NASr-21(02)

MEMORANDUM
RM-4653-NASA
JULY 1965

GPO PRICE $ ________

CFSTI PRICE(S) $ ________

Hard copy (HC) 2.00

Microfiche (MF) .50

ff 653 July 65

HARD LIMITING OF THREE AND FOUR SINUSOIDAL SIGNALS

William Sollfrey

PREPARED FOR:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

The RAND Corporation
SANTA MONICA • CALIFORNIA

MEMORANDUM
RM-4653-NASA
JULY 1965

HARD LIMITING OF THREE AND FOUR SINUSOIDAL SIGNALS

William Sollfrey

This research is sponsored by the National Aeronautics and Space Administration under Contract No. NASr-21. This report does not necessarily represent the views of the National Aeronautics and Space Administration.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

Published by The RAND Corporation

PREFACE

This Memorandum is a result of RAND's continuing study of Communication Satellite Technology for the National Aeronautics and Space Administration. It presents an analysis of the behavior of hard limiters for certain special analytically solvable conditions involving three or four input sinusoidal signals. It should be of particular interest to engineers concerned with the theoretical or experimental behavior of hard limiters for use in multiple access operation of communication satellites.

SUMMARY

29748

An analysis has been performed of the effect of hard limiting on a sum of three or four sinusoidal signals. Expressions are obtained for the output amplitudes for three input signals, two of equal amplitude, and for four signals, amplitudes equal in pairs. The answers are compared with experiment and display excellent agreement.

The results indicate the general character of the reduction of the suppressive effects of limiting as the number of signals increases. Also, "negative suppression" occurs for certain amplitude ranges.

Author

ACKNOWLEDGMENTS

The author is indebted to W. Doyle for the original suggestion of this problem, and for the use of his theoretical results. Also, Messrs. R. Davies and W. Wood of the Philco Corporation Western Development Laboratories very kindly gave permission to quote their previously unpublished experimental results.

CONTENTS

LIST OF FIGURES

I. INTRODUCTION

The effect of hard limiting on a sum of signals causes the limiter output to contain both signal and intermodulation products. The detailed investigation of the dependence of the relative output signal levels on the relative input levels is an exceedingly complex problem. The theory has been developed for Gaussian signals,[1,2] and for one or two sinusoids plus noise.[3-5] The problem of three or more sinusoids has generally been regarded as too difficult for analytic investigation, though at least one attempt toward its solution has been made.[6]

While the general problem has not been solved, this Memorandum presents analytic answers to the input-output level problem for three signals, two of equal amplitude, and for four signals, amplitudes equal in pairs. The frequencies of all signals have been assumed incommensurable and the bandwidths narrow, so no higher modulation products appear at the signal frequencies in the output.

In Section II, the theory will be developed for three sinusoidal signals, and in Section III, for four such signals. The theoretical results are compared with experimental investigations, and extremely close agreement is displayed.

In the two-signal case, limiting produces 6 db suppression of a weak signal with respect to a strong signal. For three input signals, two equal, there is again 6 db suppression when the single component is strong compared to the double components. However, when the double components are strong compared to the single component, the phenomenon of "negative suppression" occurs. When the strong to weak ratio at the input exceeds 2 db, the weak component at the output is enhanced with respect to the strong components. This behavior appears in both the theoretical and experimental results.

For four input signals, amplitudes equal in pairs, the "negative suppression" again appears when the input strong to weak ratio exceeds 6 to 8 db. Again theory and experiment display the same character.

The analysis shows that the suppressive effects of limiting decrease as the number of signals increases.

II. LIMITING OF THREE SINUSOIDAL SIGNALS, TWO OF EQUAL AMPLITUDE

It has not been possible to solve in manageable form the limiting of three signals of arbitrary amplitude. However, when two of the signals have equal amplitude, the output can be expressed as a rapidly converging series in the ratio of the weak to the strong component. This enables rapid computation of the output levels and suppression effects.

Let the input be

$$e_{in} = a \cos(\omega_1 t+\theta_1) + b[\cos(\omega_2 t+\theta_2) + \cos(\omega_3 t+\theta_3)] \tag{1}$$

where the frequencies ω_1, ω_2, and ω_3 are assumed incommensurable. The limiter characteristic is taken to be

$$\begin{aligned} e_{out} &= 1 \qquad & e_{in} > 0 \\ &= 0 \qquad & e_{in} = 0 \\ &= -1 \qquad & e_{in} < 0 \end{aligned} \tag{2}$$

Thus, the output is a rectangular wave which changes sign at the zero crossings of the input. This characteristic may be represented in the form

$$e_{out} = \frac{2}{\pi} \int_0^\infty \frac{dx}{x} \sin(x\, e_{in}) \tag{3}$$

When the expression (1) is inserted into this integral, the sine of a sum may be transformed by simple trigonometry into the sum of four products of sines. Thus

$$e_{out} = \frac{2}{\pi}\int_0^\infty \frac{dx}{x}\left[\begin{array}{l}\sin(ax \cos r_1)\cos(bx \cos r_2)\cos(bx \cos r_3)\\ + \sin(bx \cos r_2)\cos(bx \cos r_1)\cos(bx \cos r_3)\\ + \sin(bx \cos r_3)\cos(bx \cos r_1)\cos(bx \cos r_2)\\ - \sin(ax \cos r_1)\sin(bx \cos r_2)\sin(bx \cos r_3)\end{array}\right] \qquad (4)$$

where $r_1 = \omega_1 t+\theta_1$ and similarly for r_2,r_3. The sines may be expanded as Fourier series in r_1, r_2, r_3 whose coefficients are Bessel functions. Under the assumption that the frequencies are incommensurable, the expansions need include only the constant and fundamental terms to give the expressions for the signal components. To this order

$$\begin{aligned}\sin(ax \cos r_1) &\rightarrow 2\, J_1(ax)\cos r_1\\ \cos(bx \cos r_2) &\rightarrow J_0(bx)\end{aligned} \qquad (5)$$

The output signal components may, therefore, be written as

$$e_S = c \cos r_1 + d(\cos r_2 + \cos r_3) \qquad (6)$$

$$c = \frac{4}{\pi}\int_0^\infty \frac{dx}{x} J_1(ax) \; [J_0(bx)]^2 \qquad (7)$$

$$d = \frac{4}{\pi}\int_0^\infty \frac{dx}{x} J_0(ax)J_0(bx)J_1(bx) \qquad (8)$$

The evaluation of these integrals requires a lengthy sequence of transformations, but leads to a straightforward result.

