The capacity of the quantum depolarizing channel

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Abstract

The information carrying capacity of the d-dimensional depolarizing channel is computed. It is shown that this capacity can be achieved by encoding messages as products of pure states belonging to an orthonormal basis of the state space, and using measurements which are products of projections onto this same orthonormal basis. In other words, neither entangled signal states nor entangled measurements give any advantage for information capacity. The result follows from an additivity theorem for the product channel $\Delta \otimes \Psi$, where Δ is the depolarizing channel and Ψ is a completely arbitrary channel. We establish the Amosov-Holevo-Werner p-norm conjecture for this product channel for all $p \geq 1$, and deduce from this the additivity of the minimal entropy and of the Holevo quantity χ^* .

1 Background and statement of results

1.1 Introduction

This paper computes the capacity of the d-dimensional quantum depolarizing channel for transmission of classical information. The result confirms a long-standing conjecture, namely that the best rate of information transfer can be achieved without any entanglement across multiple uses of the channel. It is sufficient to choose an orthonormal basis for the state space, and then use this basis both to encode messages as product states at the input side and to perform measurements at the output side which project onto this same basis. In this sense the depolarizing channel can be treated as a classical channel.

The Holevo-Schumacher-Westmoreland theorem allows the capacity to be expressed in terms of the Holevo quantity χ^* . If the Holevo quantity is additive, then this expression implies that the capacity is actually equal to χ^* . In this paper we prove several additivity properties for the depolarizing channel, including the additivity of χ^* . One important mathematical tool used in the proof is the Lieb-Thirring inequality, which provides a bound for the non-commutative p-norm of a product of positive matrices. These notions, as well as the definition of channel capacity and its relation to the Holevo quantity, are described in the following subsections.

1.2 The depolarizing channel

The depolarizing channel is a particularly simple model for noise in quantum systems [11], and has been studied in a variety of contexts [3, 5, 6]. In d dimensions the model is implemented by a completely positive trace-preserving map Δ_{λ} , depending on one real parameter λ , which maps a state ρ on \mathbf{C}^d into a linear combination of itself and the $d \times d$ identity matrix I:

$$\Delta_{\lambda}(\rho) = \lambda \, \rho + \frac{1 - \lambda}{d} \, I \tag{1}$$

The condition of complete positivity requires that λ satisfy the bounds

$$-\frac{1}{d^2 - 1} \le \lambda \le 1\tag{2}$$

The channel Δ_{λ} maps a pure input state to a mixed output state. Because the channel is highly symmetric, all such output states are unitarily equivalent,

and have eigenvalues $\lambda + (1 - \lambda)/d$ (with multiplicity 1) and $(1 - \lambda)/d$ (with multiplicity d - 1).

1.3 Measures of noisiness

We will use three measures of noisiness for quantum channels, namely the minimal output entropy S_{\min} , the maximal output p-norm ν_p , and the Holevo quantity χ^* . First recall that a state ρ is a positive operator with trace equal to 1, and its von Neumann entropy is defined by

$$S(\rho) = -\text{Tr}\rho\log\rho\tag{3}$$

The minimal output entropy of the channel Ψ is defined as

$$S_{\min}(\Psi) = \inf_{\rho} S(\Psi(\rho)) \tag{4}$$

Recall also that the p-norm of a positive matrix A is defined for $p \geq 1$ by

$$||A||_p = \left(\text{Tr}A^p\right)^{1/p} \tag{5}$$

Amosov, Holevo and Werner [2] introduced the notion of the maximal p-norm of a channel as a way to characterize its noisiness. This quantity is defined as

$$\nu_p(\Psi) = \sup_{\rho} ||\Psi(\rho)||_p \tag{6}$$

Since the entropy of a state is the negative of the derivative of the p-norm at p = 1, it follows that for any channel Ψ ,

$$\left. \frac{d}{dp} \nu_p(\Psi) \right|_{p=1} = -S_{\min}(\Psi) \tag{7}$$

The third measure of noisiness, the Holevo quantity, is closely related to the information-carrying capacity of the channel, as will be explained in the next section. We will use the symbol \mathcal{E} to denote an ensemble of input states for the channel, that is a collection of states ρ_i together with a probability distribution π_i . The Holevo quantity is

$$\chi^*(\Psi) = \sup_{\mathcal{E}} \left[S(\Psi(\rho)) - \sum_i \pi_i S(\Psi(\rho_i)) \right]$$
 (8)

where $\rho = \sum \pi_i \rho_i$ is the average input state of the ensemble.

These three measures can be easily computed for the depolarizing channel. The values are

$$S_{\min}(\Delta_{\lambda}) = -\left(\lambda + \frac{1-\lambda}{d}\right)\log\left(\lambda + \frac{1-\lambda}{d}\right) - (d-1)\left(\frac{1-\lambda}{d}\right)\log\left(\frac{1-\lambda}{d}\right) \tag{9}$$

$$\nu_p(\Delta_\lambda) = \left[\left(\lambda + \frac{1 - \lambda}{d} \right)^p + (d - 1) \left(\frac{1 - \lambda}{d} \right)^p \right]^{1/p} \tag{10}$$

$$\chi^*(\Delta_\lambda) = \log d - S_{\min}(\Delta_\lambda) \tag{11}$$

The value (11) is achieved by choosing an ensemble \mathcal{E} consisting of pure states belonging to an orthonormal basis (it does'nt matter which one), and choosing the uniform distribution $\pi_i = 1/d$. Since the average input state for this ensemble is (1/d) I, the terms on the right side of (8) are separately maximized for this choice of ensemble, and this leads to the result (11).

1.4 Additivity conjectures

It is conjectured that S_{\min} and χ^* are both additive for product channels. This would mean that for any channels Ψ_1 and Ψ_2 ,

$$S_{\min}(\Psi_1 \otimes \Psi_2) = S_{\min}(\Psi_1) + S_{\min}(\Psi_2) \tag{12}$$

and

$$\chi^*(\Psi_1 \otimes \Psi_2) = \chi^*(\Psi_1) + \chi^*(\Psi_2)$$
(13)

Equivalently, the conjectures would imply that for product channels both of these measures of noisiness are achieved with product input states. As we explain in the next section, the Holevo quantity χ^* is related to channel capacity, and in fact (13) would imply that the channel capacity of an arbitrary channel Ψ is precisely $\chi^*(\Psi)$. These conjectures have been established for some special classes of channels, including all unital qubit channels [9] and all entanglement-breaking channels [13]. However a general proof has remained elusive.

