Asynchronous Multiple-Access Channel Capacity

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Abstract—**The capacity region for the discrete memoryless multiple-access channel without time synchronization at the transmitters and receivers is shown to be the same as the known capacity region for the ordinary multiple-access channel. The proof utilizes time sharing of two optimal codes for the ordinary multiple-access channel and uses maximum likelihood decoding over shifts of the hypothesized transmitter words.**

Introduction

A TWO-USER discrete memoryless multiple-access channel $\{\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{Y}, p(y|x_1, x_2)\}$ has two senders x_1 and x_2, and a receiver y. When two users are attempting to use the same channel, there are two kinds of cooperation that make physical sense. The first is a strategic cooperation—both the senders and the receiver agree on the code books that will be used. This is the usual assumption for the Shannon channel.

The second possible cooperation occurs when the independent messages are actually sent. If both senders are aware of each other's messages W_1 and W_2 at the beginning of the transmission, then they can send at respective rates R_1 and R_2 by using the channel cooperatively as an ordinary 1-sender Shannon channel with capacity

$$R_1 + R_2 \leq C = \max_{p(x_1, x_2)} I(X_1, X_2; Y). \tag{1}$$

However, it is more common that W_1 is known only to x_1 and W_2 only to x_2, thus allowing only convex combinations of rates (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2), \\ R_2 &\leq I(X_2; Y|X_1), \\ R_1 + R_2 &\leq I(X_1, X_2; Y), \end{aligned} \tag{2}$$

for $p(x_1, x_2) = p(x_1)p(x_2)$. This independent-user region is the multiple-access channel capacity found by Ahlswede [1] and Liao [2]. We shall only be concerned with the independent user capacity of (2).

Manuscript received July 3, 1978; revised August 21, 1980. This work was supported by NSF Grant ECS78-23334, Stanford Research Institute ARPA Contract DAHC-15-C-0187, and at the Jet Propulsion Laboratory of the California Institute of Technology by the National Aeronautics and Space Administration under Contract N157-100.

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Returning now to the strategic cooperation used to derive (1) and (2), we see that implicit use is made of time synchronization. Even simple time sharing, in which each sender is quiet while the other sends, requires a common time base. What happens to the capacity region in (2) when there is a time uncertainty for the users and the receiver? Clearly new code books may have to be constructed. Moreover, the interference of codewords from the users' respective code books cannot now be cooperatively allowed for. For example, the strategic cooperation by time sharing may be ruined by unavoidable overlapping of the transmission periods. Finally, even the receiver must revise his decoding strategy in order to look for the joint transmissions with arbitrary time shifts.

This paper would not be necessary if the union of the regions given in (2) were convex, but Bierbaum and Wallmeier [3] have found an example demonstrating that the union is not convex. We shall show that the capacity region is unaffected by lack of synchronization.

I. Definitions and Review of Multiple-Access Capacity

An $((M_1, M_2), n, P_n)$ *code* for the (discrete memoryless) multiple-access channel $\{\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{Y}, p(y|x_1, x_2)\}$ is a pair of maps x_1: $\{1,2,\cdots,M_1\} \to \mathcal{X}_1^n$, x_2: $\{1,2,\cdots,M_2\} \to \mathcal{X}_2^n$, and a map g: $\mathcal{Y}^n \to \{1,2,\cdots,M_1\} \times \{1,2,\cdots,M_2\}$. The probability of error P_n of this code is defined under the assumption that the indices I and J are drawn independently according to a uniform distribution. Thus

$$\begin{aligned} P_n &= P\{g(Y^n) \neq (I, J)\} \\ &= \frac{1}{M_1 M_2} \sum_{i,j} P\{g(Y^n) \neq (i, j)|I = i, J = j\}, \end{aligned} \tag{3}$$

where

$$p(y^n|i, j) = \prod_{k=1}^{n} p(y_k|x_{1k}(i), x_{2k}(j)). \tag{4}$$

A pair of rates (R_1, R_2) is said to be *achievable* if there exists a sequence of $((2^{nR_1}, 2^{nR_2}), n, P_n)$ codes with $P_n \to 0$. The *capacity region* C^* is the closure of the set of all achievable rates (R_1, R_2).