Replace $J_1(ax)$ in c by its Poisson integral representation[7]

$$J_1(ax) = \frac{2ax}{\pi} \int_0^{\pi/2} \sin^2 \varphi \cos(ax \cos \varphi) d\varphi \tag{9}$$

and the square of a Bessel function by the special case n = m = 0 of the Neumann integral representation (Ref. 7, p. 150)

$$J_n(z) J_m(z) = \frac{2}{\pi} \int_0^{\pi/2} J_{n+m} (2z \cos \theta) \cos(n-m)\theta d\theta \tag{10}$$

When these expressions are substituted in (7) and the order of integration changed, the integral over x may be performed by the formula (Ref. 7, p. 405)

$$\int_0^\infty dx\, J_0(\alpha x) \cos Bx = \begin{array}{ll} \sqrt{\alpha^2 - B^2} & \alpha > B \\ 0 & \alpha < B \end{array} \tag{11}$$

which then leads to

$$c = \frac{16}{\pi^3} \iint \frac{d\theta d\varphi \sin^2 \varphi}{\left[4 \frac{b^2}{a^2} \cos^2\theta - \cos^2\theta\right]^{\frac{1}{2}}} \qquad 2b \cos \theta > a \cos \varphi \tag{12}$$

The limits of integration are not specified in (12), but go over that portion of the region $0 \le \theta, \varphi \le \frac{\pi}{2}$ which satisfies the indicated inequality. The analysis now separates into two cases, depending on whether $\frac{2b}{a}$ is greater or less than one.

If $\alpha = \frac{2b}{a}$ is less than one, the inequality places no restriction on θ, but limits φ. There results

$$c = \frac{16}{\pi^3} \int_0^{\pi/2} d\theta \int_{\cos^{-1}(\alpha \cos\theta)}^{\pi/2} \frac{\sin^2 \varphi d\varphi}{\left[\alpha^2\cos^2\theta - \cos^2\varphi\right]^{\frac{1}{2}}} \tag{13}$$

The transformation

$$\cos\varphi = \alpha \cos\theta \cos\psi$$

and subsequent simplification yields

$$c = \frac{16}{\pi^3} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\psi \left[1 - \alpha^2\cos^2\theta \cos^2\psi\right]^{\frac{1}{2}} \tag{14}$$

Anticipating later results, introduce the notation

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\, n!} = \frac{(2n)!}{2^{2n}(n!)^2} \tag{15}$$

Since α is less than one, the square root can be expanded by binomial theorem, yielding

$$c = \frac{16}{\pi^3} \int\int_0^{\pi/2} d\theta d\psi \left[1 - \frac{1}{2} \sum_0^\infty \frac{a_n}{n+1} \alpha^{2n+2} \cos^{2n+2}\theta \cos^{2n+2}\varphi\right] \tag{16}$$

Since it will be used in several forms, the general relation is now stated[8]

$$\int_0^{\pi/2} \sin^{2x}\varphi \cos^{2y}\varphi \, d\varphi = \frac{\Gamma(x+\frac{1}{2})\Gamma(y+\frac{1}{2})}{2\Gamma(x+y+1)} \tag{17}$$

Specializing to x = 0, y = n + 1 yields

$$\int_0^{\pi/2} \cos^{2n+2}\theta d\theta = \frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{3}{2})}{2\Gamma(n+2)} = \frac{\pi a_{n+1}}{2} \tag{18}$$

Using this relation in (16) yields

$$c = \frac{2}{\pi}\left[2 - \sum_0^\infty \frac{a_n a^2_{n+1}}{n+1}\left(\frac{2b}{a}\right)^{2n+2}\right] \qquad \frac{2b}{a} < 1 \tag{19}$$

For n large, a_n is approximately $(\pi n)^{-\frac{1}{2}}$, whence the series converges as $n^{-5/2}$. Even for $\frac{2b}{a} = 1$, the series can be evaluated very quickly with only a desk calculator or slide rule.

If $\alpha = \frac{2b}{a}$ is greater than one, a more complicated sequence of transformations is required. Now θ is restricted by the inequality in (12), while φ is not. The integral becomes

$$c = \frac{16}{\pi^3}\int_0^{\pi/2} \sin^2\varphi d\varphi \int_0^{\sin^{-1}\sqrt{1-\cos^2\varphi/\alpha^2}} \frac{d\theta}{\left[\alpha^2\cos^2\theta - \cos^2\varphi\right]^{\frac{1}{2}}} \tag{20}$$

The transformation

$$\sin\theta = \sqrt{1 - \frac{\cos^2\varphi}{\alpha^2}}\,\sin\psi \tag{21}$$

brings this into the form

$$c = \frac{16}{\pi^3\alpha}\int_0^{\pi/2} \sin^2\varphi d\varphi \int_0^{\pi/2} \frac{d\psi}{\left[1-(1-\cos^2\varphi/\alpha^2)\sin^2\psi\right]^{\frac{1}{2}}} \tag{22}$$

The integration over ψ yields the complete elliptic integral of the first kind,[(9)]

$$c = \frac{16}{\pi^3 \alpha} \int_0^{\pi/2} \sin^2\varphi \; K\left(\sqrt{1 - \cos^2\varphi/\alpha^2}\right) d\varphi \tag{23}$$

The key to the solution is to expand the elliptic integral K(k) in powers of the "complementary modulus" $k^1 = \sqrt{1 - k^2}$. The resulting expression involves a logarithm. The first four terms of the expansion are given in Ref. 9, and the complete expansion may be determined by using the relations between elliptic integrals and hypergeometric functions. It proves convenient to introduce the additional notation

$$\begin{aligned} b_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \quad , \quad b_0 = 0 \\ &= \log 2 + \frac{1}{2}\,\psi(n+\tfrac{1}{2}) - \frac{1}{2}\,\psi(n+1) \end{aligned} \tag{24}$$

where ψ denotes the logarithmic derivative of the gamma function. The expansion for K is now

$$K\left(\sqrt{1 - \cos^2\varphi/\alpha^2}\right) = \sum_0^\infty a_n^2 \frac{\cos^{2n}\varphi}{\alpha^{2n}} \left[\log \frac{4\alpha}{\cos\varphi} - 2b_n\right] \tag{25}$$

This is substituted into (23) and the series is then integrated term by term. Those terms free from logarithms may be evaluated by using (17) with x = 1, y = n, yielding

$$\int_0^{\pi/2} \sin^2\varphi \cos^{2n}\varphi d\varphi = \frac{\Gamma(3/2)\,\Gamma(n+\frac{1}{2})}{2\,\Gamma(n+2)} = \frac{\pi}{4}\,\frac{a_n}{n+1} \tag{26}$$

The logarithmic term is found by differentiating (17) with respect to y and then setting y = n. The result, which involves the logarithmic derivative of the gamma function, may be expressed in terms of a_n and b_n, giving

$$\int_0^{\pi/2} \sin^2\varphi \cos^{2n}\varphi \log \cos \varphi d\varphi = -\frac{\pi}{4}\frac{a_n}{n+1}\left[\log 2 - b_n + \frac{1}{2(n+1)}\right] \tag{27}$$

