_{0[v]}\left(k\right)\right|^2\right\} \\ Q_{mp}\left(k\right) = F_m\left(k\right)\overline{F_p\left(k\right)}$$

where \overline{x} is the complex conjugate of x, and $MSE(x, y) = \mathbb{E}\{(x - x_0)(\overline{y - y_0})\}$, with x_0 , y_0 the true values. An estimate of the MSE of $\hat{G}_{m[r,v]}(j\omega_l)$ is obtained from the residuals of the local polynomial approximation (see [33] for the details).

REFERENCES

- Y. Van Ingelgem, E. Tourwé, J. Vereecken and A. Hubin, "Application of multisine impedance spectroscopy, FE-AES and FE-SEM to study the early stages of copper corrosion," *Electrochemica Acta*, vol. 53, no. 25, pp. 7523-7530, 2008.
- [2] S. Y. Chen, and J. S. Yuan, "Adaptive gate bias for power amplifier temperature compensation," *IEEE Trans. on Device and Material Reliability*, vol. 11, no. 3, pp. 442-449, 2011.
- [3] K. Chen, and X. Li, "Accurate determination of battery discharge characteristics - A comparison between two battery temperature control methods," *Journal of Power Sources*, vol. 247, pp. 961-966, 2014.
- [4] J. R. M. Aerts and J. J. J. Dirckx, "Nonlinearity in eardrum vibration as a function of frequency and sound pressure," *Hearing Research*, vol. 263, no. 1-2, pp. 26-32, 2010.
 [5] B. Bigland-Ritchie, "EMG/force relations and fatigue of human
- [5] B. Bigland-Ritchie, "EMG/force relations and fatigue of human voluntary contractions," *Exercise and Sport Sciences Reviews*, vol. 9, no. 1, pp. 75-117, 1981.
- [6] G. Dimitriadis and J. E. Cooper, "Flutter prediction from flight flutter test data," *Journal of Aircraft*, vol. 38, no. 2, pp. 355-367, 2001.
- [7] B. Sanchez, E. Louarroudi, E. Jorge, J. Cinca, R. Bragos, and R. Pintelon, "A new measuring and identification approach for timevarying bioimpedance using multisine electrical impedance spectroscopy," *Physiological Measurement*, vol. 34, no. 3, pp. 339-357, 2013.
- [8] B. Sanchez, E. Louarroudi, R. Bragos, and R. Pintelon, "Harmonic impedance spectra identification from time-varying bioimpedance: theory and validation," *Physiological Measurement*, vol. 34, no. 10, pp. 1217-1238, 2013.
- [9] C. W. Lee, and D. J. Han, "Strength of modes in rotating machinery," *Journal of Sound and Vibration*, vol. 313, no. 1-2, pp. 268-289, 2008.
- [10] A. Jhinaoui, L. Mevel, and J. Morlier, "A new SSI algorithm for LPTV systems: Application to a hinged-bladed helicopter," *Mechanical Systems and Signal Processing*, vol. 42, no. 1-2, pp. 152-166, 2014.
- [11] B. J. B. Grant, J. M. Fitzpatrick, and B. B. Lieber, "Time-varying pulmonary arterial compliance," *Journal of Applied Physiology*, vol. 70, no. 2, pp. 575-583, 1991.
- [12] E. Möllerstedt, and B. Bernhardsson, "Out of control because of harmonics – An analysis of the harmonic response of an inverter locomotive," *IEEE Control Systems Magazine*, vol. 20, no. 4, pp. 70-81, 2000.
- [13] R. P. Aguilera, B. I. Godoy, J. C. Agüero, G. C. Goodwin, and J. I. Yuz, "An EM-based identification algorithm for a class of hybrid systems with application to power electronics," *International Journal of Control*, vol. 87, no. 7, pp. 1339-1351, 2014.
- [14] J. D. Hamilton, "Analysis of time series subject to changes in regime," *Journal of Econometrics*, vol. 45, no. 1-2, pp. 39-70, 1990.
- [15] G. G. Yin, S. Kan, L. Y. Wang, and C. Z. Xu, "Identification of systems with regime switching and unmodeled dynamics," *IEEE Trans. Autom. Contr.*, vol. 54, no. 1, pp. 34-47, 2009.
- [16] S. Paoletti, A. Lj. Juloski, G. Ferrari-Trecate, and R. Vidal, "Identification of hybrid systems – A tutorial," *European Journal of Control*, vol. 13, no. 2-3, pp. 242-260, 2007.
- [17] L. A. Zadeh, "The determination of the impulsive response of variable networks," *Journal of Applied Physics*, vol. 21, pp. 642-645, 1950.
- [18] L. A. Zadeh, "Frequency analysis of variable networks," *Proceedings of the I.R.E.*, vol. 38, pp. 291-299, 1950.
- [19] R. Tóth, Modeling and identification of linear parameter-varying systems. Berlin: Springer Verlag, 2010.

