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***Abstract*—This article considers the temporal logic defined over the class of all lexicographic products of dense linear orders without endpoints and provides a complete axiomatization for it.**

I. Introduction

Given modal logics L_1 and L_2 in languages respectively based on $\square_1$ and $\square_2$, their "Cartesian" product is a multimodal logic in the language based on both $\square_1$ and $\square_2$. Its semantics is based on the product $\mathcal{F}_1 \times \mathcal{F}_2 = (W, S_1, S_2)$ of structures $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$ defined by: $W = W_1 \times W_2$, $(x_1, x_2)\ S_1\ (y_1, y_2)$ iff $x_1\ R_1\ y_1$ and $x_2 = y_2$ and $(x_1, x_2)\ S_2\ (y_1, y_2)$ iff $x_1 = y_1$ and $x_2\ R_2\ y_2$. The above product of structures has been considered within the context of reasoning about knowledge [7]. See [9] for a detailed study of the axiomatization of the corresponding modal logics.

Given modal logics L_1 and L_2 in languages respectively based on $\square_1$ and $\square_2$, it also makes sense to consider their "lexicographic" product defined as a multimodal logic in the language based on both $\square_1$ and $\square_2$. Its semantics is based on the product $\mathcal{F}_1 \rhd \mathcal{F}_2 = (W, S_1, S_2)$ of structures $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$ defined by: $W = W_1 \times W_2$, $(x_1, x_2)\ S_1\ (y_1, y_2)$ iff $x_1\ R_1\ y_1$ and $(x_1, x_2)\ S_2\ (y_1, y_2)$ iff $x_1 = y_1$ and $x_2\ R_2\ y_2$. The above product of structures has been considered within the context of reasoning about time [1]. See [2] for a first step towards the axiomatization of the corresponding modal logics.

This article considers the temporal logic defined over the class of all lexicographic products of dense linear orders without endpoints and gives its complete axiomatization. Its section-by-section breakdown is as follows. Section II defines the lexicographic product of dense linear orders without endpoints and studies its elementary properties. In section III, we introduce the syntax and the semantics of the temporal logic we will be working with. Section IV gives its axiomatization. In section V and section VI, a method is presented for proving the completeness of this axiomatization. Section VII pays particular attention to the pure future fragment of our temporal language.

II. Lexicographic products of linear orders

Let $(S, <_S)$ and $(T, <_T)$ be dense linear orders without endpoints. Their lexicographic product is the structure $F = (\mathcal{R}, \prec_1, \prec_2)$ where $\mathcal{R} = S \times T$ and $\prec_1$ and $\prec_2$ are the binary relations on $\mathcal{R}$ defined by $(s, t) \prec_1 (s', t')$ iff $s <_S s'$ and $(s, t) \prec_2 (s', t')$ iff $s = s'$ and $t <_T t'$. The effect of the operation of lexicographic product may be described informally as follows: F is the structure obtained from $(S, <_S)$ and $(T, <_T)$ by replacing each element of $(S, <_S)$ by a copy of $(T, <_T)$. See [3] or [6] for a discussion about the global intuitions underlying such an operation. In order to characterize its elementary properties, we introduce a first-order language. Let Var denote a countable set of individual variables (with typical members denoted x, y, etc). The set of all well-formed formulas (with typical members denoted ϕ, ψ, etc) of the first-order language is given by the rule

- $\phi := x <_1 y \mid x <_2 y \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \forall x\ \phi \mid x = y$.

The intended meanings of $x <_1 y$ and $x <_2 y$ are as follows: "x precedes but is not infinitely close to y" and "x precedes and is infinitely close to y". We adopt the standard definitions for the remaining Boolean operations and for the existential quantifier. Another construct can be defined in terms of the primitive ones as follows:

- $x < y := x <_1 y \vee x <_2 y$.

The intended meaning of $x < y$ is as follows: "x precedes y". The notion of a subformula is standard. We adopt the standard rules for omission of the parentheses. Formulas in which every individual variable in an atomic subformula is in the scope of a corresponding quantifier are called sentences. Models for the first-order language are flows $F = (\mathcal{R}, \prec_1, \prec_2)$ where $\mathcal{R}$ is a nonempty set of instants and $\prec_1$ and $\prec_2$ are binary relations on $\mathcal{R}$. We define the binary relation $\prec$ on $\mathcal{R}$ by $t \prec u$ iff either $t \prec_1 u$, or $t \prec_2 u$ for each $t, u \in \mathcal{R}$. An assignment on F is a function $f\colon Var \mapsto \mathcal{R}$. Satisfaction is a 3-place relation $\models$ between a flow $F = (\mathcal{R}, \prec_1, \prec_2)$, an assignment f on F and a formula ϕ. It is inductively defined as usual. In particular,

- $F \models_f x <_1 y$ iff $f(x) \prec_1 f(y)$ and
- $F \models_f x <_2 y$ iff $f(x) \prec_2 f(y)$.

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- $F \models_f x < y$ iff $f(x) \prec f(y)$.

Obviously, every lexicographic product of dense linear orders without endpoints satisfies the following sentences:

$IRRE$
- $\forall x\ x \not<_1 x$,
- $\forall x\ x \not<_2 x$,

$DISJ$
- $\forall x\ \forall y\ (x \not<_1 y \vee x \not<_2 y)$,

$TRAN$
- $\forall x\ \forall y\ (\exists z\ (x <_1 z \wedge z <_1 y) \rightarrow x <_1 y)$,
- $\forall x\ \forall y\ (\exists z\ (x <_1 z \wedge z <_2 y) \rightarrow x <_1 y)$,
- $\forall x\ \forall y\ (\exists z\ (x <_2 z \wedge z <_1 y) \rightarrow x <_1 y)$,
- $\forall x\ \forall y\ (\exists z\ (x <_2 z \wedge z <_2 y) \rightarrow x <_2 y)$,

$DENS$
- $\forall x\ \forall y\ (x <_1 y \rightarrow \exists z\ (x <_1 z \wedge z <_1 y))$,
- $\forall x\ \forall y\ (x <_1 y \rightarrow \exists z\ (x <_1 z \wedge z <_2 y))$,
- $\forall x\ \forall y\ (x <_1 y \rightarrow \exists z\ (x <_2 z \wedge z <_1 y))$,
- $\forall x\ \forall y\ (x <_2 y \rightarrow \exists z\ (x <_2 z \wedge z <_2 y))$,

$SERI$
- $\forall x\ \exists y\ x <_1 y$,
- $\forall x\ \exists y\ x <_2 y$,
- $\forall x\ \exists y\ y <_1 x$,
- $\forall x\ \exists y\ y <_2 x$ and

$UNIV$
- $\forall x\ \forall y\ (x = y \vee x <_1 y \vee x <_2 y \vee y <_1 x \vee y <_2 x)$.

