Entropy and the timing capacity of discrete queues

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Abstract — **Queueing systems which map Poisson input processes to Poisson output processes have been well-studied in classical queueing theory. This paper considers two discrete-time queues whose analogs in continuous-time possess the Poisson-in-Poisson-out property. It is shown that when packets arriving according to an arbitrary ergodic stationary arrival process are passed through these queueing systems, the corresponding departure process has an entropy rate no less (some times strictly more) than the entropy rate of the arrival process. Some useful by-products are discrete-time versions of: (i) a proof of Burke's Theorem [4], (ii) a proof of the uniqueness, amongst renewal inputs, of the Poisson process as a fixed point for exponential server queues [1], and (iii) connections with the timing capacity of queues [2].**

I. Introduction

Several queueing systems have the Poisson-in-Poisson-out property: If the arrival process to such a queueing system is Poisson, and it is stable (arrival rate < service rate), then the equilibrium departure process from the queueing system is also Poisson. These systems include the first-come-first-served (FCFS) exponential server queue, and a queue which dispenses i.i.d. services with a general distribution and uses the last-come-first-served with pre-emptive resume service discipline.

We shall show that the discrete versions of these queueing systems are *entropy increasing* in the following sense: When an arbitrary stationary and ergodic arrival process is passed through such a queueing system, the corresponding equilibrium departure process has an entropy rate no less (and some times strictly more) than that of the arrival process. We also explore the connection of entropy increasing properties with the timing capacity of queues, as considered in [2], [3] and [5].

A Notation

Suppose that time is slotted and arrivals take place just at the beginning of time slots and departures take place just before the end of time slots. Let a_n be the arrival time of the n^{th} packet. Suppose that $a_i < a_j$ for $i < j$ and $a_0 < 0 \leq a_1$. Let $A_n = a_{n+1} - a_n$ be the n^{th} interarrival time, and let $\mathbf{A}$ be the process $\{A_n, n \in \mathcal{Z}\}$. Let s_n be the service time requirement of the n^{th} packet. For stability, it is assumed that $E(s_1) < E(A_1)$. For $n > 0$, let d_n be the *time of the n^{th} departure* from the queue at or after time 0, and for $n \leq 0$, let d_n be the time of the $-(n-1)^{th}$ departure from the queue before time 0. Thus $-\infty < \cdots < d_0 < 0 \leq d_1 < \cdots < \infty$. Let $D_n = d_{n+1} - d_n$ be the interdeparture times.

Let $H_{ER}(\mathbf{A}) = \lim_{N\to\infty} H(A_1, \ldots, A_N)/N$ be the entropy rate of the arrival process $\mathbf{A}$, and let $H_{ER}(\mathbf{D}) = \lim_{N\to\infty} H(D_1, \ldots, D_N)/N$ be the entropy rate of the departure process.

II. Theorems

FCFS queue with geometric services. Consider a single server FCFS queue with independent and geometrically distributed services of mean $1/\mu$; i.e., $P(s_0 = k) = \mu(1-\mu)^{k-1}$, for $k \geq 1$. The arrival process $\mathbf{A}$ is assumed to be stationary and ergodic, with $E(A_1) > E(s_1)$, and is independent of the service times $\{s_n, n \in \mathcal{Z}\}$.

Theorem 1 *Suppose the arrival process $\mathbf{A}$ is stationary and ergodic, with $E(A_1) > E(s_1)$, and is independent of the service times $\{s_n, n \in \mathcal{Z}\}$. Then $H_{ER}(\mathbf{A}) \leq H_{ER}(\mathbf{D})$. Further, if the interarrival times are i.i.d., then $H_{ER}(\mathbf{A}) = H_{ER}(\mathbf{D})$ iff A_1 is geometric: $P(A_1 = k) = \lambda(1-\lambda)^{k-1}$ for $k \geq 1$, where $\lambda < \mu$.*

LCFS queue with arbitrary services. The service times $\{s_n\}$ are non-negative integer-valued, i.i.d., arbitrarily distributed with mean equal to $1/\mu$. The service discipline is last-come-first-served (LCFS) with preemptive resume. A stable discrete-time $\cdot$/GI/1-LCFS queue is said to satisfy the Q-condition if the equilibrium queue-length has a finite first moment. It is said to satisfy the B-condition if the equlibrium busy period straddling the time instant 0 has a finite first moment.

Theorem 2 *For a stable $\cdot$/GI/1-LCFS queue satisfying the Q- and B-conditions, $H_{ER}(\mathbf{A}) \leq H_{ER}(\mathbf{D})$.*

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