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ASYMPTOTIC EFFICIENCIES OF TRUNCATED SEQUENTIAL TESTS

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Truncation of a sequential test with constant boundaries is considered for the problem of testing a location hypothesis, $f(x-\theta_0)$ versus $f(x-\theta_1)$. A test design procedure is developed by using bounds for the error probabilities under the hypothesis and alternative. By viewing the truncated sequential test as a mixture of a sequential probability ratio test and a fixed sample size test, its boundaries and truncation point can be obtained once the degree of mixture is specified. Asymptotically correct approximations for the

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operating characteristic function and the average sample number function of the resulting test are derived. Numerical results show that an appropriately designed truncated sequential test performs favorably as compared to both the fixed sample size test and the sequential probability ratio test with the same error probabilities. The average sample number function of the truncated test is uniformly smaller than that of the fixed sample size test while the truncated test maintains average sample sizes under the hypothesis and the alternative that are close to those optimum values achieved by Wald's sequential probability ratio test. Moreover, the truncated test is more favorable than the sequential probability ratio test in the sense that it has smaller average sample size when the actual location parameter is between θ_0 and θ_1. This behavior becomes more pronounced as the error probabilities become smaller, implying that the truncated sequential test becomes more favorable as the error probabilities become smaller.

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by

Sawasd Tantaratana and H. Vincent Poor

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FOREWORD

This report is a preprint of a paper with the same title which is scheduled to appear in the January 1983 issue of the <u>IEEE Transactions on Information Theory</u> (vol. IT-29).

I. INTRODUCTION

Let $x_1, x_2, \ldots$ be observations of independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$. Consider testing an hypothesis H_0, under which X_i has a probability density function $f(x-\theta_0)$, against a shifted alternative H_1; that is, consider the hypothesis pair

$$\text{versus}\quad \begin{array}{l} H_0:\ X_i \sim f(x-\theta),\ \theta = \theta_0, \\ H_1:\ X_i \sim f(x-\theta),\ \theta = \theta_1 > \theta_0. \end{array} \tag{1}$$

The Neyman-Pearson fixed sample size (FSS) test for (1) is obtained by taking M samples and testing [1]

$$\sum_{i=1}^{M} z_i \begin{cases} \geq \tau \Rightarrow H_1 \\ < \tau \Rightarrow H_0 \end{cases} \tag{2}$$

where z_i is the observed realization of the random variable $Z_i = \ell n(f(X_i-\theta_1)/f(X_i-\theta_0))$, and the sample size M and the threshold τ are pre-chosen so that the test has error probabilities P(choosing $H_1 | H_0$ true) and P(choosing $H_0 | H_1$ true) of α and $1-\beta$, respectively. (Since we are mainly interested in asymptotic properties here, randomization of the test is not included in (2).) Alternately, Wald's [2] sequential probability ratio test (SPRT) is obtained by testing, at the n-th sample,

$$\sum_{i=1}^{n} z_i \begin{cases} \geq a \Rightarrow H_1 \\ \leq b \Rightarrow H_0 \\ \in (b,a) \Rightarrow \text{take another sample,} \end{cases} \tag{3}$$

where the boundaries a and b are chosen so that the error probabilities are α and $1-\beta$. The sample size $N = \min\{n : \sum_{i=1}^{n} Z_i \notin (b,a)\}$ is now a random variable, and the average sample number (ASN) (i.e., the expected value of N) depends on the actual distribution of X_i, i.e., on the actual value of θ.

It is well-known that the SPRT (3) has the smallest ASN under H_0 and H_1 among all tests with error probabilities no larger than α and $1-\beta$. However, because the test is not truncated an occasional long test can result, which is undesirable. Moreover, if the parameter θ is not the assumed value θ_0 or θ_1, the ASN of the SPRT can be very large. In particular, if the density f(x) is symmetric and if $\alpha = 1-\beta$, then $\max_\theta E(N|\theta)$ occurs when $\theta = (\theta_0 + \theta_1)/2$, where $E(N|\theta)$ denotes the expected value of N given that each X_i has the density function $f(x-\theta)$. This maximum value becomes worse when α and $1-\beta$ are smaller [3]. For example, if $\alpha = 1-\beta < 0.008$, which is the case in many signal detection problems, then $\max_\theta E(N|\theta)$ is larger than the sample size M of an FSS test with the same α and $1-\beta$. Truncation of the SPRT can be used to prevent this problem; however, one or both of the error probabilities will be made larger as a result of such truncation. Quantitative analysis is needed to study the effect of truncation on the error probabilities and to find a simple design scheme for a truncated SPRT which gives error probabilities as required. A preliminary study of such effects is given in [4] where a bound for the probability of terminating before the truncation point and a bound for the resulting ASN have been obtained. Also, Anderson [5] has studied a truncated test with two converging boundaries so that the maximum ASN is reduced. However, the converging boundaries are difficult to design and must be chosen from the results of simulation. Read [6] has studied a related test in which a fixed number of samples is taken first, and then, after this fixed number, one additional sample is taken at a time and the test statistic is tested sequentially with two constant boundaries. It is shown in [6] that the maximum (over $\theta \in [\theta_0, \theta_1]$) ASN is reduced by this technique. However, such a scheme still has occasional undesirably large sample sizes since the test is not truncated. In [7], the idea of converging boundaries has been applied to the test of [6]; namely, the test has two converging boundaries from the start up to a fixed number and then the boundaries become constant after this fixed number. Similar reduction in maximum ASN as in [6] is observed in [7], but the test still retains the disadvantage of occasional long sample sizes.

In this paper, we study further the truncated SPRT by extending the analysis given in [4]. It is observed here that the truncated test can be viewed as a mixture of an SPRT and an FSS test. Depending on the chosen degree of mixture, the truncation point and the constant boundaries can be easily designed such that the resulting test has approximate error probabilities no larger than given nominal values α and $1-\beta$. In Section II we describe the procedure for choosing the boundaries and the truncation point when the required error probabilities are α and $1-\beta$ and when Gaussian statistics are assumed. Approximate expressions (which are asymptotically correct as θ_1 approaches θ_0) for the ASN and operating characteristic (OC) functions and for the sample-size variance of the truncated test are given in the same section. These expressions are evaluated in Section III, and the advantages of the truncated sequential test become obvious. Regularity conditions under which the results of Sections II and III hold for non-Gaussian data are given in Section IV. These regularity conditions are fairly mild and are satisfied by a large class of commonly used densities. In Section V, truncated sequential testing with quantized data is considered, and similar results are found to hold in this case as well. Further, exact results are computed for the particular case of two-level quantization, and these are seen to agree closely with results computed using the approximations of Section II.

Notation:

At this point, we define the following notation which will be used throughout the paper:

$$\mu_\theta \triangleq E(Z_i|\theta) = \int \ell n[f(x-\theta_1)/f(x-\theta_0)]f(x-\theta)dx, \tag{4a}$$

$$m_\theta \triangleq E(Z_i^2|\theta) = \int \{\ell n[f(x-\theta_1)/f(x-\theta_0)]\}^2 f(x-\theta)dx, \tag{4b}$$

$$\sigma_\theta^2 \triangleq m_\theta - \mu_\theta^2, \tag{4c}$$

and

$$\mu_0 \triangleq \mu_{\theta_0},\ \mu_1 \triangleq \mu_{\theta_1},\ \sigma_0^2 \triangleq \sigma_{\theta_0}^2,\ \sigma_1^2 \triangleq \sigma_{\theta_1}^2\ . \tag{4d}$$

Thus, μ_θ and σ_θ^2 are the mean and variance of the random variable Z_i from (2) and (3) when the random variables X_i have density function $f(x-\theta)$.

II. TRUNCATED SEQUENTIAL TESTING FOR THE GAUSSIAN CASE

In this section we assume that the density of the data is Gaussian, namely $f(x)=(1/(2\pi)^{\frac{1}{2}}K)\exp(-x^2/2K^2)$ where K is known, and we will describe how to choose the boundaries and the truncation point of a truncated sequential test correspondingly. Then expressions for the resulting ASN and OC functions and for the sample-size variance will be given.

For the Gaussian case, the log-likelihood ratio, its mean, second moment, and variance as defined in (4) are given by

$$Z_i = (\theta_1-\theta_0)(X_i-(\theta_1+\theta_0)/2)/K^2, \tag{5a}$$

$$\mu_\theta = (\theta_1-\theta_0)(\theta-(\theta_1+\theta_0)/2)/K^2=(\theta_1-\theta_0)^2(r-\tfrac{1}{2})/K^2, \tag{5b}$$

$$\mu_0 = -(\theta_1-\theta_0)^2/2K^2 = -\mu_1 , \tag{5c}$$

$$m_\theta = (\theta_1-\theta_0)^2/K^2 + (\theta_1-\theta_0)^4(r-\tfrac{1}{2})^2/K^4, \tag{5d}$$

and

$$\sigma_\theta^2 = (\theta_1-\theta_0)^2/K^2 \triangleq \sigma^2, \tag{5e}$$

where $r = (\theta-\theta_0)/(\theta_1-\theta_0)$ is the ratio of the difference between the actual parameter θ and θ_0 and the difference between θ_1 and θ_0. In subsequent analysis, we will often use this parameter r instead of θ so that, when we consider limits as θ_1 approaches θ_0, we allow θ to approach θ_0 in a way such that r is constant.

The FSS test (2) with error probabilities α and $1-\beta$ has a sample size M and threshold τ given by

$$M = [\Phi^{-1}(\alpha)+\Phi^{-1}(1-\beta)]^2(\sigma^2/(\mu_1-\mu_0)^2) \tag{6a}$$

and

$$\tau = M^{\frac{1}{2}}[\mu_1\Phi^{-1}(\alpha)+\mu_0\Phi^{-1}(1-\beta)](\sigma/(\mu_0-\mu_1)) , \tag{6b}$$

where $\Phi(\cdot)$ is the standard normal distribution function and $\Phi^{-1}(\cdot)$ is its inverse. The ASN function and the operating characteristic (OC) function, $L(\theta) \overset{\Delta}{=} P(\text{choosing } H_0|\theta)$, of the SPRT (3) are given by [2]

$$E(N|\theta) = \begin{cases} \dfrac{(1-L(\theta))a + L(\theta)b}{\mu_\theta} + o(1), & \mu_\theta \neq 0 \\[2ex] -ab/\sigma_\theta^2 + o(1) & , \mu_\theta = 0 \end{cases} \tag{7}$$

and

$$L(\theta) = \begin{cases} \dfrac{e^{ah(\theta)} - 1}{e^{ah(\theta)} - e^{bh(\theta)}} + o(1), & h(\theta) \neq 0 \\[2ex] \dfrac{a}{a-b} + o(1) & , h(\theta) = 0 , \end{cases} \tag{8}$$

where $\lim\limits_{\theta_1 \to \theta_0} o(1) = 0$, and $h(\theta)$ satisfies

$$\int_{-\infty}^{\infty} [f(x-\theta_1)/f(x-\theta_0)]^{h(\theta)} f(x-\theta)dx = 1 , \tag{9}$$

which gives $h(\theta_0) = 1$ and $h(\theta_1) = -1$. For the Gaussian case we have $h(\theta) = 1 - 2(\theta - \theta_0)/(\theta_1 - \theta_0)$. The $o(1)$ terms in above expressions arise from the excess over a boundary when the SPRT terminates. That this excess diminishes as $\theta_1 \to \theta_0$ follows from [2, Appendices A2 and A3].

