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CONTROL SYSTEMS LABORATORY

DETECTION OF FLUCTUATING PULSED SIGNALS
IN THE PRESENCE OF NOISE

Report R-87

January, 1957

Contract DA-36-039-SC-56695
D/A Sub-Task 3-99-06-111

UNIVERSITY OF ILLINOIS · URBANA · ILLINOIS

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Prepared by

Peter Swerling

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Abstract

This paper treats the detection of pulsed signals in the presence of receiver noise for the case of randomly fluctuating signal strength. The system considered consists of a predetection stage, a square-law envelope detector, and a linear post-detection integrator. The main problem is the calculation of the probability density function of the output of the post-detection integrator. The analysis is carried out for a large family of probability density functions of the signal fluctuations, and for very general types of correlation properties of the signal fluctuations. The effects of nonuniform beam shape and of nonuniform weighting of pulses by the post-detection integrator are also taken into account. The function which is actually evaluated is the Laplace transform of the probability density function of the integrator output. In many of the cases treated, the resulting Laplace transform has an inverse of known form. In such cases the evaluation of the probability density function would require the computation of a finite number of constants; in practice this would usually require the use of computing machinery, but would be perfectly feasible with presently available computing machinery.

An extensive treatment of detection theory for pulsed signals in noise, for the case where the signal amplitude is constant, was given by Marcum[(1),(2)]. This analysis has been extended by other investigators[(3),(4),(5)] to some cases where the signal pulse amplitudes are randomly modulated. However, almost all analyses to date have dealt with just two cases, insofar as the correlation properties of the signal fluctuations are concerned: the signal amplitudes fluctuate independently from pulse to pulse; or, the signal amplitudes are constant during the integration time of the receiver, but are independent from one integration period to the next. Fluctuations not conforming to one of these assumptions have been treated only in very special cases (see e.g. Ref. 4). The purpose of this paper is to extend the analysis to much more general cases, insofar as the correlation properties of the fluctuations are concerned. The analysis is carried out for a large family of probability density functions for the signal fluctuations. Also, since no additional work is involved, we treat the case where the post-detection integrator forms a weighted sum of the input pulses.

We shall consider a receiver consisting of a pre-detection stage, a square-law envelope detector, and a linear post-detection integrator. The receiver noise at the detector input is assumed to be additive Gaussian (with zero mean), completely correlated for times of the order of one pulse width, and completely uncorrelated from one pulse to the next. The detector is assumed to be a square-law envelope detector. For mathematical convenience, the detector output is assumed to be normalized as follows: detector output equals input envelope squared divided by twice the input mean square receiver noise voltage. This normalization was used by Marcum[1],[2] and followed by Swerling[3]; it simplifies some of the formulas, but results in no actual loss of generality.

Denoting by v_i the (normalized) i^{th} pulse emerging from the detector, we assume the post-detection integrator to form the following weighted sum:

$$(1) \qquad y = \text{integrator output} = \sum_{i=1}^{N} \alpha_i v_i$$

where α_i are positive real numbers.

We assume that the detection procedure requires the integrator output y to exceed a threshold Y_b in order for detection of a signal to be announced. Here Y_b is a dimensionless quantity, because of the normalization of the detector output described above.

If G(y) represents the probability density function (p.d.f.) for y, and if $G_o(y)$ represents the p.d.f. in the case where receiver noise only is present, then

$$(2) \qquad \text{Probability of detection} = \int_{Y_b}^{\infty} G(y)\, dy$$

$$(3) \qquad \text{Probability of false alarm} = \int_{Y_b}^{\infty} G_o(y)\, dy$$

In most applications, the probability of false alarm is set at some desired level and Y_b is calculated from (3); then probability of detection is calculated from (2).

In case $\alpha_i = 1$, all i, then $G_o(y)$ is[2]

(4) $$G_o(y) = \frac{1}{(N-1)!}\, y^{N-1}\, e^{-y} \qquad (\text{for } \alpha_i = 1, \text{ all } i)$$

In the general case, the Laplace transform of $G_o(y)$ is given by

(5) $$\int_o^\infty e^{-py}\, G_o(y)\, dy = \prod_{i=1}^{N} \frac{1}{\alpha_i p+1} \qquad (\text{Real part of } p \geqq 0)$$

This holds, of course, independent of any assumptions as to the signal fluctuations. (The integral need be taken only from zero to infinity since $G(y)$ vanishes for $y < 0$.)

In order to calculate $G(y)$ in the presence of signal, it is necessary to specify the statistical nature of the signal fluctuations. For present purposes, no attempt will be made to derive the following formulation of the signal fluctuation from physical considerations. Its justification is simply, that from it one can construct a wide variety of p.d.f.'s and correlation properties for the signal fluctuation.

