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The Resolution of Point Sources of Light as Analyzed by Quantum Detection Theory

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ABSTRACT

The resolvability of point sources of incoherent light is analyzed by quantum detection theory in terms of two hypothesis-testing problems. In the first, the observer must decide whether there are two sources of equal radiant power at given locations, or whether there is only one source of twice the power located midway between them. In the second problem, either one, but not both, of two point sources is radiating, and the observer must decide which it is. The decisions are based on optimum processing of the electromagnetic field at the aperture of an optical instrument. In both problems the density operators of the field under the two hypotheses do not commute. The error probabilities, determined as functions of the separation of the points and the mean number of received photons, characterize the ultimate resolvability of the sources.

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Two point sources of light appear as one when they are very close; one function of a telescope or a microscope is to separate their images to the point of distinguishability. How well it does so is measured by its resolving power. When all the aberrations of the lens system have been eliminated, diffraction of the light at the aperture of the instrument remains to spread the images and cause them to overlap. According to the commonly accepted Rayleigh criterion, the two images are said to be resolved when the peak illuminance of the diffraction pattern of one falls on the first minimum of the diffraction pattern of the other. Alternative measures of image resolvability and instrumental resolving power have been proposed, and methods such as apodization for maximizing them have been studied.[1,2] These strive essentially to raise as far as possible the likelihood that an observer will see two close point sources as indeed two rather than one. However ingeniously his optical system may be designed, his perception is subject to error because of the stochastic, quantum-mechanical nature of the light. By studying resolution from the standpoint of hypothesis testing, we can determine how the observer can decide most reliably whether there are two sources or only one, and in this way we can bring out the fundamental limitations on the resolvability of two luminous points.

Two such decision problems will be studied in this paper. In the first, the observer is to decide whether a single point source of known power is present in an object plane, or whether two sources are present, each emitting half the power. In the second, he is to decide which of two point sources is radiating during a certain interval, only one being allowed to radiate at a time. The sources, at known locations, emit incoherent, quasimonochromatic light of given spectral density. The decisions are to be based on the electromagnetic field at the aperture A of an optical instrument during a fixed observation interval (0, T). The instrument is to process that field in such a manner that the decisions can be made most reliably. The minimum attainable probability of error characterizes the resolvability of the sources.

The optimum processing of the aperture field is determined by quantum detection theory [3-5]. It requires us to find the eigenvectors and eigenvalues of the operator $\rho_1 - \Lambda\rho_0$, where ρ_0 and ρ_1 are the quantum-mechanical density operators of the aperture field under the two hypotheses in question, and Λ is a constant. In neither of our problems do the density operators commute. Except for choices between pure states, there are few physically significant pairs of noncommuting density operators for which exact sets of eigenvalues and eigenvectors have been calculated. In order to obtain exact solutions, it has been necessary to assume the absence of background light, errors in the decisions arising only because of the quantum nature of the light from each source. We determine in this way an absolute limit to their resolution.

In attacking each problem we must first find an expansion of the aperture field in terms of which the density operators take the most convenient forms.[6,7] Then the eigenvalue equation is written down in the coherent-state representation,[8] which leads to especially simple eigenvectors, constructed from the eigenvectors of ρ_0 and ρ_1 individually. The vanishing of a determinant of the coefficients of these eigenvectors gives us a quadratic equation for the eigenvalues. From the eigenvalues and eigenvectors the probabilities of error can be calculated. They depend on the separation ε of the two sources, on the form and size of the aperture, and on the total average number N_s of photons received during the observation interval.

1. Two Sources or One? The Aperture Modes.

The observer is to choose between two hypotheses: (H_0) a single point source with a total radiant power P_s is present at the origin $\underset{\sim}{u} = \underset{\sim}{0}$ of an object plane at distance R, and (H_1) two point sources, each with radiant power $\frac{1}{2} P_s$, are present at points $\underset{\sim}{u} = -\frac{1}{2}\underset{\sim}{\varepsilon}$ and $\underset{\sim}{u} = +\frac{1}{2}\underset{\sim}{\varepsilon}$ in that plane. The sources radiate quasimonochromatic incoherent light with a mean angular frequency Ω, wavelength $\lambda = 2\pi c/\Omega$ (c is the velocity of light), and a spectral density $X(\omega - \Omega)$ whose width $2\pi W$ is much less than Ω.

The decision is to be based on observations of the electromagnetic field at the aperture A of an optical instrument during an interval (0, T). The aperture plane lies parallel to the object plane. The field can be treated as a scalar function $\psi(\underset{\sim}{r}, t)$ of aperture coordinates $\underset{\sim}{r}$ and time t, provided--as we assume--that the sources are close enough together that the rays from them are paraxial, $|\underset{\sim}{\varepsilon}|/R \ll 1$, and $W \ll \Omega$. The field is broken into its positive-frequency part $\psi^{(+)}(\underset{\sim}{r}, t)$ and its negative-frequency part $\psi^{(-)}(\underset{\sim}{r}, t)$,

$$\psi(\underset{\sim}{r}, t) = \psi^{(+)}(\underset{\sim}{r}, t) + \psi^{(-)}(\underset{\sim}{r}, t).$$

Classically, because the sources radiate natural, incoherent light, $\psi^{(+)}(\underset{\sim}{r}, t)$ is a complex Gaussian spatio-temporal random process. Quantum-mechanically the field is described by its density operators ρ_0 and ρ_1 under the two hypotheses. In the coherent-state or P-representation[8] these have Gaussian forms depending only on the mutual coherence functions of the light, which are the quantum-mechanical averages[9]

$$\tfrac{1}{2} \operatorname{Tr} \rho_i \, \psi^{(-)}(\underset{\sim}{r}_1, t_1)\, \psi^{(+)}(\underset{\sim}{r}_2, t_2) =$$

$$\varphi_s^{(i)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)\, \chi(t_1 - t_2) \exp[-i\Omega(t_1 - t_2)], \quad i = 0, 1. \quad (1)$$

Now $\psi^{(+)}(\underset{\sim}{r}, t)$ is an operator in the Hilbert space of states of the field, $\psi^{(-)}(\underset{\sim}{r}, t)$ is its Hermitian conjugate, and Tr stands for the trace; $\chi(\tau)$, the temporal coherence function, is the Fourier transform of $X(\omega)$, normalized so that $\chi(0) = 1$.

The spatial coherence function $\varphi_s^{(i)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$ is proportional to the spatial Fourier transform of the radiance distribution $B(\underset{\sim}{u})$ in the object plane.[9] Under hypothesis H_0 $B(\underset{\sim}{u}) = P_s\, \delta(\underset{\sim}{u})$, and the spatial coherence function

$$\varphi_s^{(0)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = (E_s/AT)\ Fr_1\ Fr_2^* \tag{2}$$

is constant except for the ubiquitous Fresnel factors Fr_1 and Fr_2^*, which are defined as

$$Fr_i = \exp(ik\underset{\sim}{r}_i^2/2R), \qquad k = 2\pi/\lambda, \qquad i = 1, 2. \tag{3}$$

Here $E_s = P_s AT/4\pi R^2$ is the total average energy received at the aperture, whose area is A, during the observation interval. Under hypothesis H_1

$$B(\underset{\sim}{u}) = \tfrac{1}{2} P_s[\delta(\underset{\sim}{u} - \tfrac{1}{2}\underset{\sim}{\varepsilon}) + \delta(\underset{\sim}{u} + \tfrac{1}{2}\underset{\sim}{\varepsilon})],$$

and the spatial coherence function is

$$\begin{aligned}\varphi_s^{(1)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2) &= \tfrac{1}{2}(E_s/AT)\ \{\exp[ik\underset{\sim}{\varepsilon}\cdot(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)/2R] + \\ &\qquad \exp[-ik\underset{\sim}{\varepsilon}\cdot(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)/2R]\}\ Fr_1\ Fr_2^* \\ &= (E_s/AT)\cos\mu(x_1 - x_2)\ Fr_1\ Fr_2^*,\end{aligned}$$

$$\underset{\sim}{r}_j = (x_j, y_j), \qquad j = 1, 2, \qquad \mu = k\varepsilon/2R, \qquad \varepsilon = |\underset{\sim}{\varepsilon}|. \tag{4}$$

The x-axis has been set parallel to the line between the two sources.

The optimum strategy for processing the incident light and making the decision is most expediently derived from a decomposition of the aperture field $\psi^{(+)}(\underset{\sim}{r}, t)$ into a countable set of spatio-temporal modes,[6,7] represented by products of spatial mode functions $\eta_p(\underset{\sim}{r})$ and temporal mode functions $\gamma_m(t)\ e^{-i\Omega t}$.