Theorem 1 establishes the capacity region C^*. An alternative proof to those in [1], [2] will be given as a model for the subsequent proof that C^* remains unchanged when there is no time reference, i.e., no synchronization.

0018-9448/81/0700-0409$00.75

For the multiple-access channel without synchronization, the error criterion is more stringent—the decoding must be correct for all shifts of $\boldsymbol{x}_i(i)$ and $\boldsymbol{x}_2(j)$, where the shifts are imbedded in arbitrary sequences from $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. This is the appropriate error criterion. Imbedding the codewords in arbitrary sequences of input symbols yields the most difficult recognition task for the receiver. The arbitrary part of the transmission can be chosen to mimic some other codeword.

We shall first define a code for a channel in which both shifts are known not to exceed d. Then a rate region independent of d will be defined.

An $((M_1, M_2), n, d, P_n)$ *code for a multiple-access channel with maximum relative delay* d is a pair of maps

$$\boldsymbol{x}_1\colon \{1,2,\cdots,M_1\} \to \mathcal{X}_1^n,$$
$$\boldsymbol{x}_2\colon \{1,2,\cdots,M_2\} \to \mathcal{X}_2^n, \tag{5}$$

and a map

$$g\colon \mathcal{Y}^{n+d} \to \{1,2,\cdots,M_1\} \times \{1,2,\cdots,M_2\}.$$

The probability of error of this code is

$$P_n = \frac{1}{M_1 M_2} \cdot \sum_{i,j} \max_{\substack{1\le d_1 \le d\\ 1\le d_2\le d\\ \tilde{\boldsymbol{x}}_1, \tilde{\tilde{\boldsymbol{x}}}_1, \tilde{\boldsymbol{x}}_2, \tilde{\tilde{\boldsymbol{x}}}_2}} P\{g(Y^{n+d}) \neq (i,j) | d_1, d_2, I=i, J=j\}, \tag{6}$$

where

$$p(y^{n+d}|i,j,d_1,d_2) = \prod_{k=1}^{n+d} p(y_k | x_{1,k-d_1}(i), x_{2,k-d_2}(j)),$$

and the first d_1 symbols and last $d-d_1$ symbols of $\boldsymbol{x}_1(i)$ are arbitrary sequences $\tilde{\boldsymbol{x}}_1 \in \mathcal{X}^{d_1}$, $\tilde{\tilde{\boldsymbol{x}}}_1 \in \mathcal{X}_1^{d-d_1}$ (with a parallel condition for d_2 and $\boldsymbol{x}_2(j)$).

We shall explain this definition. Here $\mathcal{X}_1$ and $\mathcal{X}_2$ are the channel input alphabets for the two users. The integers M_1 and M_2 are equal to 2^{nR_1} and 2^{nR_2}, where R_1 and R_2 are the rates of the two users. The mappings $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the encoding mappings which produce codewords of length n from the index message. The probability $p(y|x_1, x_2)$ is the transition probability for the channel, which specifies the probability of output y when x_1 and x_2 are the inputs of the two users.

The set $\mathcal{Y}$ is the output alphabet. We assume that there is one receiver trying to reconstruct the two inputs.

The relative delay d is the maximum amount by which the two messages are assumed to be out of synchronization relative to a known or prearranged time. For many applications, we can assume such a d exists. The map g is the decoding operation which can commence when $n+d$ symbols have been received.

The error probability P_n is an average word error probability. It is defined under the assumption that all M_1M_2 pairs (i, j) of inputs are equally likely. For each pair, the term to be averaged is the maximum over d of the probability of incorrect decoding of i or j or both, given that the delay of the ith user relative to the prearranged start of the codeword block is d_i. Here d_1 and d_2 are constrained to be at most the maximum relative delay d. The maximum is also taken over all possible "head" and "tail" sequences from prior or subsequent codewords. These intrude into the $n+d$ symbols observed by the receiver to pad out the length from n, the code block length, to $n+d$, the block length plus the maximum relative delay.