We now describe a truncated sequential test with constant boundaries a^* and b^* and truncation performed at $n = M^*$, as follows: At each observation $n < M^*$ test

$$\sum_{i=1}^{n} z_i \begin{cases} \geq a^* \Rightarrow H_1 \\ \leq b^* \Rightarrow H_0 \\ \in (b^*, a^*) \Rightarrow \text{take another sample,} \end{cases} \tag{10a}$$

and at $n = M^*$, test

$$\sum_{i=1}^{M^*} z_i \begin{cases} \geq t^* \Rightarrow H_1 \\ < t^* \Rightarrow H_0 \end{cases} \tag{10b}$$

where t^* is a fixed threshold. Let α^* and $1-\beta^*$ be the error probabilities under H_0 and H_1 of the test (10). Although α^* and $1-\beta^*$ can be approximated, the expressions are complicated as we shall see later in this section. Therefore, designing a^*, b^*, M^*, and t^* from these expressions is prohibitive. However, we can turn to simple bounds for α^* and $1-\beta^*$ and use them for designing the truncated test (10). It was shown in [4] that

$$\alpha^* \leq \alpha_{SPRT} + \alpha_{FSS} \tag{11a}$$

and

$$1-\beta^* \leq (1-\beta_{SPRT}) + (1-\beta_{FSS}), \tag{11b}$$

where α_{SPRT} and $(1-\beta_{SPRT})$ are the error probabilities of an SPRT with thresholds a^* and b^*, namely

$$\alpha_{SPRT} \approx \frac{1-\exp(b^*)}{\exp(a^*) - \exp(b^*)} \tag{12a}$$

$$1-\beta_{SPRT} \approx \frac{\exp(-a^*) - 1}{\exp(-a^*) - \exp(-b^*)}, \tag{12b}$$

and where α_{FSS} and $(1-\beta_{FSS})$ are the error probabilities of an FSS test with sample size M^* and threshold t^*, namely

$$\alpha_{FSS} = 1 - \Phi((t^* - \mu_0 M^*)/\sigma(M^*)^{1/2}) \tag{13a}$$

$$1-\beta_{FSS} = \Phi((t^* - \mu_1 M^*)/\sigma(M^*)^{1/2}). \tag{13b}$$

Again, the approximations in (12) arise from neglecting the excess over the threshold boundary at termination of the test.

The bounds of (11) can be viewed as mixtures of the error probabilities of an SPRT and those of an FSS test. In order to design a truncated test with error probabilities less than α and $1-\beta$, we then can set the bounds in (11) to be α and $1-\beta$, namely

$$\alpha_{SPRT} + \alpha_{FSS} = \alpha \tag{14a}$$

$$(1-\beta_{SPRT}) + (1-\beta_{FSS}) = 1-\beta \tag{14b}$$

Thus, we have freedom to choose the degree of mixture between the SPRT error probabilities and the FSS error probabilities. The choice of the mixture will determine the truncation point M^*, the threshold t^*, and the constant boundaries a^* and b^*. It will also reflect whether the performance (ASN and OC functions) of the resulting test will be closer to that of an SPRT or closer to that of an FSS test or intermediate to these two, as we shall see later. Note that the values α and $1-\beta$ are used as nominal values in designing a^*, b^*, M^*, and t^* so that $\alpha^* \leq \alpha$ and $1-\beta^* \leq 1-\beta$. It is very unlikely that either equality, $\alpha^* = \alpha$ or $1-\beta^* = 1-\beta$, will result and no attempt is made in the design to achieve the equalities. Therefore, as numerical results in Section III will indicate, the resulting error probabilities α^* and $1-\beta^*$ will usually be smaller than the nominal values α and $1-\beta$ used in the design.

Now let c_1 and c_2 be two constants between 0 and 1 that determine the mixture implied by (14); i.e., let

$$\alpha_{FSS} = c_1\alpha \text{ and } \alpha_{SPRT} = (1 - c_1)\alpha , \tag{15a}$$

and

$$(1-\beta_{FSS}) = c_2(1-\beta) \text{ and } (1-\beta_{SPRT}) = (1 - c_2)(1-\beta), \tag{15b}$$

Note that, if c_1 and c_2 are both zero, then the resulting test (10) is the SPRT. This is equivalent to saying that the truncation is at $M^* = \infty$. On the other hand if $c_1 = c_2 = 1$, then the test (10) is reduced to the FSS test or, equivalently, the boundaries a^* and b^* are ∞ and $-\infty$, respectively. If c_1 and c_2 are both 0.5, then the test (10) can be thought of as being half mixed between an SPRT and an FSS test. Using (12) and (15), we set

$$a^* = \ell n\left[\frac{1 - (1 - c_2)(1-\beta)}{(1 - c_1)\alpha}\right] \quad \text{and } b^* = \ell n\left[\frac{(1 - c_2)(1-\beta)}{1 - (1 - c_1)\alpha}\right] , \tag{16}$$

and using (13) and (15), we set

$$M^* = [\Phi^{-1}(c_1\alpha) + \Phi^{-1}(c_2(1-\beta))]^2(\sigma/(\mu_1 - \mu_0))^2 \tag{17a}$$

and

$$t^* = [M^*]^{\frac{1}{2}}[\mu_1\Phi^{-1}(c_1\alpha) + \mu_0\Phi^{-1}(c_2(1-\beta))](\sigma/(\mu_0 - \mu_1)) . \tag{17b}$$

Since σ, μ_1, and μ_0 are known, once c_1 and c_2 are chosen the test (10) can be determined by calculating a^*, b^*, M^*, and t^* from (16) and (17). Good choices for c_1 and c_2 will be discussed in Section III where numerical results are presented.

Denote the ASN and OC functions of the truncated test (10) by $E(N^*|\theta)$ and $L^*(\theta)$, respectively. It is then obvious from the design that

$$L^*(\theta_1) = 1-\beta^* \le 1-\beta\,, \tag{18a}$$

$$1-L^*(\theta_0) = \alpha^* \le \alpha\,, \tag{18b}$$

and

$$E(N^*|\theta) \le \min\{M^*, \tilde{E}(N|\theta)\}\,, \tag{19}$$

where $\tilde{E}(N|\theta)$ is the ASN function of an SPRT with boundaries a^* and b^*, and is given by (7)-(9) with a and b replaced by a^* and b^*, respectively.

In addition to the upper bounds of (18) and (19), approximate expressions for $L^*(\theta)$ and $E(N^*|\theta)$ can also be obtained by using a Brownian motion approximation to the relevant test statistic. In particular, the random process $B_\theta(t) = (Z_1 + \ldots + Z_{[M^*t]} - \mu_\theta[M^*t])/\sigma_\theta(M^*)^{1/2}$, $0 \le t \le T$, converges weakly to a standard Brownian motion as M^* goes to infinity [8, p. 137] for each finite $T > 0$. (Here, $[Mt]$ denotes the largest integer less than or equal to Mt.) Therefore, for large M^*, approximations using Brownian motion results are justified.

From [9], the distribution of the first passage time $T^* = \inf\{M^*t: (\sigma_\theta(M^*)^{\frac{1}{2}}B_\theta(t) + \mu_\theta[M^*t]) \notin (b^*, a^*)\}$ is given by

$$F_\theta(u) = P(T^* \le u|\theta)$$

$$= 1 + \frac{2\sigma_\theta^2\pi}{(a^*-b^*)^2} \sum_{j=1}^{\infty} \frac{j(-1)^j}{\left(\frac{\mu_\theta}{\sigma_\theta}\right)^2 + \left(\frac{\sigma_\theta j\pi}{a^*-b^*}\right)^2} \,\cdot$$

$$\left\{\exp\left(\frac{\mu_\theta b^*}{\sigma_\theta^2}\right)\sin\left(\frac{j\pi a^*}{a^*-b^*}\right) - e\,\exp\left(\frac{\mu_\theta a^*}{\sigma_\theta^2}\right)\cdot\sin\left(\frac{j\pi b^*}{a^*-b^*}\right)\right\}\cdot \exp\left\{-\frac{u}{2}\left(\left(\frac{\mu_\theta}{\sigma_\theta}\right)^2 + \left(\frac{\sigma_\theta j\pi}{a^*-b^*}\right)^2\right)\right\},\ 0 \le u \le 1. \tag{20}$$

Using this expression the ASN of the test (10), $E(N^*|\theta)$, can be approximated by

$$E(N^*|\theta) \approx E(T^*|\theta)$$

$$= \int_0^{M^*} u\,dF_\theta(u) + M^*(1 - F_\theta(M^*))$$

$$= \frac{-2\pi}{\mu_1(a^*-b^*)^2} \sum_{j=1}^{\infty} \zeta(j)(\gamma(j))^{-2} \left(1-\exp(-\eta(j))\right) \tag{21}$$

where

$$\gamma(j) = (r-\tfrac{1}{2})^2 + \left(\frac{j\pi}{a^*-b^*}\right)^2 , \tag{22a}$$

$$\zeta(j) = j(-1)^j \left[e^{(r-\frac{1}{2})b^*} \sin\left(\frac{j\pi a^*}{a^*-b^*}\right) - e^{(r-\frac{1}{2})a^*} \sin\left(\frac{j\pi b^*}{a^*-b^*}\right)\right], \tag{22b}$$

$$\eta(j) = \tfrac{1}{2} \left[\Phi^{-1}(c_1\alpha) + \Phi^{-1}(c_2(1-\beta))\right]^2 \left[(r-\tfrac{1}{2})^2 + \left(\frac{j\pi}{a^*-b^*}\right)^2\right], \tag{22c}$$

and $r=(\theta-\theta_0)/(\theta_1-\theta_0)$.

The ASN of (21) can also be evaluated using a result of Anderson's [5, (5.7)], which yields

$$E(N^*|\theta) \approx \frac{1}{\mu_1}\{ [\mathcal{E}(a^*,b^*,r-\tfrac{1}{2}) + \mathcal{E}(-b^*,-a^*,\tfrac{1}{2}-r)] + \tfrac{1}{2}[\Phi^{-1}(c_1\alpha) + \Phi^{-1}(c_2(1-\beta))]^2$$
$$[1 - G(a^*,b^*,r-\tfrac{1}{2}) - G(-b^*,-a^*,\tfrac{1}{2}-r)]\} \qquad (2\ )$$

where, with $d = -(\Phi^{-1}(c_1\alpha) + \Phi^{-1}(c_2(1-\beta)))$, $\varphi(x) = (1/(2\pi)^{\frac{1}{2}})\exp(-x^2/2)$ (i.e., φ is the unit normal density), $\mathcal{E}(\cdot)$ is defined by

$$
\mathcal{E}(a,b,c) = \begin{cases}
\frac{1}{2c} \sum_{j=0}^{\infty} \Big\{ \Big[\Phi\Big(-cd + \frac{2jb-(2j+1)a}{d}\Big)\cdot e^{-2c[jb-(j+1)a]} \\
\quad - \Phi\Big(cd + \frac{2jb-(2j+1)a}{d}\Big)\cdot e^{-2jc(a-b)}\Big](2jb-(2j+1)a) \\
\quad - \Big[\Phi\Big(-cd + \frac{2(j+1)b-(2j+1)a}{d}\Big)\cdot e^{-2c(j+1)(b-a)} \\
\quad - \Phi\Big(cd + \frac{2(j+1)b-(2j+1)a}{d}\Big)\cdot e^{-2c[ja-(j+1)b]}\Big] 2(j+1)b-(2j+1)a)\Big\}, \\
\qquad\qquad c \neq 0 \quad (24a) \\
\sum_{j=0}^{\infty} \Big\{ \Big[(2(j+1)b-(2j+1)a)\Phi\Big(\frac{2(j+1)b-(2j+1)a}{d}\Big) \\
\quad + d\varphi\Big(\frac{2(j+1)b-(2j+1)a}{d}\Big)\Big](2(j+1)b-(2j+1)a) \\
\quad - \Big[(2jb-(2j+1)a)\Phi\Big(\frac{2jb-(2j+1)a}{d}\Big) + d\varphi\Big(\frac{2jb-(2j+1)a}{d}\Big)\Big] \\
\quad \cdot (2jb-(2j+1)a)\Big\}, \quad c = 0,
\end{cases}
$$

and G is defined by

$$
\begin{aligned}
G(a,b,c) = \Phi\Big(cd - \frac{a}{d}\Big) + \sum_{j=1}^{\infty} \Big\{ &\Phi\Big(\frac{2(j-1)b-(2j-1)a}{d} - cd\Big)\cdot e^{-2c((j-1)b-ja)} \\
&- \Phi\Big(\frac{2jb-(2j-1)a}{d} - cd\Big)\cdot e^{-2cj(b-a)} \\
&- \Phi\Big(\frac{2jb-(2j-1)a}{d} + cd\Big)\cdot e^{-2c((j-1)a-jb)} \\
&+ \Phi\Big(\frac{2jb-(2j+1)a}{d} + cd\Big)\cdot e^{-2jc(a-b)}\Big\} . \qquad (24b)
\end{aligned}
$$

Expressions (21) and (23) give the same result. However, (23) is preferable for numerical evaluation since (21) normally converges much more slowly than does (23). Usually, only the first few terms of $\mathcal{E}(\cdot)$ and $G(\cdot)$ are needed for numerical evaluations.