The $\gamma_m(t)$ are taken as the eigenfunctions of the temporal coherence function $\chi(\tau)$, as given by the integral equation

$$\chi_m \, \gamma_m(t_1) = T^{-1} \int_0^T \chi(t_1 - t_2) \, \gamma_m(t_2) \, dt_2; \tag{5}$$

they are orthonormal with respect to the observation interval (0, T). The eigenvalues χ_m, which sum to 1, give the average fraction of the light going into each temporal mode.[9] When WT >> 1 there are roughly WT such modes containing a significant portion of the light. By virtue of (5) the temporal modes are statistically independent, and the density operators ρ_0 and ρ_1 can be factored into products, of which each factor refers to a different temporal mode; we have here a spatio-temporal counterpart of the Karhunen-Loève expansion. For simplicity we assume at first that only a single temporal mode is excited; its mode function can be taken as $\gamma_1(t) = T^{-\frac{1}{2}}$, with $\chi_1 = 1$. Later we shall without difficulty extend our results to a large number (WT >> 1) of statistically independent temporal modes.

The spatial mode functions $\eta_p(\underset{\sim}{r})$ are orthonormal over the aperture,

$$\int_A \eta_q{}^*(\underset{\sim}{r}) \, \eta_p(\underset{\sim}{r}) \, d^2\underset{\sim}{r} = \delta_{pq}. \tag{6}$$

In the simplest binary detection problems[9] these can be simultaneously eigenfunctions of both spatial coherence functions $\varphi_s^{(i)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$, i = 0, 1, and the spatial modes are statistically independent under both hypotheses H_0 and H_1, but that is impossible here.

In order to limit the number of spatial modes that must be considered, we postulate that the aperture A is symmetrical with respect to the x- and y-axes. Our results will be illustrated for a rectangular aperture having one side of length a in the x-direction and centered at the origin. The two eigenfunctions of $\varphi_s^{(1)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$, which is proportional to $\cos \mu x_1 \cos \mu x_2 + \sin \mu x_1 \sin \mu x_2$, are

$$\eta_1(\underset{\sim}{r}) = C_1 \cos \mu x \; Fr, \tag{7}$$

$$\eta_2(\underset{\sim}{r}) = C_2 \sin \mu x \; Fr, \tag{8}$$

where Fr is again a Fresnel Factor, and C_1 and C_2 are normalization constants. We call $\eta_1(\underset{\sim}{r})$ the cosine mode, $\eta_2(\underset{\sim}{r})$ the sine mode. The associated eigenvalues h_1 and h_2, defined by[7]

$$h_p \; \eta_p(\underset{\sim}{r}_1) = A^{-1} \int_A \varphi_s^{(1)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2) \; \eta_p(\underset{\sim}{r}_2) \; d^2\underset{\sim}{r}_2, \tag{9}$$

are for the rectangular aperture

$$h_1 = \tfrac{1}{2}(1 + \text{sinc}\; \sigma), \qquad h_2 = \tfrac{1}{2}(1 - \text{sinc}\; \sigma),$$
$$\sigma = \mu a/\pi = \varepsilon a/\lambda R. \tag{10}$$

They determine what fraction of the light goes into each mode under hypothesis H_1, and $h_1 + h_2 = 1$. The remaining eigenfunctions of $\varphi_s^{(1)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$ are orthogonal to $\eta_1(\underset{\sim}{r})$ and $\eta_2(\underset{\sim}{r})$ and have zero eigenvalues.

Under hypothesis H_0 all the light goes into the planar mode $\eta_0'(\underset{\sim}{r}) = A^{-\frac{1}{2}}$ Fr, but this is not orthogonal to $\eta_1(\underset{\sim}{r})$. We therefore introduce what we shall call the zero mode,

$$\eta_0(\underset{\sim}{r}) = (C_0' + C_0'' \cos \mu x) \; Fr, \tag{11}$$

choosing C_0' and C_0'' so that $\eta_0(\underset{\sim}{r})$ is both orthogonal to $\eta_1(\underset{\sim}{r})$ and normalized. As an even function of x it is also orthogonal to $\eta_2(\underset{\sim}{r})$, which is odd in x.

The remaining spatial modes are generated by the Gram-Schmidt procedure in such a way as to be orthogonal to $\eta_0(\underset{\sim}{r})$, $\eta_1(\underset{\sim}{r})$, and $\eta_2(\underset{\sim}{r})$. They are eigenfunctions of both $\varphi_s^{(0)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$ and $\varphi_s^{(1)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$, but with zero eigenvalues. As none of them is excited by the incident light under either hypothesis, we can disregard them henceforth.

The aperture field $\psi^{(+)}(\underset{\sim}{r}, t)$ is expanded in a series of spatio-temporal modes $\eta_p(\underset{\sim}{r}) \; \gamma_m(t)$ with expansion coefficients proportional to[10]

$$a_{pm} = (2\hbar\Omega)^{-\frac{1}{2}} \int_0^T \int_A \eta_p^*(\underset{\sim}{r})\ \gamma_m^*(t)\ \psi^{(+)}(\underset{\sim}{r}, t)\ d^2\underset{\sim}{r}dt, \tag{12}$$

where $\hbar$ is Planck's constant $h/2\pi$. Quantum-mechanically a_{pm} is the annihilation operator for photons in the (pm) mode. The density operators ρ_0 and ρ_1 of the field under the two hypotheses depend only on the covariance matrices $\underset{\sim}{\varphi}^{(i)}$ of these coefficients; the elements of the covariance matrices can be expressed through (1) in terms of the mutual coherence functions of the field,

$$\varphi^{(i)}_{pq,mn} = \mathrm{Tr}\ \rho_i\ a^+_{qn}\ a_{pm} =$$

$$(T/\hbar\Omega)\ \chi_m\ \delta_{mn} \int_A \int_A \eta_p^*(\underset{\sim}{r}_1)\ \varphi_s^{(i)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)\ \eta_q(\underset{\sim}{r}_2)\ d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2, \tag{13}$$

where we have used (5). Because we are for the time being treating only a single temporal mode, $\gamma_1(t)$, we put $\chi_1 = 1$ and drop the subscripts m and n referring to the temporal modes. If we now use our mode functions from (7), (8), and (11) and the spatial covariance functions in (2) and (4), we find that the covariance matrices are, with p and q equal to 0, 1, or 2,

$$\varphi^{(0)}_{pq} = N_s\ y_p\ y_q\ (1 - \delta_{p2})\ (1 - \delta_{q2}), \tag{14}$$

$$\varphi^{(1)}_{pq} = N_s\ h_p\ \delta_{pq}\ (1 - \delta_{p0}) \tag{15}$$

where δ_{pq} is the Kronecker delta, equal to 1 for $p = q$ and to zero for $p \neq q$,

$$N_s = E_s/\hbar\Omega \tag{16}$$

is the mean number of photons received,

$$y_i = A^{-\frac{1}{2}} \int_A \eta_i(\underset{\sim}{r})\ d^2\underset{\sim}{r},$$

and h_1 and h_2 are the eigenvalues defined by (9). For the rectangular aperture,

$$y_1^2 = 2\ \mathrm{sinc}^2(\tfrac{1}{2}\ \sigma)/(1 + \mathrm{sinc}\ \sigma),$$

$$y_0^2 = 1 - y_1^2, \quad \sigma = \epsilon a/\lambda R, \tag{17}$$

and h_1 and h_2 are given in (10). The density operators in the P-representation are[8]

$$\rho_i = \iiint P_i(\underset{\sim}{\alpha}) \prod_{k=0}^{2} |\alpha_k\rangle\langle\alpha_k| \; (d^2\alpha_k/\pi),$$

$$\underset{\sim}{\alpha} = (\alpha_0, \alpha_1, \alpha_2), \quad \alpha_k = \alpha_{kx} + i\alpha_{ky}, \quad d^2\alpha_k = d\alpha_{kx}\, d\alpha_{ky},$$

with

$$P_i(\underset{\sim}{\alpha}) = \pi^{-3} \left|\det \underset{\sim}{\varphi}^{(i)}\right|^{-1} \exp\left[-\sum_{p=0}^{2}\sum_{q=0}^{2} \alpha_p{}^*(\underset{\sim}{\varphi}^{(i)^{-1}})_{pq}\, \alpha_q\right],$$

$$i = 0, 1. \tag{18}$$

Each integral is taken over the entire complex α-plane. The density operators ρ_0 and ρ_1 do not commute.

