Multiple-Antennas and Isotropically-Random Unitary Inputs: The Received Signal Density in Closed-Form

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Abstract — **Computing the capacity of a multiple antenna wireless link subject to flat Rayleigh block-fading, with no channel-state information available either to the transmitter or to the receiver, is an important open problem. The isotropically-random (i.r.) unitary matrix–having orthonormal columns, and a probability density that is invariant to pre-multiplication by an independent unitary matrix–plays a central role in the calculation of capacity and in some special cases is capacity-achieving. In this paper we take an important step towards computing this capacity by obtaining, in *closed-form*, the probability density of the received signal when transmitting i.r. unitary matrices. This enables us to evaluate the mutual information for any case of interest, something that could previously only be done for single transmit and receive antennas. Simulation results show that at high SNR the mutual information is maximized for $M = \min(N, T/2)$ transmit antennas, where N is the number of receive antennas and T is the length of the coherence interval, whereas at low SNR the mutual information is maximized by allocating all transmit power to a single antenna.**

I. Summary

Consider a single-user block-fading multiple antenna link with M transmit and N receive antennas described by a propagation matrix that is constant during coherence intervals of length T symbols, after which it jumps to a new independent value. During any coherence interval, a $T \times M$ complex matrix S is transmitted, and a $T \times N$ complex matrix X, is received

$$X = \sqrt{\frac{\rho}{M}} SH + W, \qquad (1)$$

where H is a $M \times N$ propagation matrix, and W is a $T \times N$ additive noise matrix. The elements of H and W are independent, zero-mean, circularly symmetric, complex Gaussian with unit variance, and are unknown to both the transmitter and the receiver. The entries of S are assumed to have, on the average, unit variance so that ρ is equal to the SNR.

The assumption of no CSI–as opposed to CSI available at the receiver–converts the problem of computing capacity from a straightforward problem in Shannon theory into a rather difficult, albeit more realistic, one [1, 2]. The isotropically-random (i.r.) unitary matrix plays a central role in this problem and is capacity-achieving in certain limiting regimes.

The contribution of this paper is a closed-form expression for the probability density of the received signal when transmitting i.r. inputs. This allows us to compute mutual information for any case of interest.

Theorem 1 *Consider the channel model (1). Then if $S = \sqrt{T}\Phi$, where Φ is an isotropically-distributed unitary matrix, we have*

$$I(X;\Phi) = \rho TN \log e - \log \frac{\Gamma(T)\cdots\Gamma(T+1-M)}{\Gamma(M)\cdots\Gamma(1)} - \mathrm{E} \log \det F,$$

where F is the $M \times M$ Hankel matrix with entries

$$F_{mn} = \sum_{k=1}^{K} \frac{(-1)^{m+n-2}}{\prod_{l \neq k}(\alpha\sigma_k - \alpha\sigma_l)} \sum_{q=0}^{\infty} \frac{(\alpha\sigma_k)^q}{(q+T-(m+n+K)+2)!}$$

where $\alpha = \frac{\rho TM}{1+\rho TM}$, $K = \min(T, N)$, and the σ_k are the nonzero eigenvalues of $XX^\dagger$.

Analytic evaluation of the expectation in the mutual information expression of Theorem 1 appears formidable. However, numerical evaluation is straightforward via Monte-Carlo integration. An example is shown in the figure below.
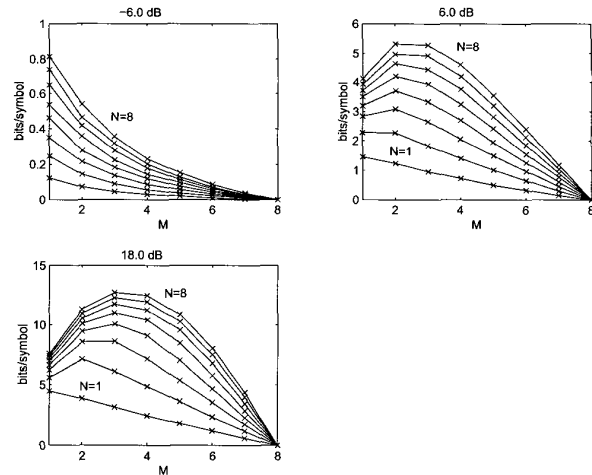
References

[1] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE Trans. Info. Theory*, vol. 45, pp. 139-157, 1999.

[2] L. Zheng and D. Tse, "Packing spheres in the Grassman manifold: a geometric approach to the noncoherent multi-antenna channel," *submitted to IEEE Trans. Info. Theory*, 2000.