A pair of rates (R_1, R_2) will be said to be *achievable* if there exists a sequence $d_n \to \infty$ and a sequence of $((2^{nR_1}, 2^{nR_2}), n, d_n, P_n)$ codes with $P_n \to 0$ as $n \to \infty$. This means that we can guarantee arbitrarily low word error probabilities at these rates, no matter how large the relative delay bound may be, as long as we know a bound for the relative delay. A stronger sense of achievability independent of knowledge of a relative delay bound will be demonstrated in Section III. Finally the *capacity region* C is as usual the closure of the set of achievable rates.

Example (The Binary Erasure Multiple-Access Channel): Let $\mathcal{X}_1 = \mathcal{X}_2 = \{0,1\}$, $\mathcal{Y} = \{0,1,2\}$. Consider the deterministic channel $y = x_1 + x_2$. For obvious reasons, $y=1$ is called an *erasure*. The capacity region (see Theorem 1) is given by $R_1 \le 1$, $R_2 \le 1$, $R_1 + R_2 \le 1.5$. See Gaarder and Wolf [4].

A new proof of the direct part of the following theorem will be given and used as the model for the proof in the next section. This is the known result for the synchronized multiple-access channel.

Theorem 1 (Ahlswede, Liao): The capacity region of the multiple-access channel is given by the set of all rates in the convex closure of the set of rates (R_1, R_2) satisfying

$$\begin{aligned} R_1 &< I(X_1; Y|X_2),\\ R_2 &< I(X_2; Y|X_1),\\ R_1 + R_2 &< I(X_1, X_2; Y), \end{aligned} \tag{7}$$

for some $p(x_1, x_2, y) = p(x_1)p(x_2)p(y|x_1, x_2)$.

Proof: Fix $p(x_1)$, $p(x_2)$. Let $p(x_1, x_2) = p(x_1)p(x_2)$. Choose a random code of 2^{nR_1} words $\boldsymbol{x}_1 \in \mathcal{X}_1^n$ independent identically distributed (i.i.d.) $\sim \prod_{i=1}^n p(x_{1i})$, and independently choose 2^{nR_2} words $\boldsymbol{x}_2 \in \mathcal{X}_2^n$ i.i.d. $\sim \prod_{i=1}^n p(x_{2i})$.

Let S denote a subset of $\{X_1, X_2, Y\}$ and let s be the associated set of n-sequences in $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}\}$. Define the set A_ϵ^n of ϵ-typical $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y})$ triples by

$$A_\epsilon^n = \Big\{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}^n\colon \cdot \Big| -\frac{1}{n}\log p(\boldsymbol{s}) - H(S)\Big| < \epsilon, \quad \text{for all } S \subseteq \{X_1, X_2, Y\}\Big\}. \tag{8}$$

See [5] for more detail on joint typicality. We note that (X_{1i}, X_{2i}, Y_i) are i.i.d. $\sim p(x_1)p(x_2)p(y|x_1,x_2)$. Thus by the law of large numbers, $-(1/n)\log p(\cdot) \to H(\cdot)$ with probability one, for each of the eight constraints in (8). Hence there exists an n_0 such that, for $n \ge n_0$, $P(A_\epsilon^n) \ge 1 - \epsilon$. Also, it can be seen from (8) that the cardinality of the set A_ϵ^n is bounded by

$$\|A_\epsilon^n\| \le 2^{n(H(X_1, X_2, Y)+\epsilon)}. \tag{9}$$

For decoding, given y, simply choose the pair (i, j) such that

$$(x_1(i), x_2(j), y) \in A_\epsilon^n, \tag{10}$$

if such an $(i, j) \in \{1, 2, \cdots, 2^{nR_1}\} \times \{1, 2, \cdots, 2^{nR_2}\}$ exists and is unique—otherwise declare an error.