When the expressions are assembled and simplified, there results

$$c = \frac{2a}{\pi^2 b}\sum_0^\infty \frac{a_n^3}{(n+1)}\left(\frac{a}{2b}\right)^{2n}\left[\log\frac{2b}{a} + 3(\log 2 - b_n) + \frac{1}{2(n+1)}\right] \qquad \frac{2b}{a} = \alpha > 1 \tag{28}$$

A similar technique may be used to evaluate the coefficient d. The product of Bessel functions of the same argument is replaced by a Neumann integral, which now involves a J_1 function. The new J_1 function is replaced by a Poisson integral, and the x integration is performed. At this point

$$d = \frac{16\alpha}{\pi^3}\iint \frac{d\theta d\varphi \cos^2\theta \sin^2\varphi}{\left[1-\alpha^2\cos^2\theta\cos^2\varphi\right]^{\frac{1}{2}}} \qquad \alpha\cos\theta\cos\varphi < 1 \tag{29}$$

For $\alpha < 1$, there are no restrictions on either integration variable. Expanding by binomial theorem, and then using (17) yields

$$d = \frac{4b}{\pi a}\sum_0^\infty \frac{a_n^2 a_{n+1}}{(n+1)}\left(\frac{2b}{a}\right)^{2n} \qquad \frac{2b}{a} < 1 \tag{30}$$

When $\alpha > 1$, the integral is best evaluated by using the method of rotations on the surface of a unit sphere expounded in Chapter 12 of

Watson's treatise.[7] Since this method has not appeared very frequently in the literature, the intermediate steps will be presented. First, replacing θ by $\frac{\pi}{2}$ - θ in (29) yields

$$d = \frac{16\alpha}{\pi^3} \iint \frac{d\theta d\varphi \sin^2\theta \sin^2\varphi}{\left[1-\alpha^2\sin^2\theta \cos^2\varphi\right]^{\frac{1}{2}}} \qquad 0 \leq \theta, \varphi \leq \frac{\pi}{2}, \alpha \sin\theta \cos\varphi < 1 \tag{31}$$

Now view θ and φ as spherical coordinates on the surface of a unit sphere. The direction cosines on the surface are defined by

$$\ell = \sin\theta \cos\varphi \tag{32}$$

$$m = \sin\theta \sin\varphi \tag{33}$$

$$n = \cos\theta \tag{34}$$

and the element of surface area is

$$d\Omega = \sin\theta d\theta \, d\varphi \tag{35}$$

The integral may now be written in the form

$$d = \frac{16\alpha}{\pi^3} \int \frac{d\,\Omega\, m^2}{(1-n^2)^{\frac{1}{2}}(1-\alpha^2\ell^2)^{\frac{1}{2}}} \qquad \ell,m,n > 0, \quad \alpha\ell < 1 \tag{36}$$

The point of the method is that the integral is invariant with respect to a cyclic permutation of the direction cosines, which is equivalent to relabeling the coordinate axes. Thus, on performing the interchange $\ell \to n$, $m \to \ell$, $n \to m$, there results

$$d = \frac{16\alpha}{\pi^3} \int \frac{d\,\Omega\, \ell^2}{(1-m^2)^{\frac{1}{2}}(1-\alpha^2 n^2)^{\frac{1}{2}}} \qquad \ell,m,n > 0, \quad \alpha n < 1 \tag{37}$$

The rotation has arranged that the coordinate restriction applies to only one variable. Since $\alpha > 1$, restoration of the θ, φ expressions now yields

$$d = \frac{16\alpha}{\pi^3} \int_0^{\pi/2} \cos^2\varphi d\varphi \int_{\cos^{-1} 1/\alpha}^{\pi/2} \frac{\sin^3\theta d\theta}{(1-\sin^2\theta \sin^2\varphi)^{\frac{1}{2}}(1-\alpha^2\cos^2\theta)^{\frac{1}{2}}} \tag{38}$$

The transformation

$$\alpha \cos \theta = \cos \psi \tag{39}$$

and subsequent simplification gives

$$d = \frac{16}{\pi^3} \int_0^{\pi/2} d\psi \int_0^{\pi/2} \frac{d\varphi\left(1- \frac{\cos^2\psi}{\alpha^2}\right)\cos^2\varphi}{\left[1-\left(1- \frac{\cos^2\psi}{\alpha^2}\right)\sin^2\varphi\right]^{\frac{1}{2}}} \tag{40}$$

This is again an elliptic integral, yielding

$$d = \frac{16}{\pi^3} \int_0^{\pi/2} d\psi \left[E\left(\sqrt{1- \frac{\cos^2\psi}{\alpha^2}}\right) - \frac{\cos^2\psi}{\alpha^2} K\left(\sqrt{1- \frac{\cos^2\psi}{\alpha^2}}\right)\right] \tag{41}$$

where $E(k)$ denotes the complete elliptic integral of the second kind. This may be expanded by a relation similar to (25)

$$E\left(\sqrt{1- \frac{\cos^2\psi}{\alpha^2}}\right) = 1 + \sum_0^\infty a_n a_{n+1} \left(\frac{\cos \psi}{\alpha}\right)^{2n+2} \left[\log \frac{4\alpha}{\cos \psi} - b_n - b_{n+1}\right] \tag{42}$$

When (25) and (42) are substituted into (41), the resulting expression may be integrated term by term. After many tedious simplifications, there results the expression

$$d = \frac{4}{\pi^2}\left[2-\sum_0^{\infty}\frac{a_n^2 a_{n+1}}{(n+1)}\left(\frac{a}{2b}\right)^{2n+2}\left\{\log\frac{2b}{a} + 3\log 2 - 2b_n - b_{n+1} + \frac{1}{2(n+1)}\right\}\right] \quad \frac{2b}{a} > 1 \tag{43}$$

The four expressions (19), (28), (30), and (43) give the amplitude of the single and double output components for all values of the amplitudes of the input components. They are plotted against the ratio b/a, the double to single ratio, in Fig. 1. For b/a small, that is, weak double component, the single component tends to $4/\pi$ while the double component vanishes as $2b/\pi a$. For b/a large, that is, weak single component, the double component tends to $8/\pi^2$ while the single component vanishes as $\frac{2a}{\pi^2 b}\left(\log\frac{16b}{a} + \frac{1}{2}\right)$. At b/a = 1, or three equal components, the output amplitude is .6683. The total signal power output in this case is $\frac{3}{2}(.6683)^2 = .670$. The limited signal power output for three equal signals has been computed by W. Doyle,* using a digital computer simulation program. He obtains the value .669, displaying essentially perfect agreement.