Obviously, the sentences as above have not the finite model property. By Löwenheim-Skolem theorem, they have models in each infinite power. A flow $F = (\mathcal{R}, \prec_1, \prec_2)$ is said to be standard iff it satisfies the sentences as above. Let $F = (\mathcal{R}, \prec_1, \prec_2)$ be a flow, R be a binary relation on $\mathcal{R}$ and $\mathcal{L}$ be a sublanguage of our first-order language. We shall say that R is definable with $\mathcal{L}$ in F iff there exists a formula $\phi(x, y)$ in $\mathcal{L}$ such that for all assignments f on F, $f(x)\ R\ f(y)$ iff $F \models_f \phi(x, y)$.

Proposition 1: (1) $=$ is not definable with $<_1$ in any standard flow; (2) $=$ is definable with $<_2$ in any standard flow; (3) $\prec_2$ is not definable with $=$ and $<_1$ in any standard flow; (4) $\prec_1$ is not definable with $=$ and $<_2$ in any standard flow; (5) $\prec_1$ is not definable with $=$ and $<$ in any standard flow; (6) $\prec_2$ is not definable with $=$ and $<$ in any standard flow.

The following proposition illustrates the value of countable standard flows.

Proposition 2: Let $F = (\mathcal{R}, \prec_1, \prec_2)$ and $F' = (\mathcal{R}', \prec_1', \prec_2')$ be standard flows. If F is countable then F is elementary embeddable in F'.

As a corollary of proposition 2 we obtain that any two standard flows are elementary equivalent. The first-order theory HY of standard flows has the following list of proper axioms: $IRRE$, $DISJ$, $TRAN$, $DENS$, $SERI$ and $UNIV$. There are several results about HY:

Proposition 3: (1) HY is countably categorical; (2) HY is not categorical in any uncountable power; (3) HY is maximal consistent; (4) HY is complete with respect to the lexicographic product of any dense linear orders without endpoints.

The membership problem in HY is this: given a sentence ϕ, determine whether ϕ is in HY. The results are summarized in the following proposition:

Proposition 4: (1) HY is decidable; (2) The membership problem in HY is $PSPACE$-complete.

See [1] for the proofs of the above results.

III. A TEMPORAL LOGIC

It is now time to meet the temporal logic we will be working with.

A. Syntax

Let At be a countable set of atomic formulas (with typical members denoted p, q, etc). We define the set of formulas of the temporal language (with typical members denoted ϕ, ψ, etc) as follows:

- $\phi := p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid G_1\phi \mid G_2\phi \mid H_1\phi \mid H_2\phi$,

the formulas $G_1\phi$ and $G_2\phi$ being read "ϕ will be true at each instant within the future of but not infinitely close to the present instant" and "ϕ will be true at each instant within the future of and infinitely close to the present instant" and the formulas $H_1\phi$ and $H_2\phi$ being read "ϕ has been true at each instant within the past of but not infinitely close to the present instant" and "ϕ has been true at each instant within the past of and infinitely close to the present instant". We adopt the standard definitions for the remaining Boolean connectives. As usual, we define

- $F_i\phi := \neg G_i\neg\phi$ and
- $P_i\phi := \neg H_i\neg\phi$

for each $i \in \{1, 2\}$. The notion of a subformula is standard. It is usual to omit parentheses if this does not lead to any ambiguity.

B. Semantics

A Kripke model is a structure $\mathcal{M} = (\mathcal{R}, \prec_1, \prec_2, V)$ where $(\mathcal{R}, \prec_1, \prec_2)$ is a flow and $V\colon \mathcal{R} \mapsto 2^{At}$ is a function. $V^{-1}\colon At \mapsto 2^{\mathcal{R}}$ will denote the function such that $V^{-1}(p) = \{s \in \mathcal{R}\colon p \in V(s)\}$. Satisfaction is a 3-place relation $\models$ between a Kripke model $\mathcal{M} = (\mathcal{R}, \prec_1, \prec_2, V)$, an instant $t \in \mathcal{R}$ and a formula ϕ. It is inductively defined as usual. In particular, for all $i \in \{1, 2\}$,

- $\mathcal{M} \models_t G_i\phi$ iff $\mathcal{M} \models_u \phi$ for each instant $u \in \mathcal{R}$ such that $t \prec_i u$ and
- $\mathcal{M} \models_t H_i\phi$ iff $\mathcal{M} \models_u \phi$ for each instant $u \in \mathcal{R}$ such that $u \prec_i t$.

As a result, for all $i \in \{1, 2\}$,

- $\mathcal{M} \models_t F_i\phi$ iff $\mathcal{M} \models_u \phi$ for some instant $u \in \mathcal{R}$ such that $t \prec_i u$ and
- $\mathcal{M} \models_t P_i\phi$ iff $\mathcal{M} \models_u \phi$ for some instant $u \in \mathcal{R}$ such that $u \prec_i t$.

Let ϕ be a formula. We shall say that ϕ is true in a Kripke model $\mathcal{M} = (\mathcal{R}, \prec_1, \prec_2, V)$, in symbols $\mathcal{M} \models \phi$, iff $\mathcal{M} \models_t \phi$ for each instant $t \in \mathcal{R}$. ϕ is said to be valid in a flow $(\mathcal{R}, \prec_1, \prec_2)$, in symbols $(\mathcal{R}, \prec_1, \prec_2) \models \phi$, iff $\mathcal{M} \models \phi$ for each Kripke model $\mathcal{M} = (\mathcal{R}, \prec_1, \prec_2, V)$ based on $(\mathcal{R}, \prec_1, \prec_2)$. We shall say that ϕ is valid in a class $\mathcal{C}$ of flows, in symbols

$\mathcal{C} \models \phi$, iff $(\mathcal{R}, \prec_1, \prec_2) \models \phi$ for each flow $(\mathcal{R}, \prec_1, \prec_2)$ in $\mathcal{C}$. The class of all standard flows will be denoted more briefly as $\mathcal{C}_s$ whereas the class of all countable standard flows will be denoted more briefly as $\mathcal{C}_s^c$.