By the symmetry of the random code construction, the probability of error (averaged over the random code) is independent of the index (i, j) sent. Thus without loss of generality assume that $(i, j) = (1, 1)$ is sent. Consider the events

$$E_{ij} = \{(X_1(i), X_2(j), Y) \in A_\epsilon^n\}. \tag{11}$$

Then by the union, bound

$$P_n = P\left(E_{11}^c \cup \left(\bigcup_{(i,j)\neq(1,1)} E_{ij}\right)\right) \le P(E_{11}^c) + \sum_{i\neq 1} P(E_{i1}) + \sum_{j\neq 1} P(E_{1j}) + \sum_{i\neq 1, j\neq 1} P(E_{ij}), \tag{12}$$

where E^c denotes the complement of the event E. Assume henceforth that $n \ge n_0$. Thus $P(E_{11}^c) \le \epsilon$.

Next, for $i \neq 1$

$$\begin{aligned} P(E_{i1}) &= \sum_{(x_1, x_2, y)\in A} p(x_1, x_2, y) \\ &= \sum_A p(x_1) p(x_2, y) \\ &\le \|A\| 2^{-n(H(X_1)-\epsilon)} 2^{-n(H(X_2, Y)-\epsilon)} \\ &\le 2^{-n(I(X_1; X_2, Y)-3\epsilon)} \\ &= 2^{-n(I(X_1; Y|X_2)-3\epsilon)}. \end{aligned} \tag{13}$$

where the first equality is by definition of E_{i1}, the second from the independence of X_1 from (X_2, Y) which follows from $i \neq 1$, the third from the definition of $A = A_\epsilon^n$, the fourth from (9), and the last from the independence of X_1 and X_2 and the identity $I(X_1; Y|X_2) = I(X_1; Y, X_2) - I(X_1; X_2)$.

Similarly for $j \neq 1$,

$$P(E_{1j}) \le 2^{-n(I(X_1; Y|X_2)-3\epsilon)}, \tag{14}$$

and for $i \neq 1, j \neq 1$,

$$P(E_{ij}) \le 2^{-n(I(X_1, X_2; Y)-3\epsilon)}. \tag{15}$$

Hence, returning to (12) we have

$$\begin{aligned} P_n \le \epsilon &+ 2^{nR_1} 2^{-n(I(X_1; Y|X_2)-3\epsilon)} \\ &+ 2^{nR_2} 2^{-n(I(X_2; Y|X_1)-3\epsilon)} \\ &+ 2^{n(R_1+R_2)} 2^{-n(I(X_1, X_2; Y)-3\epsilon)}. \end{aligned} \tag{16}$$

Thus for $\epsilon > 0$ sufficiently small, the conditions of the theorem cause each term to tend to zero as $n \to \infty$.

Time sharing (allowable because of time synchronization) achieves any (R_1, R_2) in the convex hull, and the direct part of the theorem is proved.

The converse is well-known and will not be repeated.

II. Capacity Without Synchronization

We shall show that the same sequence of random codes causing $P_n \to 0$ in the previous section will also cause $P_n \to 0$ if the words are not synchronized. The construction is a form of time sharing that works in the absence of synchronization. We thus obtain the same capacity region as if we had time synchronization between the two users.

Let d_1 and d_2 be fixed nonnegative integers unknown to the receiver. Sender k, $k = 1, 2$, sends an arbitrary sequence of d_k symbols from alphabet $\mathcal{X}_k$ follows by codeword $x_k(i_k)$ of block length n, followed by more arbitrary symbols from $\mathcal{X}_k$.

We shall first assume that the receiver knows a bound d on the delays, i.e., $d_1, d_2 \le d$. Hence the receiver inspects $y \in \mathcal{Y}^{n+d}$ for the presence of $x_1(i_1)$, $x_2(i_2)$ imbedded with arbitrary shifts in arbitrary transmitter sequences. Later we shall remove the receiver's knowledge of d.