To compare theory with experiment, signal suppression will be considered. This is denoted by γ, and is defined as the quotient of the weak to strong component ratio at the limiter output to the weak to strong ratio at the input. Thus

$$\gamma = \frac{c/d}{a/b} \qquad a/b < 1 \tag{44}$$

$$= \frac{d/c}{b/a} \qquad a/b > 1 \tag{45}$$

The first case corresponds to a weak single component and strong double components, the second to weak double and strong single. In Fig. 2,

---

*Private communication from W. Doyle, Consultant to The RAND Corporation.

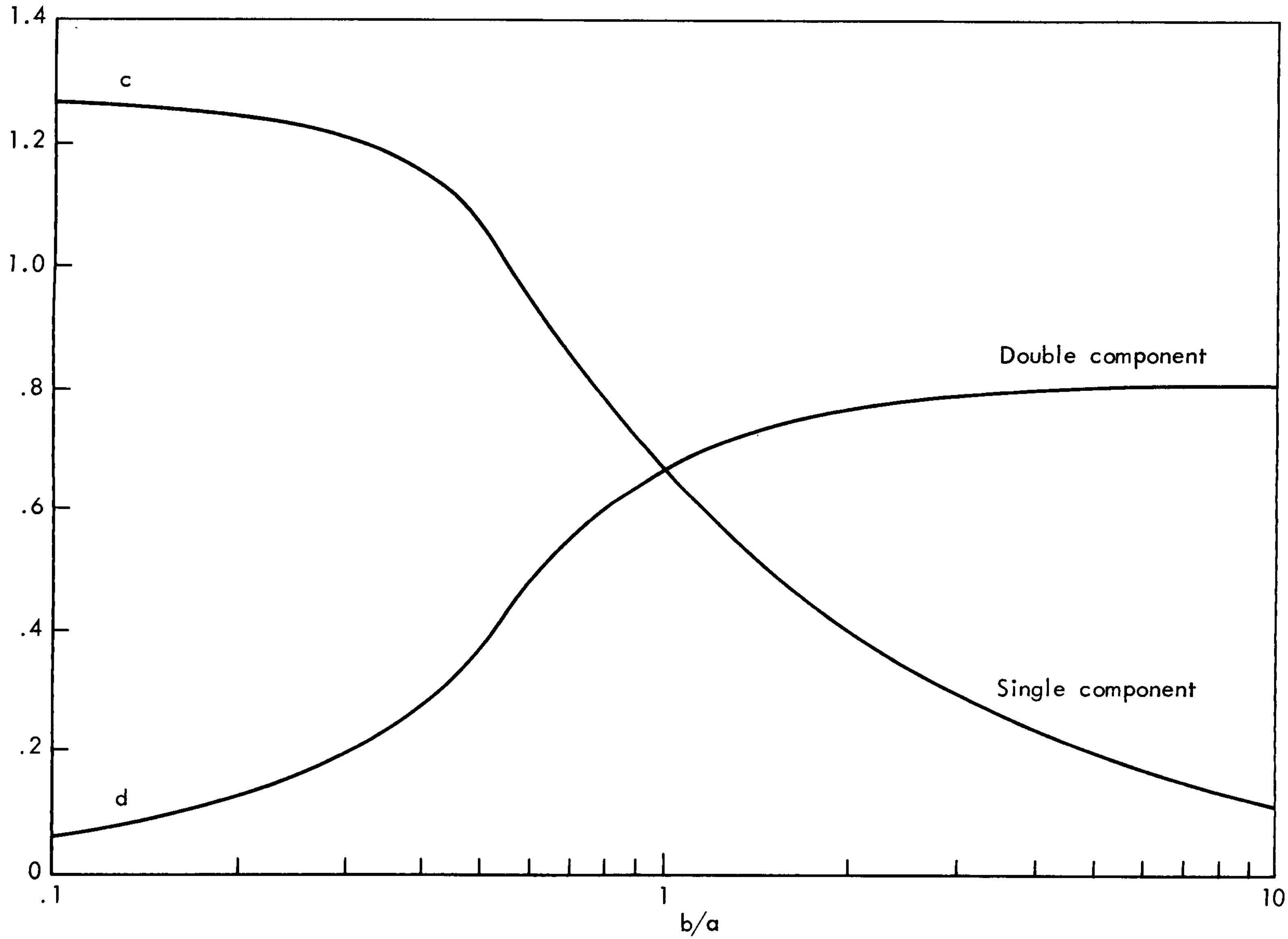

Fig. 1—Limited signal output components, three inputs

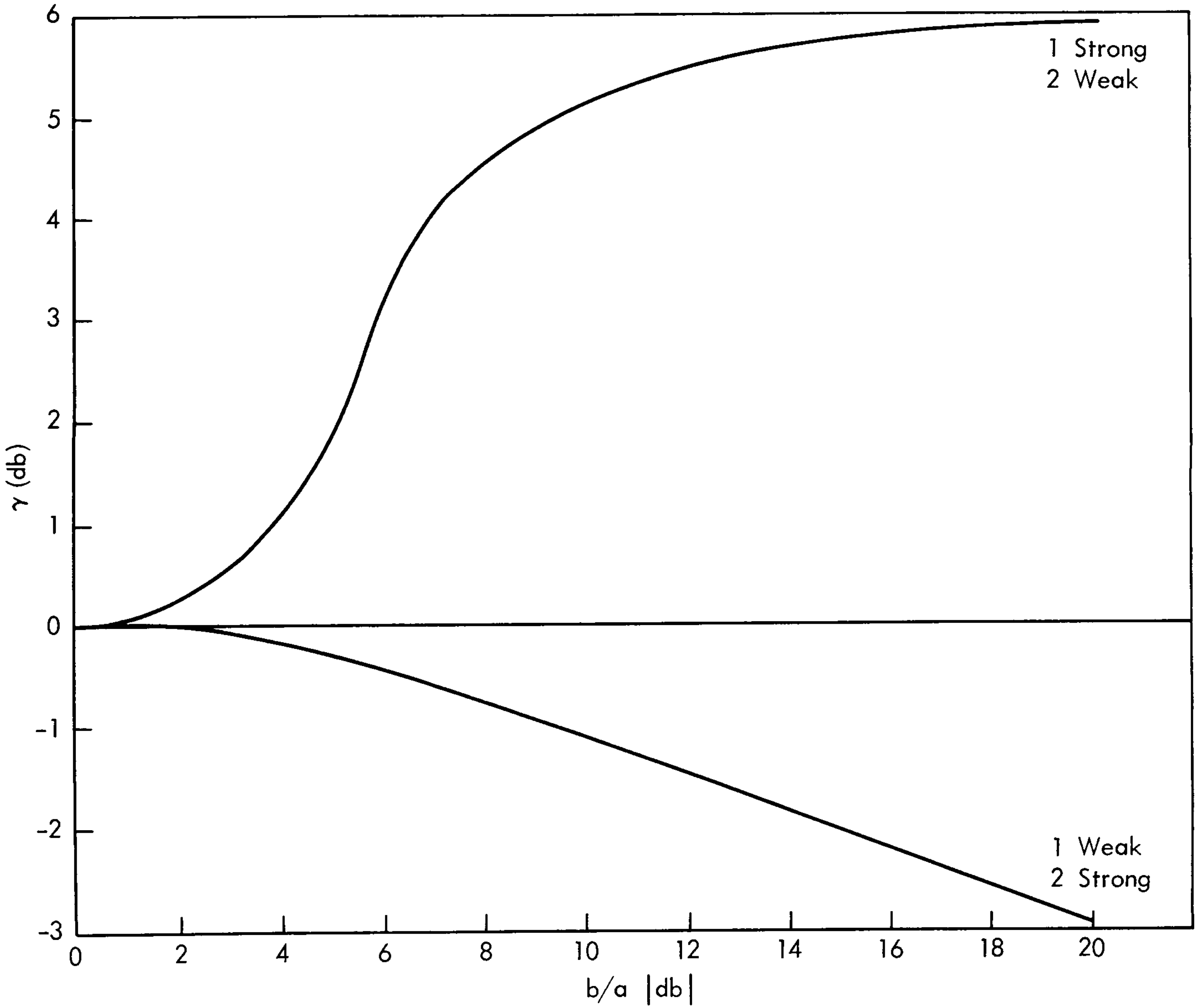

Fig.2—Three signal suppression (theoretical)

the suppression ratio γ is plotted in decibels against the input ratio b/a in absolute decibels. Positive decibel values of γ correspond to reduction of the weak component.

The behavior of the ratio γ is conventional for the one strong-two weak case. It starts at zero db for equal components and rises rapidly as the single component increases. For the single component very strong compared to the double component, γ tends to 6 db (or a voltage factor of 2). This limiting 6 db behavior has been known for many years, and has been used as a description of the interference-suppressing properties of a limiter.

However, the behavior of the ratio γ in the one weak-two strong case is most unusual. The ratio rises very slightly, then reverses, crosses zero again at an input ratio of 2.2 db, and goes slowly to negative values. For b/a large, the suppression ratio behaves asymptotically as

$$\gamma(\mathrm{db}) \to -\ 20\ \log_{10}\left[.818 + .576\ \log_{10} \frac{b}{a}\right] \tag{46}$$

and tends very slowly to $-\infty$. Even for an input ratio $b/a = 10^6$, the suppression ratio is only -12.5 db.