2. Two Sources or One? The Optimum Strategy.

According to quantum detection theory the optimum strategy for choosing between two hypotheses requires measuring the projection operator[3-5]

$$\Pi = \sum_{n,r} |\eta_{nr}\rangle\langle\eta_{nr}| \; U(\eta_{nr}), \tag{19}$$

where U(x) is the unit step function, and $|\eta_{nr}\rangle$ is an eigenvector of the operator $\rho_1 - \Lambda\rho_0$ with eigenvalue η_{nr} given by the operator equation

$$(\rho_1 - \Lambda\rho_0) \quad |\eta_{nr}\rangle = \eta_{nr}|\eta_{nr}\rangle; \tag{20}$$

the sum in (19) is over all the eigenstates. (The paired subscripts, which are non-negative integers, are used for later convenience.) The result of measuring Π is either the number 0, which brings the decision for hypothesis H_0, or the number 1, which brings the decision for H_1. Here Λ is a parameter depending on the decision criterion. If the average error probability is to be minimum, Λ is the ratio $\zeta/(1 - \zeta)$ of the prior probabilities of hypotheses H_0 and H_1. If the Neyman-Pearson criterion has been adopted, Λ is set to yield a pre-assigned false-alarm probability. The average probability of error is

$$\begin{aligned} P_e &= \zeta\, Q_0 + (1 - \zeta)\,(1 - Q_d) \\ &= (1 - \zeta) \left[1 - \sum_{n,r} \eta_{nr}\; U(\eta_{nr})\right], \end{aligned} \tag{21}$$

the false-alarm probability that H_1 is chosen when H_0 is true is

$$Q_0 = \sum_{n,r} \langle\eta_{nr}|\rho_0|\eta_{nr}\rangle \; U(\eta_{nr}), \tag{22}$$

and the detection probability--the probability of saying there are two sources when there really are two--is

$$Q_d = \sum_{n,r} \langle \eta_{nr} | \rho_1 | \eta_{nr} \rangle \quad U(\eta_{nr}). \tag{23}$$

Both Q_0 and Q_d are functions of Λ.

Our principal task is to solve (20). By expressing the eigenvectors $|\eta_k\rangle$ in terms of the coherent states as[11]

$$F_{nr}(\underset{\sim}{\alpha}^*) = \langle \underset{\sim}{\alpha} | \eta_{nr} \rangle \exp(\tfrac{1}{2} |\underset{\sim}{\alpha}|^2),$$

$$|\underset{\sim}{\alpha}\rangle = |\alpha_0\rangle \, |\alpha_1\rangle \, |\alpha_2\rangle, \qquad |\underset{\sim}{\alpha}|^2 = |\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2, \tag{24}$$

we can write (20) as an integral equation[3]

$$\int [R_1(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) - \Lambda R_0(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma})] \exp(-|\underset{\sim}{\gamma}|^2) \, F_{nr}(\underset{\sim}{\gamma}^*) \prod_{i=0}^{2} (d^2\gamma_i/\pi)$$

$$= \eta_{nr} \, F_{nr}(\underset{\sim}{\beta}^*), \tag{25}$$

where

$$R_i(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) = \langle \underset{\sim}{\beta} | \rho_i | \underset{\sim}{\gamma} \rangle \exp \tfrac{1}{2}(|\underset{\sim}{\beta}|^2 + |\underset{\sim}{\gamma}|^2) =$$

$$\det(\underset{\sim}{I} + \underset{\sim}{\varphi}^{(i)})^{-1} \exp \sum_{p=0}^{2} \sum_{q=0}^{2} \beta_p^* [(\underset{\sim}{I} + \underset{\sim}{\varphi}^{(i)})^{-1} \underset{\sim}{\varphi}^{(i)}]_{pq} \gamma_q,$$

$$i = 0,\ 1, \tag{26}$$

with $\underset{\sim}{I}$ the 3×3 identity matrix. Here, by (14) and (15),

$$R_1(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) = (1 - v_1)(1 - v_2) \exp(v_1\beta_1^*\gamma_1 + v_2\beta_2^*\gamma_2),$$

$$v_j = N_j/(N_j + 1), \qquad N_j = h_j \, N_s, \qquad j = 1,\ 2, \tag{27}$$

and

$$R_0(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) = (1 - v_0) \exp[v_0(y_0\beta_0^* + y_1\beta_1^*)(y_0\gamma_0 + y_1\gamma_1)],$$

$$v_0 = N_s/(N_s + 1). \tag{28}$$

A typical eigenvalue of the density operator ρ_1 is the probability

$$P^{(1)}_{nr} = (1 - v_1)(1 - v_2) \, v_1^{\,n} \, v_2^{\,r} \tag{29}$$

of finding n photons in the cosine mode $\eta_1(\underset{\sim}{r})$ and r photons in the sine mode

$\eta_2(\underset{\sim}{r})$; the associated eigenvector is, in the coherent-state representation, proportional to $\gamma_1^{*n}\ \gamma_2^{*r}$. For $r > 0$ this is also an eigenvector of $\rho_1 - \Lambda\rho_0$,

$$F_{nr}(\gamma^*) = (n!\ r!)^{-\frac{1}{2}}\ \gamma_1^{*n}\ \gamma_2^{*r},$$
$$\eta_{nr} = P_{nr}^{(1)}, \qquad r > 0, \tag{30}$$

because $\rho_0|\eta_{nr}\rangle = 0$. This means that the observer should measure the number of photons, or the energy, in the sine mode, which can be done without disturbing the other modes; and if he finds any photons there at all, $r > 0$, he chooses hypothesis H_1. We need, therefore, to consider further only the strategy to be followed when no photons are found in the sine mode ($r = 0$).

The density operator ρ_0 has the eigenvalues

$$P_n^{(0)} = (1 - v_0)\ v_0^{\ n}, \tag{31}$$

which are the probabilities of finding various numbers of photons in the planar mode $\eta_0'(\underset{\sim}{r})$; the associated eigenvectors are in the coherent-state representation proportional to $(y_0\gamma_0^* + y_1\gamma_1^*)^n$. As a solution of the integral equation (25) we therefore try

$$F_{n0}(\gamma^*) = x_0(y_0\gamma_0^* + y_1\gamma_1^*)^n + x_1\gamma_1^{*n}, \tag{32}$$

with x_0 and x_1 constants yet to be evaluated. After substituting it and carrying out the integration, we find by the method outlined in the appendix

$$(x_1 + y_1^{\ n}\ x_0)\ P_{n0}^{(1)}\ \beta_1^{*n} - \Lambda(x_1y_1^{\ n} + x_0)\ P_n^{(0)}(y_0\beta_0^* + y_1\beta_1^*)^n$$
$$= \eta_{n0}[x_0(y_0\beta_0^* + y_1\beta_1^*)^n + x_1\beta_1^{*n}]. \tag{33}$$

Equating the coefficients of the two types of terms, we obtain the linear homogeneous equations

$$(P_{n0}^{(1)} - \eta_{n0})\ x_1 + y_1^{\ n}\ P_{n0}^{(1)}\ x_0 = 0,$$
$$-\Lambda y_1^{\ n}\ P_n^{(0)}\ x_1 - (\Lambda P_n^{(0)} + \eta_{n0})\ x_0 = 0, \tag{34}$$

and setting the determinant of the coefficients of x_0 and x_1 equal to zero yields a quadratic equation for the eigenvalues η_{n0},

$$\eta_{n0}^2 - (P_{n0}^{(1)} - \Lambda P_n^{(0)})\ \eta_{n0} - \Lambda P_{n0}^{(1)}\ P_n^{(0)} (1 - y_1{}^{2n}) \quad = \quad 0. \tag{35}$$

For n = 0 the two eigenvalues are zero and

$$\eta_{00} \quad = \quad P_{00}^{(1)} - \Lambda P_0^{(0)}. \tag{36}$$

Whether η_{00} is positive or negative depends on Λ; for $\Lambda = 1$, $\eta_{00} < 0$. Of the remaining pairs of eigenvalues one is positive and one is negative, $\eta_{n0}^{(+)} > 0$, $\eta_{n0}^{(-)} < 0$. Putting these values back into one or the other part of (34) enables the ratio x_1/x_0 to be determined. The values of x_0 and x_1 are then found from the normalization requirement

$$\langle\eta_{n0}|\eta_{n0}\rangle \quad = \quad 1 \quad = \iiint |F_{n0}(\gamma^*)|^2 \exp(-|\gamma|^2) \prod_{i=0}^{2} (d^2\gamma_i/\pi)$$

$$= \quad n!(x_1{}^2 + 2y_1{}^n x_0 x_1 + x_0{}^2). \tag{37}$$

By using (30) and

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} P_{nr}^{(1)} \quad = \quad 1, \tag{38}$$

we can write the average error probability from (21) as

$$P_e \quad = \quad \zeta\ Q_0 + (1 - \zeta)\ Q_d \quad =$$

$$(1 - \zeta) \left\{ \min[P_{00}^{(1)},\ \Lambda P_0^{(0)}] + \sum_{n=1}^{\infty} (P_{n0}^{(1)} - \eta_{n0}^{(+)}) \right\}. \tag{39}$$

To find the false-alarm probability we use (22), in which the sum can be restricted to terms with r = 0 because

$$\langle\eta_{nr}|\rho_0|\eta_{nr}\rangle \quad = \quad 0, \quad r > 0. \tag{40}$$