In general, for a multiple-access channel without synchronization, it is necessary to form the convex combination of rate points (R_1, R_2) and (R_1', R_2') to achieve the point $(R_1^0, R_2^0) = \alpha(R_1, R_2) + \bar{\alpha}(R_1', R_2')$, $0 < \alpha < 1$, $\bar{\alpha} = 1 - \alpha$. This time sharing is necessitated by the possible lack of convexity of the union of the set of (R_1, R_2) satisfying (2). Let $p_1(x_1, x_2) = p_1(x_1)p_1(x_2)$ induce a region given in (2) that has (R_1, R_2) as an extreme point, and let $p_2(x_1, x_2) = p_2(x_1)p_2(x_2)$ induce a region that has (R_1', R_2') as an extreme point. Using the random coding procedure of Section I, generate a random $((2^{\alpha n R_1}, 2^{\alpha n R_2}), \alpha n)$ code according to p_1 and a random $((2^{\bar{\alpha} n R_1'}, 2^{\bar{\alpha} n R_2'}), \bar{\alpha} n)$ code according to p_2. The sent and received sequences will then appear as in Fig. 1 for some d_1, d_2.

The crucial point is that $x_1(i)$ and $x_2(j)$ will have substantial overlap, and the region of overlap can be prespecified. This overlap will be sufficient to detect typicality and reject atypicality. In fact inspection of Fig. 2 shows the overlap regions to be of lengths at least $\alpha n - d$ and $\bar{\alpha} n - d$, independent of d_1, d_2 for $0 \le d_1, d_2 \le d$.

The decoding is as follows. We must look for codewords under all possible shifts, up to the maximum delay d. Let the maximal delay d be fixed and known. Let τ^k denote a cyclic shift k units to the right of a given $(n + d)$-tuple. Let $W_{\alpha n - d} = W$ denote the window function that inspects only the values of the vector in the first window specified in Fig. 2. Note that no dummy symbols could be in the window. Define the set of $p_1(x_1, x_2, y)$-typical sequences A_ϵ^1 *only* over the $(\alpha n - d)$ coordinates specified in the first window in Fig. 2. Thus, for example, there are at most $2^{(\alpha n - d)(H_1(X_1, X_2, Y)+\epsilon)}$ jointly typical triples in the first window, and each triple in A_ϵ^1 has probability $\le 2^{-(\alpha n - d)(H_1(X_1, X_2, Y)-\epsilon)}$. The second window will be treated by similar techniques.

Again, without loss of generality assume that $(1, 1)$ was sent and that the delays were d_1, d_2, where $1 \le d_1, d_2 \le d$. To place an upper bound on the probability of error P_n^1 in the first code for $1 \le k_1, k_2 \le d$, $i \in \{1, 2, \cdots, 2^{nR_1}\}$, $j \in \{1, 2, \cdots, 2^{nR_2}\}$, define $E_{k_1, k_2, i, j}$ to be the event that $(W(\tau^{k_1} X_1(i)), W(\tau^{k_2} X_2(j)), W(Y)) \in A_\epsilon$. That is, the event

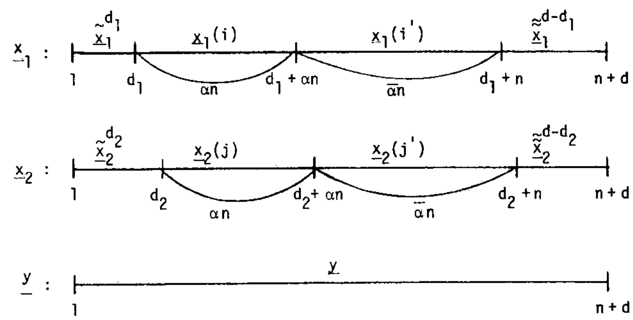

Fig. 1. Shifted codewords.

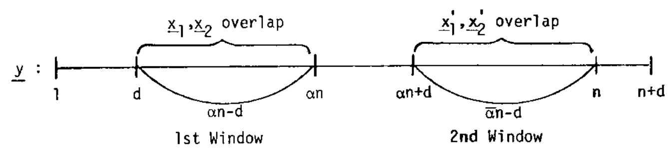

Fig. 2. Overlap regions.

