# Deviation Bounds for Wavelet Shrinkage 

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#### Abstract

We analyse the wavelet shrinkage algorithm of Donoho and Johnstone in order to assess the quality of the reconstruction of a signal obtained from noisy samples. We prove deviation bounds for the maximum of the squares of the error, and for the average of the squares of the error, under the assumption that the signal comes from a Hölder class, and the noise samples are independent, of 0 mean, and bounded. Our main technique is Talgrand's isoperimetric theorem. Our bounds refine the known expectations for the average of the squares of the error.


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## 1 Introduction

We address the classical problem of the reconstruction of signal samples from noisy samples. We consider an original signal of bounded duration $f: t \in[0,1] \rightarrow f(t) \in \mathbb{R}$. We also have additive noise $e:[0,1] \rightarrow \mathbb{R}$. Thus, the observed noisy signal at time $t$ is $y(t)=f(t)+e(t)$.

We sample the noisy signal at $n$ uniformly spaced instants and we denote the sample values by $y_{i}=f_{i}+e_{i}=f\left(\frac{i}{n}\right)+e\left(\frac{i}{n}\right)$ (for $1 \leq i \leq n$ ). Our goal is to recover a good approximation of the original signal samples $\left(f_{1}, \ldots, f_{n}\right)$ from the noisy signal samples $\left(y_{1}, \ldots, y_{n}\right)$. For this to be possible we need some assumptions that distinguish the signal from the noise:

- The original signal $f$ has a certain degree of "smoothness", i.e., $f$ belongs to a Hölder class $\Lambda^{\alpha}(M)$ for some $\alpha>0$ and $M>0$.
- The noise is "random", i.e., $\left(e_{1}, \ldots, e_{n}\right)$ consists of $n$ independent Borel random variables.

The Hölder classes are defined as follows:
For $0<\alpha \leq 1, \Lambda^{\alpha}(M)=\left\{h \in \mathbb{R}^{[0,1]}:\left(\forall x_{1}, x_{2} \in[0,1]\right),\left|h\left(x_{1}\right)-h\left(x_{2}\right)\right| \leq M\left|x_{1}-x_{2}\right|^{\alpha}\right\}$.
For $1<\alpha, \Lambda^{\alpha}(M)=\left\{h \in \mathbb{R}^{[0,1]}:(\forall x \in[0,1])\left|h^{\prime}(x)\right| \leq M, h^{\lfloor\alpha\rfloor}\right.$ exists, and

$$
\left.\left(\forall x_{1}, x_{2} \in[0,1]\right)\left|h^{\lfloor\alpha\rfloor}\left(x_{1}\right)-h^{\lfloor\alpha\rfloor}\left(x_{2}\right)\right| \leq M\left|x_{1}-x_{2}\right|^{\alpha-\lfloor\alpha\rfloor}\right\} .
$$

Let $\left(\tilde{y}_{1}, \ldots, \tilde{y}_{n}\right)$ be an approximation of $\left(f_{1}, \ldots, f_{n}\right)$, obtained from $\left(y_{1}, \ldots, y_{n}\right)$. Most commonly, the closeness of this approximation is measured by $\frac{1}{n} \sum_{i=1}^{n}\left(\tilde{y}_{i}-f_{i}\right)^{2}$ or by the expectation $\mathbf{E}\left[\frac{1}{n} \sum_{i=1}^{n}\left(\tilde{y}_{i}-f_{i}\right)^{2}\right]$ (which makes sense since the $e_{i}$, and hence the $\tilde{y}_{i}$, are random variables).