We find for each term

$$\langle\eta_{n0}|\rho_0|\eta_{n0}\rangle \quad =$$

$$\iiint F_{n0}^{*}(\beta^{*})\ R_0(\beta^{*},\ \gamma)\ F_{n0}(\gamma^{*})\ \exp(-|\beta|^2 - |\gamma|^2) \prod_{i=0}^{2} (d^2\beta_i d^2\gamma_i/\pi^2)$$

$$= n!(x_0 + x_1 y_1{}^n)^2\ P_n^{(0)}, \tag{41}$$

whereupon the false-alarm probability becomes

$$Q_0 = \sum_{n>0} P_n^{(0)} \frac{(x_0 + x_1 y_1{}^n)^2}{x_1{}^2 + 2x_0 x_1 y_1{}^n + x_0{}^2} + P_0^{(0)}\ U(P_{00}^{(1)} - \Lambda P_0^{(0)})$$

$$= \sum_{n>0} P_n^{(0)}\ [n_{n0}^{(+)} - P_{n0}^{(1)}(1 - y_1{}^{2n})]/R_n + P_0^{(0)}\ U(P_{00}^{(1)} - \Lambda P_0^{(0)}), \tag{42}$$

where

$$R_n = [(P_{n0}^{(1)} - \Lambda P_n^{(0)})^2 + 4\Lambda P_{n0}^{(1)}\ P_n^{(0)}(1 - y_1{}^{2n})]^{\frac{1}{2}}\ . \tag{43}$$

By using (39), (42), and rather much algebra, we find for the detection probability

$$Q_d = 1 - \sum_{n>0} P_{n0}^{(1)} \frac{x_0{}^2(1 - y_1{}^{2n})}{x_0{}^2 + 2x_0 x_1 y_1{}^n + x_1{}^2} - P_{00}^{(1)}\ U(\Lambda P_0^{(0)} - P_{00}^{(1)})$$

$$= 1 - \sum_{n>0} P_{n0}^{(1)}\ [\Lambda P_n^{(0)} y_1{}^{2n} - (P_{n0}^{(1)} - n_{n0}^{(+)})]/R_n$$

$$- P_{00}^{(1)}\ U(\Lambda P_0^{(0)} - P_{00}^{(1)}). \tag{44}$$

If thermal background light is present, the covariance matrices in (14) and (15) have additional terms $\mathcal{N}\delta_{pq}$, where $\mathcal{N}$ is the mean number of background photons per mode. The eigenfunctions $F_{nr}(\gamma^{*})$ are sums of trinomials in $\gamma_0{}^{*}$, $\gamma_1{}^{*}$, $\gamma_2{}^{*}$ with various powers, but the equations for their coefficients and the determinantal equation for the eigenvalues are much more complicated.

Multiple Spatio-temporal Modes

Now we suppose that the field is divided among a great many statistically independent temporal modes, each of which bears three spatial modes $\eta_0(\underset{\sim}{r})$, $\eta_1(\underset{\sim}{r})$, and $\eta_2(\underset{\sim}{r})$. We shall show that (39), (42), and (44) still hold, provided that the probabilities $P_{n0}^{(1)}$ and $P_n^{(0)}$ are replaced by Poisson probabilities,

$$P_n^{(0)} = N_s{}^n \exp(-N_s)/n!, \tag{45}$$

$$P_{n0}^{(1)} = h_1{}^n P_n^{(0)}. \tag{46}$$

Let us consider the finite number $\nu >> WT$ of temporal modes having as temporal eigenvalues $\chi_1, \chi_2, \ldots, \chi_\nu$, the ν largest eigenvalues of $\chi(\tau)$. Later we let ν and WT go to infinity.

The first step in the decision strategy is to determine whether there are any photons in any of the ν sine modes $\eta_2(\underset{\sim}{r})\ \gamma_m(t)\cdot e^{-i\Omega t}$. If so, hypothesis H_1 is chosen at once, for we know that this would be impossible under hypothesis H_0. The only eigenvectors of $\rho_1 - \Lambda\rho_0$ that need to be considered further, as before, are those in which there are no photons in any of the ν sine modes.

We add a subscript m to the complex variables γ and $\underset{\sim}{\beta}$ to indicate the temporal mode they refer to, and we recognize that the coherent-state representations R_0 and R_1 of the density operators ρ_0 and ρ_1 are now products of functions like those in (27) and (28), with a factor for each temporal mode. The eigenfunctions of the integral equation now have the form, replacing (32), of

$$F(\{n_m, 0\}; \underset{\sim}{\gamma}^*) = x_0 \prod_{m=1}^{\nu} (y_0\ \gamma_{m0}{}^* + y_1\ \gamma_{m1}{}^*)^{n_m} + x_1 \prod_{m=1}^{\nu} \gamma_{m1}^{*n_m}\ , \tag{47}$$

where we have replaced $F_{nr}(\underset{\sim}{\gamma}^*)$ by $F(\{n_m, r_m\}; \underset{\sim}{\gamma}^*)$, $\underset{\sim}{\gamma} = \{\gamma_{m0}, \gamma_{m1}\}$, and shall replace η_{nr} by $\eta(\{n_m, r_m\})$. When this eigenfunction is substituted into the multimode version of the integral equation (25), we obtain, much as before,

$$(x_1 + y_1^n x_0) \prod_m P_{n_m 0}^{(1)} \beta_{m1}^{*n_m}$$

$$-\Lambda(x_1 y_1^n + x_0) \prod_m P_{n_m}^{(0)} (y_0 \beta_{m0}^* + y_1 \beta_{m1}^*)^{n_m}$$

$$= \eta(\{n_m, 0\}) F(\{n_m, 0\}; \underset{\sim}{\gamma}^*) , \tag{48}$$

where

$$n = \sum_m n_m \tag{49}$$

is the total number of photons in the states contributing to the given eigenfunction. By equating coefficients of like terms on both sides of (48), we obtain a pair of linear equations like those in (34), except for the replacements

$$P_{n0}^{(1)} \rightarrow P^{(1)}(\{n_m, 0\}) = \prod_m P_{n_m 0}^{(1)} = \prod_m (1 - v_{1m})(1 - v_{2m}) v_{1m}^{n_m},$$

$$P_n^{(0)} \rightarrow P^{(0)}(\{n_m\}) = \prod_m P_{n_m}^{(0)} = \prod_m (1 - v_{0m}) v_{0m}^{n_m} ,$$

$$v_{jm} = h_j \chi_m N_s / (h_j \chi_m N_s + 1) , \; j = 1,2,$$

$$v_{0m} = \chi_m N_s / (\chi_m N_s + 1) . \tag{50}$$

As the time-bandwidth product WT increases, the temporal eigenvalues χ_m become smaller and smaller, and we can approximate the denominators in the expressions for v_{0m}, v_{1m}, and v_{2m} by 1. Then for $\nu >> WT >> 1$,

$$P^{(0)}(\{n_m\}) \rightarrow \prod_m \exp(-\chi_m N_s) (\chi_m N_s)^{n_m} = \exp(-N_s) \prod_m (\chi_m N_s)^{n_m}, \tag{51}$$

$$P^{(1)}(\{n_m, 0\}) \rightarrow \prod_m \exp(-h_1 \chi_m N_s) \exp(-h_2 \chi_m N_s) (h_1 \chi_m N_s)^{n_m}$$

$$= h_1^n \exp(-N_s) \prod_m (\chi_m N_s)^{n_m} = h_1^n P^{(0)}(\{n_m\}), \tag{52}$$

since $h_1 + h_2 = 1$ and the eigenvalues χ_m sum to 1.

From (35) we see that the eigenvalue $\eta(\{n_m, 0\})$ is now proportional to $P^{(0)}(\{n_m\})$ through a factor η_n' depending only on the sum n of the n_m's,

$$\eta(\{n_m, 0\}) = \eta_n' \, P^{(0)}(\{n_m\}) \;;$$

that factor is a root of the equation

$$\eta_n'^2 - (h_1{}^n - \Lambda)\eta_n' - \Lambda h_1{}^n(1 - y_1{}^{2n}) = 0 \;.$$

Furthermore, by the new version of (34) arising from (48), the coefficients x_0 and x_1 stand in a ratio depending only on the sum n and not on the individual n_m's. If we go through the rest of the calculation of the error probabilities, we find that the sums in (39), (42), and (44) consist of terms with $P^{(1)}(\{n_m, 0\})$ or $P^{(0)}(\{n_m\})$ multiplied by factors depending only on the sum n. We can therefore combine terms having the same value of $n = \sum_m n_m$, replacing $P^{(0)}(\{n_m\})$ by the total probability of n photons under hypothesis H_0, which when $WT \gg 1$ is the Poisson probability $P_n^{(0)}$ in (45), and by virtue of (52) replacing the terms $P^{(1)}(\{n_m, 0\})$ by $h_1{}^n \, P_n^{(0)}$.

We have used a scalar theory of the electromagnetic field, which seems to require that the light be linearly polarized. Unpolarized light can be divided into two statistically independent linearly polarized components, each of which is then broken up into temporal modes. The only change in our analysis is a doubling of the number of temporal modes, and as the number of such modes eventually goes to infinity, our results for $WT \gg 1$ must hold for unpolarized light as well.