It was further conjectured in [2] that the quantity ν_p should be multiplicative for product channels for all $p \geq 1$, reflecting the idea that the p-norm of a product channel would be maximized with product states. This conjecture

would imply (12), since the entropy can be obtained from the derivative of the p-norm at p=1. Again this conjecture has been established for some special classes of channels, in particular for unital qubit channels [9]. However it is now known that the conjecture does not hold in general – this was demonstrated recently by the discovery of a family of counterexamples for values $p \geq 5$ [16]. Nevertheless, in this paper we will prove that the AHW conjecture is true for any product channel of the form $\Delta_{\lambda} \otimes \Psi$ where Ψ is arbitrary, and we will show how this property implies the additivity of S_{\min} and of χ^* for such a product channel.

1.5 Channel capacity

In order to relate these additivity results to the channel capacity problem, we recall first the definition of the capacity for a general channel Ψ . Again we denote by \mathcal{E} an ensemble of input states, and we also denote by \mathcal{M} a measurement, or POVM, at the output side of the channel. Recall that a POVM is a collection of positive operators $\{E_j\}$ which sum to the identity matrix. When a state ρ is measured using a POVM $\{E_j\}$, the outcome j is obtained with probability $\text{Tr}(\rho E_j)$. This notion generalizes the familiar von Neumann measurement, which is the special case where the operators E_j are orthogonal projections.

The ensemble \mathcal{E} , the POVM \mathcal{M} and the channel Ψ together define a classical noisy channel, whose transition matrix is

$$p_{ij} = \text{Tr}\Big[\Psi(\rho_i)E_j\Big] \tag{14}$$

If we write X for a random input signal with distribution $\pi_i = P(X = i)$, then the output signal Y from this classical channel has distribution

$$P(Y = j) = \sum_{i} \pi_{i} p_{ij} \tag{15}$$

The Shannon capacity of a classical noisy channel measures the maximum rate at which information can be reliably transmitted through the channel. Shannon's formula computes this capacity as the maximum of the mutual information I(X,Y) between an input signal X and its corresponding output signal Y given by (15), where the maximum is evaluated over all choices of distribution $\{\pi_i\}$ for X. For the case of a quantum channel, we are interested in the maximum rate that can be achieved using the optimal choices of input states $\{\rho_i\}$ and of output

measurements $\{E_j\}$. Therefore we are led to define the Shannon capacity of the quantum channel Ψ as

$$C_{\text{Shan}}(\Psi) = \sup_{\mathcal{E}, \mathcal{M}} I(X, Y)$$
 (16)

where the distribution of the input signal X is determined by the ensemble \mathcal{E} , that is $P(X = i) = \pi_i$, and where the output signal Y is determined by (15). This can be easily evaluated for the depolarizing channel. The mutual imformation I(X,Y) in (16) is maximized by choosing an ensemble consisting of projections onto an orthonormal basis, and using the same basis for the measurement. The result is

$$C_{\text{Shan}}(\Delta_{\lambda}) = \chi^*(\Delta_{\lambda}) \tag{17}$$

where the capacity $\chi^*(\Delta_{\lambda})$ is given by (11).

If two copies of the channel Ψ are available then it may be possible to achieve a higher rate of transmission by sharing the signals across the two channels. This possibility exists because quantum channels have an additional resource which is not available for classical channels, namely entangled states which can be used to encode signals for the product channel. It is also possible to make measurements at the output side using a POVM which projects onto entangled states. With these resources the best rate that can be achieved using two copies of the channel is

$$\frac{1}{2}C_{\text{Shan}}(\Psi \otimes \Psi) \tag{18}$$

It is known that in general (18) is larger than (16) [8, 4]. This observation leads to the question of finding the asymptototic capacity which would be achieved by sharing the input signals across an unlimited number of copies of Ψ . This ultimate capacity is given by

$$C_{\text{ult}}(\Psi) = \lim_{n \to \infty} \frac{1}{n} C_{\text{Shan}}(\Psi^{\otimes n})$$
(19)

(a standard subadditivity argument shows the existence of this limit).

At the present time it is an open problem to determine $C_{\text{ult}}(\Psi)$ for an arbitrary channel Ψ . However it can be expressed in terms of the Holevo quantity (8). Recall that the Holevo bound implies that

$$C_{\text{Shan}}(\Psi) \le \chi^*(\Psi),$$
 (20)

and hence that

$$C_{\text{ult}}(\Psi) \le \lim_{n \to \infty} \frac{1}{n} \chi^*(\Psi^{\otimes n}) \tag{21}$$

Furthermore the Holevo-Schumacher-Westmoreland theorem [7, 14] shows that the rate $\chi^*(\Psi)$ can be achieved with multiple copies of the channel, by restricting to product states for the input signals, but allowing entangled measurements at the outputs. Applying this theorem to the product channel $\Psi^{\otimes n}$ implies that the rate $(1/n) \chi^*(\Psi^{\otimes n})$ is achieved with input signals which may be entangled across n uses of the channel. Allowing n to be arbitrarily large leads to the result

$$C_{\text{ult}}(\Psi) = \lim_{n \to \infty} \frac{1}{n} \chi^*(\Psi^{\otimes n})$$
 (22)

1.6 Statement of results

Our first result is the evaluation of (19) for the depolarizing channel.

Theorem 1 The capacity of the d-dimensional depolarizing channel is

$$C_{\text{ult}}(\Delta_{\lambda}) = \chi^*(\Delta_{\lambda}) = C_{\text{Shan}}(\Delta_{\lambda}) = \log d - S_{\min}(\Delta_{\lambda})$$
 (23)

where $S_{\min}(\Delta_{\lambda})$ is evaluated in (9).