The wavelet shrinkage algorithm of Donoho and Johnstone [6], [7] is a very efficient tool for finding good estimates $\tilde{y}$. In outline, the algorithm works as follows:
(Step 0) Choose a wavelet system with $N$ vanishing moments ( $N \geq \alpha$ ); choose a level of coarseness $J_{0} \geq 0$ ( $J_{0}$ will depend on $\alpha$ ), and consider the multi-resolution chain of Hilbert spaces $V_{J_{0}} \subset V_{J_{0}+1} \subset \ldots \subset V_{j} \subset \ldots$.
(Step 1) Apply the Discrete Wavelet Transform (DWT) to the noisy signal samples $\left(y_{1}, \ldots, y_{n}\right)$, where $n \geq 2^{J_{0}}$. This yields the "empirical wavelet coefficients" $\left(\xi_{1}, \ldots, \xi_{n}\right)$.
(Step 2) Fix a "threshold" $\lambda_{n}(>0)$ and apply either "hard" or "soft thresholding" to $\left(\xi_{1}, \ldots, \xi_{n}\right)$.

Hard thresholding consists of replacing each $\xi_{i}$ by 0 when $\left|\xi_{i}\right| \leq \lambda_{n}$, and keeping $\xi_{i}$ unchanged when $\left|\xi_{i}\right|>\lambda_{n}$.

Soft thresholding consists of transforming each $\xi_{i}$ as follows: $\xi_{i}$ is replaced by 0 if $\left|\xi_{i}\right| \leq \lambda_{n}$; if $\xi_{i}>\lambda_{n}, \xi_{i}$ is replaced by $\xi_{i}-\lambda_{n}$; if $\xi_{i}<-\lambda_{n}, \xi_{i}$ is replaced by $\xi_{i}+\lambda_{n}$.
(Step 3) Apply the inverse DWT to the result of (2). This yields the estimate ( $\tilde{y}_{1}, \ldots, \tilde{y}_{n}$ ).
To what extent does wavelet shrinkage depend on the smoothness conditions of the signal $f$ and on the randomness conditions of the noise samples $e_{i}$, and how do the estimators $\tilde{y}_{i}$ approximate the original signal $f$ ? In [6], [7] it was assumed that the $e_{i}$ are iid Gaussian variables with distribution $N\left(0, \sigma^{2}\right)$, and the threshold was chosen to be $\lambda_{n}=\sigma \sqrt{2 \frac{\log n}{n}}$. Assuming that $f \in \Lambda^{\alpha}(M)$ (the Hölder class) with $\alpha>0$, it is proved in [6], [7] that $\mathbf{E}\left[\frac{1}{n} \sum_{i=1}^{n}\left(\tilde{y}_{i}-f_{i}\right)^{2}\right]<$
$C \cdot\left(\frac{1}{n} \log n\right)^{\frac{2 \alpha}{1+2 \alpha}}$, where $C$ depends only on $M$ and on the wavelet system used. It was observed in [6], [7] (the proofs are due to Lepskii (9] and to Brown and Low (3]) that this upper bound is optimal over all possible algorithms, if the parameters $\alpha$ and $M$ are not known. For the optimality of the wavelet shrinkage algorithm it is important that the threshold be of the form $c \cdot \sqrt{\frac{\log n}{n}}$ (where $c$ does not depend on $n$ ).

Since the publication of [6], [7] there has been further progress on wavelet shrinkage (chapter 6 of [13] is an excellent reference up to 1999). Most recently, Averkamp and Houdré (1]) [2] expanded the scope of wavelet shrinkage by allowing the noise samples $e_{i}$ to have different distributions $F_{i}$, chosen from a wide class of distributions. They show in [1] (page 32) that the error expectation of the wavelet shrinkage algorithm for bounded noise is roughly the same as for Gaussian noise, if the parameters $\alpha$ and $M$ of the Hölder class of the signal are not known. They also discuss various choices of thresholds.

All the results on wavelet shrinkage in the literature so far evaluate the quality of the approximation by bounding the expectation $\mathbf{E}\left[\frac{1}{n} \sum_{i=1}^{n}\left(\tilde{y}_{i}-f_{i}\right)^{2}\right]$, to the best of our knowledge. In this paper we study deviation bounds (rather than just the expectation) of $\frac{1}{n} \sum_{i=1}^{n}\left(\tilde{y}_{i}-f_{i}\right)^{2}$ and of $\max \left\{\left(\tilde{y}_{i}-f_{i}\right)^{2}: 1 \leq i \leq n\right\}$.