The Error Probabilities

We evaluate all the probabilities of error for observation over a rectangular aperture. In Fig. 1 is plotted, as a function of the parameter $\sigma = \epsilon a/\lambda R$ and for various values of N_s, the average error probability for choices between hypotheses H_0 and H_1, equal prior probabilities being assigned to each. As the two point sources separate, this error probability approaches $\frac{1}{2} \exp(-N_s)$. When the sources are very far apart, the observer can count the numbers of photons in the planar mode $\eta_0'(\tilde{r})$ and in all modes orthogonal to it. Finding any at all in the planar mode, he chooses H_0; finding any at all in the rest, he chooses H_1. The only possibility of an error arises when no photons at all are counted, whereupon he chooses H_0 and H_1 at random with equal probabilities 1/2. The probability of this event is $\exp(-N_s)$. The limiting value of $\frac{1}{2} \exp(-N_s)$ is reached when $\sigma = 2$, with only small deviations above it for $\sigma > 2$. If we say that when $\sigma = 2$ the two point sources are resolved as well as they ever can be for a given average number N_s of received photons, we require a separation of $2\lambda R/a$, which is twice that specified by the Rayleigh criterion.

In Fig. 2 is plotted the detection probability Q_d versus the false-alarm probability Q_0 for $N_s = 2$ and various values of $\sigma = \epsilon a/\lambda R$. These curves, along each of which Λ is a parameter, are sometimes called the operating characteristics of the system. The portions between the dashed lines, where $\Lambda = 1$, are straight lines and represent the use of a randomized strategy each time no photons at all are observed, hypothesis H_1 then being chosen with a certain probability f and H_0 with probability 1 - f. As f varies from 0 to 1, the straight lines are traversed from left to right. To the right of the dashed lines is the region $\Lambda < 1$ ($\zeta < 1/2$); to the left is the region $\Lambda > 1$ ($\zeta > 1/2$).

Zero false-alarm probability can be attained with a finite detection probability

$$Q_d = 1 - \exp[-N_s(1 - h_1 y_1^2)] =$$

$$1 - \exp\{-N_s(1 - \mathrm{sinc}^2 \tfrac{1}{2}\sigma)\}; \qquad Q_0 = 0, \; \Lambda = \infty. \tag{53}$$

The strategy achieving this chooses hypothesis H_1 if any photons are observed in any mode orthogonal to the planar mode $\eta_0'(\underset{\sim}{r})$. Since this is impossible when a single source is present (H_0), $Q_0 = 0$. An error occurs under hypothesis H_1 only when no photons happen to appear in those orthogonal modes. The average number of photons in the planar mode under H_1 is

$$\underset{\sim}{E}(n_0|H_1) = N_s A^{-2} \iint_A \cos \mu(x_1 - x_2)\, d^2\underset{\sim}{r}_1 d^2\underset{\sim}{r}_2 = N_s h_1 y_1^2. \tag{54}$$

The average number in the rest of the modes is $N_s(1 - h_1 y_1^2)$, and the probability that no photons are observed in them and H_0 is chosen is $\exp[-N_s(1 - h_1 y_1^2)]$, whence we obtain (53). The complementary probability $1 - Q_d$ has been plotted in Fig. 3 versus σ for various mean numbers N_s of received photons. A larger detection probability can be attained by accepting a positive false-alarm probability and adopting a much more complicated strategy.

Infallible detection ($Q_d = 1$) can be achieved by accepting a false-alarm probability

$$Q_0 = \exp(-N_s y_0^2), \quad Q_d = 1, \quad \Lambda = 0,$$

where $y_0^2 = 1 - y_1^2$ is given by (17). For $\sigma << 1$, $y_0^2 \doteq (\pi\sigma)^4/720$; thus the false-alarm probability is close to 1 unless N_s is very large or the sources are well separated. To achieve this pair (Q_0, $Q_d = 1$) hypothesis H_1 is chosen when there are no photons in the zero mode $\eta_0(\underset{\sim}{r})$ given by (11); otherwise H_0 is chosen. The mean number of photons in the zero mode under hypothesis H_0 is given by (14) as $N_s y_0^2$, and the probability that there is none is $\exp(-N_s y_0^2)$. Under H_1 there will never be any photons in the zero mode, and it will always be correctly chosen.

The Circular Aperture

We list here for reference the formulas for the constants appearing in our solution when the aperture is circular, centered at the origin and having radius a. With $m = \mu a = k\varepsilon a/2R = \pi\varepsilon a/\lambda R$,

$$
\begin{aligned}
y_1^2 &= 8m^{-1}[m + J_1(2m)]^{-1}\,[J_1(m)]^2, \\
h_1 &= \tfrac{1}{2}\,[1 + m^{-1}\,J_1(2m)], \\
y_0^2 &= 1 - y_1^2,\ h_2 = 1 - h_1, \\
h_1 y_1^2 &= 4m^{-2}[J_1(m)]^2\ ;
\end{aligned}
\tag{55}
$$

the last describes the Airy pattern for a circular aperture. Here $J_1(x)$ is the Bessel function of order 1.

3. One Point Source or the Other?

An optical communication system could transmit binary information by turning on one or the other of two point sources for intervals of duration T. If the sources are very close, a distant observer will make errors in deciding which one is on during a given interval. Both sources can emit radiant power P_S, and they are separated by $\underset{\sim}{\varepsilon}$. We can put one at the origin $\underset{\sim}{u} = \underset{\sim}{0}$ without loss of generality when the sources are close to the optic axis normal to the aperture at its center ($\varepsilon << R$). Under hypothesis H_0 the emitting source is at $\underset{\sim}{u} = \underset{\sim}{0}$, and the spatial covariance of the aperture field is that given by (2); under H_1 it is at $\underset{\sim}{u} = \underset{\sim}{\varepsilon}$, and the spatial covariance function is

$$\varphi_S^{(1)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2) =$$

$$(E_S/AT) \exp[i\, k\, \underset{\sim}{\varepsilon} \cdot (\underset{\sim}{r}_1 - \underset{\sim}{r}_2)/R]\, Fr_1\, Fr_2^* \tag{56}$$

Otherwise the sources are as before. The first task is to find suitable spatial modes for the aperture field.

The planar mode $\eta_0'(\underset{\sim}{r}) = A^{-\frac{1}{2}} Fr$ is an eigenfunction of $\varphi_S^{(0)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$; an eigenfunction of $\varphi_S^{(1)}(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$ is similarly

$$A^{-\frac{1}{2}} \exp(i\, k\, \underset{\sim}{\varepsilon} \cdot \underset{\sim}{r}/R)\, Fr\ .$$

We therefore take as our basic modes $\eta_0'(\underset{\sim}{r})$ and

$$\eta_1'(\underset{\sim}{r}) = [C_1' + C_1'' \exp(i\mu' x)]\, Fr,$$

$$\mu' = k\varepsilon/R,\ \varepsilon = |\underset{\sim}{\varepsilon}|\ , \tag{57}$$

choosing C_1' and C_1'' so that $\eta_1'(\underset{\sim}{r})$ and $\eta_0'(\underset{\sim}{r})$ are orthogonal and properly normalized. We find

$$C_1'' = [(1 - q_0^2)A]^{-\frac{1}{2}},\ C_1' = -C_1''\, q_0\ ,$$

$$q_0 = A^{-1} \int_A \exp(i\mu' x)\, d^2\underset{\sim}{r}\ . \tag{58}$$

The aperture A may have an arbitrary shape, and without loss of generality q_0 can be taken as real. The remaining spatial modes are made orthogonal to $\eta_0'(\underset{\sim}{r})$ and $\eta_1'(\underset{\sim}{r})$ by the Gram-Schmidt process. They are unexcited by the source under either hypothesis and can be disregarded. As before we start by assuming that $WT \ll 1$ and only a single temporal mode needs to be considered. Later we again extend our formulas to cover a great many statistically independent temporal modes.

The two spatio-temporal modes are represented quantum-mechanically by harmonic oscillators, and the density operators ρ_0 and ρ_1 under the two hypotheses have Gaussian P-representations with mode correlation matrices given by (13) as

$$\varphi_{pr}^{(0)} = N_s\, \delta_{p0}\, \delta_{r0} ,$$

$$\varphi_{pr}^{(1)} = N_s\, q_p\, q_r ,$$

$$(p,\ r) = (0,\ 1),\ q_1 = (1 - q_0^2)^{\frac{1}{2}} ; \tag{59}$$

N_s is the mean number of photons received from either source during $(0, T)$.