The fact that $C_{\text{ult}}(\Delta_{\lambda}) = C_{\text{Shan}}(\Delta_{\lambda})$ means that as far as the information-carrying properties of the depolarizing channel are concerned, there is no advantage gained by using either entangled input states or using entangled measurements. The optimal rate can be achieved by choosing an orthonormal basis (because Δ_{λ} is symmetric it does'nt matter which one) and using this basis to encode the signals and also to measure them. In this sense the channel behaves like a classical channel, and entanglement does not play any role in its capacity.

The basic ingredient in the proof of Theorem 1 is the additivity of the Holevo quantity χ^* for the depolarizing channel, which we state in the next Theorem.

Theorem 2 For any channel Ψ ,

$$\chi^*(\Delta_\lambda \otimes \Psi) = \chi^*(\Delta_\lambda) + \chi^*(\Psi) \tag{24}$$

Theorem 1 follows easily from this, as we now demonstrate. The result $C_{\text{ult}}(\Delta_{\lambda}) = \chi^*(\Delta_{\lambda})$ in Theorem 1 follows immediately from Theorem 2 by choosing $\Psi = \Delta_{\lambda}^{\otimes n}$ in (24) and applying (22). The second equality $\chi^*(\Delta_{\lambda}) = C_{\text{Shan}}(\Delta_{\lambda})$ was derived in (17).

Finally we state the AHW conjecture for the depolarizing channel, which underlies all the other results. Since the derivative of $\nu_p(\Psi)$ at p=1 is equal to $-S_{\min}(\Psi)$, the additivity of S_{\min} is a special case of the AHW conjecture. We state both results next in Theorem 3.

Theorem 3 For any channel Ψ , and any $p \geq 1$,

$$\nu_p(\Delta_\lambda \otimes \Psi) = \nu_p(\Delta_\lambda) \,\nu_p(\Psi) \tag{25}$$

and hence

$$S_{\min}(\Delta_{\lambda} \otimes \Psi) = S_{\min}(\Delta_{\lambda}) + S_{\min}(\Psi) \tag{26}$$

Special cases of Theorem 3 were previously established, namely for integer values of p in all dimensions d [1], and for all $p \ge 1$ in dimension d = 2 [9].

1.7 Organization

The paper is organised as follows. Section 2 outlines the proof of Theorem 3, and states two key results which are used, namely the convex decomposition of the depolarizing channel, and the bound for the phase-damping channel. These results are then established in Sections 3 and 4, and finally Theorem 2 is proved in Section 5. Section 3 also describes in detail the convex decompositions for the two-dimensional qubit depolarizing channel. Section 6 contains some discussion of the nature of the proof, and some ideas about further directions to pursue.

2 Outline of the proof

2.1 Definition of phase-damping channel

As discussed above, Theorem 1 follows immediately from Theorem 2, using the HSW Theorem (22). Theorem 2 itself is a slight extension of Theorem 3, and will be proved in Section 5. Most of the work in this paper goes into the proof

of the AHW conjecture (25) in Theorem 3. The proof presented here develops further the methods introduced in [9] where the same result was established for unital qubit channels. The basic idea is similar: we express the depolarizing channel Δ_{λ} as a convex combination of simpler channels, and then we prove a bound for these simpler channels which implies the result (25). In [9] the simpler channels were unitarily equivalent to phase-damping channels, and we use the same name for the channels here, which are defined as follows.

Definition 4 Let $\mathcal{B} = \{|\psi_i\rangle\}$ be an orthonormal basis, and let $E_i = |\psi_i\rangle\langle\psi_i|$. The phase-damping channel corresponding to \mathcal{B} is the one-parameter family of maps

$$\Phi_{\lambda}(\rho) = \lambda \rho + (1 - \lambda) \sum_{i=1}^{d} E_i \rho E_i$$
(27)

where the parameter λ satisfies the bounds

$$-\frac{1}{d-1} \le \lambda \le 1 \tag{28}$$

The parameter range (28) is required by the condition of complete positivity. If we write $\rho = (\rho_{ij})$ as a matrix in the basis $|\psi_i\rangle$ then the channel (27) acts by scaling the off-diagonal entries and leaving unchanged the diagonal entries, that is

$$\Phi_{\lambda}(\rho)_{ij} = \begin{cases} \rho_{ij} & \text{if } i = j\\ \lambda \rho_{ij} & \text{if } i \neq j \end{cases}$$
 (29)

We will express Δ_{λ} as a convex combination of phase-damping channels, all with the same parameter λ . Furthermore these phase-damping channels will all share a common property, which is expressed by the following definition.

Definition 5 We say that a vector $\mathbf{v} = (v_1, \dots, v_d)$ in \mathbf{C}^d is uniform if $|v_i| = |v_j|$ for all $i, j = 1, \dots, d$.

It will turn out that the phase-damping channels which arise in the convex decomposition are constructed from orthonormal bases $\mathcal{B} = \{|\psi_i\rangle\}$ where all the vectors $|\psi_i\rangle$ are uniform. Since each vector $|\psi_i\rangle$ is normalized, it follows that

all its entries have absolute value $1/\sqrt{d}$. As a consequence, if D is any diagonal matrix, then for any uniform state $|\psi\rangle$

$$\operatorname{Tr}\left[|\psi\rangle\langle\psi|D\right] = \langle\psi|D|\psi\rangle = \frac{1}{d}\operatorname{Tr}D\tag{30}$$

Definition 6 Let Φ be the phase-damping channel corresponding to the orthonormal basis \mathcal{B} . We say that Φ is **uniform** if $|\psi_i\rangle$ is uniform for every $|\psi_i\rangle \in \mathcal{B}$.

2.2 Three lemmas

There are three steps in the proof of Theorem 3. The goal is to find a bound for $||(\Delta_{\lambda} \otimes \Psi)(\tau_{12})||_p$ which will lead to (25), where Ψ is any other channel, and τ_{12} is any state. The first step is a partial diagonalization of the state τ_{12} . This step uses the following invariance property of the depolarizing channel.

Lemma 7 Let $\tau_1 = \text{Tr}_2(\tau_{12})$ denote the reduced density matrix of τ_{12} , and let U be a unitary matrix. Define $\tau'_{12} = (U \otimes I)\tau_{12}(U^* \otimes I)$. Then for all $p \geq 1$

$$||(\Delta_{\lambda} \otimes \Psi)(\tau_{12})||_{p} = ||(\Delta_{\lambda} \otimes \Psi)(\tau'_{12})||_{p}$$
(31)

Proof: the definition of Δ_{λ} in (1) implies that the unitary matrix $U \otimes I$ can be pulled through the channel $\Delta_{\lambda} \otimes \Psi$, and then the invariance of the *p*-norm implies (31). **QED**

The second step uses the following result which expresses Δ_{λ} as a convex combination of phase-damping channels. This result will be derived in Section 3.