This behavior indicates that in the one weak-two strong case, hard limiting enhances the weaker component with respect to the stronger components at large input ratios. It is not enhanced in absolute value, but tends to zero as shown in Fig. 1. Therefore, under these circumstances the limiter displays "negative suppression."

The explanation of this effect may be as follows: Since the frequencies are incommensurable, there will be times when the two strong components are nearly 180° out of phase. The weak component then exerts an inordinate effect on the zero crossings of the input signal, and the limiter squares up the waveform to enhance the weak component. Since the 180° phase condition is relatively rare, the total enhancement is moderate. However, it is sufficient to reverse the normal behavior of the two weak-one strong case, and to produce slight negative suppression.

To demonstrate that this effect is not purely mathematical, an experimental curve of three-signal suppression by limiting is presented in Fig. 3. This curve was obtained by members of the Philco Corporation Western Development Laboratories,* who very kindly gave permission to reproduce it in this report. The theoretical and experimental curves agree within ½ db for all input values, and are much closer for most values. The negative suppression is very clearly displayed in the experimental curve, and may be regarded as established.

Figures 1 and 2, with the corroborating experimental curve of Fig. 3, give a complete presentation of the three-signal output amplitudes when there is no crosstalk and two of the three input amplitudes are equal. The experimental conditions were carefully adjusted to meet such circumstances, and the experimenters were not aware of the existence of the theory presented in this Memorandum at the time they performed their experiments. Also, the theory was developed before the author had seen the experimental results, which were originally communicated to him by N. Feldman of The RAND Corporation.

---

*Private communication from R. S. Davies and W. Wood of the Philco Corporation, Western Development Laboratories.