C. Bounded morphisms

Let $(\mathcal{R}, \prec_1, \prec_2)$ and $(\mathcal{R}', \prec_1', \prec_2')$ be flows. A function f: $\mathcal{R} \mapsto \mathcal{R}'$ is a bounded morphism from $(\mathcal{R}, \prec_1, \prec_2)$ to $(\mathcal{R}', \prec_1', \prec_2')$ iff the following conditions are satisfied for each $i \in \{1, 2\}$:

- for all $t \in \mathcal{R}$ and for all $u' \in \mathcal{R}'$, $f(t) \prec_i' u'$ iff there exists $u \in \mathcal{R}$ such that $t \prec_i u$ and $f(u) = u'$ and
- for all $t \in \mathcal{R}$ and for all $u' \in \mathcal{R}'$, $u' \prec_i' f(t)$ iff there exists $u \in \mathcal{R}$ such that $u \prec_i t$ and $f(u) = u'$.

If there is a surjective bounded morphism from $(\mathcal{R}, \prec_1, \prec_2)$ to $(\mathcal{R}', \prec_1', \prec_2')$ then we say that $(\mathcal{R}', \prec_1', \prec_2')$ is a bounded morphic image of $(\mathcal{R}, \prec_1, \prec_2)$.

Lemma 1: Let $(\mathcal{R}, \prec_1, \prec_2)$ and $(\mathcal{R}', \prec_1', \prec_2')$ be flows. If $(\mathcal{R}', \prec_1', \prec_2')$ is a bounded morphic image of $(\mathcal{R}, \prec_1, \prec_2)$ then for all formulas ϕ, if $(\mathcal{R}, \prec_1, \prec_2) \models \phi$ then $(\mathcal{R}', \prec_1', \prec_2') \models \phi$.

Proof: Use the bounded morphism lemma [4]. ∎

IV. Axiomatization

A temporal logic is defined to be any normal logic in the temporal language that contains the formulas

- $\phi \to G_i P_i \phi$ and
- $\phi \to H_i F_i \phi$

as proper axioms for each $i \in \{1, 2\}$. Notice that these formulas come in pairs of "mirror images" obtained by interchanging future and past connectives. Let HTL be the smallest temporal logic that contains the formulas

4
- $F_1 F_1 \phi \to F_1 \phi$,
- $F_1 F_2 \phi \to F_1 \phi$,
- $F_2 F_1 \phi \to F_1 \phi$,
- $F_2 F_2 \phi \to F_2 \phi$,

d
- $F_1 \phi \to F_1 F_1 \phi$,
- $F_1 \phi \to F_1 F_2 \phi$,
- $F_1 \phi \to F_2 F_1 \phi$ and
- $F_2 \phi \to F_2 F_2 \phi$

and the formulas

D
- $F_1 \top$,
- $F_2 \top$,

L
- $F_1 \phi \wedge F_1 \psi \to F_1(\phi \wedge \psi) \vee F_1(\phi \wedge F_1 \psi) \vee F_1(\phi \wedge F_2 \psi) \vee F_1(\psi \wedge F_1 \phi) \vee F_1(\psi \wedge F_2 \phi)$,
- $F_1 \phi \wedge F_2 \psi \to F_2(\psi \wedge F_1 \phi)$,
- $F_2 \phi \wedge F_1 \psi \to F_2(\phi \wedge F_1 \psi)$ and
- $F_2 \phi \wedge F_2 \psi \to F_2(\phi \wedge \psi) \vee F_2(\phi \wedge F_2 \psi) \vee F_2(\psi \wedge F_2 \phi)$

and their mirror images as proper axioms.

Proposition 5: Let ϕ be a formula. If $\phi \in HTL$ then $\mathcal{C}_s \models \phi$.

Proof: Left to the reader. ∎

A flow $(\mathcal{R}, \prec_1, \prec_2)$ is said to be prestandard iff it satisfies $TRAN$, $DENS$, $SERI$ and the following sentences:

$LINE$
- $\forall x\ \forall y\ (\exists z\ (z <_1 x \wedge z <_1 y) \to x = y \vee x <_1 y \vee x <_2 y \vee y <_1 x \vee y <_2 x)$,
- $\forall x\ \forall y\ (\exists z\ (z <_1 x \wedge z <_2 y) \to y <_1 x)$,
- $\forall x\ \forall y\ (\exists z\ (z <_2 x \wedge z <_1 y) \to x <_1 y)$,
- $\forall x\ \forall y\ (\exists z\ (z <_2 x \wedge z <_2 y) \to x = y \vee x <_2 y \vee y <_2 x)$,
- $\forall x\ \forall y\ (\exists z\ (x <_1 z \wedge y <_1 z) \to x = y \vee x <_1 y \vee x <_2 y \vee y <_1 x \vee y <_2 x)$,
- $\forall x\ \forall y\ (\exists z\ (x <_1 z \wedge y <_2 z) \to x <_1 y)$,
- $\forall x\ \forall y\ (\exists z\ (x <_2 z \wedge y <_1 z) \to y <_1 x)$ and
- $\forall x\ \forall y\ (\exists z\ (x <_2 z \wedge y <_2 z) \to x = y \vee x <_2 y \vee y <_2 x)$.

The class of all prestandard flows will be denoted more briefly as $\mathcal{C}_p$ whereas the class of all countable prestandard flows will be denoted more briefly as $\mathcal{C}_p^c$.

Proposition 6: Let ϕ be a formula. If $\mathcal{C}_p \models \phi$ then $\phi \in HTL$.

Proof: It suffices to observe that the proper axioms 4 and d and the proper axioms D and L and their mirror images are Sahlqvist formulas and correspond to sentences in a very precise way: 4 corresponds to $TRAN$, d corresponds to $DENS$, D and its mirror image correspond to $SERI$ and L and its mirror image correspond to $LINE$. Then use Sahlqvist completeness theorem [4]. ∎

Obviously, every standard flow is prestandard. Conversely, the importance of prestandard flows lies in the fact that every countable prestandard flow satisfying $UNIV$ is a bounded morphic image of every countable standard flow. A proof of this fact will be found in section VI.

V. Preliminary lemmas

Let $(\mathcal{R}, \prec_1, \prec_2)$ be a standard flow and $(\mathcal{R}', \prec_1', \prec_2')$ be a prestandard flow. Suppose $\mathcal{R}$ and $\mathcal{R}'$ are countable. The four following lemmas constitute the heart of our method.

