Asymmetries in Relativistic Information Flow

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Abstract—In the so-called "twin paradox" of relativity theory, one spaceship leaves another at velocity $v=\beta c$ and returns to find that the other ship has aged more by a factor $\gamma=(1-\beta^2)^{-1/2}$. The asymmetry is first isolated in order to resolve the paradox. Then the principle of relativity is used to derive the relative aging factor γ and the relativistic Doppler shift. Using the Doppler factor, asymmetries in information transmission between moving spaceships are investigated. An additive white Gaussian noise channel with Shannon capacity $C=W\log(1+P/NW)$ is considered. After accounting for the effect of the relativistic Doppler shift on signal power and bandwidth, it is found that for a given transmission rate and bandwidth, the traveler needs γ times the energy of the stationary spaceship to transmit $1/\gamma$ times as much information. The asymmetry in efficiency is thus γ^2. A simple proof is given that the round trip asymmetry in efficiency for constant-rate transmission is always the square of the relative aging factor for all trajectories regardless of accelerations and the presence of gravitational fields.

I. Introduction

The TWIN paradox provides a suitable foundation for a discussion of relativistic information flow. We present a resolution of the twin paradox in Section II, introduce the necessary concepts of relativity, and derive the pertinent relativistic Doppler shift factors. We shall consider communication over an additive white Gaussian noise channel with receiver-noise spectral density N, transmitter power P, and transmitter bandwidth W. Capacity formulas and results for constant-Doppler problems are presented in Sections III and IV.

In Section V we consider time-dependent problems such as the twin problem in which the Doppler shift α is time-varying. We show that the instantaneous transmission capacity is $C(t)=W\log(1+\alpha(t)P(t)/NW)$, where $\alpha(t)$ is the Doppler shift appropriate for signals transmitted at time t. Thus $\alpha(t)$ incorporates knowledge of the current trajectory of the transmitter and the future trajectory of the receiver. Also in Section V, we prove that for a round trip where terminals A and B start and finish together,

$$\int_0^{T_A} dt/\alpha_A(t)=T_B, \quad \text{and} \quad \int_0^{T_B} d\tau/\alpha_B(\tau)=T_A,$$

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where T_A and T_B are the respective travel times of A and B.

These are the basic equations from which we develop the fundamental rate equations in Section VI. It can be shown that if A and B transmit at the same constant rate and bandwidth and if γ_0 is the relative aging factor T_A/T_B, then B will need γ_0 times as much energy as A to transmit $1/\gamma_0$ times as many bits during the round trip. Regardless of acceleration or gravitational fields, the trajectory dependence of the asymmetry in efficiency reduces to the square of the aging factor.

Next we demand that $P(t)$ be constant, and we show that traveler B needs more energy per bit sent. Alternatively, if we place a constraint on the total transmitted energy and calculate the number of bits that can be sent during the round trip, a round trip energy constraint of the form $\int_0^T P(t)\,dt=E$ yields a simple water-filling solution for the optimal form of $P(t)$.

In Section IX we apply the previously derived general results for time-dependent Doppler factors to the linear round trip or twin problem. This example illustrates the asymmetries that result from the formulations of the communication problem in Sections VI, VII, and VIII. All asymmetries favor the stationary transmitter.

II. The Twin Paradox

In the so-called twin paradox, a traveling spaceship leaves another at a velocity $v=\beta c$, turns around after traveling some distance and returns to the starting point. Comparing clocks at the end of the journey reveals that the nontraveling spaceship has aged by a factor $\gamma=(1-\beta^2)^{-1/2}$ more than the traveler. It has been argued that there should be no age difference, because either spaceship could be considered the traveler. However, there is a difference between the two ships—the traveler has undergone an initial acceleration and a final deceleration that could easily be detected by experimental apparatus. Nonetheless, the aging difference does not take place during these accelerations; rather, the accelerations serve to place the spaceships in different reference frames for different times. The Appendix provides an analogy to clarify this point. It is the different time averages of the same inbound and outbound relativistic Doppler shifts that result in the asymmetric aging factor γ.

We use symmetry arguments to derive the relative aging factor. (See Oliver [1], Darwin [2], and Bondi [3] for alternative derivations.) Consider the following numerical example.

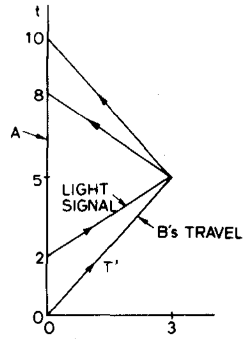

Fig. 1. Space–time diagram of B's trip with A at rest. B sees A recede for T' years. A sees B recede for eight years. B receives A's transmissions sent during the first two of A's years. A receives B's transmissions sent during B's journey of T' years. Reception rates $T'/8$ and $2/T'$ must be equal.

Traveler A stays at home. Traveler B travels at velocity $v = \beta c$ ($\beta = 3/5$) to a star three light years away. Thus A knows that B reaches the star after five of A's years. Since light from the star requires three years to return to A, however, A does not see B turn around until eight of A's years have elapsed. Traveler B actually returns to A after ten of A's years. Finally, only light emanating from A in the first two of A's years will reach B in his journey toward the star. This scenario is depicted in the space–time diagram in Fig. 1.

