Evaluation of Signal-Plus-Noise Detection Error in an Envelope Detector with Logarithmic Compression

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***Abstract*—A correction factor is derived for the amplitude of the detected output of a modulated sinusoidal signal with added Gaussian noise, as processed by an envelope detector with logarithmic compression. Supporting experimental data are presented that were obtained using a typical system having such a detector.**

I. INTRODUCTION

Signal detection techniques serve to convert a signal into a relatively slowly varying voltage that is intelligible and useful. One price paid in this process is that no one signal detection technique will so convert all different signal waveforms without introducing some intrinsic amount of error, possibly a major one. For example, a root-mean-square (rms) detector will accurately represent power of the signal, but if peak amplitude of a short rectangular pulse of low duty cycle is desired, a correction factor must be applied to remove the error. This factor is a function of both the signal characteristics and the detector properties.

One very common and useful detection system is that whose envelope detects a received signal and then logarithmically compresses the result. Such a system is fast, stable, economical, and sensitive; has wide frequency and dynamic ranges, and provides a convenient output expressed in decibels. This detection system is used in a large number of commercially available spectrum analyzers and frequency-selective voltmeters. Here we will analyze the detection error caused by adding Gaussian noise to a modulated sinusoidal carrier. A typical problem in which this effect is encountered is the measurement of the ratio of carrier power to noise power density in a satellite communications link using an automatic spectrum analyzer.

II. FORMULATION OF THE PROBLEM

A rigorous generalization of the envelope function has been obtained in terms of Hilbert transforms [1]. If $h(t)$ is a real time series, its corresponding preenvelope (or predetection) function $z(t)$ is given by

$$z(t) = h(t) + j\hat{h}(t) \tag{1}$$

where $\hat{h}(t)$ is the Hilbert transform of $h(t)$. The envelope function is the absolute value of $z(t)$. It is useful to express the waveform power (averaged over a period short compared with envelope variation) in terms of the envelope function. This function is given [2] by

$$p(t) = \frac{|z(t)|^2}{2}. \tag{2}$$

The probability density function (PDF) for the envelope of Gaussian noise added to a signal composed of a sum of sinusoids is given by the Rice–Nakagami relation [3], [4],

$$P(x) = \begin{cases} \dfrac{x}{\psi_0} I_0\left[\dfrac{x|z(t)|}{\psi_0}\right] \exp\left[\dfrac{x^2 + |z(t)|^2}{-2\psi_0}\right], & x > 0 \quad \text{(3a)} \\ 0, & x < 0 \quad \text{(3b)} \end{cases}$$

where $P(x)\,dx$ is the probability that the random function falls in the interval $(x, x + dx)$; ψ_0 is the Gaussian noise power; and I_0 is the modified Bessel function of the first kind and order zero.

For convenience, $|z(t)|$ is written henceforth as $|z|$. Using (2), we find the signal-to-noise power ratio to be

$$r = \frac{|z|^2}{2\psi_0}. \tag{4}$$

If we let x represent the random variable of the detected signal, i.e., the envelope voltage, then the logarithmically compressed output is given by

$$y = \ln x. \tag{5}$$

Since y has a one-to-one correspondence with x, its PDF can be obtained [5] from the PDF of x as

$$P(y)\,dy = P(x)\,dx. \tag{6}$$

The observed average value of the output voltage is given by the first moment of $P(y)$. By multiplying the right side of (6) by $\ln x$ and integrating, the first moment of y can be written in the form

$$\langle y \rangle = \frac{e^{-r}}{2}\left\{\ln(2\psi_0)\int_0^\infty I_0(2\sqrt{wr})\,e^{-w}\,dw + \int_0^\infty \ln w I_0(2\sqrt{wr})\,e^{-w}\,dw\right\}, \tag{7}$$

where $w = x^2/2\psi_0$.

The total quasi-instantaneous detected power (before logarithmic compression) can be obtained from the second moment of $P(x)$, which integrates to

$$\langle x^2 \rangle = 2\left(\psi_0 + \frac{|z|^2}{2}\right). \tag{8}$$

By using this result, evaluating the integrals in (7) [2], and performing some manipulation, we arrive at the following expression

$$\ln\frac{\langle x^2 \rangle}{2} = (2\langle y \rangle - \ln 2) + [\gamma + \ln(1 + r) - \zeta(r)] \tag{9}$$

where γ is the Euler's constant = 0.5772156649, and

$$\zeta(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} r^n}{n n!}.$$

Each term of (9) is in nepers. The left side corresponds to the true signal-plus-noise power. The first term in parentheses on the right side is the power indicated by the detector, and the term in square brackets is the desired correction term $F(r)$, which depends only upon the signal-to-noise power ratio:

$$F(r) = \gamma + \ln(1 + r) - \zeta(r) \quad \text{(Np)}. \tag{10a}$$

This can also be written as

$$F(r) = \ln\left[\frac{1 + r}{r}\right] - E_1(r) \tag{10b}$$

where $E_1(r)$ is the real limit of the exponential integral [6].