Lemma 8 For $n = 1, ..., 2d^2(d+1)$, there are positive numbers c_n , unitary matrices U_n and uniform phase-damping channels $\Phi_{\lambda}^{(n)}$ such that for any state ρ

$$\Delta_{\lambda}(\rho) = \sum_{n=1}^{2d^{2}(d+1)} c_{n} U_{n}^{*} \Phi_{\lambda}^{(n)}(\rho) U_{n}$$
(32)

The third step in the proof uses the following bound for the phase-damping channels. This bound will be derived in Section 4.

Lemma 9 Let Φ_{λ} be a phase-damping channel defined as in (27), with corresponding orthogonal projectors $E_i = |\psi_i\rangle\langle\psi_i|$. For an arbitrary bi-partite state ρ_{12} define

$$\rho_2^{(i)} = \operatorname{Tr}_1 \left[(E_i \otimes I) \rho_{12} \right] \tag{33}$$

where Tr_1 is the trace over the first factor. Recall the factor $\nu_p(\Delta_\lambda)$ from (10). Then for all $p \geq 1$,

$$||(\Phi_{\lambda} \otimes I)(\rho_{12})||_{p} \le d^{(1-1/p)} \nu_{p}(\Delta_{\lambda}) \left[\sum_{i=1}^{d} \operatorname{Tr}(\rho_{2}^{(i)})^{p}\right]^{1/p}$$
 (34)

2.3 Proof of Theorem 3

We will now prove Theorem 3 using these three lemmas. Since the left side of (25) is at least as big as the right side, it is sufficient to prove that for any bipartite state τ_{12}

$$||(\Delta_{\lambda} \otimes \Psi)(\tau_{12})||_{p} \le \nu_{p}(\Delta_{\lambda}) \,\nu_{p}(\Psi) \tag{35}$$

The first step is to use Lemma 1 to partially diagonalize the state τ_{12} . Let U be a unitary matrix which diagonalizes the reduced density matrix $\tau_1 = \text{Tr}_2(\tau_{12})$, and let $\tau'_{12} = (U \otimes I)\tau_{12}(U^* \otimes I)$, so that $\tau'_1 = U\tau_1U^*$ is diagonal. By Lemma 7 we can replace τ_{12} by τ'_{12} without changing the left side of (35). Therefore we will assume henceforth without loss of generality that τ_1 is diagonal.

The second step is to apply the convex decomposition (32) on the left side of (35):

$$(\Delta_{\lambda} \otimes \Psi)(\tau_{12}) = \sum_{n=1}^{2d^2(d+1)} c_n \left(U_n^* \otimes I \right) \left(\Phi_{\lambda}^{(n)} \otimes \Psi \right) (\tau_{12}) \left(U_n \otimes I \right)$$
 (36)

For the third step, notice that by convexity of the *p*-norm it is sufficient to prove the bound (35) for each term $(\Phi_{\lambda}^{(n)} \otimes \Psi)(\tau_{12})$ appearing on the right side of (36), namely

$$||(\Phi_{\lambda}^{(n)} \otimes \Psi)(\tau_{12})||_{p} \le \nu_{p}(\Delta_{\lambda}) \nu_{p}(\Psi)$$
(37)

In order to derive (37), we apply (34) with

$$\rho_{12} = (I \otimes \Psi)(\tau_{12}), \quad \rho_2^{(i)} = \Psi(\tau_2^{(i)}) = \Psi(\operatorname{Tr}_1[(E_i \otimes I)\tau_{12}])$$
(38)

Therefore (34) gives

$$||(\Phi_{\lambda}^{(n)} \otimes \Psi)(\tau_{12})||_{p} \le d^{(1-1/p)} \nu_{p}(\Delta_{\lambda}) \left[\sum_{i=1}^{d} \operatorname{Tr}(\Psi(\tau_{2}^{(i)}))^{p}\right]^{1/p}$$
 (39)

Now the definition of the p-norm $\nu_p(\Psi)$ implies that for each i,

$$\left[\operatorname{Tr}(\Psi(\tau_2^{(i)}))^p\right]^{1/p} \le \nu_p(\Psi) \operatorname{Tr}(\tau_2^{(i)}) \tag{40}$$

From the definition of $\tau_2^{(i)}$ it follows that

$$Tr(\tau_2^{(i)}) = Tr(E_i \tau_1) \tag{41}$$

Furthermore in the first step we chose the state τ_1 to be diagonal, and from Lemma 8 the phase-damping channels appearing on the right side of (36) are all uniform (recall Definition 6). Hence from (30) we get

$$Tr(E_i\tau_1) = \frac{1}{d}Tr\tau_1 = \frac{1}{d}$$
(42)

Inserting into (39) gives

$$||(\Phi_{\lambda}^{(n)} \otimes \Psi)(\tau_{12})||_{p} \leq d^{(1-1/p)} \nu_{p}(\Delta_{\lambda}) \nu_{p}(\Psi) \left[d\left(\frac{1}{d}\right)^{p}\right]^{1/p}$$

$$(43)$$

$$= \nu_p(\Delta_\lambda) \, \nu_p(\Psi) \tag{44}$$

which completes the proof. **QED**

3 The convex decomposition

3.1 Proof of Lemma 9

The derivation proceeds in two stages. First we define a new channel Ω_{λ} which appears in an intermediate role:

$$\Omega_{\lambda}(\rho) = \Delta_{\lambda}(\rho) + \frac{1-\lambda}{d} \left[\rho - \operatorname{diag}(\rho) \right]$$
(45)

where $\operatorname{diag}(\rho)$ is the diagonal part of the matrix ρ . It is not hard to see that Ω_{λ} is completely positive and trace-preserving for all λ in the range (28). For example it can be re-written as