Lemma 2: Let $s \in \mathcal{R}$ and $s' \in \mathcal{R}'$. The partial function f: $\mathcal{R} \mapsto \mathcal{R}'$ defined by $dom(f) = \{s\}$ and $f(s) = s'$ is a partial homomorphism with finite nonempty domain.

Proof: Obvious. ∎

The partial function f: $\mathcal{R} \mapsto \mathcal{R}'$ defined by lemma 2 is called initial function with respect to s and s'.

Lemma 3: Let $s \in \mathcal{R}$ and f: $\mathcal{R} \mapsto \mathcal{R}'$ be a partial homomorphism with finite nonempty domain. There exists a partial homomorphism g: $\mathcal{R} \mapsto \mathcal{R}'$ with finite nonempty domain such that $dom(g) = dom(f) \cup \{s\}$ and $g(t) = f(t)$ for each $t \in dom(f)$.

Proof: Since $dom(f)$ is finite and nonempty, then there exists a positive integer k and there exists $w_1, \ldots, w_k \in \mathcal{R}$ such that $\{w_1, \ldots, w_k\} = dom(f)$. Let us remind that $(\mathcal{R}, \prec_1, \prec_2)$ is standard. Hence, without loss of generality, we may assume that $w_1 \prec \ldots \prec w_k$. Now, consider the four following cases.

1) Suppose there exists a positive integer l such that $l \leq k$ and $s = w_l$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f)$ and $g(t) = f(t)$ for each $t \in dom(f)$.
2) Suppose there exists a positive integer l such that $1 \leq l-1$, $l \leq k$, $w_{l-1} \prec s$ and $s \prec w_l$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DISJ$, then $w_{l-1} \prec_i s$ for exactly one $i \in \{1, 2\}$ and $s \prec_j w_l$ for exactly one $j \in \{1, 2\}$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $TRAN$, f: $\mathcal{R} \mapsto \mathcal{R}'$ is a partial homomorphism and $(\mathcal{R}', \prec_1', \prec_2')$ satisfies $DENS$, then there exists $s' \in \mathcal{R}'$ such that $f(w_{l-1}) \prec_i' s'$ and $s' \prec_j' f(w_l)$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{s\}$, $g(t) = f(t)$ for each $t \in dom(f)$ and $g(s) = s'$.
3) Suppose $s \prec w_1$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DISJ$, then $s \prec_i w_1$ for exactly one $i \in \{1, 2\}$. Since $(\mathcal{R}', \prec_1', \prec_2')$ satisfies $SERI$, then there exists $s' \in \mathcal{R}'$ such that $s' \prec_i' f(w_1)$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{s\}$, $g(t) = f(t)$ for each $t \in dom(f)$ and $g(s) = s'$.
4) Suppose $w_k \prec s$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DISJ$, then $w_k \prec_i s$ for exactly one $i \in \{1, 2\}$. Since $(\mathcal{R}', \prec_1', \prec_2')$ satisfies $SERI$, then there exists $s' \in \mathcal{R}'$ such that $f(w_k) \prec_i' s'$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{s\}$, $g(t) = f(t)$ for each $t \in dom(f)$ and $g(s) = s'$.

The reader may easily verify that g: $\mathcal{R} \mapsto \mathcal{R}'$ is a partial homomorphism with finite nonempty domain such that $dom(g) = dom(f) \cup \{s\}$ and $g(t) = f(t)$ for each $t \in dom(f)$. ■

The partial function g: $\mathcal{R} \mapsto \mathcal{R}'$ defined by lemma 3 is called forward completion of f with respect to s.

Lemma 4: Let $s \in \mathcal{R}$, $t' \in \mathcal{R}'$, $i \in \{1, 2\}$ and f: $\mathcal{R} \mapsto \mathcal{R}'$ be a partial homomorphism with finite nonempty domain such that $s \in dom(f)$ and $f(s) \prec_i' t'$. There exists $t \in \mathcal{R}$ and there exists a partial homomorphism g: $\mathcal{R} \mapsto \mathcal{R}'$ with finite nonempty domain such that $s \prec_i t$, $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.

Proof: Since $dom(f)$ is finite, then $dom(f) \cap \{t \in \mathcal{R}$: $s \prec t\}$ is finite. Hence, there exists a nonnegative integer k and there exists $t_1, \ldots, t_k \in \mathcal{R}$ such that $\{t_1, \ldots, t_k\} = dom(f) \cap \{t \in \mathcal{R}$: $s \prec t\}$. Let us remind that $(\mathcal{R}, \prec_1, \prec_2)$ is standard. Hence, without loss of generality, we may assume that $s \prec t_1 \ldots \prec t_k$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DISJ$, then $s \prec_{j_1} t_1 \ldots \prec_{j_k} t_k$ for exactly one k-tuple $(j_1, \ldots, j_k) \in \{1, 2\}^k$. Since f: $\mathcal{R} \mapsto \mathcal{R}'$ is a partial homomorphism, $s \in dom(f)$ and $\{t_1, \ldots, t_k\} \subseteq dom(f)$, then $f(s) \prec_{j_1}' f(t_1) \ldots \prec_{j_k}' f(t_k)$. Now, we proceed by induction on k.

Basis. Suppose $k = 0$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $SERI$, then there exists $t \in \mathcal{R}$ such that $s \prec_i t$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.

Step. Suppose $k > 1$. Now, consider the four following cases.