The optimum strategy is again specified by the eigenvectors of $\rho_1 - \Lambda\rho_0$, which can be determined from an integral equation like (25), except that only two modes instead of three are involved, and a single subscript suffices to label the eigenvalues and eigenvectors. Here

$$R_0(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) = (1 - v_0) \exp v_0\, \beta_0^*\, \gamma_0 \tag{60}$$

$$R_1(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) = (1 - v_0) \exp[v_0(q_0\beta_0^* + q_1\beta_1^*)(q_0\gamma_0 + q_1\gamma_1)],$$

$$v_0 = N_s/(N_s + 1) . \tag{61}$$

The eigenfunctions in the coherent-state representation of (24) are now

$$F_n\, (\underset{\sim}{\gamma}^*) = x_0\, \gamma_0^{*n} + x_1(q_0\, \gamma_0^* + q_1\, \gamma_1^*)^n , \tag{62}$$

which when substituted into (25) yields

$$P_n[(x_0 q_0^n + x_1)\ (q_0\beta_0^* + q_1\beta_1^*)^n$$

$$-\Lambda(x_0 + x_1\ q_0^n)\ \beta_0^{*n}] = \eta_n\ F_n(\tilde{\beta}^*)\ ,$$

$$P_n = (1 - v_0)\ v_0^n\ , \tag{63}$$

and equating coefficients of like terms gives the homogeneous equations

$$P_n\ q_0^n\ x_0 + (P_n - \eta_n)\ x_1 = 0,$$

$$-(\Lambda P_n + \eta_n)\ x_0 - \Lambda P_n\ q_0^n\ x_1 = 0, \tag{64}$$

the determinant of which, when set equal to zero, provides the quadratic equation for the eigenvalues,

$$\eta_n^2 - (1 - \Lambda)\ P_n\ \eta_n - \Lambda(1 - q_0^{2n})P_n^2 = 0. \tag{65}$$

When the prior probabilities of the two hypotheses are equal, $\Lambda = 1$, and

$$\eta_n = \pm\ P_n(1 - q_0^{2n})^{\frac{1}{2}}. \tag{66}$$

The error probability, given by (21), is now

$$P_e = \frac{1}{2}\left\{P_0 + \sum_{n=1}^{\infty} P_n[1 - (1 - q_0^{2n})^{\frac{1}{2}}]\right\}. \tag{67}$$

An argument like that in Section 2 permits us to include multiple temporal modes in the limit $WT \gg 1$ by simply replacing P_n by the Poisson probability

$$P_n = N_s^n \exp(-N_s)/n!. \tag{68}$$

In Fig. 4 we have plotted the resulting error probability, postulating a rectangular aperture of width a in the direction parallel to the line between the sources. The separation ε is embodied in the parameter $\sigma = \varepsilon a/\lambda R$, and $q_0 =$ sinc σ. The limiting value is again $\frac{1}{2}\exp(-N_s)$, attained when $\sigma = 1$, which corresponds to the image separation prescribed by the Rayleigh criterion when a diffraction-limited optical system is used. Now, however, the image of only one point source will be present at a time.

Appendix

Some Coherent-State Calculus

Let $\underset{\sim}{\alpha} = (\alpha_0, \alpha_1, \ldots)$ be a column vector of coherent-state amplitudes, let $\underset{\sim}{\alpha}^+ = (\alpha_0^*, \alpha_1^*, \ldots)$ be its Hermitian conjugate row vector, and let $\underset{\sim}{K}$ be an $n \times n$ Hermitian matrix. Then the multivariate Gaussian integral in its most convenient form for calculations with the coherent-state representation is

$$\int \cdots \int \exp(-\underset{\sim}{\alpha}^+ \underset{\sim}{K} \underset{\sim}{\alpha} + \underset{\sim}{\beta}^+ \underset{\sim}{\alpha} + \underset{\sim}{\alpha}^+ \underset{\sim}{\gamma}) \prod_{i=1}^{n} (d^2\alpha_i/\pi) = (\det \underset{\sim}{K})^{-1} \exp(\underset{\sim}{\beta}^+ \underset{\sim}{K}^{-1} \underset{\sim}{\gamma}), \tag{69}$$

where $d^2\alpha_i = d\alpha_{ix}\, d\alpha_{iy}$, $\underset{\sim}{\beta}$ and $\underset{\sim}{\gamma}$ are constant column vectors, and the integration is carried out over the entire 2n-dimensional space of $\{\alpha_{ix}, \alpha_{iy}\}$.

In order to derive the coherent-state representation of the density operator (18), as given in (26), one applies equation (9.11) of Glauber's paper[8] to (18), with $\underset{\sim}{\varphi}^{(i)} = \underset{\sim}{\varphi}$,

$$\begin{aligned} R_i(\underset{\sim}{\beta}, \gamma) &= |\det \underset{\sim}{\varphi}|^{-1} \int \cdots \int \exp(-\underset{\sim}{\alpha}^+ \underset{\sim}{\varphi}^{-1} \underset{\sim}{\alpha} + \underset{\sim}{\beta}^+ \underset{\sim}{\alpha} + \underset{\sim}{\alpha}^+ \underset{\sim}{\gamma} - \underset{\sim}{\alpha}^+ \underset{\sim}{\alpha}) \prod_i (d^2\alpha_i/\pi) \\ &= |\det (\underset{\sim}{\varphi} + \underset{\sim}{I})|^{-1} \exp[\underset{\sim}{\beta}^+ (\underset{\sim}{\varphi}^{-1} + \underset{\sim}{I})^{-1} \underset{\sim}{\gamma}] \end{aligned} \tag{70}$$

by (69) with $\underset{\sim}{K} = \underset{\sim}{\varphi}^{-1} + \underset{\sim}{I}$, where $\underset{\sim}{I}$ is the identity matrix.

The integrals required when substituting the eigenfunctions of (32) into (25) are most easily derived from the generating function

$$\begin{aligned} f_i(\underset{\sim}{\beta}^*, \underset{\sim}{\kappa}) &= \iiint R_i(\underset{\sim}{\beta}^*, \gamma) \exp\left[\sum_{i=0}^{2} (\kappa_i \gamma_i^* - |\gamma_i|^2)\right] \prod_{i=0}^{2} (d^2\gamma_i/\pi) \\ &= |\det (\underset{\sim}{\varphi}^{(i)} + \underset{\sim}{I})|^{-1} \int \cdots \int \exp[\underset{\sim}{\beta}^+ (\underset{\sim}{\varphi}^{(i)-1} + \underset{\sim}{I})^{-1} \underset{\sim}{\gamma} \\ &\qquad + \underset{\sim}{\gamma}^+ \underset{\sim}{\kappa} - \underset{\sim}{\gamma}^+ \underset{\sim}{\gamma}] \prod_i (d^2\gamma_i/\pi) \end{aligned}$$

$$= |\det (\underset{\sim}{\varphi}^{(i)} + \underset{\sim}{I})|^{-1} \exp[\underset{\sim}{\beta}^{+}(\underset{\sim}{\varphi}^{-1} + \underset{\sim}{I})^{-1} \underset{\sim}{\kappa}], \quad i = 0, 1. \tag{71}$$

We see that the effect of the integration is to replace $\underset{\sim}{\gamma}$ by $\underset{\sim}{\kappa}$. Thus

$$f_1(\underset{\sim}{\beta}^*, \underset{\sim}{\kappa}) = (1 - v_1)(1 - v_2) \exp(v_1\beta_1{}^*\kappa_1 + v_2\beta_2{}^*\kappa_2), \tag{72}$$

$$f_0(\underset{\sim}{\beta}^*, \underset{\sim}{\kappa}) = (1 - v_0) \exp[v_0(y_0\beta_0{}^* + y_1\beta_1{}^*)(y_0\kappa_0 + y_1\kappa_1)]. \tag{73}$$

Putting $\kappa_0 = \varphi\, y_0$, $\kappa_1 = \varphi\, y_1$, $\kappa_2 = 0$ in these, we get the generating function

$$f_i{}'(\underset{\sim}{\beta}^*, \varphi) = \iiint R_i(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) \exp[\varphi(y_0\gamma_0{}^* + y_1\gamma_1{}^*) - |\underset{\sim}{\gamma}|^2] \prod_i (d^2\gamma_i/\pi), \quad i = 0, 1, \tag{74}$$

with

$$f_1{}'(\underset{\sim}{\beta}^*, \varphi) = (1 - v_1)(1 - v_2) \exp(\varphi\, v_1y_1\beta_1{}^*) \tag{75}$$

$$f_0{}'(\underset{\sim}{\beta}^*, \varphi) = (1 - v_0) \exp[\varphi\, v_0(y_0\beta_0{}^* + y_1\beta_1{}^*)], \tag{76}$$

since $y_0{}^2 + y_1{}^2 = 1$. The required multivariate integrals are then $(i = 0, 1)$

$$\iiint \gamma_1^{*n} R_i(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) \exp(-|\underset{\sim}{\gamma}|^2) \prod_{i=0}^{2} (d^2\gamma_i/\pi) = \partial^n f_i(\underset{\sim}{\beta}^*, \underset{\sim}{\kappa})/\partial\kappa_1{}^n|_{\underset{\sim}{\kappa}=\underset{\sim}{0}}, \tag{77}$$

$$\iiint (y_0\gamma_0{}^* + y_1\gamma_1{}^*)^n R_i(\underset{\sim}{\beta}^*, \underset{\sim}{\gamma}) \exp(-|\underset{\sim}{\gamma}|^2) \prod_{i=0}^{2} (d^2\gamma_i/\pi) = \partial^n f_i{}'(\underset{\sim}{\beta}^*, \varphi)/\partial\varphi^n|_{\varphi=0}, \tag{78}$$

from which (33) follows immediately. Equation (63) is similarly derived.