$$\Omega_{\lambda}(\rho) = \left(\lambda + \frac{1-\lambda}{d}\right)\rho + \frac{(d-1)(1-\lambda)}{d}\frac{1}{d-1}\left[I - \operatorname{diag}(\rho)\right],\tag{46}$$

and the map $\left[I - \operatorname{diag}(\rho)\right]/(d-1)$ is easily seen to be completely positive and trace-preserving. Next let G be the diagonal unitary matrix with entries

$$G_{kk} = \exp\left(\frac{2\pi i k}{d}\right), \quad 1 \le k \le d$$
 (47)

$$G_{kl} = 0, \qquad k \neq l \tag{48}$$

Lemma 10 For any matrix ρ ,

$$\Delta_{\lambda}(\rho) = \frac{\lambda d}{1 + (d-1)\lambda} \Omega_{\lambda}(\rho) + \frac{1-\lambda}{1 + (d-1)\lambda} \frac{1}{d} \sum_{k=1}^{d} (G^*)^k \Omega_{\lambda}(\rho) G^k$$
(49)

Proof: A straightforward computation shows that for any matrix ρ

$$\frac{1}{d} \sum_{k=1}^{d} (G^*)^k \rho G^k = diag(\rho)$$
 (50)

Applying the definitions of Δ_{λ} and Ω_{λ} from (1) and (45), the result now follows easily. **QED**

For the second stage in the derivation of (32) we express the channel Ω_{λ} itself as a convex combination of phase-damping channels. To this end define the diagonal unitary matrix H by

$$H_{kk} = \exp\left(\frac{2\pi i k^2}{2d^2}\right), \quad 1 \le k \le d \tag{51}$$

$$H_{kl} = 0, k \neq l (52)$$

and define the following pure state $|\theta\rangle$:

$$|\theta\rangle = \frac{1}{\sqrt{d}} \begin{pmatrix} 1\\1\\\vdots\\1 \end{pmatrix} \tag{53}$$

For each $k=1,\ldots,d$ and $a=1,\ldots,2d^2$ we define the pure state

$$|\psi_{k,a}\rangle = G^k H^a |\theta\rangle \tag{54}$$

and the corresponding orthogonal projection

$$E_{k,a} = |\psi_{k,a}\rangle\langle\psi_{k,a}| \tag{55}$$

For each fixed a, the states $\{|\psi_{k,a}\rangle\}$ form an orthonormal basis. We denote by $\Phi_{\lambda}^{(a)}$ the corresponding family of phase-damping channels, that is

$$\Phi_{\lambda}^{(a)}(\rho) = \lambda \rho + (1 - \lambda) \sum_{k=1}^{d} E_{k,a} \rho E_{k,a}, \qquad a = 1, \dots, 2d^{2}$$
 (56)

Lemma 11

$$\Omega_{\lambda} = \frac{1}{2d^2} \sum_{a=1}^{2d^2} \Phi_{\lambda}^{(a)} \tag{57}$$

Proof: Using the definitions of Ω_{λ} , Δ_{λ} and $\Phi_{\lambda}^{(a)}$, it suffices to show that for any state ρ

$$\frac{1}{2d} \sum_{a=1}^{2d^2} \sum_{k=1}^d E_{k,a} \rho E_{k,a} = I + \rho - \operatorname{diag}(\rho)$$
 (58)

For each x = 1, ..., d, let $|x\rangle$ be the unit vector with entry 1 in position x, and 0 elsewhere. Then it suffices to show that for all x, y

$$\frac{1}{2d} \sum_{a=1}^{2d^2} \sum_{k=1}^d \langle x | E_{k,a} \rho E_{k,a} | y \rangle = \begin{cases} 1 & \text{if } x = y \\ \langle x | \rho | y \rangle & \text{if } x \neq y \end{cases}$$
 (59)

The $(a,k)^{\text{th}}$ term on the left side of (59) can be written as

$$\langle x|E_{k,a}\rho E_{k,a}|y\rangle = \langle x|\psi_{k,a}\rangle\langle\psi_{k,a}|\rho|\psi_{k,a}\rangle\langle\psi_{k,a}|y\rangle$$

$$= \sum_{u,v=1}^{d} \langle x|\psi_{k,a}\rangle\langle\psi_{k,a}|u\rangle\langle u|\rho|v\rangle\langle v|\psi_{k,a}\rangle\langle\psi_{k,a}|y\rangle$$
(60)

Furthermore

$$\langle x|\psi_{k,a}\rangle = \langle x|G^kH^a|\theta\rangle = \frac{1}{\sqrt{d}}\exp\left[\frac{2\pi ikx}{d}\right]\exp\left[\frac{2\pi iax^2}{2d^2}\right]$$
 (61)

Substituting into (60) gives

$$\langle x|E_{k,a}\rho E_{k,a}|y\rangle$$
 (62)

$$= \frac{1}{d^2} \sum_{u,v=1}^{d} \langle u | \rho | v \rangle \exp \left[\frac{2\pi i k}{d} (x + v - y - u) \right] \exp \left[\frac{2\pi i a}{2d^2} (x^2 + v^2 - y^2 - u^2) \right]$$

When the right side of (62) is substituted in (59), the sum over k gives zero unless x + v - y - u is an integer multiple of d. Since x, v, y, u vary between 1 and d, the only possible values are

$$x + v - y - u = 0, d, -d (63)$$

Similarly, the sum over a gives zero unless $x^2 + v^2 - y^2 - u^2$ is an integer multiple of $2d^2$. In this case the only possibility is

$$x^2 + v^2 - y^2 - u^2 = 0 ag{64}$$

Consider first the case that (63) gives

$$x + v - y - u = d \tag{65}$$

Let $\gamma = y + u$, then $x + v = \gamma + d$, and hence

$$2 \le \gamma \le d \tag{66}$$

Elementary bounds then lead to

$$x^2 + v^2 > \gamma^2 > y^2 + u^2 \tag{67}$$

which shows that there can be no simultaneous solution of (64) and (65). A similar argument holds for the case

$$x + v - y - u = -d \tag{68}$$

The remaining case in (63) can be written as

$$x - u = y - v \tag{69}$$

and also (64) can be written as

$$(x-u)(x+u) = (y-v)(y+v)$$
(70)