The series $\zeta(r)$ is absolutely and uniformly convergent for finite r, and furthermore, may be truncated with any desired maximum error because it is alternating. Computation for very large r, however, may still prove difficult. The magnitude of the terms of the series will increase from r (for the first term) to the order of $e^r r^{-3/2}/\sqrt{2\pi}$ (for the $(r-2)$th or $(r-1)$th term) and only thereafter converge. Inasmuch as the computation process requires subtraction of successive terms, this could demand reten-

Manuscript received January 8, 1980. This work was supported in part by the U.S. Satellite Communications Agency.

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tion of an enormous number of significant figures for only modest accuracy in the result. Thus for computational ease, asymptotic approximations for large and small r can be obtained as

$$F(r) = \frac{1}{r}, \qquad r \gg 1, \tag{11}$$

and

$$F(r) = \gamma - \frac{r^2}{4} + \frac{5r^3}{18} - \cdots, \qquad r \ll 1. \tag{12}$$

To express this correction in decibels,

$$F_{\text{dB}}(r) = (10 \log_{10} e) F(r) \tag{13}$$

and as a multiplicative ratio correction, when the power is in absolute units,

$$\epsilon(r) = e^{F(r)}. \tag{14}$$

Alternative computational forms are available using polynomial expansions for the exponential integral form of $F(r)$ in (10b) [7].

III. Experimental Results

Commercial automatic spectrum analyzers are typical instruments using the detection system analyzed in this correspondence. Response data were obtained with one of the most advanced such instruments, using as input a variable noise source (solid-state diode followed by a calibrated attenuator) and a nominally calibrated signal generator. These data are shown in Fig. 1. The calculated curve (solid line) was obtained from (11) for $r > 10$, (12) for $r < 0.1$, and (10a) for the intermediate range $0.1 < r < 10$.

IV. Using the Correction Factor

In order to use the correction factor derived in this correspondence, it is necessary to make a pair of readings between which some known relation exists. Most commonly, this would be a measurement of an unknown signal plus received noise and a second measurement of the noise condition alone with the signal removed. With this pair of readings, the signal-to-noise power ratio of the first measurement is given by solving the implicit relationship

$$r = \left[\frac{P_{S+N}}{P_N}\right]\left[\frac{\epsilon(r)}{\epsilon(0)}\right] - 1 \tag{15}$$

for r, where the term in the first parentheses is the difference in decibels of the two readings converted to a power ratio and $\epsilon(0) = 1.781072$. Because $\epsilon(r)$ is a slowly varying function of r, two or three iterations will normally achieve convergence of the calculation.

Other reference combinations of signal and noise can be used, but the one described above is probably the simplest and most widely useful. It is apparent that using the system noise level as a reference serves two additional purposes: first, it enables an absolute level calibration to be obtained if a standard noise source is available; second, transference to the condition when the signal and noise powers are of comparable magnitudes places the least stringent requirements upon system linearity.

Acknowledgment

The authors are indebted to R. J. Matheson of the National Telecommunication and Information Agency's Institute for Telecommunication Sciences for the use of his automatic spectrum analyzer and for his help in setting up and conducting the experiment.

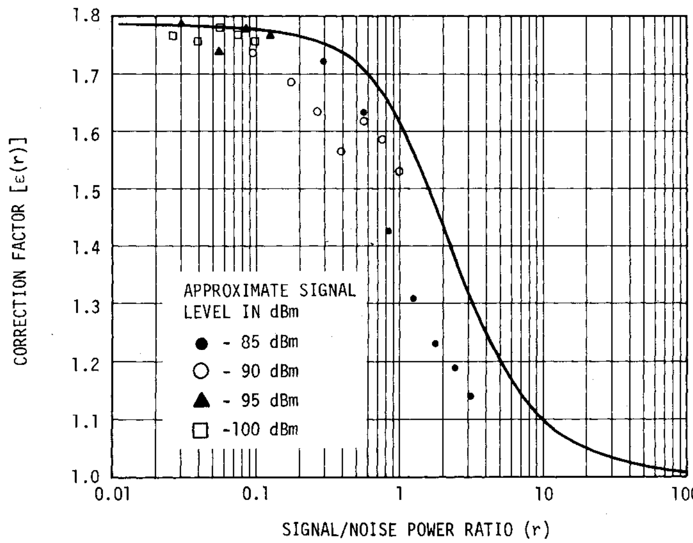

Fig. 1. Multiplicative correction error.

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On the Identifiability of Finite Mixtures of Distributions

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Abstract—**Finite mixtures of the following ten families of univariate distributions are shown to be identifiable: logarithmic series, discrete rectangular, rectangular, first law of Laplace, noncentral χ^2, logistic, generalized logistic, generalized hyperbolic-secant, inverse Gaussian, and random walk. A generalized version of a theorem given by Teicher is used to show that the finite mixtures of the following multivariate distributions are also identifiable: negative binomial, logarithmic series, Poisson, normal, inverse Gaussian, and random walk.**

Introduction

Let

$$\mathcal{F}_{n,m} = \{F(x;\alpha): x \in R^n, \alpha \in R_1^m\} \tag{1}$$

Manuscript received October 31, 1979; revised May 7, 1980. The multivariate findings were presented at the 5th International Congress for Statistics, Computer Science, Social and Demographic Research, Cairo, Egypt, March 29–April 3, 1980.

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0018-9448/81/0900-0664$00.75 ©1981 IEEE