The second moment of N* can be approximated by calculating $E((T^*)^2|\theta)$ using (20), and thereby we obtain

$$E((N^*)^2|\theta) \approx E((T^*)^2|\theta)$$

$$= \int_0^{M^*} u^2 \, dF_\theta(u) + (M^*)^2 \, (1-F_\theta(M^*))$$

$$= \frac{-4\pi}{\mu_1^2(a^*-b^*)^2} \sum_{j=1}^{\infty} \zeta(j)(\gamma(j))^{-3} \, (1-[1+\eta(j)] \exp(-\eta(j)). \quad (25)$$

where ζ. γ, and η are as defined above. The variance of N* is given by

$$\mathrm{Var}_\theta(N^*) = E((N^*)^2|\theta) - E^2(N^*|\theta) \quad (26)$$

and can be calculated using (21) or (23) and (25).

To find an approximate expression for the OC function $L^*(\theta)$ of the truncated test, we again use Anderson's result [5, (4.68)] and obtain

$$L^*(\theta) \approx B(a^*,b^*,r) \quad (27a)$$

where, with r and d as previously defined and with $A = -rd - \Phi^{-1}(\alpha c_1)$, the function $B(\cdot)$ is given by

$$B(a,b,r) = \Phi(A) - \sum_{j=1}^{\infty} \{ [\Phi(A+\frac{2(j(b-a)-b)}{d})\cdot e^{2b(r-\frac{1}{2})} - \Phi(A+\frac{2j(b-a)}{d})]\cdot e^{-2j(b-a)(r-\frac{1}{2})}$$

$$+ [\Phi(\frac{2j(b-a)}{d}-A) - \Phi(\frac{2(j(b-a)+a)}{d}-A)\cdot e^{2a(r-\frac{1}{2})}]\cdot e^{-2j(a-b)(r-\frac{1}{2})}\}. \quad (27b)$$

Under H_0 and H_1 we have $\alpha^* \approx 1-B(a^*,b^*,0)$ and $1-\beta^* \approx B(a^*,b^*,1)$, respectively.

If we consider limiting values of the ASN functions by letting θ_1 approach θ_0, the ASN values approach infinity. However, if we consider $((\theta_1-\theta_0)^2/2K^2)$ times the ASN (i.e., $\mu_1 \cdot$ASN) instead, we find that limiting values exist. Therefore, we will subsequently evaluate and compare the limits of $((\theta_1-\theta_0)^2/2K^2)$ times the ASN or sample size. With μ_0, μ_1, and σ_θ^2 given by (5c) and (5e), we have, from (6a),

$$\lim_{\theta_1 \to \theta_0} [(\theta_1-\theta_0)^2 M/2K^2] = \tfrac{1}{2}[\Phi^{-1}(\alpha) + \Phi^{-1}(1-\beta)]^2. \tag{28}$$

From (7) and (8), with $h(\theta) = 1 - 2(\theta-\theta_0)/(\theta_1-\theta_0) = 1 - 2r$, it follows that, for the SPRT,

$$\lim_{\theta_1 \to \theta_0} [(\theta_1-\theta_0)^2 E(N|\theta)/2K^2] = \begin{cases} \dfrac{1}{2r-1}\left[a + (b-a)\,\dfrac{e^{a(1-2r)}-1}{e^{a(1-2r)}-e^{b(1-2r)}}\right], & r \neq \tfrac{1}{2} \\[2ex] -ab/2, & r = \tfrac{1}{2}\,. \end{cases} \tag{29}$$

Since approximation by Brownian motion is asymptotically correct as M^* approaches infinity (which is the case when θ_1 approaches θ_0), we can argue that (21) and (23) lead to

$$\lim_{\theta_1 \to \theta_0} [(\theta_1-\theta_0)^2 E(N^*|\theta)/2K^2] = \frac{-2\pi}{(a^*-b^*)^2} \sum_{j=1}^{\infty} \zeta(j)(\gamma(j))^{-2} [1-\exp(-\eta(j))] , \qquad (30a)$$

or

$$\lim_{\theta_1 \to \theta_0} [(\theta_1-\theta_0)^2 E(N^*|\theta)/2K^2] = \mathcal{E}(a^*,b^*,r-\tfrac{1}{2}) + \mathcal{E}(-b^*,-a^*,\tfrac{1}{2}-r)$$

$$+ \tfrac{1}{2}[\Phi^{-1}(c_1\alpha)+\Phi^{-1}((1-\beta)c_2)]^2 [1 - G(a^*,b^*,r-\tfrac{1}{2})$$

$$-G(-b^*,-a^*,\tfrac{1}{2}-r)] \qquad (30b)$$

where $r = (\theta - \theta_0)/(\theta_1 - \theta_0)$ as previously defined, and $\mathcal{E}(\cdot)$ and $G(\cdot)$ are given by (23) and (24). Similarly, from (25) we can write the asymptotic second moment of N^* as

$$\lim_{\theta_1 \to \theta_0} [(\theta_1-\theta_0)^4 E((N^*)^2|\theta)/4K^4] = \frac{-4\pi}{(a^*-b^*)^2} \sum_{j=1}^{\infty} \zeta(j)(\gamma(j))^{-3} (1-[1+\eta(j)]\exp(-\eta(j))). \qquad (31)$$

The power function of the FSS test (2) is given by

$$\beta_{FSS}(r) = 1 - \Phi\left(\frac{\tau - (r-\frac{1}{2})[\Phi^{-1}(\alpha)+\Phi^{-1}(1-\beta)]^2}{-[\Phi^{-1}(\alpha)+\Phi^{-1}(1-\beta)]}\right), \tag{32}$$

where $\tau = \frac{1}{2}[(\Phi^{-1}(\alpha))^2 - (\Phi^{-1}(1-\beta))^2]$, which is obtained from (6b). The SPRT has a limiting power function

$$\lim_{\theta_1 \to \theta_0} \beta_{SPRT}(r) = \lim_{\theta_1 \to \theta_0} [1 - L(\theta)] = \begin{cases} \dfrac{1 - e^{b(1-2r)}}{e^{a(1-2r)} - e^{b(1-2r)}}, & r \neq \frac{1}{2} \\ a/(a-b), & r = \frac{1}{2}. \end{cases} \tag{33}$$

We can also obtain a limiting power function for the truncated sequential test as

$$\lim_{\theta_1 \to \theta_0} \beta_T(r) = \lim_{\theta_1 \to \theta_0} [1 - L^*(\theta)] = 1 - B(a^*, b^*, r) \tag{34}$$

where $B(\cdot)$ is given by (27b).

In the next section we will evaluate numerically the expressions given by (28) through (34) in order to compare the three tests, the FSS, the SPRT, and the truncated sequential test.

III. NUMERICAL RESULTS

In this section we evaluate the performance of the truncated test in order to compare it with the SPRT and the FSS test. We wish to compare average sample sizes when each test is designed to have given α and $1-\beta$. As noted in the previous section, the ASN is a function of $\theta_1-\theta_0$ and it goes to infinity as θ_1 approaches θ_0. In order to avoid the parameter $(\theta_1-\theta_0)$, we defined $r = (\theta - \theta_0)/(\theta_1 - \theta_0)$ and obtained asymptotic expressions for $(\theta_1 - \theta_0)^2/2K^2$ times the ASN (i.e., $\mu_1 \cdot$ASN), namely expressions (28), (29) and (30). Since each ASN is multiplied by the same factor, $(\theta_1 - \theta_0)^2/2K^2$, ratios of two quantities among (28), (29), and (30) are the limiting ratios of ASN functions as $\theta_1 \to \theta_0$. For example, the quantity

$$\lim_{\theta_1 \to \theta_0} \frac{[(\theta_1 - \theta_0)^2 E(N^*|\theta)/2K^2]}{[(\theta_1 - \theta_0)^2 M/2K^2]} = \lim_{\theta_1 \to \theta_0} \frac{E(N^*|\theta)}{M}$$

is a measure of the asymptotic efficiency of the FSS test relative to the truncated sequential test.

Instead of plotting asymptotic relative efficiencies $\lim_{\theta_1 \to \theta_0} [E(N^*|\theta)/M]$ and $\lim_{\theta_1 \to \theta_0} [E(N^*|\theta)/E(N|\theta)]$, we will plot expressions (28), (29), and (30) on the same graph. In Fig. 1, (28), (29) and (30) are plotted for $\alpha = 1-\beta = 0.01$ with the mixture constants c_1 and c_2 for the truncated sequential test both equal to 0.9. It can be seen that the SPRT has uniformly smaller ASN than the FSS test and that the truncated test has larger ASN than the SPRT except when r is near 0.5. As one would expect, the truncated test has performance between that of the SPRT and the FSS test. Under H_0 where $r = 0$ and H_1 where $r = 1$, the truncated test exhibits significant savings over the FSS test (about 40%). The upper bound for the truncated test given by (19) is also plotted in Fig. 1. It should be noted that the absolute maximum sample size for the truncated test is only about 3% more than the sample size of the FSS test, while the average sample size is uniformly smaller than the sample size of the FSS test. Figure 2 shows the same quantities for the alternate case $\alpha = 1\text{-}\beta = 0.001$. For these smaller values of error probabilities, the nice features of the truncated test become more apparent. Now a disadvantage of the SPRT shows, namely the ASN becomes larger than the FSS sample size for r between 0.4 and 0.6. However, the truncated test retains uniformly smaller ASN than the FSS test while significant savings (close to that of the SPRT) near $r = 0$ and $r = 1$ are still observed and while the truncation point can be kept at a sample size only a few percent larger than the FSS sample size. Two truncated sequential tests' results are shown in Fig. 2, namely tests with $c_1 = c_2 = 0.5$ and $c_1 = c_2 = 0.9$. Note that the ASN for $c_1 = c_2 = 0.5$ is smaller than that for $c_1 = c_2 = 0.9$ but the truncation point is larger. Therefore, there is a trade-off between the truncation point and the ASN. Choices of c_1 and c_2 which result in larger truncation points seem to result in (not necessarily uniformly over θ) smaller ASN's. Figure 3 shows similar behavior for the case $\alpha = 0.0001$ and $1\text{-}\beta = 0.0005$. In this case, we note that $\max_\theta E(N|\theta)$ of the SPRT becomes worse. Further, two truncated tests, with $c_1 = 0.83$, $c_2 = 0.1$ and $c_1 = c_2 = 0.9$, still show ASN's uniformly smaller than the FSS sample size. However, more savings in ASN than in previous cases are observed under H_0 and H_1. Note that the graphs are skewed in this case because $\alpha \neq 1\text{-}\beta$. Note also that the truncated test with $c_1 = 0.83$ and $c_2 = 0.1$ has a larger truncation point than the truncated test with $c_1 = c_2 = 0.9$. However, its ASN is not uniformly smaller than that of the other case. From these numerical results, we conclude that a truncated test can be designed (with error probabilities less than α and $1\text{-}\beta$) by suitable choices of c_1 and c_2 so that it retains the advantage of savings in sample sizes near $r = 0$ and $r = 1$ while the ASN is uniformly smaller than the sample size of a corresponding FSS test and while the truncation point is only slightly larger than the sample size of the FSS test.