On B's outbound journey, we see that B receives two years worth of pulses in T' of B's years. The time T' of B's outbound leg is still to be determined. Thus B receives pulses at rate $R' = 2/T'$. The situation for A is similar. Since A sees B outbound for eight years, A receives T' pulses in eight years for a rate of $R = T'/8$ pulses per year.

Invoking the principle of relativity (for the uniform velocity segment of the journey), the rates R and R' must be equal. Otherwise, a preferred reference frame would be revealed. But $R = R'$ implies $T'/8 = 2/T'$, or $T' = 4$ years. Thus B ages four years on the outbound leg.

Similar calculations show that B ages four years on the inbound leg. While A ages ten years, B therefore ages eight years, and the age ratio is $10/8 = (1-(3/5)^2)^{-1/2}$, as predicted by the Special Theory. The corresponding values are calculated in Fig. 2 for arbitrary velocity $v = \beta c$ and arbitrary distance x.

Here $R' = ((x/\beta) - x)/T'$, and $R = T'/((x/\beta) + x)$. Finally, $R = R'$ implies $T' = x((1/\beta^2) - 1)^{1/2}$. Thus the ratio of travel times is

$$T_A/T_B = 2T/2T' = (2x/\beta)/2x(\beta^{-2} - 1)^{1/2} = \gamma, \quad (1)$$

and the desired result is obtained.

Also, since the Doppler frequency ratio ν'/ν is the ratio R of received clock rate to transmitted clock rate, the relativistic Doppler shift is a consequence of this calculation. Thus the outgoing Doppler shift α_- is given by

$$\begin{aligned}\alpha_- &= \nu'/\nu = R = T'/((x/\beta) + x)\\ &= x(\beta^{-2} - 1)^{1/2}/((x/\beta) + x)\\ &= \sqrt{(1-\beta)/(1+\beta)}\\ &= (1-\beta)\gamma, \end{aligned} \quad (2)$$

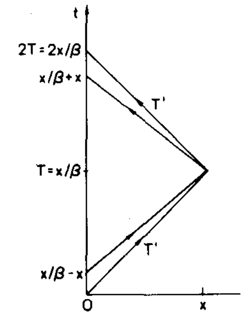

Fig. 2. Space–time diagram for velocity βc and distance x.

for spaceships traveling away from each other at velocity $v = \beta c$. Similarly, the Doppler shift α_+ on the incoming leg is

$$\alpha_+ = (1+\beta)\gamma, \quad (3)$$

where

$$\gamma = 1/\sqrt{1-\beta^2}. \quad (4)$$

III. The Additive White Gaussian Noise (AWGN) Channel with Doppler Shift

The AWGN channel with Doppler shift is illustrated in Fig. 3. The transmitter sends a signal process X with average power P and bandwidth W. Information is transmitted at rate R. These three quantities are all measured in the transmitter's frame. The receiver noise process Z is white Gaussian receiver noise with power spectral density N measured in the receiver's frame.

Suppose that the receiver sees the transmitter moving. Then the received signal is Doppler shifted by a factor α. (The receiver noise is not affected.) The exact value of the Doppler factor α as a function of velocity is unimportant here. For simplicity we make the narrow-beam assumption that the entire signal from the transmitter is intercepted by the receiver. The path loss is assumed to be independent of the distance the signal travels. Without loss of generality, we assume that the path loss is zero.

We need to know the characteristics of the Doppler shifted received signal in order to establish a constraint on the transmitted signal power, rate, and bandwidth. First, the reception rate R' is a frequency. (Throughout this paper we use primes to denote quantities pertaining to reception.) Since the Doppler factor α is the ratio of received frequency to transmitted frequency, we immediately see that $R' = \alpha R$. Second, the component frequencies of the signal are all shifted by the factor α. Consequently the received signal bandwidth is $W' = \alpha W$. Third, the received signal power is $P' = \alpha^2 P$ according to relativistic electromagnetic theory [4]. The following simple heuristic argument illustrates this equality.

Suppose that the transmitted signal is monochromatic. It then consists of a stream of photons of frequency ν being emitted at rate λ. The transmitted signal power P equals the energy per photon $h\nu$ times the photon emission rate λ. The received signal consists of a monochromatic stream of

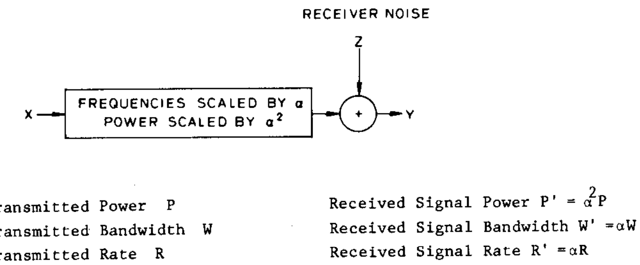

Fig. 3. The additive white Gaussian noise channel with Doppler shift.

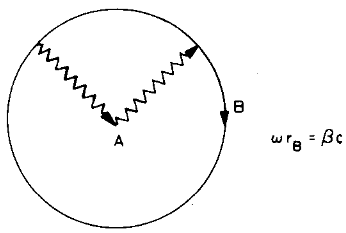

Fig. 4. Geometry of the circular trajectory problem.

photons of frequency ν' arriving at rate λ'. Assuming that the Doppler factor is $\alpha=\nu'/\nu=\lambda'/\lambda$, we see that $P'=\lambda' h\nu'=\alpha^2\lambda h\nu=\alpha^2 P$.