- [20] R. Pintelon, E. Louarroudi, and J. Lataire, "Time-variant frequency response function measurement on weakly nonlinear, arbitrarily timevarying systems excited by periodic inputs," *IEEE Trans. Instrum. and Meas.*, vol. 64, no. 10, pp. 2829-2837, 2015.
- [21] R. Pintelon, E. Louarroudi, and J. Lataire, "Detection and quantification of the influence of time variation in closed-loop frequencyresponse-function measurements," *IEEE Trans. Instrum. and Meas.*, vol. 62, no. 4, pp. 853-863, 2013.
- [22] P. E. Wellstead, "Reference signals for closed-loop identification," Int. J. Control, vol. 26, no. 6, pp. 945–962, 1977.
- [23] P. E. Wellstead, "Non-parametric methods of system identification," *Automatica*, vol. 17, no. 1, pp. 55-69, 1981.
- [24] L. A. Zadeh, "On stability of linear varying-parameter systems," *Journal of Applied Physics*, vol. 22, no. 4, pp. 402-405, 1951.
- [25] B. Boashash, ed., Time Frequency Signal Analysis and Processing: A Comprehensive Reference. Amsterdam: Elsevier, 2003.
- [26] R. Pintelon, E. Louarroudi, and J. Lataire, "Nonparametric timevariant frequency response function estimates using arbitrary excitations," *Automatica*, vol. 51, no. 1, pp. 308-317, 2015.
- [27] M. Abramowitz, and I. A. Stegun, Handbook of Mathematical Functions. New York: Dover Publications, 1970.
- [28] R. Tóth, P. S. C. Heuberger, and P. M. J. van den Hof, "Asymptotically optimal orthonormal basis functions for LPV system identification," *Automatica*, vol. 45, no. 6, pp. 1359-1370, 2009.
- [29] A. Ralston, and P. Rabinowitz, A First Course in Numerical Analysis. New York: McGraw-Hill, 1984.
- [30] R. Pintelon, E. Louarroudi, and J. Lataire, "Quantifying the timevariation in FRF measurements using random phase multisines with nonuniformly spaced harmonics," *IEEE Trans. Instrum. and Meas.*, vol. 63, no. 5, pp. 1384-1394, 2014.
- [31] E. Louarroudi, R. Pintelon, and J. Lataire, "Nonparametric tracking of the time-varying dynamics of weakly nonlinear periodically timevarying systems using periodic inputs," *IEEE Trans. on Instrum. and Meas.*, vol. 61, no. 5, pp. 1384-1394, 2012.
- [32] E. Louarroudi, J. Lataire, R. Pintelon, P. Janssens, and J. Swevers, "Frequency domain, parametric estimation of the evolution of the time-varying dynamics of periodically time-varying systems from noisy input-output observations," *Mechanical Systems and Signal Processing*, vol. 47., no. 1-2, pp. 151-174, 2014.
- [33] R. Pintelon, J. Schoukens, G. Vandersteen, and K. Barbé, "Estimation of nonparametric noise and FRF models for multivariable systems -Part I: theory," *Mechanical Systems and Signal Processing*, vol. 24, no. 3, pp. 573-595, 2010.
- [34] R. Pintelon, J. Schoukens, G. Vandersteen, and K. Barbé, "Estimation of nonparametric noise and FRF models for multivariable systems - Part II: extensions, applications," *Mechanical Systems and Signal Processing*, vol. 24, no. 3, pp. 596-616, 2010.
- [35] M. K. Bowen, and R. Smith, "Derivative formulae and errors for non-uniformly spaced points," *Proceedings of the Royal Society A*, vol. 461, no. 2059, pp. 1975-1997, 2005.
- [36] R. Pintelon, and J. Schoukens, System Identification: A Frequency Domain Approach, 2nd edition. Hoboken: John Wiley - IEEE press, 2012.
- [37] H. Kwakernaak, and R. Sivan, *Modern Signals and Systems*. Englewood Cliffs: Prentice-Hall, 1991.
- [38] J. Lataire, R. Pintelon and E. Louarroudi, "Non-parametric estimate of the system function of a time-varying system," *Automatica*, vol. 48, no. 4, pp. 666-672, 2012.