In order to see how the choices of c_1 and c_2 affect the truncation point and the ASN function, we now evaluate $\mu_1 \cdot$ASN under H_0, $\mu_1 \cdot$ASN under H_1, $\max_\theta[\mu_1 E(N^*|\theta)]$, and the truncation point of a truncated test, and plot their values versus $c_1 = c_2$. The values $\mu_1 M$ of the FSS test and $\mu_1 E(N|\theta_0)$ and $\mu_1 E(N|\theta_1)$ of the SPRT are also plotted in Fig. 4(b) for $\alpha = 0.05$, $1-\beta = 0.01$, in Fig. 5(b) for $\alpha = 1-\beta = 0.001$, and in Fig. 6(b) for $\alpha = 1-\beta = 0.0001$. Ratios of (average) sample sizes are plotted in Figs. 4(a), 5(a) and 6(a). These results indicate that as c_1 and c_2 approach zero, $E(N^*|\theta_0)$ and $E(N^*|\theta_1)$ approach $E(N|\theta_0)$ and $E(N|\theta_1)$ of the SPRT, respectively, as expected. On the other hand, as c_1 and c_2 approach unity, $E(N^*|\theta)$ approaches M of the FSS test for each value of θ. From these graphs, we can choose $c_1 = c_2$ between 0 and 1 so that M^*/M is not too large, $\{\max_\theta E(N^*|\theta)\}/M$ is near its minimum value, and $E(N^*|\theta_0)$ and $E(N^*|\theta_1)$ are as close to $E(N|\theta_0)$ and $E(N|\theta_1)$ as needed. Of course, the actual choices of c_1 and c_2 depend on the designer's judgment as to what is more important to minimize, M^*/M, $\{\max_\theta E(N^*|\theta)\}/M$, or $E(N^*|\theta_0)/E(N|\theta_0)$ and $E(N^*|\theta_1)/E(N|\theta_1)$. Since $\max_\theta E(N^*|\theta)$ seems to be less sensitive to c_1 and c_2 for c_1 and c_2 between 0.3 and 0.7, the primary tradeoff is between M^* and $E(N^*|\theta_0)$ or $E(N^*|\theta_1)$. Figures 5 and 6 indicate that good c_1 and c_2 choices seem to be between 0.3 and 0.6 for these two cases.

Further numerical investigation shows that the boundary a^* is more sensitive than the boundary b^* to changes in c_1. A result is that changing c_1 will cause more change in $E(N^*|\theta)$ for θ near θ_1 than change in $E(N^*|\theta)$ for θ near θ_0. With c_2 fixed, increasing c_1 will also increase $E(N^*|\theta_1)$. On the other hand, b^* is more sensitive to change in c_2, and increasing c_2

with c_1 fixed results in an increase in $E(N^*|\theta_0)$. Both c_1 and c_2 have the same effect on $\max_\theta E(N^*|\theta)$. Of course, c_1 and c_2 can be chosen to have different values. Optimum choices of c_1 and c_2 depend on a given criterion. For example, M^* can be set to a maximum allowable value and then c_1 and c_2 can be chosen to minimize $E(N^*|\theta_0)$, $E(N^*|\theta)$, or a weighted average of these three. Since there are many possible criteria, we will not pursue the search for optimum choices of c_1 and c_2 here.

The behavior of the variance of N* is also informative. Thus, we now compare $\lim (\mu_1^2 \operatorname{Var}_\theta(N^*))$ of the truncated test with $\lim (\mu_1^2 \operatorname{Var}_\theta(N))$ of the SPRT, where the limits are taken as θ_1 approaches θ_0. The first limit can be calculated from (30) and (31) since $\operatorname{Var}_\theta(N^*) = E((N^*)^2|\theta) - E^2(N^*|\theta)$. The second limit, that for the SPRT, can be evaluated using Wald's results [2, Appendix A.5] which give approximate formulas for moments of N (the sample size). As with the approximate formulas in Section II, these values are asymptotically correct as $\theta_1 \to \theta_0$. With these formulas, we obtain Table I, corresponding to those cases of Figures 1, 2, and 3, namely $\alpha = 1-\beta = 0.1$, $\alpha = 1-\beta = 0.001$, and $\alpha = 0.0001$ and $1-\beta = 0.0005$, respectively. Results show that the untruncated SPRT has large sample size variance when $r = (\theta-\theta_0)/(\theta_1-\theta_0)$ is near 0.5. This is due to the fact that the test terminates with very large sample size most of the time under this condition. In contrast, the truncated test has very small sample size variance when r is near 0.5. This is so because the truncated test terminates most of the time near or at the truncation point M* for r near 0.5. Note further that, under H_0 and H_1, $\mu_1^2 \operatorname{Var}_\theta(N^*)$ of the truncated test and $\mu_1^2 \operatorname{Var}_\theta(N)$ of the SPRT are only slightly different from one another. These phenomena indicate an additional favorable property of the truncated test as compared to the SPRT.

It is also of interest to investigate numerical results for the power functions of (32)-(34)A typical comparison is shown in Fig. 7, where $\alpha = 1-\beta = 0.1$. The SPRT and FSS power functions coincide at $r = 0$, 0.5, and 1.0. Between $r = 0$ and 0.5, the power function of the SPRT is smaller than that of the FSS while it is larger between $r = 0.5$ and 1.0. However, the difference between these two power functions is not significant. The power function for a truncated sequential test with $c_1 = 0.4$ and $c_2 = 0.6$ (from (34)) is plotted in the same figure. We see that this function is smaller than the other two for $r < 0.5$ and larger for $r > 0.5$. At $r = 0$ we have $\alpha^* = 1-\beta^* \approx 0.067$, which is smaller than 0.1; this is due to the fact that $\alpha = 1-\beta = 0.1$ are nominal values used for the design and they serve only as upper bounds for the actual error probabilities α^* and $1-\beta^*$. To see how close α^* and $1-\beta^*$ are to α and $(1-\beta)$, we evaluate these using (27) for various values of α, $1-\beta$, c_1 and c_2. These values are tabulated in Table II, from which it can be seen that α^* and $1-\beta^*$ are between 88% to 96% of α and $1-\beta$ for the cases considered.

IV. THE CASE OF NON-GAUSSIAN DATA

Although we have concentrated thus far on the case where $f(x)$ is a Gaussian density, the asymptotic results of Sections II and III hold for non-Gaussian densities as well. In this section, we define a class of non-Gaussian density functions and show that the previous results apply when the observation statistics are described by a member of this class. We use the same notation as in Section II.

Assume the following conditions on $f(x)$:

A1: $f(x)$ is continuous with finite Fisher's information number $I(f) = \int (f'/f)^2 f$, and $f'(x)$ exists and is continuous with a possible exception at $x = 0$, where $f'(x)$ denotes the derivative of $f(x)$.

A2: The mean and second moment of the log-likelihood ratio, μ_θ and m_θ, exist.

A3: There exists a $\Delta > 0$ such that, for $t \in [-\Delta, \Delta]$,

$$\frac{f'(x)}{f(x)} f(x+t), \quad \frac{f'(x)}{f(x)} f'(x+t), \quad \left(\frac{f'(x)}{f(x)}\right)^2 f(x+t), \text{ and } \left(\frac{f'(x)}{f(x)}\right)^2 f'(x+t)$$

are uniformly integrable. (A function $f(x,t)$ is uniformly integrable for $t \in [-\Delta, \Delta]$ if there exists an integrable function $g(x)$ such that $|f(x,t)| \le g(x)$ for all $t \in [-\Delta, \Delta]$.)

Within these assumptions, it can be shown that

$$\mu_\theta = (\theta_1 - \theta_0)^2 (r - 0.5) I(f) + o((\theta_1 - \theta_0)^2) \tag{35}$$

and

$$\sigma_\theta^2 = (\theta_1 - \theta_0)^2 I(f) + o((\theta_1 - \theta_0)^2), \tag{36}$$

where $\lim o((\theta_1 - \theta_0)^2)/(\theta_1 - \theta_0)^2 = 0$ as $\theta_1 \to \theta_0$. To prove (35) and (36), arguments parallel to those of the Appendix in [3] can be used. From (35) and (36), it follows that

$$\lim_{\theta_1 \to \theta_0} (\mu_\theta/\sigma_\theta^2) = r - \tfrac{1}{2} \tag{37a}$$

$$\lim_{\theta_1 \to \theta_0} (\sigma_0^2/(\mu_1-\mu_0)) = \lim_{\theta_1 \to \theta_0} (\sigma_1^2/(\mu_1-\mu_0)) = 1 \;. \tag{37b}$$

To maintain constant error probabilities under H_0 and H_1 as $\theta_1 \to \theta_0$, the sample size M and the truncation point M^* must approach infinity. Since μ_θ and σ_θ^2 are finite by Assumption A2, we have, by the Central Limit Theorem [10], that $(\sum_{i=1}^{M^*} Z_i - M^*\mu_\theta\ (M^*)^{\frac{1}{2}}\sigma_\theta$ and $(\sum_{i=1}^{M} Z_i - M\mu_\theta)/M^{\frac{1}{2}}\sigma_\theta$ converge in distribution to standard normal random variables as M^* and M go to infinity. Therefore, (6) and (17) hold asymptotically; i.e.,

$$\lim_{\theta_1 \to \theta_0} (M(\mu_1 - \mu_0)^2/\sigma_\theta^2) = [\Phi^{-1}(\alpha) + \Phi^{-1}(1-\beta)]^2 \;, \tag{38a}$$

$$\lim_{\theta_1 \to \theta_0} (\tau/M^{\frac{1}{2}}\sigma_\theta) = -\tfrac{1}{2}[\Phi^{-1}(\alpha) - \Phi^{-1}(1-\beta)] \;, \tag{38b}$$

$$\lim_{\theta_1 \to \theta_0} (M^*(\mu_1 - \mu_0)^2/\sigma_\theta^2) = [\Phi^{-1}(c_1\alpha) + \Phi^{-1}(c_2(1-\beta))]^2 \;, \tag{38c}$$

and

$$\lim_{\theta_1 \to \theta_0} (t^*/(M^*)^{\frac{1}{2}}\sigma_\theta) = -\tfrac{1}{2}[\Phi^{-1}(c_1\alpha) - \Phi^{-1}(c_2(1-\beta))] \;. \tag{38d}$$

As in Section II we let $B_\theta(t) = (\sum_{i=1}^{[M^*t]} Z_i - \mu_\theta[M^*t])/(M^*)^{\frac{1}{2}}\sigma_\theta, 0 \le t \le T$, for finite $T > 0$. Since μ_θ and σ_θ^2 are finite, the random process $B_\theta(t)$ converges weakly to a standard Brownian motion. As noted above, the operating characteristic function and the expected stopping time for this random process were found in [9]. In particular, with $T^* = \inf\{M^*t : (M^*)^{\frac{1}{2}}\sigma_\theta B_\theta(t) + \mu_\theta[M^*t] \notin (b,a)\}$, we have

$$E(T^*|\theta) = \begin{cases} [a(1-L(\theta)) + bL(\theta)]/\mu_\theta, & \mu_\theta \neq 0 \\ -ab/\sigma_\theta^2 & , \mu_\theta = 0 \;, \end{cases} \tag{39}$$

and the corresponding operating characteristic is

$$L(\theta) = \begin{cases} \dfrac{e^{-2(a\mu_\theta/\sigma_\theta^2)} - 1}{e^{-2(a\mu_\theta/\sigma_\theta^2)} - e^{-2(b\mu_\theta/\sigma_\theta^2)}} & , \mu_\theta \neq 0 \\[2ex] \dfrac{a}{a-b} & , \mu_\theta = 0 \, . \end{cases} \tag{40}$$

Now with (37), (39), (40) and the fact that $B_\theta(t)$ converges weakly to a Brownian motion, we have for the SPRT (3),

$$\lim_{\theta_1 \to \theta_0} [(\theta_1 - \theta_0)^2 I(f) E(N|\theta)/2] = \begin{cases} \dfrac{1}{2r-1}\left[a + (b-a)\,\dfrac{e^{-a(2r-1)} - 1}{e^{-a(2r-1)} - e^{-b(2r-1)}}\right] & , r \neq \frac{1}{2} \\ -ab/2 & , r = \frac{1}{2} \, , \end{cases} \tag{41}$$

which is the same as (29). By similar arguments for the test (10), $\lim_{\theta_1 \to \theta_0} [(\theta_1 - \theta_0)^2 I(f) E(N^*|\theta)/2]$ is given by the right-hand side of (30a) or (30b). We note that $I(f) = 1/K^2$ for the normal density with zero mean and variance K^2. Similarly, the asymptotic power functions (32)-(34) carry over to non-Gaussian densities satisfying assumptions A1 through A3.