This relationship is valid for each monochromatic component of any signal. Summing the components yields $P'=\alpha^2 P$ for an arbitrary signal process.

The received signal process Y is thus the sum of a white Gaussian receiver noise process with power spectral density N and a signal process with power $P'=\alpha^2 P$, bandwidth $W'=\alpha W$, and information rate $R'=\alpha R$. Applying Shannon's capacity result [5] for the AWGN channel, we can conclude that the maximum reception rate at which the receiver can reliably decode messages is

$$\begin{aligned} C' &= W'\log(1+P'/NW') \\ &= \alpha W\log(1+\alpha^2 P/N\alpha W) \\ &= \alpha W\log(1+\alpha P/NW)\ \text{bits/s.} \end{aligned} \tag{5}$$

The corresponding transmission rate (which is the reception rate scaled by the Doppler shift factor) is

$$C = C'/\alpha = W\log(1+\alpha P/NW).$$

To achieve this rate, the transmitter chooses a signaling interval $[0,T]$. He then chooses 2^{TR} signals at random from a white Gaussian random process with bandwidth W and power $(1-\epsilon)P$ with $\epsilon>0$. Then, for $R<C$ and $(T\to\infty, \epsilon\to 0)$, it can be shown that with very high probability this sequence of code books achieves an arbitrarily small probability of decoding error. Moreover, for any code book and for any rate $R>C$, the probability of receiver error tends to one.

Strictly speaking, all that we can say for short journeys and time-dependent α is that we expect the probability of error to be very small if the instantaneous rate R is less than C and very large if R exceeds C. It should be noted that although the transmitter sends the signal over a time interval of length T, the receiver sees the signal over an interval of length T/α.

In summary, for reliable communication we require transmission at a rate satisfying

$$\begin{aligned} R < C &= C'/\alpha = (W'/\alpha)\log(1+P'/NW') \\ &= W\log(1+\alpha P/NW). \end{aligned} \tag{6}$$

Equivalently, if we wish to transmit information reliably at rate R and bandwidth W, the required transmitted signal power is

$$P > \alpha^{-1} NW(\exp_2(R/W)-1) \triangleq P_0/\alpha, \tag{7}$$

where $P_0 = NW(\exp_2(R/W)-1)$ is the minimum transmitted power for reliable communication when there is no Doppler shift.

IV. The Circular Trajectory Problem

In the preceding section we derived the transmission capacity for the AWGN channel with a Doppler shift α. In the next section we extend this result to allow for a time-dependent Doppler factor $\alpha(t)$ like that found in the twin problem. We can first use the results of Section III, however, to solve steady-state communication problems in which the Doppler shift factor is constant. The most interesting constant Doppler shift example is the circular trajectory problem in which B circles A at a constant radius r_B and at a constant speed $\omega r_B=\beta c$ (Fig. 4). The centripetal force needed for B to maintain the circular trajectory can be provided by thrust or by a central mass M. Since there is no preferred direction, the Doppler factors must be constant. Using this fact, we can derive the relationship between the doppler shift factors and the relative aging factor.

Let α_A be the Doppler factor for transmission from A to B, and let α_B be the factor for signals sent from B to A. Let T_A be the orbital period as measured by A, and let T_B be the elapsed time per orbit as measured by B. These times can be established by reference to a fixed direction such as that of a distant star. Finally, let $\gamma_0=T_A/T_B$ be the relative aging factor. If there is no central mass, then $\gamma_0=\gamma=(1-\beta^2)^{-1/2}$, the aging factor in special relativity.

We can derive the transmission Doppler factors in terms of the aging factor γ_0 as follows. Suppose A sends pulses to B at rate $1/\alpha_A$ for one orbital period T_A. These pulses will be received by B at rate $\alpha_A/\alpha_A=1$ for one orbital period T_B. Since the number of pulses sent equals the number received, we have $(1/\alpha_A)T_A=(1)T_B$, or

$$\alpha_A = T_A/T_B = \gamma_0, \tag{8}$$

the aging factor.

Similarly, suppose B sends pulses to A at rate $1/\alpha_B$ for one orbital period T_B. These will be received by A at rate $\alpha_B/\alpha_B=1$ for one orbital period T_A. Pulse conservation yields $(1/\alpha_B)T_B=T_A$, or

$$\alpha_B = T_B/T_A = 1/\gamma_0, \tag{9}$$

the reciprocal of the aging factor.

With these results and the capacity result (6), we can write A's transmission capacity as

$$C_A = W \log(1 + \gamma_0 P_A/NW), \tag{10}$$

and B's transmission capacity as

$$C_B = W \log(1 + (1/\gamma_0) P_B/NW), \tag{11}$$

where A and B transmit signals with powers P_A and P_B, respectively, and with bandwidth W. For equal transmission powers, the received signal-to-noise ratio at B is γ_0^2 times the signal-to-noise ratio at A.