It follows that the only simultaneous solutions of the equations (69) and (70) are x = y, u = v and x = u, y = v. Hence if $x \neq y$ the left side of (59) gives $\langle x|\rho|y\rangle$, while if x = y the sum gives $\sum_{u}\langle u|\rho|u\rangle = \text{Tr}\rho = 1$. **QED**

Combining Lemma 10 and Lemma 11 we arrive at the convex decomposition of Δ_{λ} , which expresses the depolarizing channel in terms of the phase-damping channels $\Phi_{\lambda}^{(a)}$:

$$\Delta_{\lambda}(\rho) = \frac{\lambda}{1 + (d-1)\lambda} \frac{1}{2d} \sum_{a=1}^{2d^2} \Phi_{\lambda}^{(a)}(\rho)$$

$$+ \frac{1 - \lambda}{1 + (d-1)\lambda} \frac{1}{2d^3} \sum_{k=1}^{d} \sum_{a=1}^{2d^2} (G^*)^k \Phi_{\lambda}^{(a)}(\rho) G^k$$
(71)

Furthermore each state $|\psi_{k,a}\rangle$ defined in (54) is uniform, and hence the phase-damping channels $\Phi_{\lambda}^{(a)}$ defined in (56) are also uniform. This completes the proof of Lemma 8.

3.2 d=2: the qubit depolarizing channel

It is useful to look in detail at the familiar case d=2. A general state ρ can be written as a 2×2 hermitian matrix

$$\rho = \begin{pmatrix} a & c \\ \overline{c} & b \end{pmatrix}
\tag{72}$$

The depolarizing channel (1) acts by

$$\Delta_{\lambda}(\rho) = \begin{pmatrix} \lambda_{+}a + \lambda_{-}b & \lambda c \\ \lambda \overline{c} & \lambda_{-}a + \lambda_{+}b \end{pmatrix}, \tag{73}$$

where we have defined

$$\lambda_{\pm} = \frac{1 \pm \lambda}{2} \tag{74}$$

Also the 'intermediate' channel Ω_{λ} defined in (45) acts by

$$\Omega_{\lambda}(\rho) = \begin{pmatrix} \lambda_{+}a + \lambda_{-}b & \lambda_{+}c \\ \lambda_{+}\overline{c} & \lambda_{-}a + \lambda_{+}b \end{pmatrix}$$
 (75)

The first diagonal unitary matrix G defined in (47) is just

$$G = \begin{pmatrix} -1 & 0\\ 0 & 1 \end{pmatrix} = -\sigma_z \tag{76}$$

So the first relation (49) becomes

$$\Delta_{\lambda} = \frac{2\lambda}{1+\lambda} \Omega_{\lambda} + \frac{1-\lambda}{1+\lambda} \frac{1}{2} \left[\Omega_{\lambda} + \sigma_{z} \Omega_{\lambda} \sigma_{z} \right]$$
 (77)

which can be easily verified using (73) and (75).

The second diagonal unitary matrix H defined in (51) is now

$$H = \begin{pmatrix} \exp\begin{bmatrix} \pi i/4 \end{bmatrix} & 0\\ 0 & -1 \end{pmatrix} \tag{78}$$

There are eight phase-damping channels defined in (56). Four of these can be written in terms of the usual Pauli matrices:

$$\Phi_{\lambda}^{(2)}(\rho) = \Phi_{\lambda}^{(6)}(\rho) = \lambda_{+}\rho + \lambda_{-}\sigma_{y}\rho\sigma_{y}$$
(79)

$$\Phi_{\lambda}^{(4)}(\rho) = \Phi_{\lambda}^{(8)}(\rho) = \lambda_{+}\rho + \lambda_{-}\sigma_{x}\rho\sigma_{x} \tag{80}$$

The others can be written in terms of the following Pauli-type matrices:

$$\tau = \begin{pmatrix} 0 & \exp\left[\pi i/4\right] \\ \exp\left[-\pi i/4\right] & 0 \end{pmatrix}, \qquad \overline{\tau} = \begin{pmatrix} 0 & \exp\left[-\pi i/4\right] \\ \exp\left[\pi i/4\right] & 0 \end{pmatrix} (81)$$

The relations are

$$\Phi_{\lambda}^{(1)}(\rho) = \Phi_{\lambda}^{(5)}(\rho) = \lambda_{+}\rho + \lambda_{-}\tau\rho\tau \tag{82}$$

$$\Phi_{\lambda}^{(3)}(\rho) = \Phi_{\lambda}^{(7)}(\rho) = \lambda_{+}\rho + \lambda_{-}\overline{\tau}\rho\overline{\tau}$$
(83)

The second convex decomposition (57) now reads

$$\Omega_{\lambda} = \frac{1}{8} \sum_{a=1}^{8} \Phi_{\lambda}^{(a)} \tag{84}$$

There is a lot of redundancy in the final decomposition (71), which now has 24 terms on the right side. In fact Ω_{λ} can be written as a convex combination of just two uniform phase-damping channels, namely

$$\Omega_{\lambda} = \frac{1}{2} \left[\Phi_{\lambda}^{(2)} + \Phi_{\lambda}^{(4)} \right],\tag{85}$$

and this allows Δ_{λ} to be written as a convex combination of just four phase-damping channels. There may be a similar redundancy in (71) for d > 2.

4 The phase-damping channel

In this section we will establish Lemma 9 for the product channel $\Phi_{\lambda} \otimes I$. Without loss of generality we will choose the basis $\mathcal{B} = \{|i\rangle\}$, so that Φ_{λ} acts on a state by simply scaling all off-diagonal entries by the same factor λ , as in (29):

$$\Phi_{\lambda}(\rho) = \lambda \rho + (1 - \lambda) \sum_{i=1}^{d} |i\rangle \langle i| \langle i|\rho|i\rangle$$
(86)