Assumptions: We assume that the signal $f$ belongs to a Hölder class $\Lambda^{\alpha}(M)$, and that the noise samples $e_{i}$ are independent random variables (with possibly different distributions). The only restrictions on the distributions are that they are Borel measurable, have compact support (contained in an interval $\left[-\frac{b}{2}, \frac{b}{2}\right]$ ), and zero mean. The assumption that the distributions of the noise have bounded support is of course equivalent to assuming that the noise $e_{i}$ has bounded values $\left(\left|e_{i}\right| \leq \frac{b}{2}\right)$.

The main results of this paper are the following deviation bounds.
Theorem. For the wavelet shrinkage algorithm with threshold

$$
\lambda_{n, \delta}=C_{\varphi} b(1+2 \sqrt{(1+\delta) \ln 2}) \sqrt{\frac{\log n}{n}}
$$

(where $C_{\varphi}$ depends only on the wavelet system) we have the following deviation bounds:
There are $c_{1}, c_{2}>0$, depending only on $b, M$, and $\alpha$, such that for all $n \geq n_{0}$ and all $\delta>0$,

$$
\mathbf{P}\left(\max \left\{\left(\tilde{y}_{i}-f_{i}\right)^{2}: 1 \leq i \leq n\right\} \leq\left(c_{1}+c_{2} \delta\right)\left(\frac{\log n}{n}\right)^{\frac{2 \alpha}{1+2 \alpha}}\right) \geq 1-\frac{9}{n^{1+\delta}}
$$

As a consequence,

$$
\mathbf{P}\left(\frac{1}{n} \sum_{i=1}^{n}\left(\tilde{y}_{i}-f_{i}\right)^{2} \leq\left(c_{1}+c_{2} \delta\right)\left(\frac{\log n}{n}\right)^{\frac{2 \alpha}{1+2 \alpha}}\right) \geq 1-\frac{9}{n^{1+\delta}} .
$$

The minimum number of samples, $n_{0}$, is $2^{9}$ when $0<\alpha \leq 1$; when $\alpha>1$, $n_{0}=(4 \alpha+2)^{2 \alpha+2} \cdot\left(\log _{2}(4 \alpha+2)\right)^{2}$.

One notices that $n_{0}$ grows very rapidly with $\alpha$, when $\alpha>1$. For $\alpha=2$, we have $n_{0}=1.1 * 10^{7}$; for $\alpha=3, n_{0}=3.7 * 10^{10}$, which is impractical. So for large $\alpha$ our theorem is interesting only from an asymptotic point of view. On the other hand, in practice usually $\alpha \leq 1$.

## 2 Preliminaries

### 2.1 Wavelets

We will usually follow the notation of [5] regarding wavelets, the only exception being that we reverse the multi-resolution indices. Moreover, we only consider real-valued functions with domain $[0,1]$. So we have a sequence of real Hilbert spaces $V_{J_{0}} \subset V_{J_{0}+1} \subset \ldots \subset V_{j} \subset \ldots$, such that the closure of $\bigcup_{j} V_{j}$ is $\mathrm{L}^{2}[0,1]$. We let $V_{j+1}=V_{j} \oplus W_{j}$ (orthogonal complement). Since we are in the case of compactly supported functions each $V_{j}$ is a finite-dimensional real vector space (of dimension $2^{j}$ ), with orthonormal basis $\left\{\varphi_{j, k}: 0 \leq k \leq 2^{j}-1\right\}$, derived from a scaling function $\varphi$. Let $\psi$ be the wavelet function corresponding to $\varphi$, and let $\left\{\psi_{j, k}: 0 \leq k \leq 2^{j}-1\right\}$ be the corresponding orthonormal basis of $W_{j}$.