If A and B both wish to transmit at the same rate $C_A = C_B = C$ and with the same bandwidth W, B must use γ_0^2 times as much power as A. For each revolution, B will use γ_0 times as much energy as A to send $1/\gamma_0$ times as much information. The asymmetry in efficiency is thus

$$\frac{E_B/N_B}{E_A/N_A} = (P_B T_B / P_A T_A)(C T_A / C T_B) = P_B/P_A = \gamma_0^2. \tag{12}$$

If instead of transmitting at the same rate, A and B both transmit at the same power $P = P_A = P_B$, the ratio of transmission rates equals the ratio of energies per bit:

$$\frac{E_B/N_B}{E_A/N_A} = \left(\frac{PT_B}{PT_A}\right)\left(\frac{C_A T_A}{C_B T_B}\right)$$
$$= \frac{C_A}{C_B} = \frac{\log(1+\gamma_0 P/NW)}{\log(1+(1/\gamma_0)P/NW)}. \tag{13}$$

Note that the asymmetry in efficiency $(E_B/N_B)/(E_A/N_A)$ lies strictly between one and γ_0^2 for all N, W, and P, and approaches γ_0^2 as $P \to 0$. This supports the conclusion that the transmitter that ages less is less efficient.

V. The Fundamental Equations for Relativistic Capacity

In Section III we found the channel capacity expression as a function of α for uniform motion. We next consider a transmitter with time-dependent transmitter power $P(t)$ and transmitter rate $C(t)$. Integration of $C(t)$ over the transmission time yields the total number of bits communicated from the transmitter to the receiver. To proceed we introduce an anticipated Doppler shift $\alpha(t)$.

We begin by specifying the space–time trajectories for A and for B. From the knowledge of these trajectories, we can calculate an *anticipated* Doppler shift factor $\alpha(t)$ as follows. We know that clock pulses sent at transmitter time t will be received at some point in space–time by the receiver. Let $\alpha(t)$ be the observed Doppler shift at the receiver of this signal sent at time t. This definition of $\alpha(t)$ is essential to the consideration of problems incorporating time-varying Doppler factors.

We first derive the aging relationships from $\alpha(t)$ for later comparisons with the information flow relationships. We suppose that the entire journey for transmitter A takes time T_A. At the end of the journey when A and B are together, let B's clock read time T_B. Clearly, we have the trivial relationship

$$T_A = \int_0^{T_A} 1\, dt.$$

We can now introduce a way to view the problem that makes the relationship between Doppler shift and aging apparent. Let A transmit pulses to B at a time-varying rate $r(t) = 1/\alpha(t)$. Then the number of pulses sent by A will be given by

$$\int_0^{T_A} r(t)\, dt = \int_0^{T_A} (1/\alpha(t))\, dt.$$

But $r(t)$ has been adjusted so that the received pulse rate by B is $r'(\tau) = \alpha(t) r(t) = \alpha(t)/\alpha(t) = 1$, observed in B's time units. Thus B receives pulses with clocklike regularity at a constant rate one. Therefore we know B's age at the end of the journey simply by counting the pulses received by B. Finally, if A and B begin and end their journey together, the number of pulses received by B must equal the number of pulses sent by A. Thus

$$T_B = \int_0^{T_B} r'(\tau)\, d\tau = \int_0^{T_A} r(t)\, dt = \int_0^{T_A} dt/\alpha(t). \tag{14}$$

Similarly, if the anticipated Doppler factor is $\alpha_B(\tau)$ in B's time frame for B sending to A,

$$\int_0^{T_B} d\tau/\alpha_B(\tau) = T_A. \tag{15}$$

We now investigate channel capacity by using the anticipated Doppler shift factors $\alpha_A(t)$ and $\alpha_B(\tau)$. If A sends power $P(t)$ in a bandwidth W at A's time t, the received signal will have instantaneous power $\alpha^2(t)P(t)$ and bandwidth $\alpha(t)W$. Thus the instantaneous receiver capacity is

$$C'(\tau) = \alpha(t) W \log(1 + \alpha(t)P(t)/NW) \text{ bits/s},$$

where τ denotes B's reception time of signals sent by A at time t. Corrected for the Doppler factor, this corresponds to a transmission rate

$$C(t) = C'(\tau)/\alpha(t)$$
$$= W \log(1 + \alpha(t)P(t)/NW) \text{ (bits/s)} \tag{16}$$

measured in the transmitter's frame. This equation is the starting point for most of the results in this paper.

We collect results. Transmitter A sends at power $P_A(t)$ during $0 \leq t \leq T_A$. His anticipated Doppler shift factor is $\alpha_A(t)$. Let N_A be the number of bits sent from A to B, and let E_A be the total energy expended. Then we obtain the fundamental communication formulas:

$$\int_0^{T_A} 1\, dt = T_A \qquad \int_0^{T_B} 1\, d\tau = T_B;$$
$$\int_0^{T_A} dt/\alpha_A(t) = T_B; \qquad \int_0^{T_B} d\tau/\alpha_B(\tau) = T_A;$$
$$\int_0^{T_A} P_A(t)\, dt = E_A \qquad \int_0^{T_B} P_B(t)\, d\tau = E_B;$$
$$\int_0^{T_A} W \log(1 + \alpha_A(t) P_A(t)/NW)\, dt = N_A;$$
$$\int_0^{T_B} W \log(1 + \alpha_B(\tau) P_B(\tau)/NW)\, d\tau = N_B. \tag{17}$$

The reader should be aware that $\alpha_A(t)$ and $\alpha_B(\tau)$ are derived from the trajectories and thus are not independent functions. If significant gravitational masses are present, then general relativity is needed to derive $\alpha_A(t)$ and $\alpha_B(\tau)$. Otherwise, special relativity is sufficient. In any case (17) is valid. We are especially interested in relations among T_A, T_B, E_A, E_B, N_A, N_B that are trajectory-independent, i.e., independent of $\alpha_A(t)$ and $\alpha_B(\tau)$.