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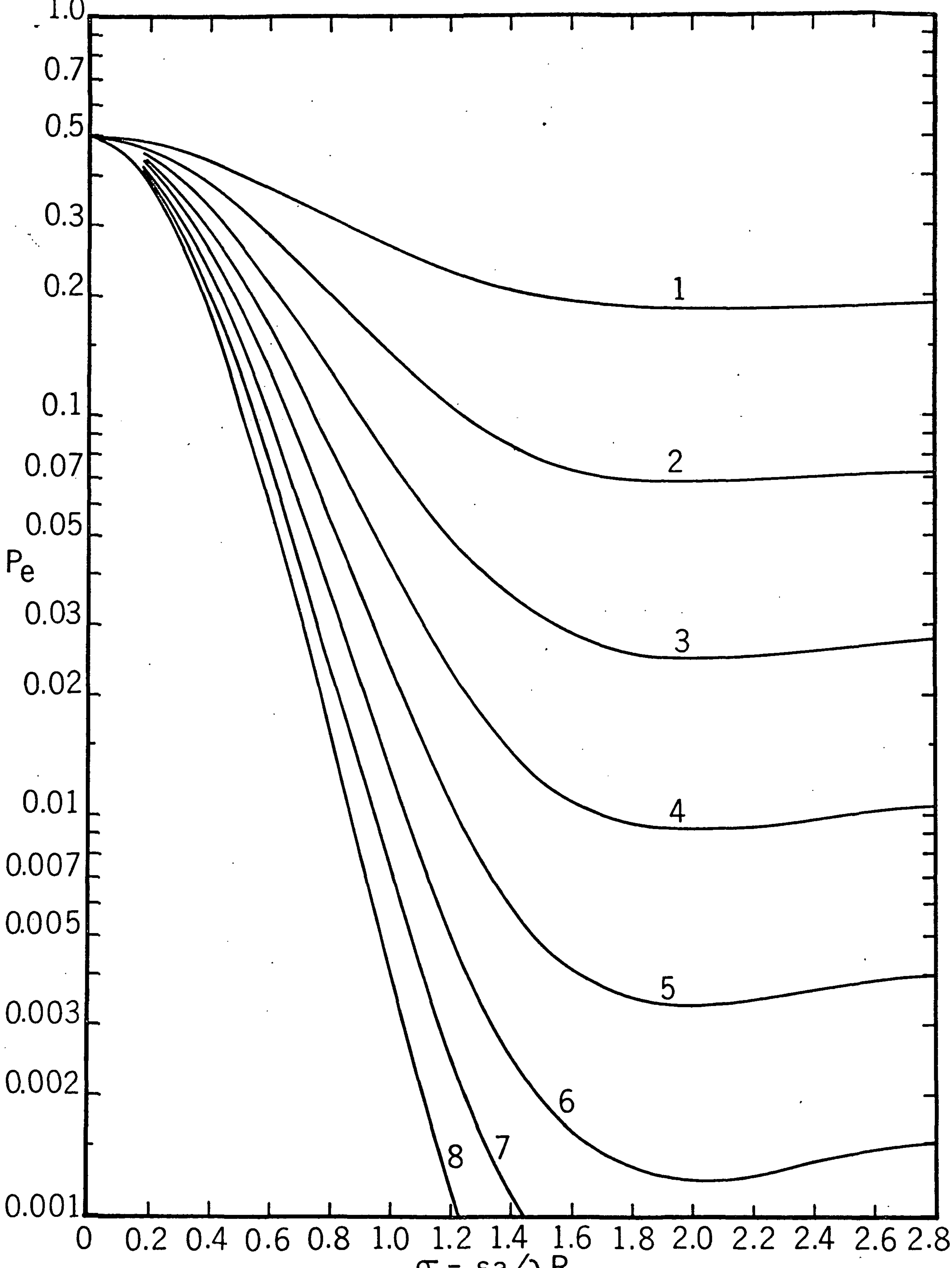
Figure Captions

Fig. 1. Average error probability P_e in deciding whether two sources or one is present, versus the separation parameter $\sigma = \varepsilon a/\lambda R$. The curves are indexed with the mean number N_s of received photons.

Fig. 2. Operating characteristics of the optimum strategy for deciding whether two sources or one is present. The curves are indexed with the separation parameter $\sigma = \varepsilon a/\lambda R$; $N_s = 2$.

Fig. 3. False-dismissal probability $(1 - Q_d)$ in decisions whether two sources are present, versus the separation parameter $\sigma = \varepsilon a/\lambda R$. The false-alarm probability is zero. The curves are indexed with the mean number N_s of received photons.

Fig. 4. Average error probability P_e in deciding which of two sources separated by ε is radiating, versus the separation parameter $\sigma = \varepsilon a/\lambda R$. The curves are indexed with the mean number N_s of received photons.



1.0
0.7
0.5
0.3
0.2
0.1
0.07
0.05
P_e
0.03
0.02
0.01
0.007
0.005
0.003
0.002
0.001
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
$\sigma = \varepsilon a/\lambda R$
1
2
3
4
5
6
7
8