For any function $g \in \mathrm{~L}^{2}[0,1]$ we define the piece-wise constant function $\bar{g}:[0,1] \rightarrow \mathbb{R}$ as follows: $\bar{g}(x)=g\left(\frac{k}{n}\right)\left(=g_{k}\right)$ if $\frac{k-1}{n}<x \leq \frac{k}{n}$ for some $k=1, \ldots, n ; \bar{g}(x)=0$ if $\left.\left.x \notin\right] 0,1\right]$. The discrete wavelet transform of a vector $\left(g_{1}, \ldots, g_{n}\right)$ can be obtained by taking the wavelet coefficients of the piecewise constant function $\bar{g}$. These wavelet coefficients are:

$$
\begin{aligned}
& c_{j, k}^{(g)}=\left\langle\bar{g}, \varphi_{j, k}\right\rangle=\int_{0}^{1} \bar{g}(x) \varphi_{j, k}(x) d x, \text { and } \\
& d_{j, k}^{(g)}=\left\langle\bar{g}, \psi_{j, k}\right\rangle=\int_{0}^{1} \bar{g}(x) \psi_{j, k}(x) d x .
\end{aligned}
$$

Then for any integer $J \geq J_{0}$ :

$$
\bar{g}(x)=\sum_{k=0}^{2^{J}-1} c_{J, k}^{(g)} \varphi_{J, k}(x)+\sum_{j=J}^{+\infty} \sum_{k=0}^{2^{j}-1} d_{j, k}^{(g)} \psi_{j, k}(x) \quad \text { a.e. }
$$

In this paper we will use two wavelet systems: The Haar wavelets (because of their simplicity, especially for programming purposes), and the interval wavelets with predefined vanishing moments, based on Daubechies wavelets (Cohen, Daubechies, Jawerth, Vial (4)).

For the Haar wavelets, the scaling function is $\varphi(x)=1$ when $0<x \leq 1$, and $\varphi(x)=0$ otherwise. Hence, $\varphi_{j, k}(x)=2^{j / 2}$ when $k 2^{-j}<x \leq(k+1) 2^{-j}$, and $\varphi_{j, k}(x)=0$ otherwise. The Haar wavelet function is $\psi(x)=1$ if $0<x \leq \frac{1}{2}, \quad \psi(x)=-1$ if $\frac{1}{2}<x \leq 1$, and $\psi(x)=0$ otherwise. Hence, $\psi_{j, k}(x)=2^{j / 2}$ if $k 2^{-j}<x \leq\left(k+\frac{1}{2}\right) 2^{-j}, \quad \psi_{j, k}(x)=-2^{j / 2}$ if $\left(k+\frac{1}{2}\right) 2^{-j}<x \leq(k+1) 2^{-j}$, and $\psi_{j, k}(x)=0$ otherwise.

For the interval wavelet system of [4], with $N$ vanishing moments, the scaling function $\varphi$ and the wavelet function $\psi$ are complicated. But all we need to know about them is the following:

- A multiresolution of $L^{2}[0,1]$ is obtained, with an orthonormal basis for $V_{j}$ when $j>J_{0}$ :
$\left\{\varphi_{j, k}: 1 \leq k<2^{j}-2 N\right\} \cup\left\{\varphi_{j, i}^{\text {left }}, \varphi_{j, i}^{\text {right }}: 0 \leq i<N\right\}$.
Each $\varphi_{j, k}$ has support $\left[k 2^{-j},(2 N-1+k) 2^{-j}\right]$, each $\varphi_{j, i}^{\text {left }}$ has support $\left[0, i 2^{-j}\right]$, and each $\varphi_{j, i}^{\text {right }}$ has support [ $1-i 2^{-j}, 1$ ].

The decomposition level $J_{0}$ is chosen so that $J_{0} \geq 1+\log _{2}(2 N-1)$. For signals in the Hölder class $\Lambda^{\alpha}(M)$ we require the number of vanishing moments to be $N \geq \alpha$.