We are now prepared to investigate the following communication problems:

1) a constant communication rate
$$C(t) = W\log(1+\alpha(t)P(t)/NW) = C_0;$$
2) a constant transmission power
$$P(t) = P_0, \qquad \text{for all t};$$
3) an energy constraint
$$\int P(t)\,dt = E.$$

VI. Constant-Rate Transmission Yields Asymmetry γ^2

A particularly striking trajectory-independent asymmetry in information flow occurs when we demand that the transmitter send at the constant rate
$$C_A = W\log(1+\alpha_A(t)P_A(t)/NW), \qquad \text{for } 0 \leq t \leq T_A. \tag{18}$$
We can motivate this problem by supposing that the transmitter is sending a real time record of his life processes. Inspection of (18) shows that the transmitter must vary his power $P_A(t)$ so that $P_A(t)\alpha_A(t) = P_A$, where P_A satisfies $C_A = W\log(1+P_A/NW)$. P_A is the power required to communicate at rate C_A between stationary terminals. Equations (17) now become
$$T_A = \int_0^{T_A} dt,$$
$$T_B = \int_0^{T_A} dt/\alpha_A(t),$$
$$E_A = \int_0^{T_A} P_A(t)\,dt = P_A\int_0^{T_A} dt/\alpha_A(t) = P_A T_B,$$
$$N_A = \int_0^{T_A} C_A\,dt = C_A T_A. \tag{19}$$
Proceeding with the same calculations when B is transmitting at constant rate C_B, we find
$$E_B = P_B\int_0^{T_B} d\tau/\alpha_B(\tau) = P_B T_A,$$
and
$$N_B = \int_0^{T_B} C_B\,d\tau = C_B T_B, \tag{20}$$
where P_B satisfies $C_B = W\log(1+P_B/NW)$.

Notice that B's energy E_B is proportional to A's time T_A and not to T_B. We now define the overall aging factor γ_0 to be the ratio of ages
$$\gamma_0 = T_A/T_B \tag{21}$$
at the end of the arbitrary journeys of A and B (subject only to the condition that A and B begin and end their journeys together). Then A has expended energy per bit transmitted
$$E_A/N_A = \frac{P_A T_B}{C_A T_A} = \frac{P_A}{C_A}\frac{1}{\gamma_0},$$
while B has expended energy per bit transmitted
$$E_B/N_B = \frac{P_B T_A}{C_B T_B} = \frac{P_B}{C_B}\gamma_0.$$
Thus the ratio of energies per bit transmitted for B and A is
$$(E_B/N_B)/(E_A/N_A) = \gamma_0^2(P_B/C_B)/(P_A/C_A), \tag{22}$$
which reduces to γ_0^2 if A and B transmit at the same rate. In fact, the entire trajectory dependence is contained in the factor γ_0^2. Setting $\gamma_0 = 1$ in (22) gives the correct result for communication betwen stationary terminals.

We conclude that the traveler who ages less (by the factor $1/\gamma_0$, with $\gamma_0 > 1$) requires γ_0^2 times as much energy per bit sent. The younger traveler has more difficulty communicating.

VII. Round Trip Average Capacity with Constant Transmitter Power

As seen in the preceding section, a constant transmission rate yields an inefficiency factor equal to the square of the age ratio. The younger traveler is less efficient. We now ask whether this general conclusion is robust, i.e., whether it is true for all reasonable communication constraints.

We are interested in particular in the most physical natural constraint that both transmitters send at constant power. With constant power transmission, it now becomes necessary to vary the instantaneous communication rate.

To establish the asymmetry in average capacities for constant power, we use special relativity to relate the anticipated Doppler shifts to the radial velocity component. We then use Jensen's inequality to bound the difference between the average capacities. We conclude that the traveler is less efficient.

Assume constant transmitter powers $P_A(t) = P_B(t) = P$ for A and B, and let N_A and N_B denote the number of bits sent. The ratio of transmitted energies is
$$E_B/E_A = (PT_B)/(PT_A) = 1/\gamma_0. \tag{23}$$
During the round trip, A sends at an average rate
$$\overline{C}_A = N_A/T_A = \frac{1}{T_A}\int_0^{T_A} W\log(1+\alpha_A(t)P/NW)\,dt. \tag{24}$$
Similarly, B sends a number of bits per unit time
$$\overline{C}_B = N_B/T_B = \frac{1}{T_B}\int_0^{T_B} W\log(1+\alpha_B(\tau)P/NW)\,d\tau \tag{25}$$
$$= \frac{1}{T_B}\int_0^{T_A} W\log(1+\alpha_B(\tau'(t))P/NW)(1/\alpha_A(t))\,dt. \tag{26}$$

The last equality uses a change of variables identical to that derived in Section V by a pulse-counting argument. Here $\tau'(t)$ is B's reception time in his frame of signals (or pulses) transmitted by A at time t.