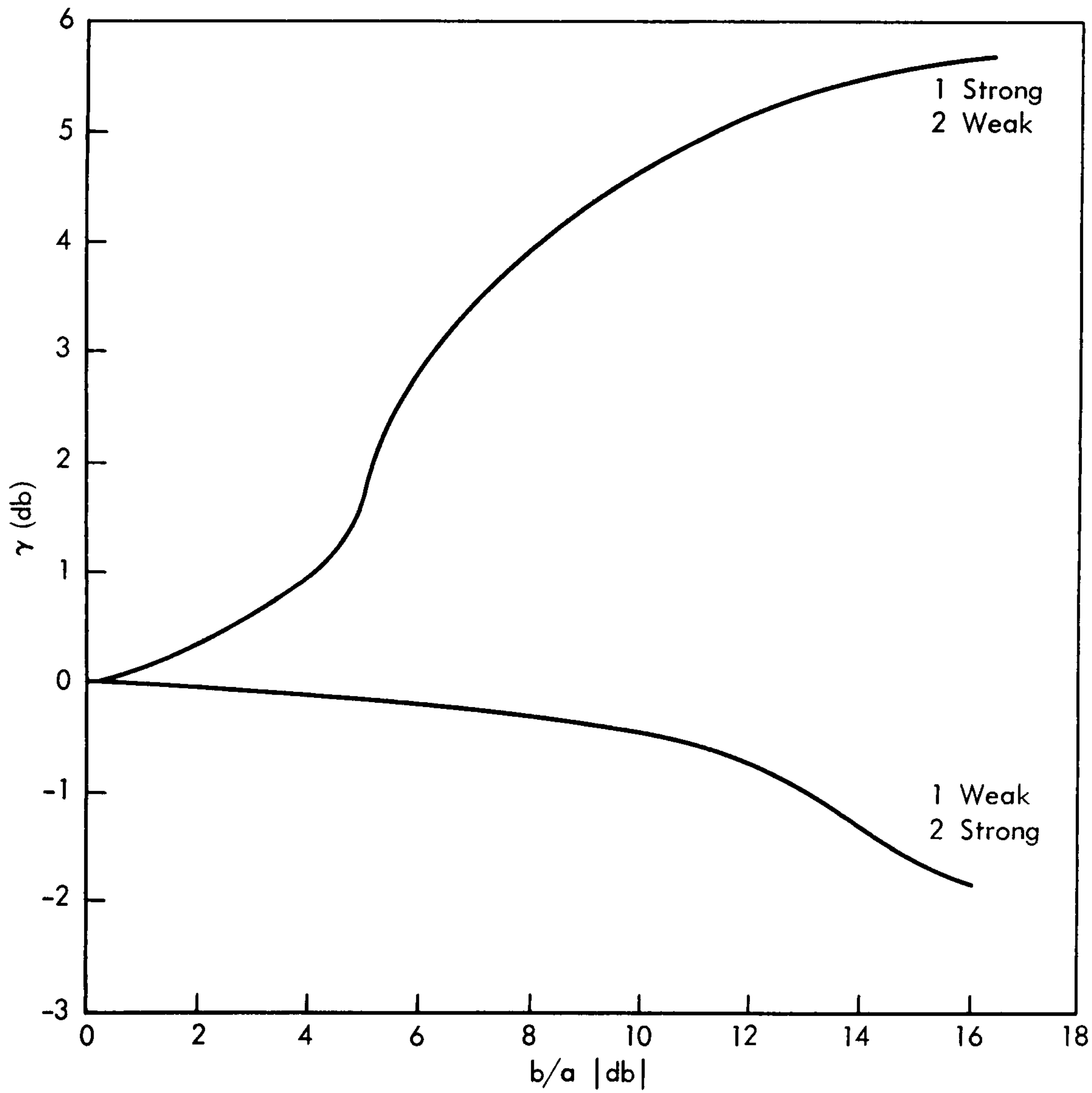

Fig.3—Three signal suppression (experimental)

III. LIMITING OF FOUR SIGNALS, AMPLITUDES EQUAL IN PAIRS

The general four-signal case has proved intractable, but it has been possible to solve the case where the amplitudes are equal in pairs using methods similar to the three-signal case. The input signal is now

$$e_{in} = a(\cos r_1 + \cos r_2) + b(\cos r_3 + \cos r_4) \qquad (47)$$

If the frequencies are incommensurable, the output signal may be written in the form

$$e_S = c(\cos r_1 + \cos r_2) + d(\cos r_3 + \cos r_4) \qquad (48)$$

By using the integral representation (3) for the limiter characteristic and expansions similar to those of Section II, the coefficients c and d are evaluated as

$$c = \frac{4}{\pi} \int_0^\infty \frac{dx}{x} J_0(ax) J_1(ax) \, (J_o(bx))^2 \qquad (49)$$

$$d = \frac{4}{\pi} \int_0^\infty \frac{dx}{x} (J_0(ax))^2 J_0(bx) J_1(bx) \qquad (50)$$

In terms of the ratio α, defined by

$$\alpha = \frac{b}{a} \qquad (51)$$

c and d are connected by

$$d(\alpha) = c(\frac{1}{\alpha}) \qquad (52)$$

Hence, only one of the integrals need by evaluated. Relation (52) is obvious from the symmetry of the input signal (47). The coefficient c is given directly as a function of α by

$$c = \frac{4}{\pi}\int_0^\infty \frac{dz}{z} J_0(z)J_1(z)\ \left(J_0(\alpha z)\right)^2 \tag{53}$$

Again, replace the square of a Bessel function by a Neumann integral involving J_0, the product J_0J_1 by a Neumann integral involving J_1, and that J_1 by a Poisson integral. The expression for c becomes a quadruple integral, and the z integration may be performed immediately by (11), yielding

$$c = \frac{32}{\pi^4}\iiint \frac{d\theta d\varphi d\psi\ \sin^2\theta\ \sin^2\varphi}{(\alpha^2\cos^2\psi-\sin^2\theta\ \cos^2\varphi)^{\frac{1}{2}}} \qquad \alpha\cos\psi > \sin\theta\cos\varphi \tag{54}$$

As before, the analysis separates into the two cases, α greater or less than 1. If $\alpha > 1$, the inequality in (54) does not restrict θ or φ, and the integral becomes

$$c = \frac{32}{\pi^4\alpha}\int_0^{\pi/2}\int d\theta d\varphi \sin^2\theta\sin^2\varphi\int_0^{\cos^{-1}(\frac{1}{\alpha}\sin\theta\cos\varphi)} \frac{d\psi}{(\cos^2\psi - \frac{1}{\alpha^2}\sin^2\theta\cos^2\varphi)^{\frac{1}{2}}} \tag{55}$$

The transformation

$$\sin\psi = \left[1 - \frac{\sin^2\theta\ \cos^2\varphi}{\alpha^2}\right]^{\frac{1}{2}} \sin\beta \tag{56}$$

brings this into the form

$$c = \frac{32}{\pi^4\alpha}\iiint_0^{\pi/2} \frac{d\theta d\varphi d\beta\ \sin^2\theta\ \sin^2\varphi}{\left[1-\left(1-\frac{\sin^2\theta\ \cos^2\varphi}{\alpha^2}\right)\sin^2\beta\right]^{\frac{1}{2}}} \tag{57}$$

The β integration is an elliptic integral. Thus

$$c = \frac{32}{\pi^4 \alpha} \int_0^{\pi/2}\!\!\int d\theta d\varphi \, \sin^2\theta \, \sin^2\varphi \, K\left(\sqrt{1 - \frac{\sin^2\theta \, \cos^2\varphi}{\alpha^2}}\right) \tag{58}$$

This elliptic integral is expanded in power series by a form analogous to (25), yielding

$$c = \frac{32}{\pi^4 \alpha} \sum_0^\infty \frac{a_n^2}{\alpha^{2n}} \int_0^{\pi/2}\!\!\int d\theta d\varphi \sin^{2n+2}\theta \sin^2\varphi \cos^{2n}\varphi \left[\log \frac{4\alpha}{\sin\theta\cos\varphi} - 2b_n\right] \tag{59}$$

All the integrals appearing here may be found from (17) or its derivative. After much algebraic reduction

$$c = \frac{4}{\pi^2 \alpha} \sum_0^\infty \frac{a_n^3 \, a_{n+1}}{(n+1)\alpha^{2n}} \left[\log 16\,\alpha - 3b_n - b_{n+1} + \frac{1}{2(n+1)}\right] \quad \alpha > 1 \tag{60}$$

For $\alpha < 1$, the inequality in (54) affects all three variables. To effect the integration, again use the method of rotation on a unit sphere. Introducing direction cosines by (32), (33), and (34), and the area element by (35), brings c into the form

$$c = \frac{32}{\pi^4} \iint \frac{d\psi d\Omega \, m^2}{(1-n^2)^{\frac{1}{2}} \, (\alpha^2\cos^2\psi - \ell^2)^{\frac{1}{2}}} \qquad \alpha \cos \psi > \ell \tag{61}$$

Cyclic interchange $\ell \to n$, $m \to \ell$, $n \to m$, followed by a return to the θ, φ representation, gives the result

$$c = \frac{32}{\pi^4} \iiint \frac{d\theta d\varphi d\psi \, \sin^3\theta \, \cos^2\varphi}{(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}} (\alpha^2\cos^2\psi - \cos^2\theta)^{\frac{1}{2}}} \qquad \alpha \cos \psi > \cos \theta \tag{62}$$