- We also have an orthonormal basis for $W_{j}$,

$$
\left\{\psi_{j, k}: 1 \leq k<2^{j}-2 N\right\} \cup\left\{\psi_{j, i}^{\text {left }}, \psi_{j, i}^{\text {right }}: 0 \leq i<N\right\}
$$

with the same supports as the corresponding $\varphi$ functions.

- $\varphi$ and $\psi$ are bounded on $[0,1]$ by a constant $C>0$, independent of $x$ and $N: \quad \forall x \in[0,1]$, $|\varphi(x)|,|\psi(x)| \leq C$.
For $0 \leq k<2^{j}-2 N$ ("inside the the interval"), $\varphi_{j, k}(x)=2^{j / 2} \varphi\left(2^{j} x-k\right)$.
At the ends of the interval $[0,1]$ we have for $0 \leq i<N$, (see (4])

$$
\varphi_{j, i}^{\text {left }}(x)=\sum_{h=1}^{2 N-1}(-h)^{i} \varphi\left(2^{j} x+h\right)
$$

A similar formula holds on the right end of the interval $[0,1]$.
Assuming that $n$ is a power of $2, n=2^{J}$, we have for the function $\bar{y}$, relative to any wavelet system: $\bar{y}(x)=\sum_{k=0}^{2^{J}-1}\left\langle\bar{y}, \varphi_{J, k}\right\rangle \varphi_{J, k}(x)$. Thus for any $J_{1}$ with $0 \leq J_{1}<J$, the DWT transforms $\left(y_{1}, \ldots, y_{n}\right)$ to $\sqrt{n}\left(c_{J_{1}, 0}^{(\bar{y})}, \ldots, c_{J_{1}, 2^{J}-1}^{(\bar{y}}, d_{J_{1}, 0}^{(\bar{y})}, \ldots, d_{J_{1}, 2^{J}-1}^{(\bar{y}}, \ldots, \ldots, d_{J-1,0}^{(\bar{y})}, \ldots, d_{J-1,2^{J}-1}^{(\bar{y})}\right)$. The DWT is an orthogonal transformation (represented by an orthogonal matrix $W$ ).

We will always assume that $n$ is a power of $2: n=2^{J}$. Throughout this paper, log will refer to $\log _{2}$, and $\ln$ will denote the natural logarithm.

Let us now return to the analysis of a noisy signal $y(t)=f(t)+e(t)$.
Lemma 2.1 With respect to the Haar wavelets, the wavelet coefficients of the function e have the following properties:
(H1) For all $j \in\left[0,2^{J}\right]$ and all $k \in\left[0,2^{j-1}-1\right]$ :

$$
c_{j, k}^{(e)}=2^{-J+j / 2} \sum_{i=0}^{2^{J-j}-1} e_{i+1+k 2^{J-j}}
$$

(H2) For all $j$ and $k$ as in (H1):

$$
d_{j, k}^{(e)}=2^{-J+j / 2} \sum_{i=0}^{2^{J-j-1}-1}\left(e_{i+1+k 2^{J-j}}-e_{i+1+\left(k+\frac{1}{2}\right) 2^{J-j}}\right)
$$

For any function $f:[0,1] \rightarrow \mathbb{R}$ belonging to $\Lambda^{(\alpha)}(M)$ with $0<\alpha \leq 1$ we have:
(H3) For all $j \in\left[0,2^{J}\right]$ and all $k \in\left[0,2^{j-1}-1\right]$ :

$$
\left|d_{j, k}^{(f)}\right|<M 2^{-j\left(\frac{1}{2}+\alpha\right)} .
$$

The proof of this lemma is just a calculation and is given in the Appendix.