If terminal A remains fixed while terminal B moves along an arbitrary trajectory in the absence of significant gravitational masses, we can use special relativity to show that the Doppler shift factors are

$$\alpha_A(t)=\gamma(\tau'(t))(1-\beta_r(\tau'(t))) \tag{27}$$

for signals transmitted by A, and

$$\alpha_B(\tau)=\left[\gamma(\tau)(1+\beta_r(\tau))\right]^{-1} \tag{28}$$

for signals transmitted by B where $\beta(\tau)c$ denotes B's velocity as measured in A's inertial frame at B's time τ, $\gamma(\tau)=(1-\beta^2(\tau))^{-1/2}$ denotes the instantaneous aging factor, and $\beta_r(\tau)$ denotes the radial component (in A's frame) of $\beta(\tau)$. Using (27) and (28) we have

$$\begin{aligned}\alpha_A(t)/\alpha_B(\tau'(t))&=\gamma^2(\tau'(t))(1-\beta_r^2(\tau'(t)))\\&=(1-\beta_r^2(\tau'(t)))/(1-\beta^2(\tau'(t))).\end{aligned}$$

Thus

$$\alpha_A(t)\geqslant\alpha_B(\tau'(t)), \tag{29}$$

with equality if and only if B's motion at time $\tau'(t)$ is purely radial in A's inertial frame.

Applying this result to (26), we have

$$N_B/T_B\leqslant\frac{1}{T_B}\int_0^{T_A}W\log(1+\alpha_A(t)P/NW)(1/\alpha_A(t))\,dt. \tag{30}$$

This yields the bound

$$\begin{aligned}&N_A/T_A-N_B/T_B\\&\geqslant\frac{1}{T_A}\int_0^{T_A}(1-\gamma_0/\alpha_A(t))W\log(1+\alpha_A(t)P/NW)\,dt\\&=\frac{1}{T_A}\int_0^{T_A}(1-\gamma_0 y)W\log(1+(P/NW)/y)\,dt,\end{aligned} \tag{31}$$

where $y=1/\alpha_A(t)$. The integrand can be shown to be a convex function of y. Thus, by Jensen's inequality,

$$N_A/T_A-N_B/T_B\geqslant(1-\gamma_0\bar{y})W\log(1+(P/NW)/\bar{y}),$$

where $\bar{y}=1/T_A\int_0^{T_A}dt/\alpha_A(t)=T_B/T_A=1/\gamma_0$. Consequently, $1-\gamma_0\bar{y}=0$, and

$$N_A/T_A\geqslant N_B/T_B. \tag{32}$$

Thus we can bound the asymmetry in efficiency:

$$\frac{E_B/N_B}{E_A/N_A}=\frac{PT_B/N_B}{PT_A/N_A}\geqslant 1. \tag{33}$$

We conclude that if A remains stationary, traveler B needs more energy per bit sent than A.

VIII. Transmission under a Round Trip Energy Constraint

In Section VII we analyzed constant power transmission. Another reasonable physical constraint on the transmission

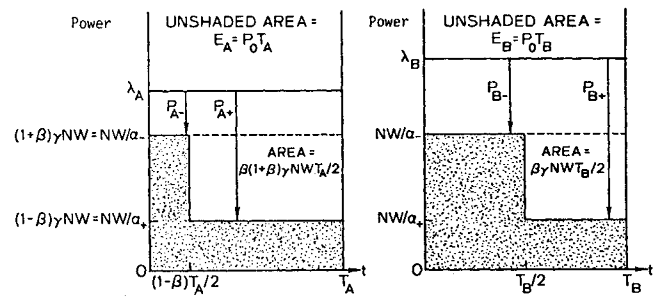

Fig. 5. Water-filling solution for the transmitted energy constraint.

power $P(t)$ is the roundtrip energy constraint

$$\int_0^T P(t)\,dt=E. \tag{34}$$

We can use straightforward variational techniques to maximize the average transmission capacity

$$\bar{C}=\frac{1}{T}\int_0^T W\log(1+\alpha(t)P(t)/(NW))\,dt. \tag{35}$$

The result is

$$P(t)=\max\left[0,\lambda-NW/\alpha(t)\right], \tag{36}$$

where λ is chosen to satisfy constraint (34). The form of $P(t)$ can be determined by a "water-filling" technique as shown in Fig. 5. (Compare Shannon's use of the same technique in the frequency domain [5].) First a graph of $NW/\alpha(t)$ from $t=0$ to $t=T$ is constructed. Then we can imagine pouring water over this graph and filling it with a total of E units of water to some level λ.

If the available energy ($E_A\triangleq P_AT_A$ for A's transmission) is sufficiently large, the water filling solution will give $\lambda>\max NW/\alpha_A(t)$. Thus we have $P_A(t)>0$ for all t and can then reduce (36) to

$$P_A(t)=\lambda-NW/\alpha_A(t). \tag{37}$$

Since $\int_0^{T_A}NW/\alpha_A(t)\,dt=NWT_B$, we obtain $E_A=\int P_A(t)\,dt=\lambda T_A-NWT_B$. Solving for λ and substituting in (37), we obtain

$$P_A(t)=P_A+NW(T_B/T_A-1/\alpha_A(t)). \tag{38}$$

We see immediately that (38) gives the correct average power P_A, since the quantity in parentheses integrates to zero. The condition that $P_A(t)>0$ reduces to

$$1/\alpha_A(t)<T_B/T_A+P_A/NW, \qquad \text{for all } t. \tag{39}$$

Equation (27) is valid if and only if condition (39) is satisfied. Otherwise the form of λ is more complicated, and $P_A(t)=0$ for some interval of time.