The inequality does not restrict φ. Since α is less than 1, there will be no values of ψ which satisfy the inequality unless $\cos\theta < \alpha$. Therefore, the integral with limits inserted becomes

$$c = \frac{32}{\pi^4}\int_0^{\pi/2}\cos^2\varphi d\varphi \int_{\cos^{-1}\alpha}^{\pi/2} \frac{\sin^3\theta d\theta}{(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}}\int_0^{\cos^{-1}(\frac{1}{\alpha}\cos\theta)} \frac{d\psi}{(\alpha^2\cos^2\psi-\cos^2\theta)^{\frac{1}{2}}} \tag{63}$$

The transformations

$$\cos\theta = \alpha\cos\beta \tag{64}$$

$$\sin\psi = \sin\beta\sin\eta \tag{65}$$

gives limits 0 and $\frac{\pi}{2}$ for both β and η, and the simplified form

$$c = \frac{32}{\pi^4}\iiint_0^{\pi/2} \frac{\sin\beta(1-\alpha^2\cos^2\beta)\cos^2\varphi d\beta d\eta d\varphi}{(1-\sin^2\beta\sin^2\eta)^{\frac{1}{2}}[1-(1-\alpha^2\cos^2\beta)\sin^2\varphi]^{\frac{1}{2}}} \tag{66}$$

Introduce a new set of spherical coordinates β, η, with appropriate direction cosines and area element. Then

$$c = \frac{32}{\pi^4}\iint \frac{d\varphi d\Omega\cos^2\varphi(1-\alpha^2n^2)}{(1-m^2)^{\frac{1}{2}}[1-(1-\alpha^2n^2)\sin^2\varphi]^{\frac{1}{2}}} \qquad \ell,m,n > 0 \tag{67}$$

Cyclic interchange $m \to n$, $n \to \ell$, $\ell \to m$, and restoration of the β, η representation brings this into the form

$$c = \frac{32}{\pi^4}\iiint_0^{\pi/2} \frac{d\beta d\eta d\varphi(1-\alpha^2\sin^2\beta\cos^2\eta)\cos^2\varphi}{[1-(1-\alpha^2\sin^2\beta\cos^2\eta)\sin^2\varphi]^{\frac{1}{2}}} \tag{68}$$

The φ integration is equivalent to that in (40), whence

$$c = \frac{32}{\pi^4} \iint_0^{\pi/2} d\beta d\eta \left[E\left(\sqrt{1-\alpha^2 \sin^2\beta\cos^2\eta}\right) - \alpha^2 \sin^2\beta\cos^2\eta \; K\left(\sqrt{1-\alpha^2\sin^2\beta\cos^2\eta}\right) \right] \tag{69}$$

The elliptic integrals again are expanded by (25) and (42), and all the integrations become special cases of (17). Finally

$$c = \frac{4}{\pi^2} \left[2 - \sum_0^\infty \frac{a_n^2 \, a_{n+1}^2}{(n+1)} \alpha^{2n+2} \left\{ \log \frac{16}{\alpha} - 2b_n - 2b_{n+1} + \frac{1}{2(n+1)} \right\} \right] \quad \alpha < 1 \tag{70}$$

Figure 4 shows the limited output components as a function of b/a. If one input pair, for example b, is small, the output for the large term c tends to $\frac{8}{\pi^2}$ and the output for the small term d tends to $\frac{2}{\pi^2 \alpha}$ $(\log 16\alpha - \frac{1}{4})$. At $\alpha = 1$, the four equal signals case, the output amplitude is .5726. The total power is then $2(.5726)^2 = .656$. This is in exact agreement with the number computed by W. Doyle using digital computer simulation.*

The signal suppression is computed directly from (60) and (70), and the result is graphed in Fig. 5. Experimental points, obtained by the Philco Corporation,** are also shown on Fig. 5. While the agreement is not as perfect as in the three input signals investigation, the experimental points still lie within 0.5 db of the theoretical curve.

Again, the signal suppression rises slowly from zero db, reaches a maximum, and goes negative. The crossover is at 8.5 db (theoretical)

---

*Private communication from W. Doyle.