1) Suppose $i = 1$ and $j_1 = 1$. Hence, $f(s) \prec_1' t'$ and $f(s) \prec_1' f(t_1)$. Since $(\mathcal{R}', \prec_1', \prec_2')$ satisfies $LINE$, then either $t' = f(t_1)$, or $t' \prec_1' f(t_1)$, or $t' \prec_2' f(t_1)$, or $f(t_1) \prec_1' t'$, or $f(t_1) \prec_2' t'$. Now, consider the five following cases.
   a) Suppose $t' = f(t_1)$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f)$ and $g(u) = f(u)$ for each $u \in dom(f)$.
   b) Suppose $t' \prec_1' f(t_1)$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DENS$, then there exists $t \in \mathcal{R}$ such that $s \prec_1 t$ and $t \prec_1 t_1$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.
   c) Suppose $t' \prec_2' f(t_1)$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DENS$, then there exists $t \in \mathcal{R}$ such that $s \prec_1 t$ and $t \prec_2 t_1$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.
   d) Suppose $f(t_1) \prec_1' t'$. Since $\{t_2, \ldots, t_k\} = dom(f) \cap \{t \in \mathcal{R}$: $t_1 \prec t\}$, then by induction hypothesis, there exists $t \in \mathcal{R}$ and there exists a partial homomorphism g: $\mathcal{R} \mapsto \mathcal{R}'$ with finite nonempty domain such that $t_1 \prec_1 t$, $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.
   e) Suppose $f(t_1) \prec_2' t'$. Since $\{t_2, \ldots, t_k\} = dom(f) \cap \{t \in \mathcal{R}$: $t_1 \prec t\}$, then by induction hypothesis, there exists $t \in \mathcal{R}$ and there exists a partial homomorphism g: $\mathcal{R} \mapsto \mathcal{R}'$ with finite nonempty domain such that $t_1 \prec_2 t$, $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.
2) Suppose $i = 1$ and $j_1 = 2$. Hence, $f(s) \prec_1' t'$ and $f(s) \prec_2' f(t_1)$. Since $(\mathcal{R}', \prec_1', \prec_2')$ satisfies $LINE$, then $f(t_1) \prec_1' t'$. Since $\{t_2, \ldots, t_k\} = dom(f) \cap \{t \in \mathcal{R}$: $t_1 \prec t\}$, then by induction hypothesis, there exists $t \in \mathcal{R}$ and there exists a partial homomorphism g: $\mathcal{R} \mapsto \mathcal{R}'$ with finite nonempty domain such that $t_1 \prec_1 t$, $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.
3) Suppose $i = 2$ and $j_1 = 1$. Hence, $f(s) \prec_2' t'$ and $f(s) \prec_1' f(t_1)$. Since $(\mathcal{R}', \prec_1', \prec_2')$ satisfies $LINE$, then $t' \prec_1' f(t_1)$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DENS$, then there exists $t \in \mathcal{R}$ such that $s \prec_2 t$ and $t \prec_1 t_1$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.
4) Suppose $i = 2$ and $j_1 = 2$. Hence, $f(s) \prec_2' t'$ and $f(s) \prec_2' f(t_1)$. Since $(\mathcal{R}', \prec_1', \prec_2')$ satisfies $LINE$, then either $t' = f(t_1)$, or $t' \prec_2' f(t_1)$, or $f(t_1) \prec_2' t'$. Now, consider the three following cases.

a) Suppose $t' = f(t_1)$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f)$ and $g(u) = f(u)$ for each $u \in dom(f)$.
b) Suppose $t' \prec'_2 f(t_1)$. Since $(\mathcal{R}, \prec_1, \prec_2)$ satisfies $DENS$, then there exists $t \in \mathcal{R}$ such that $s \prec_2 t$ and $t \prec_2 t_1$. Let g: $\mathcal{R} \mapsto \mathcal{R}'$ be the partial function defined by $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.
c) Suppose $f(t_1) \prec'_2 t'$. Since $\{t_2, \ldots, t_k\} = dom(f) \cap \{t \in \mathcal{R}$: $t_1 \prec t\}$, then by induction hypothesis, there exists $t \in \mathcal{R}$ and there exists a partial homomorphism g: $\mathcal{R} \mapsto \mathcal{R}'$ with finite nonempty domain such that $t_1 \prec_2 t$, $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.

The reader may easily verify that g: $\mathcal{R} \mapsto \mathcal{R}'$ is a partial homomorphism with finite nonempty domain such that $s \prec_i t$, $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$. ■

The partial function g: $\mathcal{R} \mapsto \mathcal{R}'$ defined by lemma 4 is called left-backward completion of f with respect to s, t' and i.

Lemma 5: Let $s \in \mathcal{R}$, $t' \in \mathcal{R}'$, $i \in \{1, 2\}$ and f: $\mathcal{R} \mapsto \mathcal{R}'$ be a partial homomorphism with finite nonempty domain such that $s \in dom(f)$ and $t' \prec'_i f(s)$. There exists $t \in \mathcal{R}$ and there exists a partial homomorphism g: $\mathcal{R} \mapsto \mathcal{R}'$ with finite nonempty domain such that $t \prec_i s$, $dom(g) = dom(f) \cup \{t\}$, $g(u) = f(u)$ for each $u \in dom(f)$ and $g(t) = t'$.

Proof: Similar to the proof of lemma 4 ■

The partial function g: $\mathcal{R} \mapsto \mathcal{R}'$ defined by lemma 5 is called right-backward completion of f with respect to s, t' and i.

VI. Completeness

We can now prove the following proposition.

Proposition 7: Let $(\mathcal{R}, \prec_1, \prec_2)$ be a standard flow and $(\mathcal{R}', \prec'_1, \prec'_2)$ be a prestandard flow. If $\mathcal{R}$ and $\mathcal{R}'$ are countable and $\mathcal{R}'$ satisfies $UNIV$ then $(\mathcal{R}', \prec'_1, \prec'_2)$ is a bounded morphic image of $(\mathcal{R}, \prec_1, \prec_2)$.

Proof: One main idea underlies our step-by-step method: we think of the construction of the surjective bounded morphism from $(\mathcal{R}, \prec_1, \prec_2)$ to $(\mathcal{R}', \prec'_1, \prec'_2)$ as a process approaching a limit via a sequence f_0: $\mathcal{R} \mapsto \mathcal{R}'$, f_1: $\mathcal{R} \mapsto \mathcal{R}'$, ... of partial homomorphisms with finite nonempty domains. Lemma 2 is used to initiate the construction whereas lemmas 3, 4 and 5 are used to make improvements at each step of the construction. Let $s_0 \in \mathcal{R}$ and $s'_0 \in \mathcal{R}'$. Consider an enumeration (t_0, u'_0, i_0), (t_1, u'_1, i_1), ... of $\mathcal{R} \times \mathcal{R}' \times \{1, 2\}$ where each item appears infinitely often. We inductively define a sequence f_0: $\mathcal{R} \mapsto \mathcal{R}'$, f_1: $\mathcal{R} \mapsto \mathcal{R}'$, ... of partial homomorphisms with finite nonempty domains as follows:

Basis. Let f_0: $\mathcal{R} \mapsto \mathcal{R}'$ be the initial function with respect to s_0 and s'_0.

Step. Let g_n: $\mathcal{R} \mapsto \mathcal{R}'$ be the forward completion of f_n with respect to t_n, h_n: $\mathcal{R} \mapsto \mathcal{R}'$ be the left-backward completion of g_n with respect to t_n, u'_n and i_n and f_{n+1}: $\mathcal{R} \mapsto \mathcal{R}'$ be the right-backward completion of h_n with respect to t_n, u'_n and i_n.