We conjecture that when terminal A is fixed, traveler B needs more energy per bit sent under the round trip energy constraint. The pointwise result

$$1/\alpha_A(t)\leqslant 1/\alpha_B(\tau'(t)) \tag{40}$$

and the water-filling or "effective noise" interpretation of Fig. 5 may be useful tools in proving the asymmetry. It can be seen by inspection that the circular trajectory problem of Section IV and the twin problem of Section II as discussed in Section IX have the desired asymmetry under the roundtrip energy constraint.

IX. Examples for the Twin Problem

In the twin problem of Section II, traveler B needs more energy per bit sent (although not always by the factor γ^2) under any one of the transmitter constraints: 1) constant rate; 2) constant power; and 3) total energy. We will specialize the results of the previous sections to the twin problem in order to obtain explicit expressions for these asymmetries.

A. Time Dependence of the Doppler Shift Factors

To analyze the problem of communication between A and B, we must first compute the Doppler shift factors for transmitted signals. Setting

$$\alpha_- = (1-\beta)\gamma = ((1-\beta)/(1+\beta))^{1/2}, \tag{41}$$

and

$$\alpha_+ = (1+\beta)\gamma = ((1+\beta)/(1-\beta))^{1/2}, \tag{42}$$

and referring to (2), (3), and Fig. 2, we see that A's transmission Doppler factor is

$$\alpha_A(t) = \begin{cases} \alpha_-, & \text{for } 0 < t < (1-\beta)T_A/2, \\ \alpha_+, & \text{for } (1-\beta)T_A/2 < t < T_A. \end{cases} \tag{43}$$

Since B spends equal time on each leg of the trip, B's transmission Doppler shift factor is

$$\alpha_B(\tau) = \begin{cases} \alpha_-, & \text{for } 0 < \tau < T_B/2, \\ \alpha_+, & \text{for } T_B/2 < \tau < T_B. \end{cases} \tag{44}$$

B. Constant Rate Transmission

If A and B both wish to transmit reliably to each other at the constant transmission rate R_0, we know from Section VI that an asymmetry in efficiency of γ^2 will result. In the twin example in Section II, it was shown that the outgoing Doppler shift is $1/2$ and the incoming Doppler shift is 2. The expended energies are $E_A = 8P_0$, and $E_B = 10P_0$, with a resulting energy ratio $E_B/E_A = 5/4 = \gamma$. The number of bits transmitted are $N_A = 10R_0$, and $N_B = 8R_0$. Thus the inefficiency ratio is $(E_B/N_B)/(E_A/N_A) = (10P_0/(8R_0))/(8P_0/(10R_0)) = 25/16 = \gamma^2$.

C. Constant Power Transmission

We recall from Section VII that the desired asymmetry for constant power transmission has been proved for general round trips. Here we restrict attention to the twin problem of Section II to obtain an explicit answer. Interestingly, the asymmetry is now bounded by a factor of two for all velocities.

To simplify the analysis and the result, we also make the infinite bandwidth assumption. Under this condition, the transmission capacities become

$$\begin{aligned} C_A(t) &= \lim_{W\to\infty} W\log(1+\alpha_A(t)P_A/(NW)) \\ &= \alpha_A(t)(P_A/N)\log e, \end{aligned}$$

and

$$C_B(\tau) = \alpha_B(\tau)(P_B/N)\log e.$$

If A transmits at this time-varying rate for time T_A while B leaves and returns at velocity $v = \beta c$, the total number of bits transmitted by A is

$$\begin{aligned} N_A &= \int_0^{T_A} C_A(t)\,dt = (P_A/N)\log e\int_0^{T_A}\alpha_A(t)\,dt \\ &= \left[\alpha_-(1-\beta)T_A/2 + \alpha_+(1+\beta)T_A/2\right](P_A/N)\log e \\ &= \left[(1-\beta)^2\gamma/2 + (1+\beta)^2\gamma/2\right](P_AT_A/N)\log e \\ &= (1+\beta^2)\gamma(P_AT_A/N)\log e. \end{aligned} \tag{45}$$

We can derive an analogous result for B

$$\begin{aligned} N_B &= (P_B/N)\log e\int_0^{T_B}\alpha_B(\tau)\,d\tau \\ &= (P_B/N)\log e(T_A/2\gamma)(\alpha_- + \alpha_+) \\ &= (P_BT_A/N)\log e. \end{aligned} \tag{46}$$

The ratio of required energies is

$$E_B/E_A = P_BT_B/(P_AT_A) = (1/\gamma)(P_B/P_A). \tag{47}$$

Equations (45)–(47) yield the ratio of energies per bit transmitted

$$(E_B/N_B)/(E_A/N_A) = (1+\beta^2) = \gamma^2(1-\beta^4). \tag{48}$$

Note that the asymmetry $(1+\beta^2)$, unlike γ^2, is bounded by two for all $-1 \leq \beta \leq 1$. We can see that for communication in the linear round trip problem, constant-power transmission does not yield results as simple as those for constant-rate transmission. Nevertheless, the relative efficiency of the traveler is strictly less than one in both cases.

D. Transmission Under an Energy Constraint

In Section VIII the asymmetry result under an energy constraint was not proved for arbitrary round trips. We now show that for the linear round trip, the traveler is less efficient under an energy constraint.

Let A and B vary their transmission power under the energy constraints

$$\begin{aligned} E_A &= \int_0^{T_A} P_A(t)\,dt = P_0T_A, \\ E_B &= \int_0^{T_B} P_B(t)\,dt = P_0T_B. \end{aligned}$$

Section VIII gives a water-filling solution for the optimal form of $P(t)$. The water-filling solutions for $P_A(t)$ and $P_B(t)$ are shown in Fig. 6.