**Private communication from R. S. Davies and W. Wood of the Philco Corporation, Western Development Laboratories.

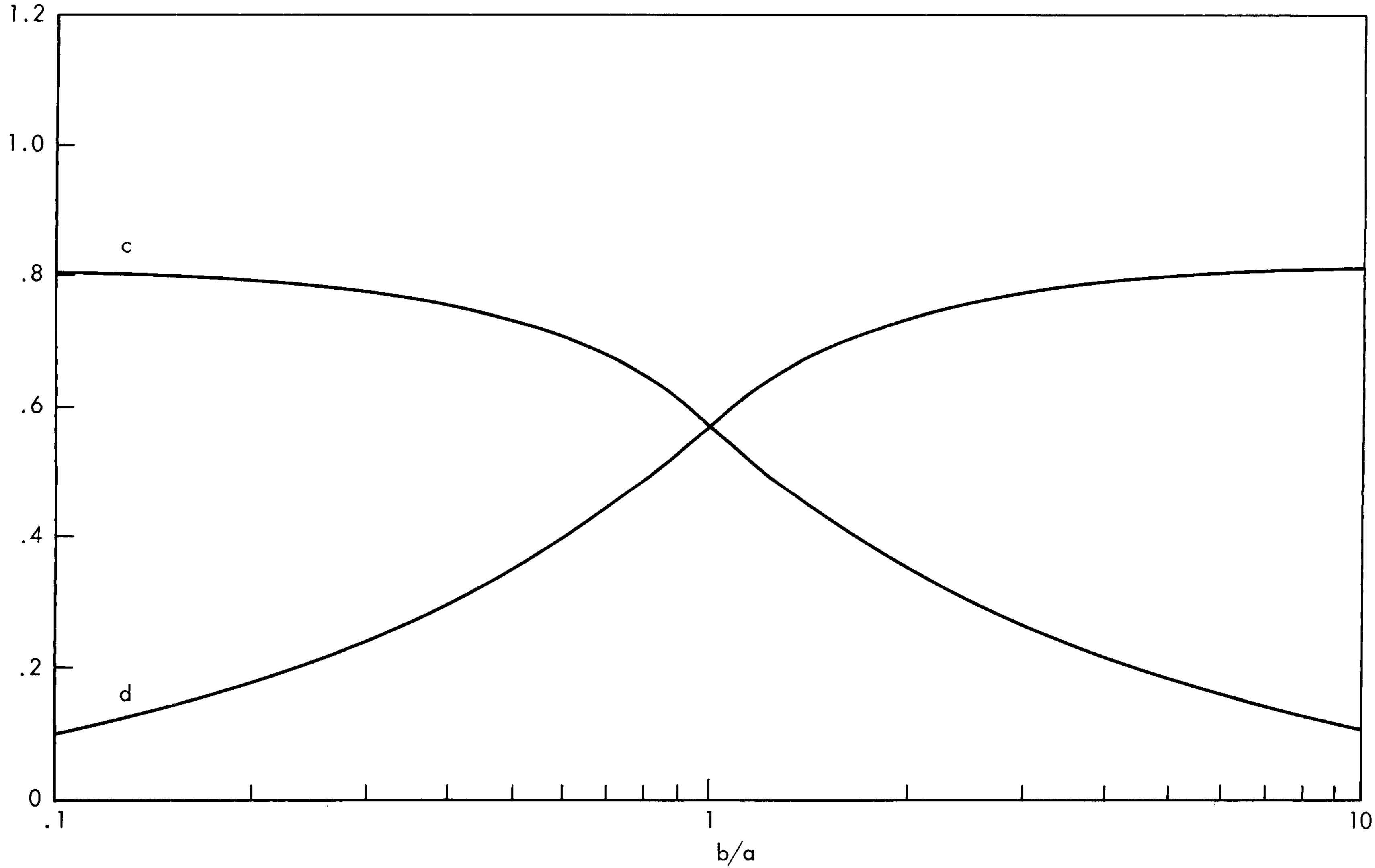

Fig.4—Limited signal output components, four inputs equal in pairs

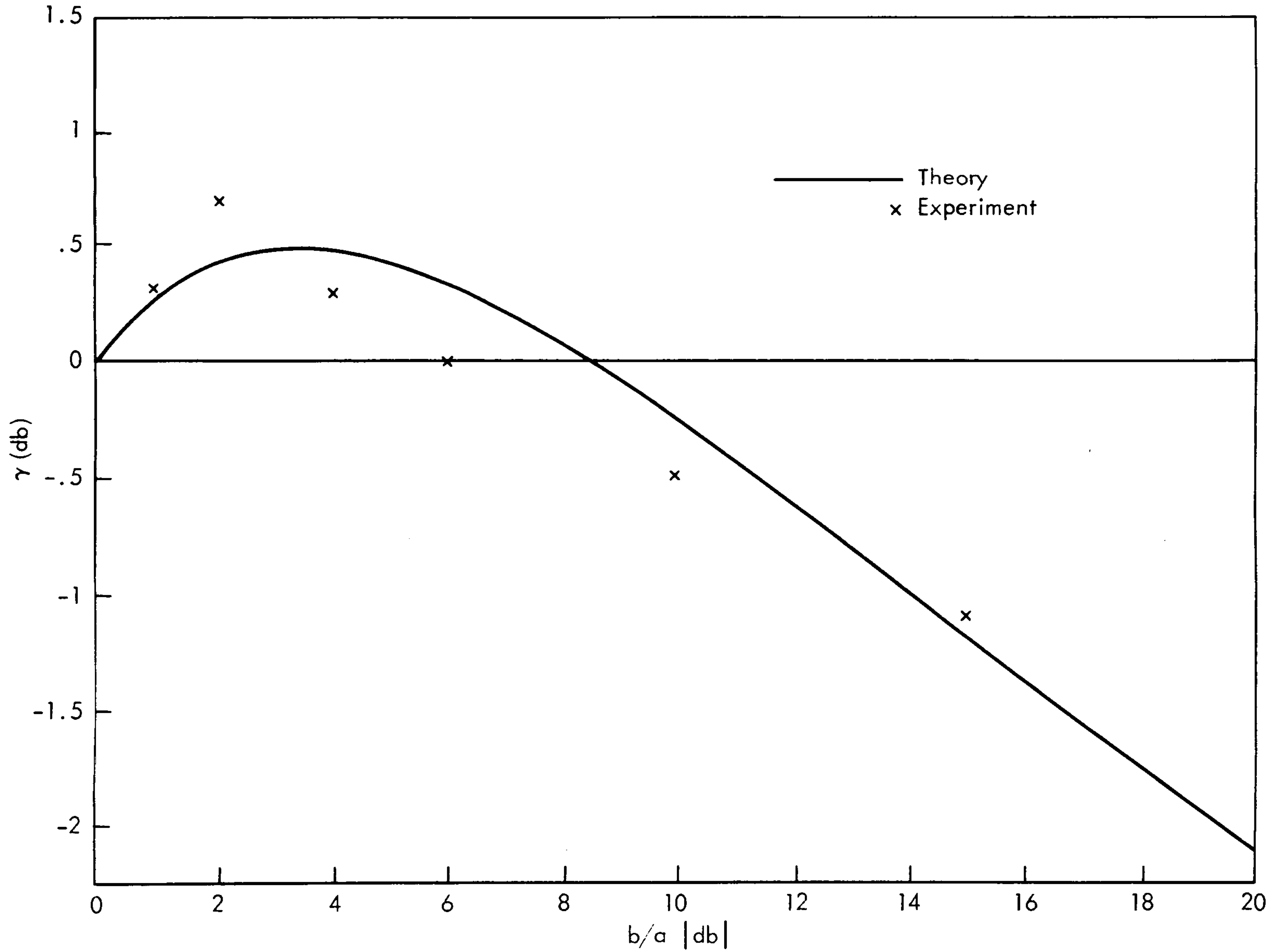

Fig.5—Signal suppression, four inputs equal in pairs

or 6 db (experimental). For large input ratios, the "negative suppression" increases very slowly toward large negative values. The explanation of this phenomenon is the same as for the three-signal case.

REFERENCES

1. Cahn, C. R., "Crosstalk Due to Finite Limiting of Frequency-Multiplexed Signals," Proc. IRE, Vol. 48, No. 1, January 1960, pp. 53-59.

2. Price, R., "A Note on the Envelope and Phase Modulated Components of Narrow-Band Gaussian Noise," IRE Trans. Info. Theory, Vol. IT-1, No. 2, September 1955, pp. 9-13.

3. Davenport, W. B., "Signal-to-Noise Ratios in Band-Pass Limiters," J. Appl. Phys., Vol. 24, No. 6, June 1953, pp. 720-727.

4. Jones, J. J., "Hard Limiting of Two Signals in Random Noise," IEEE Trans. Info. Theory, Vol. IT-9, No. 1, January 1963, p. 39.

5. Doyle, W., and I. S. Reed, "Approximate Band-Pass Limiter Envelope Distributions," IEEE Trans. Info. Theory, Vol. IT-10, No. 3, July 1964, pp. 180-184.

6. Sevy, J. L., The Effect of Multiple C.W. and F.M. Signals Passed Through a Hard Limiter or TWT, Report No. ATM-63(3111)-1, Aerospace Corporation, March 1963.

7. Watson, G. N., Theory of Bessel Functions, MacMillan Company, New York, 1948, p. 47.

8. Magnus, W., and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics, Chelsea Publishing Company, New York, 1949, p. 5.

9. Jahnke, E., and F. Emde, Tables of Functions with Formulae and Curves, Dover Publications, New York, 1945, p. 73.