The reader may easily verify that the sequence f_0: $\mathcal{R} \mapsto \mathcal{R}'$, f_1: $\mathcal{R} \mapsto \mathcal{R}'$, ... of partial homomorphisms with finite nonempty domains is such that $dom(f_0) \subseteq dom(f_1) \subseteq \ldots$, $\bigcup\{dom(f_n)$: n is a nonnegative integer$\} = \mathcal{R}$ and for all nonnegative integers n, $f_{n+1}(s) = f_n(s)$ for each $s \in dom(f_n)$. Let f: $\mathcal{R} \mapsto \mathcal{R}'$ be the function defined by $dom(f) = \mathcal{R}$ and $f(s) = f_n(s)$ for each $s \in \mathcal{R}$, n being a nonnegative integer such that $s \in dom(f_n)$. The reader may easily verify that f: $\mathcal{R} \mapsto \mathcal{R}'$ is a surjective bounded morphism from $(\mathcal{R}, \prec_1, \prec_2)$ to $(\mathcal{R}', \prec'_1, \prec'_2)$. ■

The result that emerges from the discussion above is the following theorem.

Theorem 1: Let ϕ be a formula. The following conditions are equivalent:

1) $\phi \in HTL$;
2) $\mathcal{C}_s \models \phi$;
3) $\mathcal{C}^c_s \models \phi$;
4) $\mathcal{C}_p \models \phi$;
5) $\mathcal{C}^c_p \models \phi$.

Proof: $(1) \rightarrow (2)$. Use proposition 5.
$(2) \rightarrow (3)$. Obvious.
$(3) \rightarrow (5)$. Use lemma 1, proposition 7 and the fact that every generated flow satisfying $TRAN$ and $LINE$ also satisfies $UNIV$.
$(5) \rightarrow (4)$. Use Löwenheim-Skolem theorem for modal models [4].
$(4) \rightarrow (1)$. Use proposition 6. ■

VII. Pure future formulas

ϕ is said to be a pure future formula iff it contains no occurrence of the temporal connectives H_1 and H_2. We do not know whether all standard flows validate the same pure future formulas. Nevertheless,

Proposition 8: For all pure future formulas ϕ, ϕ is valid in the lexicographic flow defined over $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ iff ϕ is valid in the lexicographic flow defined over $(\mathbb{Q}, <)$ and $(\mathbb{Q}, <)$.

Proof: Let $(\mathcal{R}, \prec_1, \prec_2)$ be the lexicographic flow defined over $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ and $(\mathcal{R}', \prec'_1, \prec'_2)$ be the lexicographic flow defined over $(\mathbb{Q}, <)$ and $(\mathbb{Q}, <)$.
Suppose $(\mathcal{R}', \prec'_1, \prec'_2) \not\models \phi$. Hence, there exists a function V': $\mathcal{R}' \mapsto 2^{At}$, there exists $t^0 \in \mathbb{Q}$ and there exists $u^0 \in \mathbb{Q}$ such that $(\mathcal{R}', \prec'_1, \prec'_2, V') \not\models_{(t^0, u^0)} \phi$. Let m be the function from $\mathbb{Q} \times \mathbb{R}$ to the set of all maximal propositionally consistent sets of formulas such that for all $t \in \mathbb{Q}$ and for all $u \in \mathbb{R}$, either $u \in \mathbb{Q}$ and $m(t, u) \supseteq \{\psi$: $(\mathcal{R}', \prec'_1, \prec'_2, V') \models_{(t,u)} \psi\}$, or $u \notin \mathbb{Q}$ and $m(t, u) \supseteq \{\psi$: there exists $u' \in \mathbb{Q}$

such that $u < u'$ and for all $u'' \in \mathbb{Q}$, if $u < u''$ and $u'' < u'$ then $(\mathcal{R}', \prec_1', \prec_2', V') \models_{(t,u'')} \psi\}$. Since $(\mathcal{R}', \prec_1', \prec_2', V')$ $\not\models_{(t^0,u^0)} \phi$, hence, $\phi \notin m(t^0, u^0)$. We define a function V: $\mathcal{R} \mapsto 2^{At}$ by $V(t, u) = m(t, u) \cap At$ for each $t \in \mathbb{Q}$ and for each $u \in \mathbb{R}$. As a simple exercise, we invite the reader to show by induction on the complexity of pure future formulas ψ that for all $t \in \mathbb{Q}$ and for all $u \in \mathbb{R}$, $(\mathcal{R}, \prec_1, \prec_2, V) \models_{(t,u)}$ ψ iff $\psi \in m(t, u)$. Since $\phi \notin m(t^0, u^0)$, then $(\mathcal{R}, \prec_1, \prec_2, V)$ $\not\models_{(t^0,u^0)} \phi$. Therefore, $(\mathcal{R}, \prec_1, \prec_2) \not\models \phi$.

Suppose $(\mathcal{R}', \prec_1', \prec_2') \models \phi$. Since HY is countably categorical, then $\mathcal{C}_s^c \models \phi$. By theorem 1, $\mathcal{C}_s \models \phi$. Hence, $(\mathcal{R}, \prec_1, \prec_2)$ $\models \phi$. ■

There is no known complete axiomatization of the set of all $\mathcal{C}_s$-valid pure future formulas. Let HTL_i denotes the restriction of HTL to the set of formulas based on the temporal connective G_i for each $i \in \{1, 2\}$.

Proposition 9: HTL_1 is equivalent to the smallest normal logic that contains, in the language based on $\Box$, the following formulas as proper axioms: $\Diamond\Diamond\phi \to \Diamond\phi$, $\Diamond\phi \to \Diamond\Diamond\phi$, $\Diamond\top$ and $\Diamond(\Box\phi \land \Diamond\psi) \to \Box(\phi \lor \Diamond\psi)$.