The product channel $\Phi_{\lambda} \otimes I$ acts on bipartite states ρ_{12} defined on $\mathbf{C}^d \otimes \mathbf{C}^{d'}$ for some dimension d'. It will be convenient to view these states as $d \times d$ block matrices, where each block is itself a $d' \times d'$ matrix. Furthermore there is a convenient factorization of these blocks, which can be derived by the following observation. Let us write $\sqrt{\rho_{12}} = (V_1 \dots V_d)$ where each V_i is a $dd' \times d'$ matrix. Then we have

$$\rho_{12} = \left(\sqrt{\rho_{12}}\right)^* \sqrt{\rho_{12}}$$

$$= \begin{pmatrix} V_1^* V_1 & \dots & V_1^* V_d \\ \vdots & \ddots & \vdots \\ V_d^* V_1 & \dots & V_d^* V_d \end{pmatrix}$$

$$(87)$$

Recall the definition of $\rho_2^{(i)}$ in (33). With our choice of basis here, the matrix $E_i \otimes I$ is simply the orthogonal projector onto the i^{th} block on the main diagonal, hence

$$\rho_2^{(i)} = V_i^* \, V_i \tag{88}$$

The key to deriving the bound (34) is to rewrite the factorization (87) as follows:

$$\rho_{12} = \begin{pmatrix} V_1^* & 0 & \dots & 0 \\ 0 & V_2^* & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & V_d^* \end{pmatrix} M \begin{pmatrix} V_1 & 0 & \dots & 0 \\ 0 & V_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & V_d \end{pmatrix}$$
(89)

where M is the $d \times d$ block matrix

$$M = \begin{pmatrix} I' & \dots & I' \\ \vdots & \ddots & \vdots \\ I' & \dots & I' \end{pmatrix} \tag{90}$$

and I' is the $dd' \times dd'$ identity matrix. Recall the state $|\theta\rangle$ defined in (53). Using this we can rewrite M as the product state

$$M = d\left(|\theta\rangle\langle\theta|\right) \otimes I' \tag{91}$$

Furthermore the simple action of the phase-damping channel Φ_{λ} implies that it acts on (89) in the following way:

$$(\Phi_{\lambda} \otimes I)(\rho_{12}) = \begin{pmatrix} V_{1}^{*}V_{1} & \dots & \lambda V_{1}^{*}V_{d} \\ \vdots & \ddots & \vdots \\ \lambda V_{d}^{*}V_{1} & \dots & V_{d}^{*}V_{d} \end{pmatrix}$$

$$= \begin{pmatrix} V_{1}^{*} & 0 & \dots & 0 \\ 0 & V_{2}^{*} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & V_{d}^{*} \end{pmatrix} (\Phi_{\lambda} \otimes I)(M) \begin{pmatrix} V_{1} & 0 & \dots & 0 \\ 0 & V_{2} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & V_{d} \end{pmatrix}$$

Let us define

$$A = \begin{pmatrix} V_1 V_1^* & 0 & \dots & 0 \\ 0 & V_2 V_2^* & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & V_d V_d^* \end{pmatrix}$$
(93)

and

$$B = (\Phi_{\lambda} \otimes I)(M) = d\left(\Phi_{\lambda}(|\theta\rangle\langle\theta|)\right) \otimes I'$$
(94)

Then $(\Phi_{\lambda} \otimes I)(\rho_{12})$ has the same spectrum as the matrix $A^{1/2}BA^{1/2}$. Therefore

$$\operatorname{Tr}\left((\Phi_{\lambda} \otimes I)(\rho_{12})\right)^{p} = \operatorname{Tr}\left(A^{1/2}BA^{1/2}\right)^{p} \tag{95}$$

Now we use the Lieb-Thirring inequality [10], which states that for all $p \geq 1$

$$\operatorname{Tr}\left(A^{1/2}BA^{1/2}\right)^{p} \le \operatorname{Tr}\left(A^{p/2}B^{p}A^{p/2}\right) = \operatorname{Tr}\left(A^{p}B^{p}\right) \tag{96}$$

The matrix A^p is block diagonal:

$$A^{p} = \begin{pmatrix} (V_{1}V_{1}^{*})^{p} & 0 & \dots & 0 \\ 0 & (V_{2}V_{2}^{*})^{p} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & (V_{d}V_{d}^{*})^{p} \end{pmatrix}$$
(97)

Furthermore

$$B^{p} = d^{p} \left(\Phi_{\lambda}(|\theta\rangle\langle\theta|) \right)^{p} \otimes I'$$
(98)

Explicit calculation shows that the diagonal entries of $d^p \left(\Phi_{\lambda}(|\theta\rangle\langle\theta|)\right)^p$ are all equal to $(1-\lambda)^p + \left[(d\lambda+1-\lambda)^p - (1-\lambda)^p\right]/d$. Comparing this with (10), and substituting (97) and (98) into the right side of (96) we get

$$\operatorname{Tr}\left(A^{p}B^{p}\right) = d^{(p-1)}\left(\nu_{p}(\Delta_{\lambda})\right)^{p} \sum_{i=1}^{d} \operatorname{Tr}(V_{i}V_{i}^{*})^{p}$$
(99)

Now recall (88), and also notice that for all i = 1, ..., d

$$\operatorname{Tr}(V_i V_i^*)^p = \operatorname{Tr}(V_i^* V_i)^p = \operatorname{Tr}\left(\rho_2^{(i)}\right)^p \tag{100}$$

Combining (96), (99) and (100) gives the bound (34). **QED**

5 The additivity of χ^*

The proof of Theorem 2 uses the representation of χ^* as a min-max of relative entropy, combined with an entropy bound derived from Lemma 9. The relative entropy representation was derived by Ohya, Petz and Watanabe [12] and Schumacher and Westmoreland [15]. Recall that the relative entropy of two states ρ and ω is defined as

$$S(\rho, \omega) = \text{Tr}\rho(\log \rho - \log \omega) \tag{101}$$

The OPWSW representation for the Holevo capacity of the channel Ψ is

$$\chi^*(\Psi) = \inf_{\omega} \sup_{\rho} S\Big(\Psi(\rho), \Psi(\omega)\Big)$$
 (102)

$$= \sup_{\rho} S\left(\Psi(\rho), \Psi(\omega^*)\right) \tag{103}$$

where the state ω^* that achieves the infimum in (102) is the optimal average input state from the channel. For the depolarizing channel this optimal average is (1/d) I, that is the totally mixed state. For the product channel $\Delta_{\lambda} \otimes \Psi$, the Holevo quantity $\chi^*(\Delta_{\lambda} \otimes \Psi)$ can be upper bounded by choosing $(1/d) I \otimes \omega^*$ as the average input state. This leads to the following inequalities:

$$\chi^*(\Delta_{\lambda}) + \chi^*(\Psi) \le \chi^*(\Delta_{\lambda} \otimes \Psi) \le \sup_{\tau_{12}} S\Big((\Delta_{\lambda} \otimes \Psi)(\tau_{12}), (1/d) \, I \otimes \Psi(\omega^*)\Big) (104)$$

In order to prove additivity we will combine this with the following result.