V. THE EFFECTS OF DATA QUANTIZATION

In this section we will show that the ASN functions of truncated sequential tests based on k-level quantized data have the same asymptotic behavior as those just studied provided that the pre-quantized data density f(x) satisfies mild regularity conditions. Consider a k-level quantizer with finite output levels $\ell_1, \ell_2, \ldots, \ell_k$ and with quantization points $-\infty < s_1 < s_2 < \ldots < s_{k-1} < \infty$ as shown in Fig. 8. For convenience let $s_0 = -\infty$ and $s_k = \infty$, and denote i.i.d. random variables $Q(X_1), Q(X_2), \ldots$ by $Y_1, Y_2, \ldots$ where

$$Q(x) = \ell_j \quad \text{if } s_{j-1} < x \leq s_j, \quad j = 1,2,\ldots,k. \tag{42}$$

The probability that Y_i takes the value ℓ_j when X_i has a density $f(x-\theta)$ is

$$p_j(\theta) = P(Y_i = \ell_j | \theta) = F(s_j - \theta) - F(s_{j-1} - \theta), \; j = 1,2,\ldots,k. \tag{43}$$

where F(x) is the distribution function corresponding to f(x)

We now consider truncation of a sequential test for (1) based on the quantized data $y_1, y_2, \ldots$ with boundaries a^* and b^* and truncation point at M^*, namely, for $n < M^*$, we test

$$\sum_{j=1}^{k} m_j(n)\ell_j = \sum_{i=1}^{n} y_i \begin{cases} \geq a^* \Rightarrow H_1 \\ \leq b^* \Rightarrow H_0 \\ \in (b^*, a^*) \Rightarrow \text{take another sample,} \end{cases} \tag{44a}$$

and, for $n = M^*$, we test

$$\sum_{j=1}^{k} m_j(n)\ell_j = \sum_{i=1}^{M^*} y_i \begin{cases} \geq t^* \Rightarrow H_1 \\ < t^* \Rightarrow H_0 \end{cases} \tag{44b}$$

where $m_j(n)$ is the number of y_i's taking the value ℓ_j, with $\sum_{j=1}^{k} m_j(n) = n$ at each stage. To obtain a truncated sequential probability ratio test on $y_1, y_2, \ldots$, we set $\ell_j = \ln(p_j(\theta_1)/p_j(\theta_0))$, for $j=1, \ldots, k$.

Define

$$\mu_\theta^k = E(Y_i|\theta) = \sum_{j=1}^{k} \ell_j \, p_j(\theta) \tag{45}$$

and

$$m_\theta^k = E(Y_i^2|\theta) = \sum_{j=1}^{k} (\ell_j)^2 \, p_j(\theta) \quad . \tag{46}$$

It is shown in the Appendix that, if $f(x)$ is continuous for all x, then

$$\mu_\theta^k = (\theta_1 - \theta_0)^2 (r - 0.5) e(k) + o((\theta_1 - \theta_0)^2) \tag{47a}$$

and

$$m_\theta^k = (\theta_1 - \theta_0)^2 \, e(k) + o((\theta_1 - \theta_0)^2) \; , \tag{47b}$$

where

$$e(k) = \sum_{j=1}^{k} \frac{(f(s_j - \theta_0) - f(s_{j-1} - \theta_0))^2}{F(s_j - \theta_0) - F(s_{j-1} - \theta_0)} \quad . \tag{48}$$

We note that $e(k)$ given by (48) is the detection efficacy of a k-level quantizer-detector [11] and can be thought of as a discrete equivalent to the Fischer information number $I(f)$.

Once we have established (47), we can argue that $\lim_{\theta_1 \to \theta_0} [(\theta_1-\theta_0)^2 e(k)E(N_k^*|\theta)/2]$ is given by expression (30a) or (30b), where N_k^* is the sample size of the truncated test (44). Thus by this result and that of Section IV, if f satisfies assumptions A1 through A3, we have

$$\lim_{\theta_1 \to \theta_0} \left[\frac{E(N_k^*|\theta)}{E(N^*|\theta)} \right] = \frac{I(f)}{e(k)} . \tag{48}$$

Therefore the truncated sequential test (44) using quantized data has the same asymptotic efficiency relative to the truncated sequential test (10) using unquantized data, as does an FSS test based on the same quantized data compared to an FSS test based on unquantized data. In other words, the percentage of (asymptotic) savings in sample size of a truncated sequential test over an FSS test with both using quantized data is the same as the (asymptotic) saving of a truncated sequential test over an FSS test with both using unquantized data. As in Section IV, the asymptotic power functions of (32)-(34) also hold for tests with k-level quantized data as well. Therefore, the conclusions of Section III carry over for FSS, SPRT, and truncated sequential tests based on quantized data, as do results in earlier works which compare quantized FSS tests to unquantized FSS tests [11].

To assess the accuracy of the approximate expressions for ASN and OC functions derived in Section II, it is interesting to consider the case in which $k = 2$ and $s_1 = (\theta_1 + \theta_0)/2$. Assuming the density $f(x)$ is symmetric about $x = 0$, we then have (from (43)) that $p_1(\theta) = 1-p_2(\theta)$ and $p_2(\theta_0) = 1-p_2(\theta_1)$. For this case, exact values of $E(N^*|\theta)$, $\mathrm{Var}_\theta(N^*)$, and $\beta_T(\theta)$ can be computed using results from [12,13]. Table III compares these exact values with the approximations based on Brownian motion for the case $p_2(\theta_1) = 0.7$ and $a^* = -b^* = 10\ \ell n\ ((1-p_2(\theta_1))/p_2(\theta_1))$ with truncation points $M^* = 25$ and $M^* = 41$. Note that θ appears here only through the value of $p_2(\theta)$. It can be seen from this table that the approximations are all reasonably good in this case, especially that for the ASN.

VI. CONCLUSIONS

In this paper we have considered truncated sequential location testing with constant boundaries and abrupt truncation. Design procedures for the two constant boundaries, the truncation point, and the threshold have been given for nominal error probabilities under H_0 and H_1. These procedures are based on treating a truncated sequential test as a mixture of a sequential probability ratio test and a fixed sample size test. Formulae for the operating-characteristic function and the average-sample-number function of the proposed tests have been given; and, although these results hold in an asymptotic sense, they may be used as approximations for the nonasymptotic case. An example comparing exact and approximate values was given in Section V. In this example the approximations were good; however, the general accuracy of these approximations is a topic for further study. If the test statistic converges rapidly to a Brownian motion, then the approximations should be good for moderate parameter values. Note for example that, if in the example of Section V f(x) is a Gaussian density with variance K^2, then $f_2(\theta_1) = 0.7$ corresponds to a signal-to-noise ratio, $(\theta_1-\theta_0)^2/K^2$, of approximately 1.1, which is moderate.

The numerical results of Section III demonstrate that a properly designed truncated sequential test can retain the advantage of sample savings of the SPRT under the hypothesis and the alternative while it eliminates the disadvantages of the SPRT of possible large sample size when the true location parameter is different from those assumed for the hypothesis and the alternative. For given error probabilities, the truncated sequential test has a uniformly smaller ASN function than a corresponding FSS test while the ASN's under H_0 and H_1 are close to those optimum values of the SPRT. Therefore, the truncated sequential test should be preferred to the SPRT if long runs cannot be tolerated and if parameter mismatching is possible, and it should be preferred to the FSS test if the small amount of additional complexity required for the truncated sequential test can be tolerated.

Before concluding, we remark that performance comparisons between SPRT's and FSS tests have been investigated in several studies including [3] and [14]-[16]. Also, the relative performance of two non-truncated sequential tests with the same constant boundaries has been investigated by Lai [17]. It was shown there that the relevant asymptotic relative efficiency is given by the ratio of the efficacies of the two test statistics, as is the case when comparing two FSS tests [18]. Note that in [17] the tests under comparison have the same decision boundaries and only the test statistics are different. However, in our study, we have compared tests with the same test statistic, namely the probability ratio, but with different decision boundaries: an FSS test with a fixed number of samples, an SPRT with two fixed boundaries, and a truncated sequential test with two fixed boundaries (different from those of the SPRT) and a truncated sample size. Asymptotic (in the sense that the alternative approaches the hypothesis) comparison between a truncated sequential test and an FSS test or an SPRT has not been previously investigated. This work is, therefore, complementary to the previous works mentioned above. Finally, we note that Berk [19] has studied asymptotic efficiencies of sequential tests in a different sense; in particular, the asymptotics in [19] are as the error probabilities approach zero.

APPENDIX: DERIVATION OF EQ. (47)

To show (47a) we write

$$\mu_\theta^k = \sum_{j=1}^{k} (p_j(\theta_0) + \Delta_{j\theta})\ell n\, (1 + \Delta_j/p_j(\theta_0)) \tag{A. 1}$$

where $\Delta_{j\theta} = (F(s_j - \theta) - F(s_j - \theta_0)) - (F(s_{j-1} - \theta) - F(s_{j-1} - \theta_0))$ and $\Delta_j = (F(s_j - \theta_1) - F(s_j - \theta_0)) - (F(s_{j-1} - \theta_1) - F(s_{j-1} - \theta_0))$. We expand $\ell n(1 + \Delta_j/p_j(\theta_0)) = (\Delta_j/p_j(\theta_0)) - 0.5(\Delta_j/p_j(\theta_0))^2 + o(\Delta_j^2)$, where $\lim o(\Delta_j^2)/\Delta_j^2 = 0$ as $\Delta_j \to 0$. Using this expansion, we can write

$$\mu_\theta^k = \sum_{j=1}^{k} \Delta_j + \sum_{j=1}^{k} \frac{\Delta_{j\theta}\Delta_j}{p_j(\theta_0)} - \tfrac{1}{2} \sum_{j=1}^{k} \frac{\Delta_j^2}{p_j(\theta_0)} - \tfrac{1}{2} \sum_{j=1}^{k} \frac{\Delta_j^2\Delta_{j\theta}}{(p_j(\theta_0))^2} + \sum_{j=1}^{k} o(\Delta_j^2). \tag{A . 2}$$