Proof: Let ϕ be a formula based on $\Box$. Obviously, as the reader is asked to show, if ϕ is derivable from the above axioms then the corresponding formula ϕ^1 based on G_1 is valid in $\mathcal{C}_s$. Reciprocally, suppose ϕ is not derivable from the above axioms. Therefore, by Sahlqvist completeness theorem, there exists a generated structure (W, R) where W is a nonempty set of instants and R is a binary relation on W such that

- for all $t, u \in W$, if there exists $v \in W$ such that $t\ R\ v$ and $v\ R\ u$ then $t\ R\ u$,
- for all $t, u \in W$, if $t\ R\ u$ then there exists $v \in W$ such that $t\ R\ v$ and $v\ R\ u$,
- for all $t \in W$, there exists $u \in W$ such that $t\ R\ u$ and
- for all $t, u, v \in W$, if $t\ R\ u$ and $t\ R\ v$ then either $\{w \in W\colon u\ R\ w\} = \{w \in W\colon v\ R\ w\}$, or $u\ R\ v$, or $v\ R\ u$,

there exists a function V: $W \mapsto 2^{At}$ and there exists $t_0 \in W$ such that $(W, R, V) \not\models_{t_0} \phi$. Let $\sim$ be the binary relation on W defined by $t \sim u$ iff either $t = u$, or not $t\ R\ u$ and not $u\ R\ t$ for each $t, u \in W$. The reader may easily verify that $\sim$ is an equivalence relation on W. Let $(\mathcal{R}', \prec_1', \prec_2')$ be the flow defined by $\mathcal{R}' = W$ and $\prec_1'$ and $\prec_2'$ are the binary relations on $\mathcal{R}'$ defined by $t' \prec_1' u'$ iff $t'\ R\ u'$ and $t' \prec_2' u'$ iff $t' \sim u'$ and V': $\mathcal{R}' \mapsto 2^{At}$ be a function such that ${V'}^{-1}(p) = V^{-1}(p)$. The reader may easily verify that $(\mathcal{R}', \prec_1', \prec_2')$ is prestandard and such that $(\mathcal{R}', \prec_1', \prec_2', V')$ $\not\models_{t_0} \phi^1$. By theorem 1, ϕ^1 is not valid in $\mathcal{C}_s$. ■

Proposition 10: HTL_2 is equivalent to the smallest normal logic that contains, in the language based on $\Box$, the following formulas as proper axioms: $\Diamond\Diamond\phi \to \Diamond\phi$, $\Diamond\phi \to \Diamond\Diamond\phi$, $\Diamond\top$ and $\Diamond\phi \land \Diamond\psi \to \Diamond(\phi \land \psi) \lor \Diamond(\phi \land \Diamond\psi) \lor \Diamond(\psi \land \Diamond\phi)$.

Proof: Let ϕ be a formula based on $\Box$. Obviously, as the reader is asked to show, if ϕ is derivable from the above axioms then the corresponding formula ϕ^2 based on G_2 is valid in $\mathcal{C}_s$. Reciprocally, suppose ϕ is not derivable from the above axioms. Therefore, by Sahlqvist completeness theorem, there exists a generated structure (W, R) where W is a nonempty set of instants and R is a binary relation on W such that

- for all $t, u \in W$, if there exists $v \in W$ such that $t\ R\ v$ and $v\ R\ u$ then $t\ R\ u$,
- for all $t, u \in W$, if $t\ R\ u$ then there exists $v \in W$ such that $t\ R\ v$ and $v\ R\ u$,
- for all $t \in W$, there exists $u \in W$ such that $t\ R\ u$ and
- for all $t, u, v \in W$, if $t\ R\ u$ and $t\ R\ v$ then $u = v$ or $u\ R\ v$ or $v\ R\ u$,

there exists a function V: $W \mapsto 2^{At}$ and there exists $t_0 \in W$ such that $(W, R, V) \not\models_{t_0} \phi$. Let $(\mathcal{R}', \prec_1', \prec_2')$ be the flow defined by $\mathcal{R}' = W \cup \{\infty\}$ where ∞ is a new instant and $\prec_1'$ and $\prec_2'$ are the binary relations on $\mathcal{R}'$ defined by $t' \prec_1' u'$ iff $u' = \infty$ and $t' \prec_2' u'$ iff either $t', u' \in W$ and $t'\ R\ u'$, or $t' = \infty$ and $u' = \infty$ and V': $\mathcal{R}' \mapsto 2^{At}$ be a function such that ${V'}^{-1}(p) = V^{-1}(p)$. The reader may easily verify that $(\mathcal{R}', \prec_1', \prec_2')$ is prestandard and such that $(\mathcal{R}', \prec_1', \prec_2', V')$ $\not\models_{t_0} \phi^2$. By theorem 1, ϕ^2 is not valid in $\mathcal{C}_s$. ■

Consider a flow $(\mathcal{R}, \prec_1, \prec_2)$ and $i, j \in \{1, 2\}$ be such that $i \neq j$. We shall say that G_i is definable with G_j in $(\mathcal{R}, \prec_1, \prec_2)$ iff there exists a formula $\phi(p)$ with G_j such that $(\mathcal{R}, \prec_1, \prec_2) \models G_i p \leftrightarrow \phi(p)$.

Proposition 11: (1) G_1 is not definable with G_2 in any standard flow; (2) G_2 is not definable with G_1 in any standard flow.

Proof: Let $(\mathcal{R}, \prec_1, \prec_2)$ be a standard flow

(1) Suppose there exists a formula $\phi(p)$ in G_2 such that $\mathcal{R} \models G_1 p \leftrightarrow \phi(p)$. Let $t, u \in \mathcal{R}$ be such that $t \prec_1 u$. We need to consider a function V: $\mathcal{R} \mapsto 2^{At}$ such that $V^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_1 s\}$ and a function V': $\mathcal{R} \mapsto 2^{At}$ such that ${V'}^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_1 s\} \setminus \{s \in \mathcal{R}\colon \text{not } s \prec_1 u\}$. Notice that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_1 p$ and $(\mathcal{R}, \prec_1, \prec_2, V') \not\models_t G_1 p$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas $\psi(p)$ in G_2 that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \psi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t \psi(p)$. Hence, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \phi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V')$ $\models_t \phi(p)$. Thus, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_1 p$ iff $(\mathcal{R}, \prec_1, \prec_2, V')$ $\models_t G_1 p$. These facts together constitute a contradiction.

(2) Suppose there exists a formula $\phi(p)$ in G_1 such that $\mathcal{R} \models G_2 p \leftrightarrow \phi(p)$. Let $t, u \in \mathcal{R}$ be such that $t \prec_2 u$. We need to consider a function V: $\mathcal{R} \mapsto 2^{At}$ such that $V^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_2 s\}$ and a function V': $\mathcal{R} \mapsto 2^{At}$ such that ${V'}^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_2 s\} \setminus \{s \in \mathcal{R}\colon \text{not } s \prec_2 u\}$. Notice that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_2 p$ and $(\mathcal{R}, \prec_1, \prec_2, V') \not\models_t G_2 p$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas $\psi(p)$ in G_1 that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \psi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t \psi(p)$. Hence, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \phi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V')$ $\models_t \phi(p)$. Thus, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_2 p$ iff $(\mathcal{R}, \prec_1, \prec_2, V')$ $\models_t G_2 p$. These facts together constitute a contradiction. ■

Let

- $G\phi := (G_1\phi \wedge G_2\phi)$,

the formula $G\phi$ being read "ϕ will be true at each instant within the future of the present instant". As a result, for all Kripke models $\mathcal{M} = (\mathcal{R}, \prec_1, \prec_2, V)$, for all instants $t \in \mathcal{R}$ and for all formula ϕ,

- $\mathcal{M} \models_t G\phi$ iff $\mathcal{M} \models_u \phi$ for each instant $u \in \mathcal{R}$ such that $t \prec u$.