We define x_i = ratio, at the detector input, of the signal power for the i^{th} pulse to the mean receiver noise power.

We assume that x_i is of the following form:

(6) $$x_i = \sum_{k=1}^{L} u_{k,i}^2 \qquad (i=1, \ldots, N)$$

where L is a positive integer, and $u_{k,i}$ are Gaussian random variables with zero mean. (The x_i are also assumed to be statistically independent of the receiver noise.)

Define the random vectors $\underline{U}^{(1)}, \ldots, \underline{U}^{(L)}$ by

$$(7) \qquad \underline{U}^{(k)} = (u_{k,1}, \ldots, u_{k,N}) \qquad (k = 1 \ldots, L)$$

We assume that the $\underline{U}^{(k)}$, $k = 1, \ldots, L$, are mutually statistically independent. Also, it is assumed that $\underline{U}^{(k)}$ has covariance matrix $(\phi_{ij}^{(k)})$ where, denoting expected values by a bar,

$$(8) \qquad \phi_{ij}^{(k)} = \overline{u_{k,i}\, u_{k,j}} \qquad \begin{array}{l} k = 1, \ldots, L \\ i,j = 1, \ldots, N \end{array}$$

In radar applications, the fluctuation in x_i is considered to be due largely to fluctuation in the scattering cross section of the target. To relate the above formulation to more familiar types of fluctuation, consider the case where $L = 2K$ and where, for each i, $\overline{u_{1,i}^2} = \overline{u_{2,i}^2} = \ldots = \overline{u_{L,i}^2}$. Then it is easily verified that x_i has a p.d.f. $w(x_i; \bar{x}_i)$ given by

$$(9) \qquad w(x_i; \bar{x}_i) = \frac{1}{(K-1)!} \frac{K}{\bar{x}_i} \left(\frac{Kx_i}{\bar{x}_i} \right)^{K-1} \exp \left(\frac{-Kx_i}{\bar{x}_i} \right), \quad x_i \geq 0$$

where exp ( ) stands for the exponential function.

Here $\bar{x}_i$ represents the average of x_i over the fluctuations. For $K = 1$, this reduces to the familiar exponential or Raleigh fluctuation for the signal power.

Note that we do not require $\bar{x}_i$ to be the same for all i. This can be considered in radar applications to reflect the effect of beam shape.

We shall now compute the Laplace transform $C(p)$ of the p.d.f. $G(y)$ of the integrator output y. This is defined as

$$(10) \qquad C(p) = \int_0^{\infty} e^{-py}\, G(y)\, dy$$

where p is a complex number with real part ≥ 0. The Laplace transform can be used because $G(y)$ vanishes for $y < 0$.

If we consider the conditional p.d.f. for y, for definite values of $x_1, \ldots, x_N$, it is well known[2] that the Laplace transform $C(p|x_1, \ldots, x_N)$ of this conditional p.d.f. is

$$(11) \qquad C(p|x_1, \ldots, x_N) = \prod_{i=1}^{N} \frac{1}{1+\alpha_i p} \exp\left[\frac{-px_i\alpha_i}{1+\alpha_i p}\right]$$

In view of (6), this can be written

$$(12) \qquad C(p|x_1, \ldots, x_N) = \prod_{i=1}^{N} \frac{1}{1+\alpha_i p} \exp\left[\frac{-\alpha_i p \sum_{k=1}^{N} u_{k,i}^2}{1+\alpha_i p}\right]$$

C(p) is simply equal to this expression averaged over the probability distribution of $\underline{U}^{(1)}, \ldots, \underline{U}^{(L)}$. Because of the mutual independence of the $\underline{U}^{(k)}$, one may write

$$(13) \qquad C(p) = \left[\prod_{i=1}^{N} \frac{1}{1+\alpha_i p}\right] \times$$

$$\int \exp\left[-\sum_{i=1}^{N} \frac{\alpha_i p}{1+\alpha_i p}\,(u_{1,i}^2 + \ldots + u_{L,i}^2)\right] \prod_{k=1}^{L} dP(\underline{U}^{(k)})$$