Lemma 2.2 With respect to the interval wavelet system [4], the wavelet coefficients of the function e have the following properties:
(D1) For all $j \in\left[0,2^{J}\right]$ and all $k \in\left[0,2^{j-1}-1\right]$ :

$$
c_{j, k}^{(e)}=2^{-J+j / 2} \sum_{i=0}^{2^{J-j}-1} \alpha_{i, j, k} e_{i+1+k 2^{J-j}}
$$

for some numbers $\alpha_{i, j, k}$ that do not depend on the noise function $e$. Moreover, $\left|\alpha_{i, j, k}\right|<C_{\varphi}$ for some constant $C_{\varphi} \geq 1$ depending only on the wavelet system.
(D2) For all $j$ and $k$ as in (D1):

$$
d_{j, k}^{(e)}=2^{-J+j / 2} \sum_{i=0}^{2^{J-j}-1} \beta_{i, j, k} e_{i+1+k 2^{J-j}}
$$

for some numbers $\beta_{i, j, k}$ that do not depend on the noise function e. Moreover, $\left|\beta_{i, j, k}\right|<C_{\varphi}$ where $C_{\varphi} \geq 1$ depends only on the wavelet system.

Suppose $f:[0,1] \rightarrow \mathbb{R}$ belongs to $\Lambda^{(\alpha)}(M)$ with $1<\alpha$, and suppose the number of vanishing moments $N$ of the wavelet system satisfies $N \geq \alpha$. Then we have:
(D3) For all $j \in\left[0,2^{J}\right]$ and all $k \in\left[0,2^{j-1}-1\right]$ :

$$
\left|d_{j, k}^{(f)}\right|<C_{\varphi} M 2^{-j\left(\frac{1}{2}+\alpha\right)}
$$

where $C_{\varphi} \geq 1$ depends only on the wavelet system.
The proof of Lemma 2.2 is just a calculation and is given in the Appendix.

### 2.2 Talagrand's isoperimetric theorems

Talagrand's isoperimetric theorems, published in 1995 [12, have had a profound impact on the probabilistic analysis of combinatorial optimization methods; Talagrand's theorems often apply quite directly, giving shorter proofs, often with dramatically better results than previously used methods (see [11], chapter 6). We will use the following result of [12].

Let $\left(\Omega, \Sigma, \mu_{i}\right)(i=1, \ldots, n)$ be Borel probability spaces, and let $\Omega^{n}$ be the product space with product measure $P=\mu_{1} \times \ldots \times \mu_{n}$. For $A \subseteq \Omega^{n}$ and $\omega=\left(\omega_{1}, \ldots, \omega_{n}\right) \in \Omega^{n}$, Talagrand's 'convex' distance is defined by
$d_{T}(\omega, A)=\sup \left\{\inf \left\{\sum_{i=1}^{n} \beta_{i} \cdot I\left(\omega_{i} \neq a_{i}\right):\left(a_{1}, \ldots, a_{n}\right) \in A\right\}:\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathbb{R}^{n}, \sum_{i=1}^{n} \beta_{i}^{2}=1\right\}$.
Notation: $I\left(\omega_{i} \neq a_{i}\right)=1$ if $\omega_{i} \neq a_{i}$, and $I\left(\omega_{i} \neq a_{i}\right)=0$ otherwise.

Theorem 2.3 (Talagrand, Theorem 4.1.1 in [19]): For any $A \subseteq \Omega^{n}$ with $P(A)>0$ :

$$
\int_{\Omega^{n}} \exp \left(\frac{1}{4} d_{T}(\omega, A)^{2}\right) d P(\omega) \leq \frac{1}{P(A)}
$$

As a corollary, for all $t>0$,

$$
P\left(d_{T}(\omega, A) \geq t\right) \leq \frac{1}{P(A)} \cdot \exp \left(-\frac{t^{2}}{4}\right)
$$