Since $\sum_{j=1}^{k} p_j(\theta_1) = \sum_{j=1}^{k} p_j(\theta_0) = 1$, we must have $\sum_{j=1}^{k} \Delta_j = 0$. Now, using a Taylor Series expansion of $F(x - \theta_1)$, we have

$$\begin{aligned}\Delta_j &= F(s_j - \theta_1) - F(s_j - \theta_0) - F(s_{j-1} - \theta_1) + F(s_{j-1} - \theta_0) \\ &= F(s_j - \theta_0) - (\theta_1 - \theta_0)f(s_j - \theta^*) - F(s_j - \theta_0) - F(s_{j-1} - \theta_0) + (\theta_1 - \theta_0)f(s_{j-1} - \theta^{**}) + \\ &\qquad + F(s_{j-1} - \theta_0) \\ &= -(\theta_1 - \theta_0)(f(s_j - \theta^*) - f(s_{j-1} - \theta^{**}))\,, \end{aligned} \tag{A. 3}$$

where $\theta^*, \theta^{**} \in (\theta_0, \theta_1)$, and, similarly,

$$\Delta_{j\theta} = -(\theta - \theta_0)(f(s_j - \theta_*) - f(s_{j-1} - \theta_{**}))\,, \tag{A. 4}$$

where $\theta_*, \theta_{**} \in (\theta_0, \theta)$. With these values we have

$$\frac{\Delta_{j\theta}\Delta_j}{p_j(\theta_0)} = r(\theta_1-\theta_0)^2 \frac{(f(s_j-\theta_*)-f(s_{j-1}-\theta_{**}))(f(s_j-\theta^*)-f(s_{j-1}-\theta^{**}))}{F(s_j-\theta_0)-F(s_{j-1}-\theta_0)}, \tag{A. 5}$$

$$\frac{\Delta_j^2}{p_j(\theta_0)} = (\theta_1-\theta_0)^2 \frac{(f(s_j-\theta^*)-f(s_{j-1}-\theta^{**}))^2}{F(s_j-\theta_0)-F(s_{j-1}-\theta_0)}, \tag{A. 6}$$

and

$$\frac{\Delta_{j\theta}\Delta_j^2}{(p_j(\theta_0))^2} = r(\theta_1-\theta_0)^3 \frac{(f(s_j-\theta_*)-f(s_{j-1}-\theta_{**}))(f(s_j-\theta^*)-f(s_{j-1}-\theta^{**}))^2}{(F(s_j-\theta_0)-F(s_{j-1}-\theta_0))^2}$$

$$= o((\theta_1-\theta_0)^2), \tag{A. 7}$$

where $r = (\theta-\theta_0)/(\theta_1-\theta_0)$. In addition, we have $o(\Delta_j^2) = o((\theta_1-\theta_0))^2$; therefore

$$\lim_{\theta_1\to\theta_0} \frac{\mu_\theta^k}{(\theta_1-\theta_0)^2} = (r-\tfrac{1}{2}) \sum_{j=1}^{k} \frac{(f(s_j-\theta_0)-f(s_{j-1}-\theta_0))^2}{F(s_j-\theta_0)-F(s_{j-1}-\theta_0)} \tag{A. 8}$$

since f(x) is continuous, and (47a) follows.

For (47b), we write

$$m_\theta^k = \sum_{j=1}^{k} (p_j(\theta_0)+\Delta_{j\theta})\,[\ell n(1+\Delta_j/p_j(\theta))]^2 \tag{A. 9}$$

With $[\ell n(1+\Delta_j/p_j(\theta_0))]^2 = (\Delta_j/p_j(\theta_0))^2 + o(\Delta_j^2)$, we have

$$m_\theta^k = \sum_{j=1}^{k} \frac{\Delta_j^2}{p_j(\theta_0)} + \sum_{j=1}^{k} \frac{\Delta_{j\theta}\Delta_j^2}{(p_j(\theta_0))^2} + o(\Delta_j^2). \tag{A.10}$$

With (A. 6) and (A. 7) for $\Delta_j^2/p_j(\theta_0)$ and $\Delta_{j\theta}\Delta_j^2/(p_j(\theta_0))^2$, (47b) follows.

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List of Tables

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$r = \frac{\theta - \theta_0}{\theta_1 - \theta_0}$	$\alpha = 1-\beta = .01$		$\alpha = 1-\beta = .001$			$\alpha = .0001 \quad 1-\beta = .0005$		
	$\mu_1^2 \mathrm{Var}_\theta(N)$	$\mu_1^2 \mathrm{Var}_\theta(N^*)$	$\mu_1^2 \mathrm{Var}_\theta(N)$	$\mu_1^2 \mathrm{Var}_\theta(N^*)$		$\mu_1^2 \mathrm{Var}_\theta(N)$	$\mu_1^2 \mathrm{Var}_\theta(N^*)$	
		$c_1 = c_2 = .9$		$c_1 = c_2 = .9$	$c_1 = c_2 = .5$		$c_1 = c_2 = .9$	$c_1 = .83,\ c_2 = .1$
0.0	8.2	7.5	13.6	15.5	14.2	15.2	18.6	15.2
0.25	32.1	7.6	81.0	24.9	36.4	106.2	43.4	61.6
0.45	71.3	4.9	345.5	14.7	34.6	717.0	25.8	70.9
0.55	71.3	4.9	345.5	14.7	34.6	733.7	20.9	57.4
0.75	32.1	7.6	81.0	24.9	36.4	121.1	37.0	57.8
1.0	8.2	7.5	13.6	15.5	14.2	18.3	20.9	21.2

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Table I (Tantaratana and Poor)

Design Values		c_1	c_2	Error Probabilities		Percentage of α and $1-\beta$
α	$1-\beta$			α^*	$1-\beta^*$	
0.01	0.01	0.9	0.9	0.0093	0.0093	93%
0.001	0.001	0.9	0.9	0.00095	0.00095	95%
0.001	0.001	0.5	0.5	0.00090	0.00090	90%
0.0001	0.0005	0.9	0.9	0.000096	0.00048	96%
0.0001	0.0005	0.83	0.1	0.000088	0.00048	88%, 96%

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Table II (Tantaratana and Poor)

	$p_2(\theta)$	Exact Values			Approximation by Brownian Motion		
		$E(N^* \mid \theta)$	$Var_\theta(N^*)$	$\beta_T(\theta)$	$E(N^* \mid \theta)$	$Var_\theta(N^*)$	$\beta_T(\theta)$
$M^*=25$	0.3	20.7	22.5	0.0175	20.6	24.7	0.0145
	0.4	23.6	10.2	0.1538	23.5	11.9	0.1537
	0.5	24.5	4.0	0.5000	24.4	5.0	0.5000
	0.6	23.6	10.2	0.8462	23.5	11.9	0.8463
	0.7	20.7	22.5	0.9825	20.6	24.7	0.9855
$M^*=41$	0.3	24.1	81.6	0.0036	24.1	85.1	0.0026
	0.4	33.5	85.6	0.0968	33.5	89.7	0.0956
	0.5	37.9	45.7	0.5000	37.8	49.2	0.5000
	0.6	33.5	85.6	0.9032	33.5	89.7	0.9044
	0.7	24.1	81.6	0.9964	24.1	85.1	0.9974

FP-7380

Table III (Tantaratana and Poor)

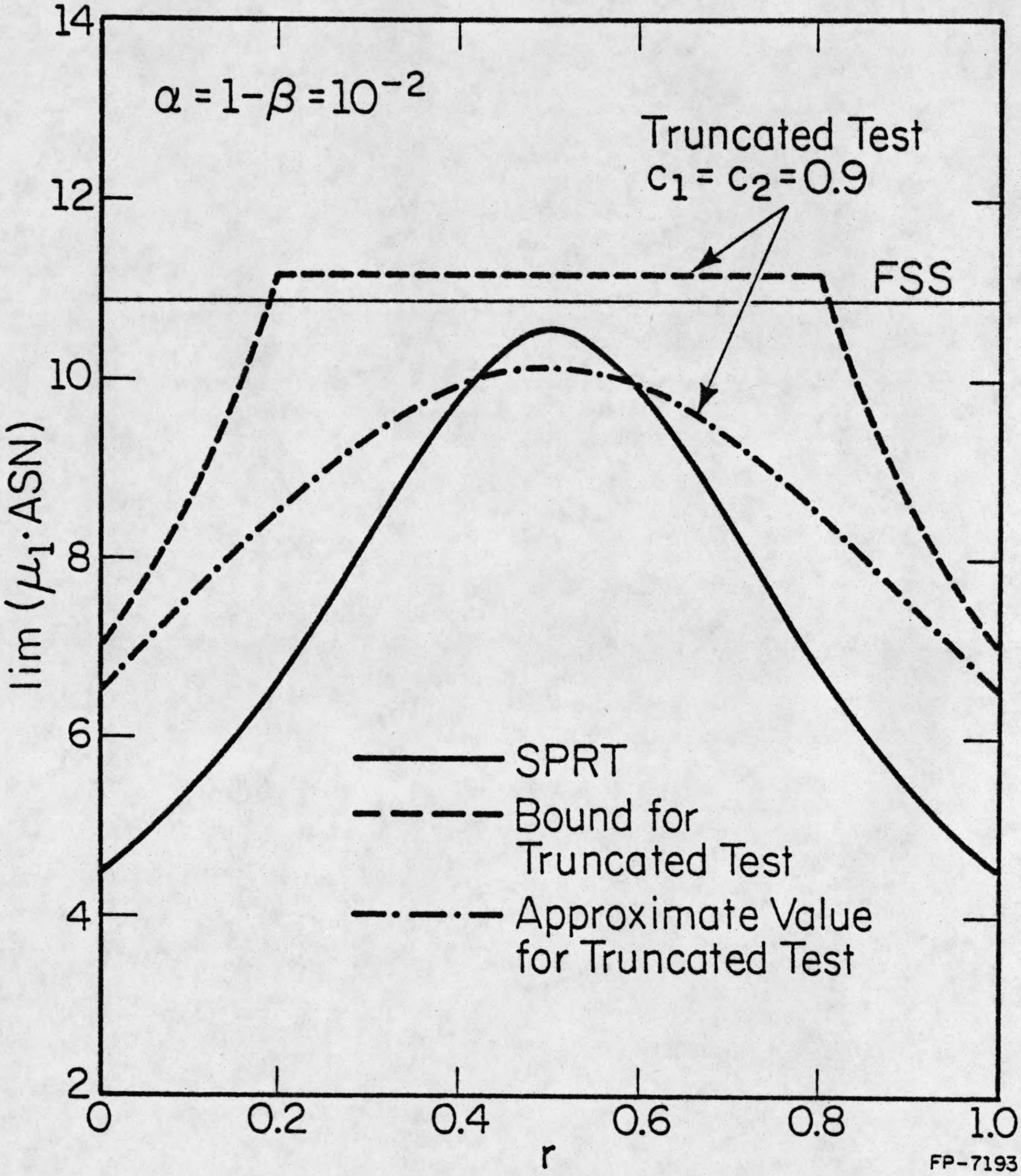

Figure 1 (Tantaratana and Poor)