$E_{k_1,k_2,i,j}$ occurs if the k_1 shift of transmitter ones's ith codeword $X_1(i)$ and the k_2 shift of transmitter two's jth codeword $X_2(j)$ are seen to be jointly typical with Y in the first window.

Since we have assumed (1, 1) was sent with delay (d_1, d_2), an error will occur if $\cup_{k_1,k_2} E_{k_1,k_2,1,1}$ does not occur (i.e., (1, 1) is not a candidate) or if for some $1 \le k_1, k_2 \le d$ and some $(i, j) \ne (1, 1)$, $E_{k_1,k_2,i,j}$ does occur (an incorrect candidate). Observe that

$$\left(\bigcup_{k_1,k_2} E_{k_1,k_2,1,1}\right)^c = \bigcap_{k_1,k_2} E^c_{k_1,k_2,1,1} \subseteq E^c_{d_1,d_2,1,1}. \quad (17)$$

Thus

$$P_n^1 \le P\left(E^c_{d_1,d_2,1,1}\right) + P\left(\bigcup_{\substack{1\le k_1,k_2\le d\\(i,j)\ne(1,1)}} E_{k_1,k_2,i,j}\right). \quad (18)$$

The first term can be made less than or equal to ϵ for n sufficiently large by the asymptotic equipartition property. Expanding and bounding the second term we have

$$P_n^1 \le \epsilon + \sum_{\substack{1\le k_1,k_2\le d\\ i=1, j\ne 1}} P\{E_{k_1,k_2,i,j}\} + \sum_{\substack{k_1,k_2\\ i\ne 1, j=1}} P(\cdot) + \sum_{\substack{k_1,k_2\\ i\ne 1, j\ne 1}} P(\cdot). \quad (19)$$

Treating the last summation first, we note the following.

i) There are $d^2(2^{(\alpha n-d)R_1}-1)(2^{(\alpha n-d)R_2}-1) \le d^2 2^{(\alpha n-d)(R_1+R_2)}$ terms in the sum. (20)

ii) Each term is upper bounded by

$$\begin{aligned} P\{E_{k_1,k_2,i,j}\} &\le \|A_\epsilon^1\| 2^{-(\alpha n-d)(H_1(X_1)-\epsilon)} \\ &\quad \cdot 2^{-(\alpha n-d)(H_1(X_2)-\epsilon)} 2^{-(\alpha n-d)(H_1(Y)-\epsilon)} \\ &\le 2^{-(\alpha n-d)(I_1(X_1,X_2;Y)-4\epsilon)}, \end{aligned} \quad (21)$$

where we have used

$$\|A_\epsilon^1\| \le 2^{-(\alpha n-d)(H_1(X_1,X_2,Y)-\epsilon)}. \quad (22)$$

Thus the last sum in (19) tends to zero if

$$\frac{1}{n}\left(\log d^2 + (\alpha n-d)(R_1+R_2) - (\alpha n-d)(I_1(X_1,X_2;Y)-4\epsilon)\right) < 0, \quad (23)$$

or equivalently,

$$R_1+R_2 < I_1(X_1,X_2;Y) - \frac{2\log d}{\alpha n-d} - 4\epsilon. \quad (24)$$

Similarly treating the first two terms, we see that these terms approach zero if

$$\begin{aligned} R_1 &< I_1(X_1;Y|X_2) - \frac{2\log d}{n} - 4\epsilon, \\ R_2 &< I_1(X_2;Y|X_1) - \frac{2\log d}{n} - 4\epsilon. \end{aligned} \quad (25)$$

A similar calculation is made for the second window, at rates $\boldsymbol{R}' = (R_1', R_2')$, where the probability of error is P_n^2, and the typical set A_ϵ^2 is defined under $p_2(\cdot,\cdot,\cdot)$.