## 3 Deviation bound for $\frac{1}{n} \sum_{i=1}^{n}\left(f_{i}-\tilde{y}_{i}\right)^{2}$

Recall that the input for wavelet shrinkage is $\left(y_{1}, \ldots, y_{n}\right)$, where $y_{i}=f_{i}+e_{i}(i=1, \ldots, n)$, the $f_{i}$ are samples from the original signal $f$, and the $e_{i}$ are additive noise. The $e_{i}$ are independent Borel random variables. We assume that the noise is bounded (with $\left|e_{i}\right| \leq \frac{b}{2}$ ), so each random variable $e_{i}$ is a Borel measurable function $e_{i}: \omega_{i} \in \Omega \mapsto e_{i}\left(\omega_{i}\right) \in\left[-\frac{b}{2}, \frac{b}{2}\right]$. Accordingly, we view $\left(e_{1}, \ldots, e_{n}\right)$ as a function $\omega=\left(\omega_{1}, \ldots, \omega_{n}\right) \in \Omega^{n} \mapsto e(\omega)=\left(e_{1}\left(\omega_{1}\right), \ldots, e_{n}\left(\omega_{n}\right)\right) \in$ $\left[-\frac{b}{2}, \frac{b}{2}\right]^{n}$. (Borel measurability is assumed in order to apply Talagrand's theorem.) To simplify the notation we often write $e_{i}(\omega)$ for $e_{i}\left(\omega_{i}\right)$.

We shall first define a subset $A$ of $\Omega^{n}$ and then show that

- $P(A)>\frac{1}{9}$ if $n$ is large enough, and
- wavelet shrinkage satisfies our deviation bounds when the noise samples are in $A$.

Then for any $\delta>0$ we define a subset $B_{\delta} \subseteq \Omega^{n}$ such that

- for any $\omega \in \Omega^{n}$, if Talagrand's distance satisfies $d_{T}(\omega, A) \leq 2 \sqrt{(1+\delta) \ln n}$ then $\omega \in B_{\delta}$;
- wavelet shrinkage satisfies our deviation bounds when the noise samples are in $B_{\delta}$.

Finally, by applying Talagrand's theorem we obtain our results.

### 3.1 The subset $A$

Recall that we assume $n=2^{J}$. For any $\omega \in \Omega^{n}$ we decompose the noise sample sequence $e(\omega)$ into blocks of length $J$, as follows:

$$
e(\omega)=\left(\ldots, \ldots, e_{k J+1}(\omega), \ldots, e_{(k+1) J}(\omega), \ldots, \ldots\right)
$$

where $k=0, \ldots, \frac{1}{J} 2^{J}-1$. Here, for simplicity we regard $\frac{1}{J} 2^{J}=2^{J-\log J}$ as an integer (i.e., we assume that $J$ is a power of 2 ).

For the Haar wavelets we define the subset $A \subset \Omega^{n}$ as follows:

$$
A=\left\{\omega \in \Omega^{n}:(\forall \ell \in[-1, J-\log J])\left(\forall k \in\left[0,2^{J-\log J-\ell}-1\right]\right)\right.
$$

$$
\left.\left|\sum_{i=0}^{J 2^{\ell-1}-1} e_{k 2^{\ell} J+i+1}(\omega)\right| \leq b J 2^{\ell / 2} \sqrt{2^{-1} \ln 2}\right\}
$$

For the interval wavelet system we define

$$
\begin{aligned}
& A=\left\{\omega \in \Omega^{n}:(\forall \ell \in[-1, J-\log J])\left(\forall k \in\left[0,2^{J-\log J-\ell}-1\right]\right),\right. \\
& \text { and } \quad\left|\sum_{i=0}^{J 2^{\ell}-1} e_{k 2^{\ell} J+i+1}(\omega) \cdot \alpha_{i, J-\log J-\ell, k}\right| \leq b J 2^{\ell / 2} \sqrt{2^{-1} \ln 2} \\
& \text { and } \left.\sum_{i=0}^{J 2^{\ell}-1} e_{k 2^{\ell} J+i+1}(\omega) \cdot \beta_{i, J-\log J-\ell, k} \mid \leq b J 2^{\ell / 2} \sqrt{2^{-1} \ln 2}\right\} .
\end{aligned}
$$