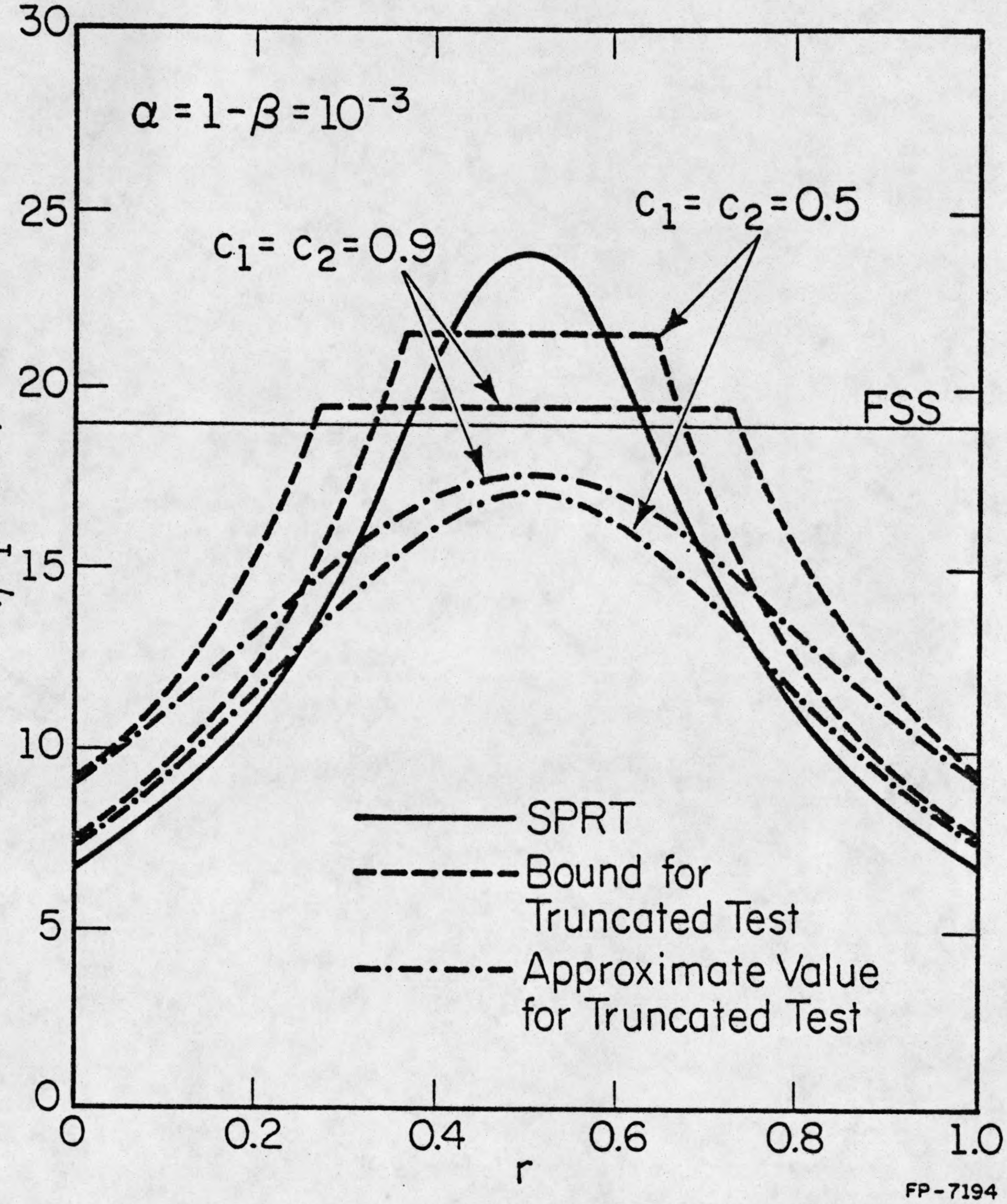

Figure 2 (Tantaratana and Poor)

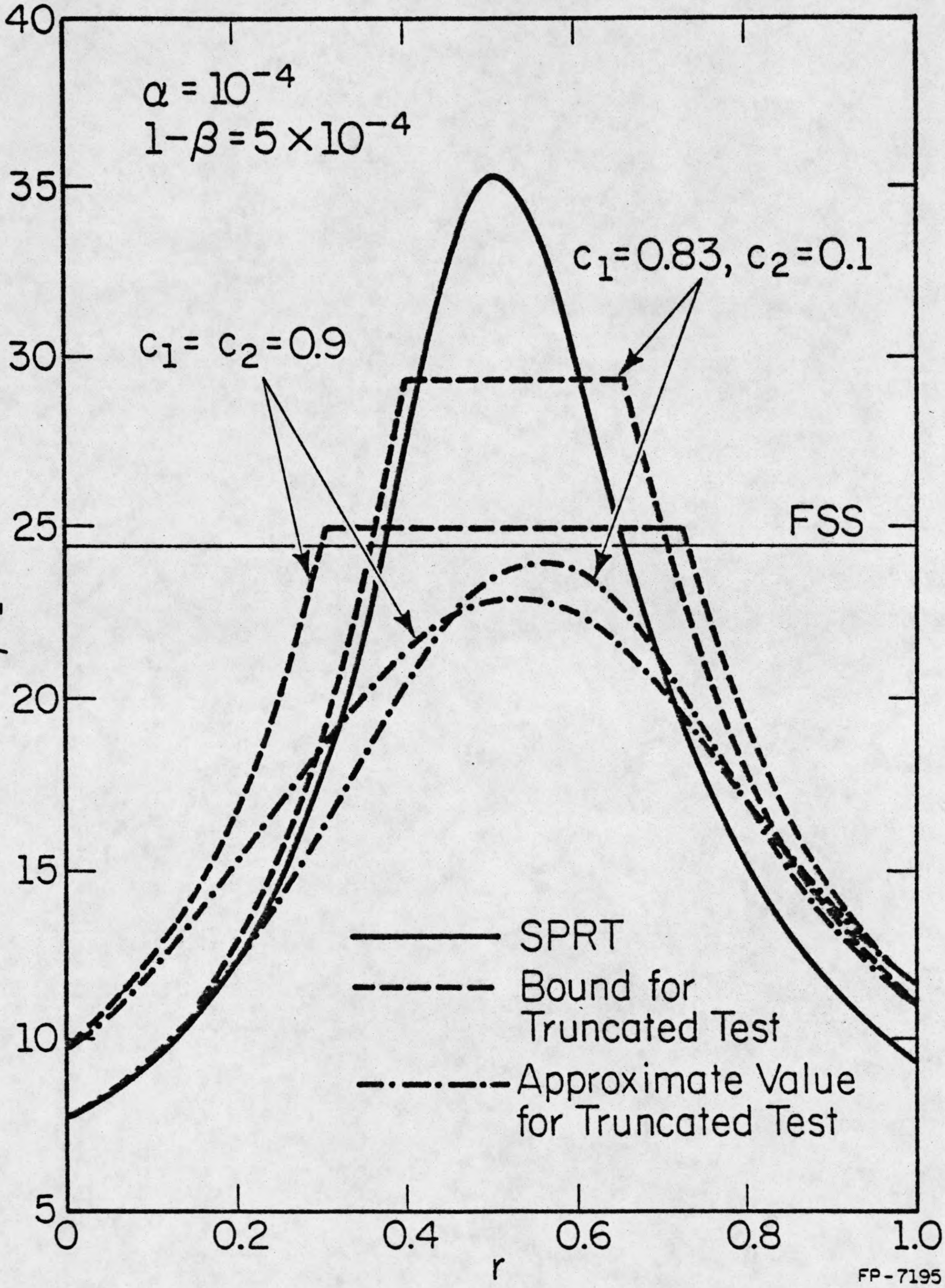

Figure 3 (Tantaratana and Poor)

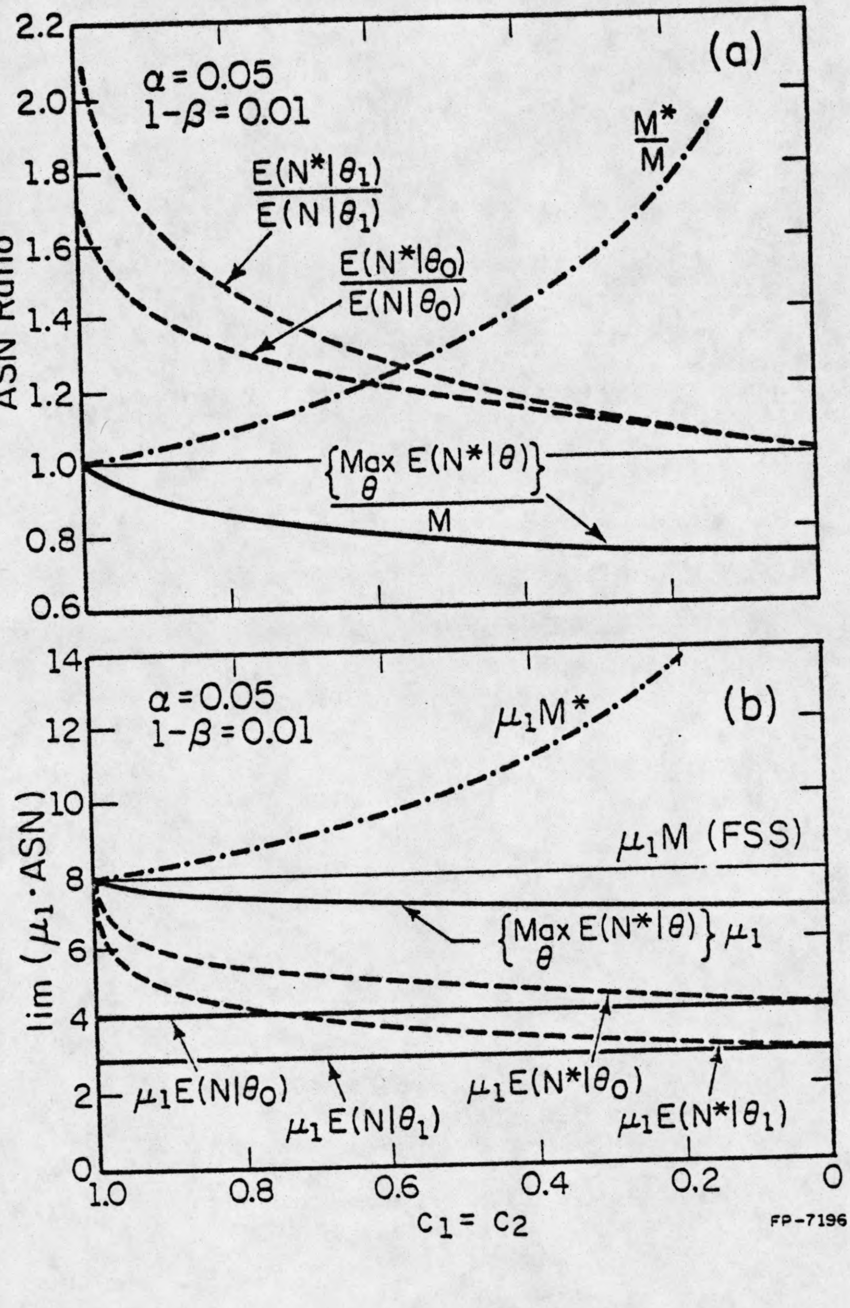

Figure 4 (Tantaratana and Poor)