Figure 1

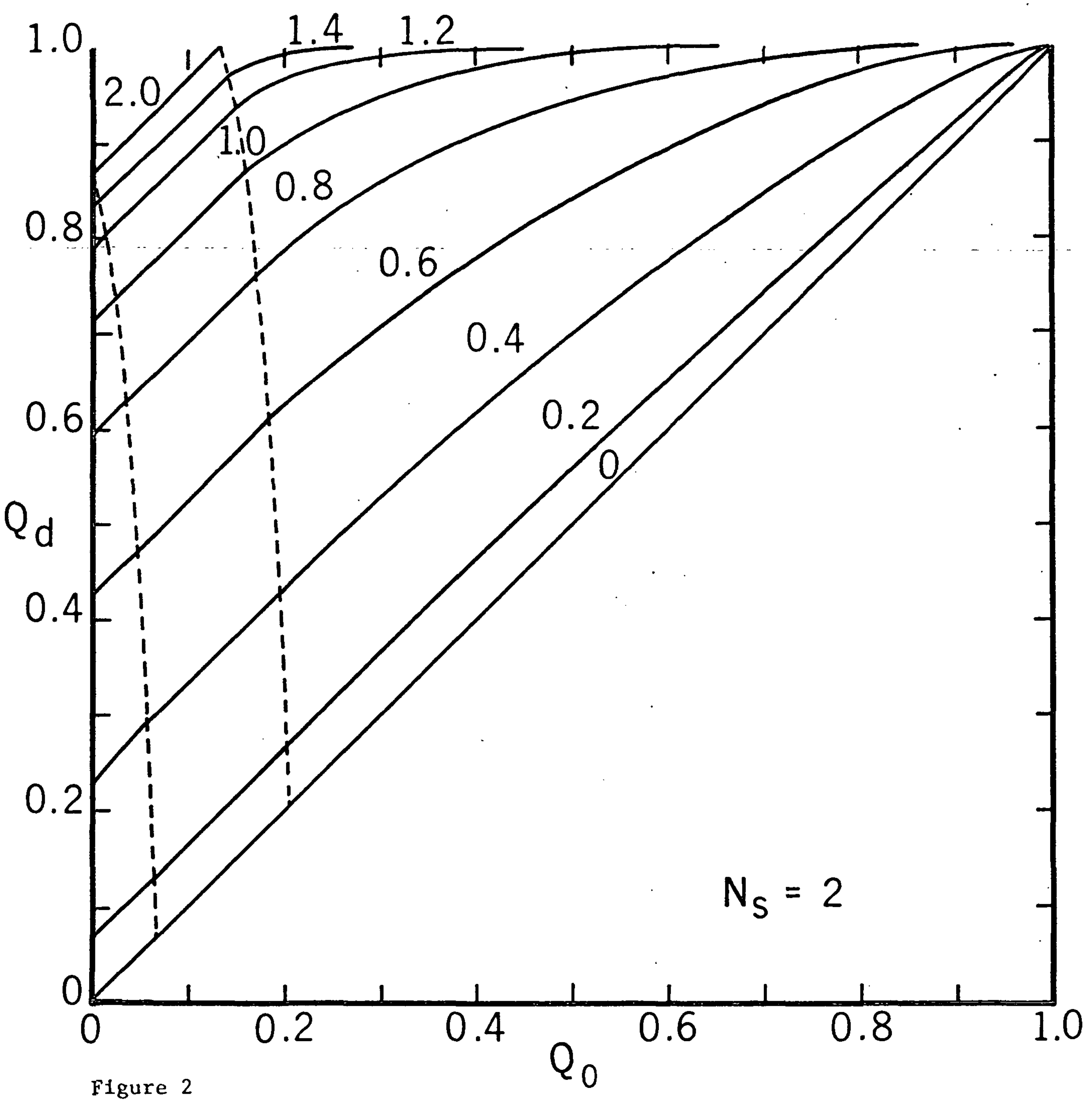


Figure 2

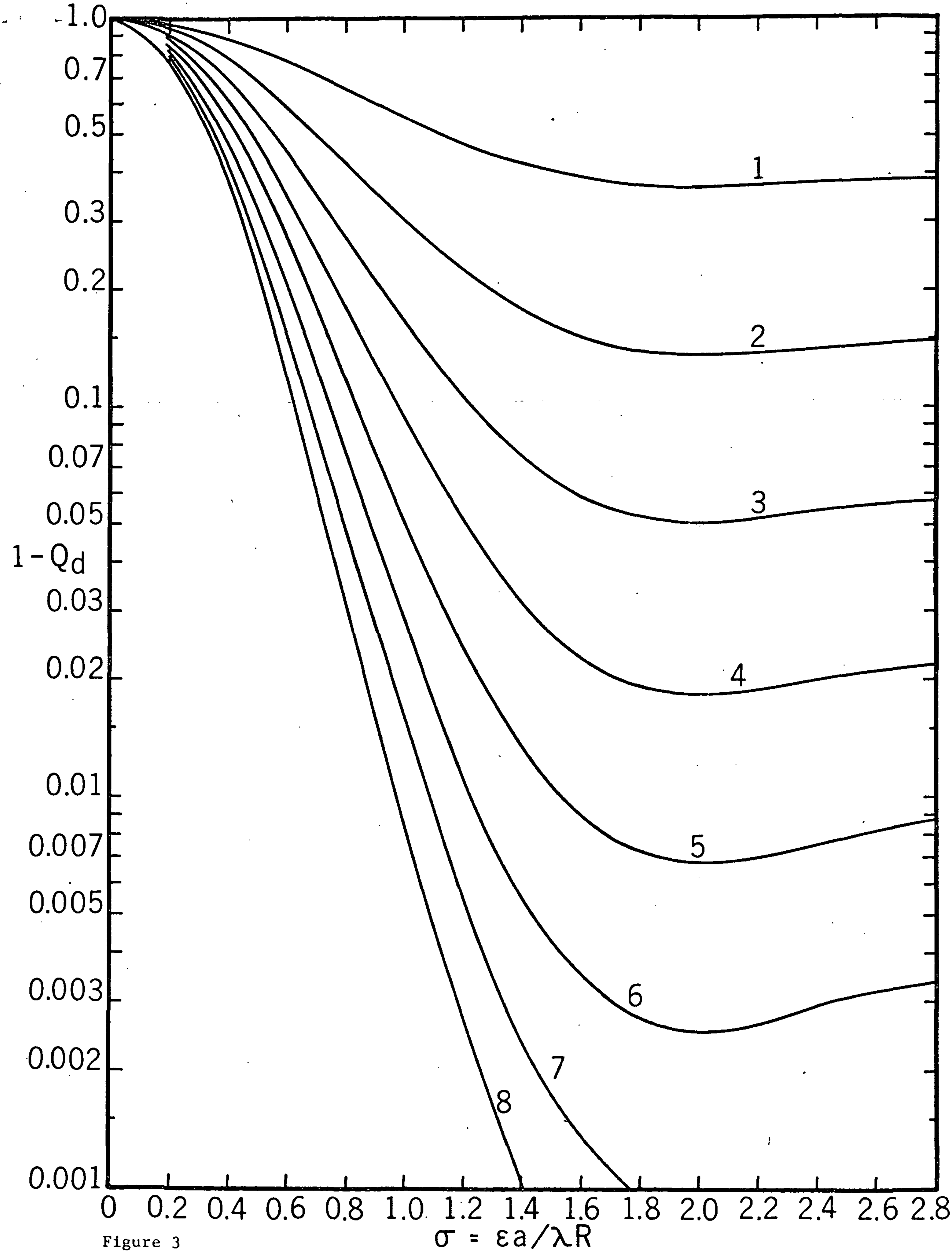

1.0
0.7
0.5
0.3
0.2
0.1
0.07
0.05
0.03
0.02
0.01
0.007
0.005
0.003
0.002
0.001
1 - Q_d
1
2
3
4
5
6
7
8
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
σ = εa/λR

Figure 3

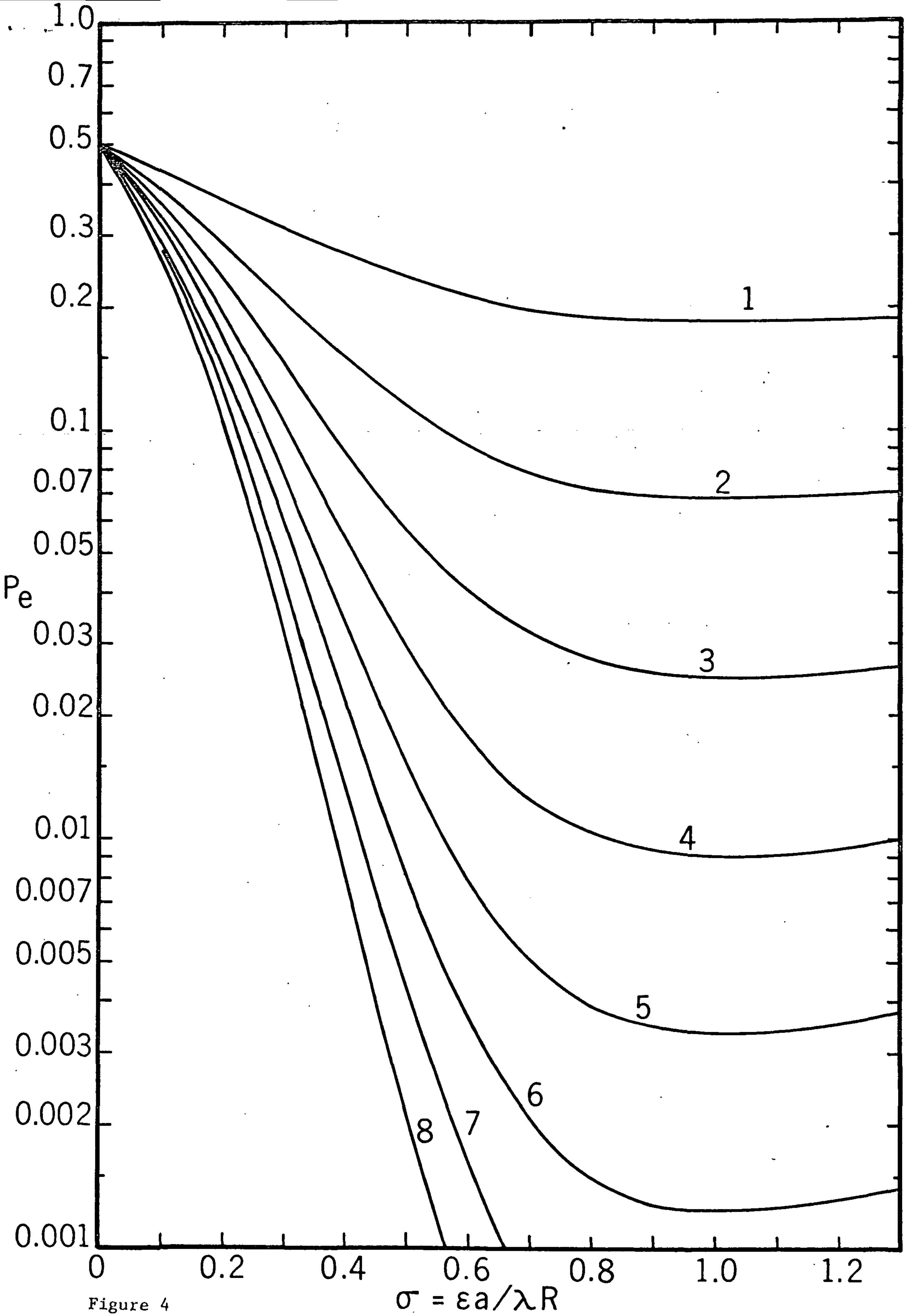

1.0
0.7
0.5
0.3
0.2
0.1
0.07
0.05
P_e
0.03
0.02
0.01
0.007
0.005
0.003
0.002
0.001
0
0.2
0.4
0.6
0.8
1.0
1.2
$\sigma = \varepsilon a/\lambda R$
1
2
3
4
5
6
7
8

Figure 4