Finally $P_n \le P_n^1 + P_n^2$, and for every d and $\epsilon > 0$, n can be chosen so that $P_n \le \epsilon$. The rate pair $\boldsymbol{R}^0$ for such a code is

$$\begin{aligned} \boldsymbol{R}^0 &= (\alpha n\boldsymbol{R} + \bar{\alpha} n\boldsymbol{R}')/(n+d) \\ &= (n/n+d)(\alpha\boldsymbol{R} + \bar{\alpha}\boldsymbol{R}') \\ &\to \alpha\boldsymbol{R} + \bar{\alpha}\boldsymbol{R}', \qquad \text{as } n\to\infty, \end{aligned} \quad (26)$$

since d is fixed. Thus we have a proof that any rate point $\boldsymbol{R}^0$ in C^* can be achieved with probability of error $P_n \to 0$.

The generalization of this problem to the case of a continuous waveform channel without synchronization would allow us to answer the more realistic question where the oscillators which generate symbol timing at each transmitter have a random phase relationship. While we have not considered this generalization here, it is expected that straightforward extensions of the techniques of this paper to the continuous case will work. We expect no loss of capacity when the two transmitters do not have a common clock, even in the continuous case.

III. Elimination of Knowledge of Delay d

For known maximal synchronization delay d and desired probability of error ϵ, there exist block codes $\mathcal{C}_1, \mathcal{C}_2$ of block lengths $n(d,\epsilon)$ achieving any rate (R_1, R_2) in the capacity region and achieving average probability of error ϵ. However, if the true delay is greater than d, the probability of error may be high.

We overcome this problem by concatenating codes of increasing block lengths $n_1, n_2, \cdots$. The ith block code is designed to have rate (R_1, R_2) and probability of error ϵ_i for all delays $\le d_i$.

For a given (R_1, R_2) in the capacity region, choose $d_i \uparrow \infty$, and let $n_i \to \infty$ in such a manner that

$$\epsilon_i \to 0, \quad (27)$$

$$d_i/n_i \to 0, \quad (28)$$

and

$$n_i / \left(\sum_1^i n_j \right) \to 1. \tag{29}$$

Moreover, in the ith block with block length n_i, *retransmit* all of the bits from the previously sent blocks.

For any d, there exists an i_0 such that $d_i \geq d$ for $i \geq i_0$. Now $(n_i)(R_1, R_2)$ bits are received in $(n_1 + n_2 + \cdots + n_i + d_i)$ transmissions for an overall rate vector of

$$(R_1^*, R_2^*) = (R_1, R_2)\left(n_i / \sum_1^i n_j + d_i \right) \to (R_1, R_2), \tag{30}$$

as $i \to \infty$, by conditions (28) and (29). Thus no bits are lost and the achievable rates are not affected.

Finally, if we add the condition

$$\sum_{i=1}^{\infty} \epsilon_i < \infty, \tag{31}$$

to (27), (28), and (29), it follows from the Borel Cantelli lemma that with probability one, only a finite number of block decoding errors will be made. At that time all previous errors will have been corrected and no future errors will be made. The choice of block lengths n_i can be made in two interesting ways.

a) $n_i/(\Sigma_1^i n_j + d_i) \to 1$, with resulting overprints on the bits already received. The result is that any given bit will eventually be correct with probability one after a finite number of changes (overprints). The problem is that the decoding delays increase very fast, resulting in $\overline{\lim}\, R_n = C$ but $\underline{\lim}\, R_n = 0$.

b) $n_i \to \infty$, $n_{i+1}/(n_i + d_i) \to 1$. Now $\overline{\lim}\, R_n = \underline{\lim}\, R_n = C$. However, bits are no longer eventually correct with probability one. On the other hand, the expected proportion of bit errors in the first n transmissions tends to zero as $n \to \infty$.

This increasing block length construction may not be completely satisfactory, however. For no time T do we know that any bits will be correctly decoded at time T. The decoding delay has been allowed to grow to infinity. Most people would not accept such a communication system. There is thus a minor gap between the results of [6] and [7] and these results which can only be closed by further research.

We do, however, have a very precise result when the delay can be bounded in advance. This may be a reasonable assumption for actual channels.

Conclusion

We have proved that lack of synchronization does not reduce the capacity region for multiple-access channels. This is achieved by codes with block lengths long compared to the delay. There is reason to believe that codes with block lengths shorter than the delay cannot in general achieve capacity.

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