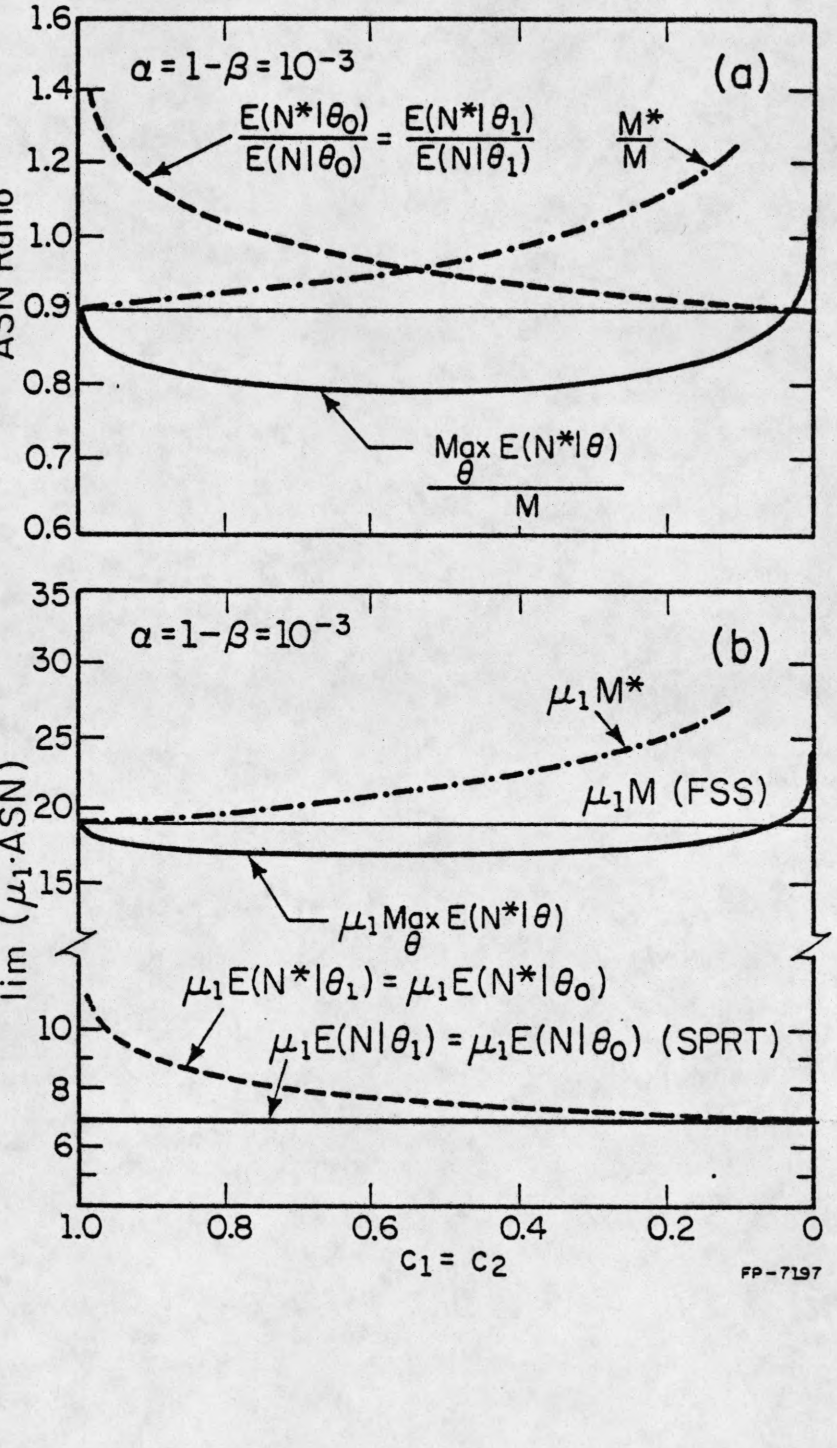

Figure 5 (Tantaratana and Poor)

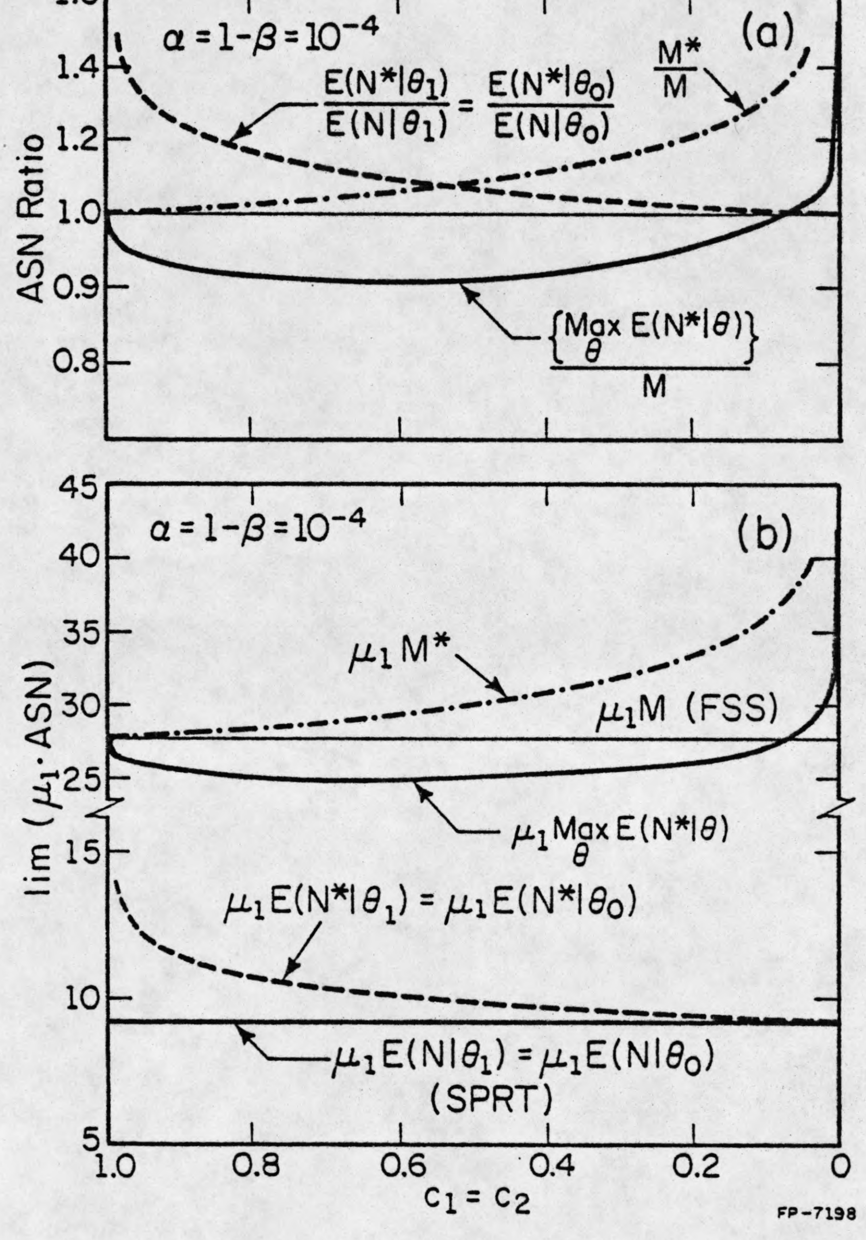

Figure 6 (Tantaratana and Poor)

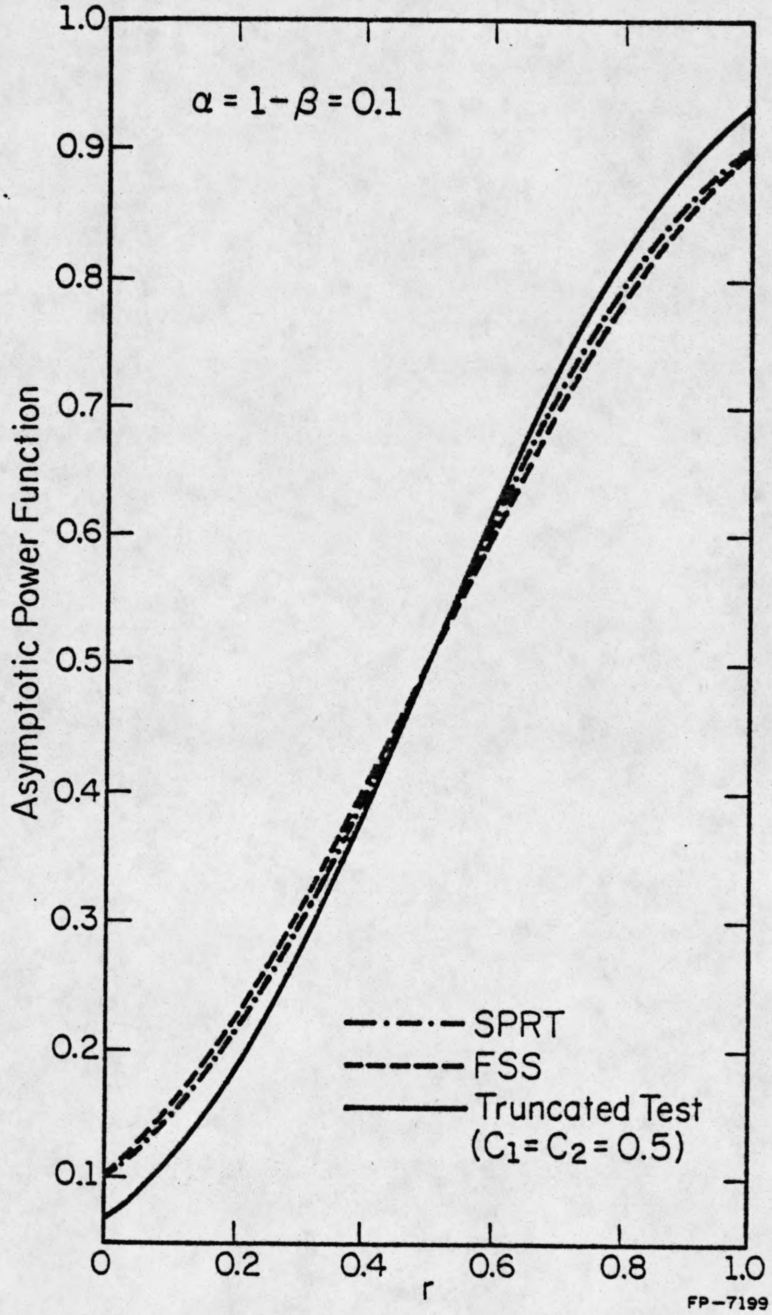

Figure 7 (Tantaratana and Poor)

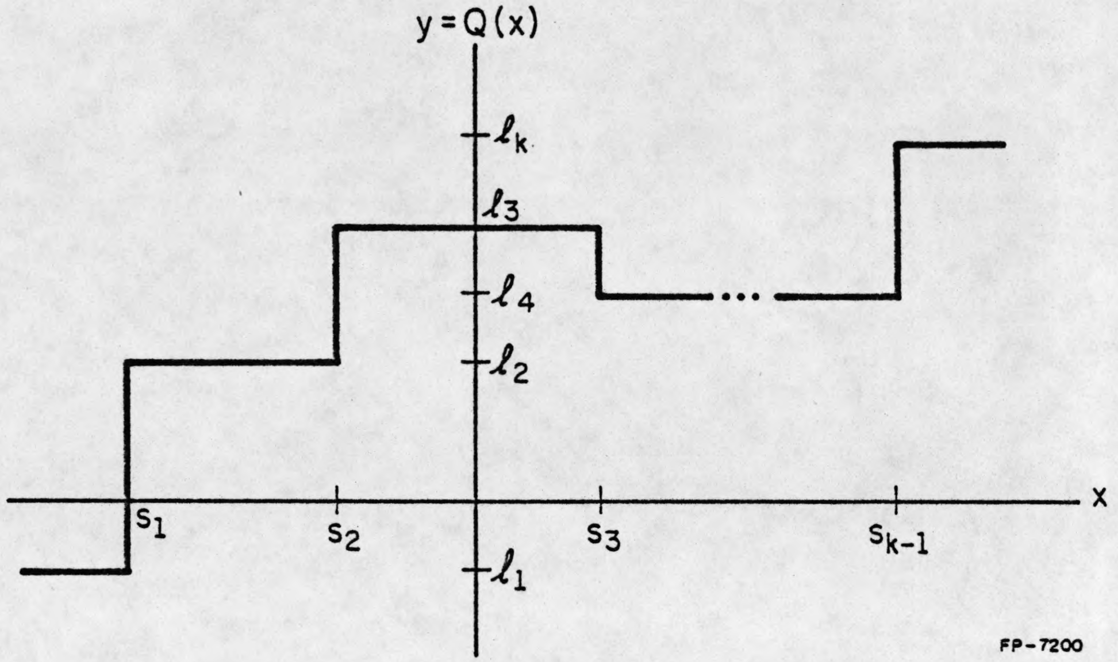

Figure 8 (Tantaratana and Poor)