Lemma 12 For all bipartite states τ_{12} ,

$$S\Big((\Delta_{\lambda} \otimes \Psi)(\tau_{12}), (1/d) I \otimes \Psi(\omega^*)\Big) \le \chi^*(\Delta_{\lambda}) + \chi^*(\Psi)$$
(105)

Proof:

The left side of (105) can be rewritten as

$$S\Big((\Delta_{\lambda} \otimes \Psi)(\tau_{12}), (1/d) I \otimes \Psi(\omega^{*})\Big) = -S\Big((\Delta_{\lambda} \otimes \Psi)(\tau_{12})\Big) + \log d - \text{Tr}\Psi(\tau_{2}) \log \Psi(\omega^{*})$$
(106)

where τ_2 is the reduced density matrix of τ_{12} . From here on we follow the steps in the proof of Theorem 3. First, by Lemma 7 we can assume without loss of

generality that $\tau_1 = \text{Tr}_2(\tau_{12})$ is diagonal. Second, notice that the channel Δ_{λ} appears on the right side of (106) only in the first term. Therefore Lemma 8 and concavity of the entropy imply that it is sufficient to establish the bound

$$S\Big((\Phi_{\lambda} \otimes \Psi)(\tau_{12}), (1/d) \, I \otimes \Psi(\omega^*)\Big) \le \chi^*(\Delta_{\lambda}) + \chi^*(\Psi) \tag{107}$$

where Φ_{λ} is a uniform phase-damping channel and where τ_1 is diagonal.

Next we apply (34) with $\rho_{12} = (I \otimes \Psi)(\tau_{12})$, and take the derivative at p = 1 to get

$$S\left((\Delta_{\lambda} \otimes \Psi)(\tau_{12})\right) \geq S_{\min}(\Delta_{\lambda}) - \log d$$

$$- \sum_{i=1}^{d} x_{i} \log x_{i} + \sum_{i=1}^{d} x_{i} S\left(\Psi\left(\frac{1}{x_{i}}\tau_{2}^{(i)}\right)\right)$$

$$(108)$$

where $x_i = \text{Tr}\left(\tau_2^{(i)}\right)$, and as usual

$$\tau_2^{(i)} = \operatorname{Tr}_1 \left[(E_i \otimes I) \tau_{12} \right] \tag{109}$$

Since τ_1 is diagonal and Φ_{λ} is uniform, it follows that

$$x_i = \frac{1}{d} \tag{110}$$

for all i = 1, ..., d, hence $\sum x_i \log x_i = -\log d$. Also recall the evaluation of $\chi^*(\Delta_{\lambda})$ in (11). Hence (108) can be written as

$$S\left((\Delta_{\lambda} \otimes \Psi)(\tau_{12})\right) \ge -\chi^*(\Delta_{\lambda}) + \log d + \frac{1}{d} \sum_{i=1}^{d} S\left(\Psi\left(d\tau_2^{(i)}\right)\right) \tag{111}$$

Furthermore, since the projections E_i in (109) constitute an orthonormal basis it follows that

$$\sum_{i=1}^{d} \tau_2^{(i)} = \text{Tr}_1 \Big[(I \otimes I) \tau_{12} \Big] = \tau_2$$
 (112)

Therefore the left side of (107) can be rewritten as in (106) to get

$$S\Big((\Phi_{\lambda} \otimes \Psi)(\tau_{12}), (1/d) I \otimes \Psi(\omega^{*})\Big) = -S\Big((\Phi_{\lambda} \otimes \Psi)(\tau_{12})\Big)$$

$$+ \log d - \frac{1}{d} \sum_{i=1}^{d} \operatorname{Tr} \Psi(d\tau_{2}^{(i)}) \log \Psi(\omega^{*})$$

$$(113)$$

Combining (113) with (111) we get

$$S\Big((\Delta_{\lambda} \otimes \Psi)(\rho_{12}), (1/d) I \otimes \Psi(\omega^{*})\Big) \leq \chi^{*}(\Delta_{\lambda})$$

$$+ \frac{1}{d} \sum_{i=1}^{d} S\Big(\Psi(d\tau_{2}^{(i)}), \Psi(\omega^{*})\Big)$$

$$(114)$$

Recall that $\text{Tr}(d\tau_2^{(i)})=1$. Therefore it follows from (102) that for each $i=1,\ldots,d$

$$S\left(\Psi(d\,\tau_2^{(i)}), \Psi(\omega^*)\right) \le \chi^*(\Psi) \tag{115}$$

and hence (114) implies (107). **QED**

6 Conclusions and discussion

We have presented the proof of a long-conjectured property of the d-dimensional depolarizing channel Δ_{λ} , namely that its capacity for transmission of classical information can be achieved with product signal states and product measurements. This result follows as a consequence of several additivity results which we prove for the product channel $\Delta_{\lambda} \otimes \Psi$ where Ψ is an arbitrary channel. The principal result is the proof of the AHW conjecture for the matrix p-norm, for all $p \geq 1$, from which we deduce the additivity of minimal entropy and of the Holevo quantity. The argument presented here is a generalization of the method used earlier by the author to prove similar results for all unital qubit channels, and involves re-writing the depolarizing channel as a convex combination of other simpler channels, which we refer to as phase-damping channels.

If the additivity conjecture for the Holevo quantity is true for all channels, then there must be a general argument which can be used to provide a proof, and presumably this would give a different method of proof for Theorem 1. However it is known that the AHW conjecture is not true in general [16], and indeed it is an interesting problem to determine the class of channels for which it does hold. As a consequence, it may be that the method of this paper gives the most direct route to the proof of the AHW property for the depolarizing channel. It is expected that the same method can be applied to prove the AHW result for a class of d-dimensional channels, and this question is under study.

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