Consider a flow $(\mathcal{R}, \prec_1, \prec_2)$ and $i \in \{1, 2\}$. We shall say that G_i is definable with G in $(\mathcal{R}, \prec_1, \prec_2)$ iff there exists a formula $\phi(p)$ with G such that $(\mathcal{R}, \prec_1, \prec_2) \models G_i p \leftrightarrow \phi(p)$.

Proposition 12: (1) G_1 is not definable with G in any standard flow. (2) G_2 is not definable with G in any standard flow.

Proof: Let $(\mathcal{R}, \prec_1, \prec_2)$ be a standard flow
(1) Suppose there exists a formula $\phi(p)$ in G such that $\mathcal{R} \models G_1 p \leftrightarrow \phi(p)$. Let $t, u \in \mathcal{R}$ be such that $t \prec_1 u$. We need to consider a function $V \colon \mathcal{R} \mapsto 2^{At}$ such that $V^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_1 s\}$ and a function $V' \colon \mathcal{R} \mapsto 2^{At}$ such that ${V'}^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_1 s\} \setminus \{s \in \mathcal{R}\colon \text{not } u \prec_1 s\}$. Notice that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_1 p$ and $(\mathcal{R}, \prec_1, \prec_2, V') \not\models_t G_1 p$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas $\psi(p)$ in G that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \psi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t \psi(p)$. Hence, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \phi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t \phi(p)$. Thus, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_1 p$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t G_1 p$. These facts together constitute a contradiction.
(2) Suppose there exists a formula $\phi(p)$ in G such that $\mathcal{R} \models G_2 p \leftrightarrow \phi(p)$. Let $t, u \in \mathcal{R}$ be such that $t \prec_2 u$. We need to consider a function $V \colon \mathcal{R} \mapsto 2^{At}$ such that $V^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_2 s\}$ and a function $V' \colon \mathcal{R} \mapsto 2^{At}$ such that ${V'}^{-1}(p) = \{s \in \mathcal{R}\colon t \prec_2 s\} \setminus \{s \in \mathcal{R}\colon \text{not } s \prec_2 u\}$. Notice that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_2 p$ and $(\mathcal{R}, \prec_1, \prec_2, V') \not\models_t G_2 p$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas $\psi(p)$ in G that $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \psi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t \psi(p)$. Hence, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t \phi(p)$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t \phi(p)$. Thus, $(\mathcal{R}, \prec_1, \prec_2, V) \models_t G_2 p$ iff $(\mathcal{R}, \prec_1, \prec_2, V') \models_t G_2 p$. These facts together constitute a contradiction. ■

VIII. Conclusion

This article considered the temporal logic defined over the class of all lexicographic products of dense linear orders without endpoints and gives its complete axiomatization. Much remains to be done.
Firstly, there is the issue of the completeness of the temporal logic characterized by the lexicographic product of two linear orderings. Could transfer results for completeness similar to the ones obtained in [10] within the context of independently axiomatizable bimodal logics be obtained in our lexicographic setting?
Secondly, there is the question of the decidability of the temporal logic characterized by the lexicographic product of two linear orderings. All extensions of $S4.3$, as proved in [5], [8], possess the finite model property and all finitely axiomatizable normal extensions of $K4.3$, as proved in [13], are decidable. Is it possible to obtain similar results in our lexicographic setting? Or could undecidability results similar to the ones obtained in [12] within the context of the products of the modal logics determined by arbitrarily long linear orders be obtained in our lexicographic setting?
Thirdly, there is the question of the complexity of the temporal logic characterized by the lexicographic product of two linear orderings. Is it possible to obtain in our lexicographic setting complexity results by following the line of reasoning suggested by [11] within the context of temporal logics?

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References

[1] Balbiani, P. Time representation and temporal reasoning from the perspective of non-standard analysis. In Brewka, G., Lang, J. (editors): Eleventh International Conference on Principles of Knowledge Representation and Reasoning. Association for the Advancement of Artificial Intelligence (2008) 695–704.

[2] Balbiani, P. Axiomatization and completeness of lexicographic products of modal logics. In Ghilardi, S., Sebastiani, R. (editors): Frontiers of Combining Systems.

[3] Van Benthem, J. The Logic of Time. Kluwer (1991).

[4] Blackburn, P., de Rijke, M., Venema, Y. Modal Logic. Cambridge University Press (2001).

[5] Bull, R. That all normal extensions of $S4.3$ have the finite model property. Zeitschrift für mathematische Logik und Grundlagen der Mathematik **12** (1966) 314–344.

[6] Endriss, U. Modal Logics of Ordered Trees. Thesis submitted to the University of London (2003).

[7] Fagin, R., Halpern, J., Moses, Y., Vardi, M. Reasoning About Knowledge. MIT Press (1995).

[8] Fine, K. The logics containing $S4.3$. Zeitschrift für mathematische Logik und Grundlagen der Mathematik **17** (1971) 371–376.

[9] Gabbay, D., Kurucz, A., Wolter, F., Zakharyaschev, M. Many-Dimensional Modal Logics: Theory and Applications. Elsevier (2003).

[10] Kracht, M., Wolter, F. Properties of independently axiomatizable bimodal logics. Journal of Symbolic Logic **56** (1991) 1469–1485.

[11] Marx, M., Mikulás, S., Reynolds, M. The mosaic method for temporal logics. In Dyckhoff, R. (editor): Automated Reasoning with Analytic Tableaux and Related Methods. Springer (2000) 324–340.

[12] Reynolds, M., Zakharyaschev, M. On the products of linear modal logics. Journal of Logic and Compution **11** (2001) 909–931.

[13] Zakharyaschev, M., Alekseev, A. All finitely axiomatizable normal extensions of $K4.3$ are decidable. Mathematical Logic Quarterly **41** (1995) 15–23.