Now, assuming nonsingularity of $(\phi_{ij}^{(k)})$ for each k,

$$(14) \qquad dP(\underline{U}^{(k)}) = \frac{1}{(2\pi)^{\frac{N}{2}}\,\Delta_k^{1/2}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{N} \xi_{ij}^{(k)}\, u_{k,i}\, u_{k,j}\right]$$

where Δ_k = determinant $(\phi_{ij}^{(k)})$;

$$(\xi_{ij}^{(k)}) = \text{matrix inverse of } (\phi_{ij}^{(k)})$$

Thus,

$$(15) \qquad C(p) = \left[\prod_{i=1}^{N} \frac{1}{1+\alpha_i p}\right] \left[\prod_{k=1}^{L} E_k\right]$$

where

$$(16) \qquad E_k = \frac{1}{(2\pi)^{\frac{N}{2}} \Delta_k^{1/2}}$$

$$\times \int \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{N} \xi_{ij}^{(k)} u_{k,i} u_{k,j} - \sum_{i=1}^{N} \frac{\alpha_i p}{1+\alpha_i p} u_{k,i}^2\right\} d\underline{U}^{(k)}$$

Define

$$(17) \quad \xi_{ij}^{(k)}(p) = \xi_{ij}^{(k)} \quad , \quad i \neq j$$

$$= \xi_{ii}^{(k)} + \frac{2\alpha_i p}{1+\alpha_i p} \quad , \quad i = j$$

and

$$(18) \qquad \Gamma_k(p) = \text{determinant}\left(\xi_{ij}^{(k)}(p)\right)$$

(Thus $\Gamma_k(0) = \frac{1}{\Delta_k}$.)

Then

$$(19) \qquad E_k = \left[\frac{\Gamma_k(0)}{\Gamma_k(p)}\right]^{1/2}$$

$$\times \left[\left\{\frac{\left[\Gamma_k(p)\right]^{1/2}}{(2\pi)^{N/2}} \int \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{N} \xi_{ij}^{(k)}(p)\, u_{k,i} u_{k,j}\right\} d\underline{U}^{(k)}\right\}\right]$$

Now assume for the time being that p is a real number $\geqq 0$. Since the right hand factor is then just the integral of a probability density and hence equal to unity,

$$(20) \qquad E_k = \left[\frac{\Gamma_k(0)}{\Gamma_k(p)} \right]^{1/2}$$

Thus, provided $(\phi_{ij}^{(k)})$ are nonsingular with matrix inverses $(\xi_{ij}^{(k)})$,

$$(21) \qquad C(p) = \left\{ \prod_{i=1}^{N} \frac{1}{1+\alpha_i p} \right\} \left\{ \prod_{k=1}^{L} \left[\frac{\Gamma_k(0)}{\Gamma_k(p)} \right]^{1/2} \right\}$$

where $\Gamma_k(p)$ is defined by (17) and (18). This has been derived assuming p to be a real number ≥ 0, but, by analytic continuation, it clearly holds for all p in the closed right half plane. The only interesting singular case is that of complete correlation of the x_i, which can be treated as a separate case[(3),(4),(5)].

It is interesting to consider some special cases. For example, suppose $\alpha_i = 1$, all i. Define

$$(22) \qquad \lambda_i^{(k)}(p) = i^{th} \text{ eigenvalue of } \left(\xi_{ij}^{(k)}(p) \right)$$

$$\lambda_i^{(k)}(0) = \lambda_i^{(k)} = i^{th} \text{ eigenvalue of } (\xi_{ij}^{(k)})$$

Then, if $\alpha_i = 1$, all i, $\lambda_i^{(k)}(p) = \lambda_i^{(k)} + \frac{2p}{1+p}$, and

$$(23) \qquad \Gamma_k(p) = \prod_{i=1}^{N} \left(\lambda_i^{(k)} + \frac{2p}{1+p}\right)$$

$$\Gamma_k(0) = \prod_{i=1}^{N} \lambda_i^{(k)}$$

Putting (23) into (21), for the case $\measuredangle_i = 1$, all i,

$$(24) \qquad C(p) = \frac{1}{(1+p)^N} \prod_{k=1}^{L} \prod_{i=1}^{N} \left[1 + \frac{2p}{(1+p)\lambda_i^{(k)}} \right]^{-1/2}$$

If the covariance matrices are all equal: $\phi_{ij}^{(k)} = \phi_{ij}$, all k, then $\lambda_i^{(k)} = \lambda_i$, all k, and

$$(25) \qquad C(p) = \frac{1}{(1+p)^N} \prod_{i=1}^{N} \left[1 + \frac{2p}{(1+p)\lambda_i} \right]^{-L/2}$$

Specializing still further, suppose that

$$(26) \quad \begin{cases} L = 2K \\ (\phi_{ij}^{(k)}) = (\phi_{ij}) \quad , \text{all } k \\ \measuredangle_i = 1 \quad , \text{all } i \\ \overline{u_{k,i}^2} = \phi_{ii} = \sigma^2, \text{ all } i \text{ and } k \end{cases}$$

In these cases, x_i are distributed according to the p.d.f. (9) and

$$(27) \qquad \overline{x}_i = \overline{x} = 2K\sigma^2 \quad , \quad \text{all } i$$

The assumptions $\measuredangle_i = 1$, all i, and $\overline{x}_i = 2K\sigma^2$, all i, amount in radar applications to assuming a uniform beam, and uniform weighting of pulses by the post-detection integrator--assumptions almost invariably made in probability of detection calculations. However, even the assumptions listed in (26) allow a large degree of freedom in the choice of correlation properties and p.d.f.'s for the signal fluctuations.