If $P_0/NW > \beta(1+\beta)\gamma/2$, both transmission problems have nontrivial solutions. Both A and B transmit with positive power for the entire trip duration. From (43) and (44) we have $\overline{\log\alpha_A} = \beta\log(1+\beta)\gamma$ and $\overline{\log\alpha_B} = 0$. Using

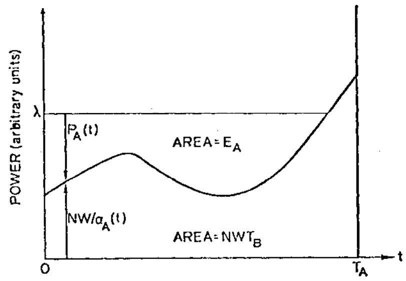

Fig. 6. Water-filling solutions for transmission power in the twin problem.

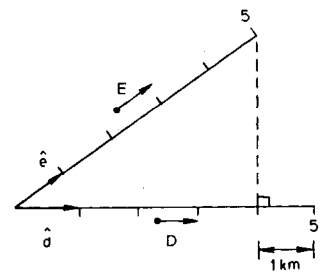

Fig. 7. D and E set off in different directions.

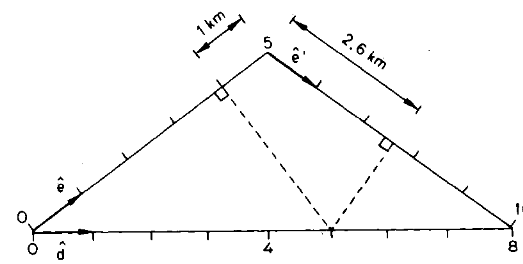

Fig. 8. D jumps ahead as E turns back toward D.

these results in (35) and (39), we find that A's average transmission capacity is

$$\bar{C}_A = W\log(1/\gamma + P_0/NW) + W\beta\log(1+\beta)\gamma. \quad (49)$$

Similarly, B's average transmission capacity is

$$\overline{C_B} = W\log(\gamma + P_0/NW). \quad (50)$$

Although the ratio of the average rates has no simple form, it is simple to show that the traveler once again needs more energy per bit sent, i.e.,

$$\frac{E_B/N_B}{E_A/N_A} = \bar{C}_A/\bar{C}_B > 1.$$

X. Summary and Conclusions

The main conclusion of this work is that if terminals A and B transmit at the same constant rate and bandwidth, the asymmetry in efficiency is equal to the square of the relative aging factor γ_0. The traveler who ages less needs more energy by the factor γ_0 and more energy per bit sent by the factor γ_0^2. This result is independent of acceleration and gravitational fields.

Instead of demanding constant transmission rates and comparing the required energies for A and B, we can demand fixed transmission powers and compare the achievable average rates. When one terminal is fixed, the traveler again is less efficient.

The analysis here supports the conjecture that under any symmetric transmission constraint, the younger traveler needs more energy per bit sent.

Appendix
An Analogy to the "Twin Paradox"

The following simple analogy replaces the space–time coordinates and relativistic velocities of the twin problem by two-dimensional space and ordinary velocities. It illustrates how turning can destroy symmetry, so that one cannot interchange the observations of A and B in the twin problem.

We consider two travelers, D and E, in two-dimensional space. They start together moving in different directions represented by unit vectors $\hat{d}$ and $\hat{e}$, such that the direction cosine is $\hat{d}\cdot\hat{e} = 4/5$ (see Fig. 7).

Each traveler moves at the constant speed of 1 km/h. At the end of five hours, D observes that E is 1 km behind him, since D measures progress along the $\hat{d}$ direction. Similarly, E observes that D is 1 km behind after 5 hours.

Now suppose that E turns back toward D's path as shown in Fig. 8.

First we consider D's point of view. Traveler E continues to fall behind D at 0.2 km/h. The result is that E arrives at the intersection point with D's path after a total of 10 hours, 2 hours after D arrives.

The situation from E's point of view is not as simple. As E turns, changing his distance measurement reference direction from $\hat{e}$ to $\hat{e}'$, traveler D appears to jump from the position 1 km behind E to a position 2.6 km ahead of E. This apparent jump is solely the result of E's coordinate change. After E's turn, traveler D again appears to fall back at 0.2 km/h. After a total of 8 hours, D reaches the intersection point, 2 hours (and 2 km) ahead of E.

The twin problem of Section II contains a discontinuity of apparent coordinates similar to the one illustrated above due to a coordinate change when traveler B turns. Failure to consider such discontinuous jumps when dealing with noninertial coordinates can easily lead to an apparent paradox.

References

[1] B. M. Oliver, "The view from the starship bridge and other observations," *IEEE Spectrum*, pp. 86–92, Jan. 1964.
[2] C. G. Darwin, "The clock paradox in relativity," *Nature*, vol. 180, pp. 976–977, November 1957.
[3] H. Bondi, "The space traveler's youth," *Discovery*, vol. 18, pp. 505–510, Dec. 1957.
[4] L. D. Landau and E. M. Lifshitz, *The Theory of Fields.* New York: Pergamon, 1971, pp. 111–112.
[5] C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, pp. 10–21, Jan. 1949.