Now define

(28) $$\mu_i = i^{th} \text{ eigenvalue of } \left(\frac{\phi_{ij}}{\sigma^2}\right) = \frac{1}{\sigma^2\lambda_i}$$

Using (25), (27), and (28), we obtain, under the conditions listed in (26),

(29) $$C(p) = (1+p)^{N(K-1)} \prod_{i=1}^{N} \frac{1}{\left[1+p\left(1+\frac{\bar{x}\mu_i}{K}\right)\right]^K}$$

where $\bar{x}$ is defined in (27), K in (26), and μ_i in (28). This can also be written

(29a) $$C(p) = \frac{1}{(1+p)^N} \prod_{i=1}^{N} \left[1 + \frac{\mu_i \bar{x} p}{K(1+p)}\right]^{-K}$$

Special cases are

(30) $$K = 1 \;:\; C(p) = \prod_{i=1}^{N} \frac{1}{1+p(1+\mu_i \bar{x})}$$

$$K = 2 \;:\; C(p) = (1+p)^N \prod_{i=1}^{N} \frac{1}{\left[1+p\left(1+\frac{\mu_i \bar{x}}{2}\right)\right]^2}$$

These formulas (30) reduce to the correct formulas[3] when the signal strength fluctuates independently from pulse to pulse, in which case $\mu_i = 1$, all i. Also, they reduce to the correct formulas[3] in the case where the signal strength is constant for the N pulses $v_1, \ldots, v_N$; in this case, $\mu_i = 0$, $i = 1, \ldots, N-1$, and $\mu_N = N$. The correct formulas are obtained even though complete correlation leads to a singular covariance matrix (ϕ_{ij}); this indicates that the validity of (29) extends to some cases where (ϕ_{ij}) is singular.

It is interesting to note what happens to C(p) in (29a) when $K \to \infty$. Since, for $K \to \infty$, the standard deviation of the fluctuations about the mean value goes to zero, one would expect C(p) to approach the form it would take for nonfluctuating signals. This is indeed what happens: from (29a), as $K \to \infty$,

$$C(p) \to \frac{1}{(1+p)^N} \exp\left[\frac{-\bar{x}p}{1+p}\sum_{i=1}^{N} \mu_i\right].$$

But $\sum_{i=1}^{N} \mu_i$ = trace of $\left(\frac{\phi_{ij}}{\sigma^2}\right) = N$, so

$$C(p) \to \frac{1}{(1+p)^N} \exp\left[\frac{-N\bar{x}p}{1+p}\right] \text{ (compare Ref. 2).}$$

One can use the expansion $(1 + \frac{a}{K})^{-K} = \exp\left[-a + \frac{a^2}{2K} - \frac{a^3}{3K^2} + \ldots\right]$,

valid for $\left|\frac{a}{K}\right| < 1$, to obtain the following expression for C(p):

$$\text{(29b)} \qquad C(p) = \frac{1}{(1+p)^N} \exp\left\{\frac{-N\bar{x}p}{1+p}\sum_{j=0}^{\infty} A_j \left[\frac{\bar{x}p}{K(1+p)}\right]^j\right\}$$

where $$A_j = \frac{(-1)^j}{j+1}\left[\frac{1}{N}\sum_{i=1}^{N} \mu_i^{j+1}\right]$$

(29b) is valid for all $p \geq 0$ if $\frac{\bar{x}\mu_i}{K} \leq 1$, all i; otherwise it is valid for

$$p < \min_i \left[\frac{1}{\frac{\bar{x}\mu_i}{K} - 1}\right]$$

It is useful to note that, in the cases represented by the assumptions listed in (26)--and in fact in more general cases--the Laplace transform $C(p)$ can be inverted in a straightforward, though possibly very tedious, manner. The simplest case is of course $K = 1$ (see (30)), in which case, if for example the μ_i are distinct, the p.d.f. $G(y)$ is of the form

$$G(y) = \sum c_i \exp \left[\frac{-y}{1+\mu_i \bar{x}} \right]$$

where the c_i are readily determined. Since digital computer programs exist for finding the eigenvalues of N x N matrices up to fairly high values of N, the above formulas can be utilized for digital machine computation of probability of detection in a fairly straightforward way.

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