We need a classical result from probability theory.
Theorem 3.1 (Hoeffding's inequality) Let $X_{1}, \ldots, X_{m}$ be independent random variables with $b_{1} \leq X_{i} \leq b_{2} \quad(i=1, \ldots, m)$. Then for all $t>0$,

$$
P\left(\left|\sum_{i=1}^{m}\left(X_{i}-E\left[X_{i}\right]\right)\right| \leq t\right) \geq 1-\exp \left(-\frac{2 t^{2}}{m\left(b_{2}-b_{1}\right)^{2}}\right)
$$

Lemma 3.2 For all $n>1, P(A) \geq 1-\frac{4}{\log n}+\frac{1}{n}$ for the Haar wavelets, and $P(A) \geq 1-\frac{8}{\log n}+\frac{2}{n}$ for the interval wavelet system.

In either case, if $n \geq 256$ then $P(A) \geq \frac{1}{128}$. If $n \geq 2^{9}$ then $P(A)>\frac{1}{9}$. Moreover, $P(A)$ tends to 1 when $n \rightarrow \infty$.

Proof: We first give the proof for the Haar wavelets. For any $\ell \in[-1, J-\log J]$ and $k \in$ $\left[0,2^{J-\log J-\ell}-1\right]$ the noise samples $e_{k 2^{\ell} J+1}, \ldots, e_{(k+1)^{\ell}{ }_{J}}$ are independent random variables, each with values in $\left[-\frac{b}{2}, \frac{b}{2}\right]$. So Hoeffding's inequality applies, and since $E\left[e_{i}\right]=0$ for all $i$, we obtain for all $t>0$,

$$
P\left(\left|\sum_{i=0}^{2^{\ell-1} J-1} e_{k 2^{\ell} J+i+1}\right| \leq t\right) \geq 1-\exp \left(-\frac{2 t^{2}}{2^{\ell} J b^{2}}\right)
$$

Letting $t=b 2^{\ell / 2} J \sqrt{2^{-1} \ln 2}$ we obtain

$$
\begin{equation*}
P\left(\left|\sum_{i=0}^{2^{\ell-1} J-1} e_{k 2^{\ell} J+i+1}\right| \leq b 2^{\ell / 2} J \sqrt{2^{-1} \ln 2}\right) \geq 1-\frac{1}{n} \tag{1}
\end{equation*}
$$

For $\ell \in[-1, J-\log J]$ and $k \in\left[0,2^{J-\log J-\ell}-1\right]$, let

$$
A_{\ell, k}=\left\{\omega \in \Omega^{n}:\left|\sum_{i=0}^{2^{\ell-1} J-1} e_{k 2^{\ell} J+i+1}(\omega)\right| \leq b 2^{\ell / 2} J \sqrt{2^{-1} \ln 2}\right\}
$$

and let $A_{\ell}=\bigcap_{k=0}^{2^{J-\log J-\ell}-1} A_{\ell, k}$.
Then by (11), $P\left(A_{\ell, k}\right) \geq 1-\frac{1}{n}$.
For the complements of these sets we have $\bar{A}_{\ell}=\bigcup_{k=0}^{2^{J-\log J-\ell}-1} \bar{A}_{\ell, k}$
hence $P\left(\bar{A}_{\ell}\right) \leq \sum_{k=0}^{2^{J-\log J-\ell-1}} \frac{1}{n}$.
Since $n=2^{J}$ we obtain $P\left(\bar{A}_{\ell}\right) \leq \frac{2^{-\ell}}{\log n}$.
Since $A=\bigcap_{\ell=-1}^{J-\log J} A_{\ell}$ we have

$$
P(A) \geq 1-\sum_{\ell=-1}^{J-\log J} P\left(\bar{A}_{\ell}\right) \geq 1-\sum_{\ell=-1}^{J-\log J} \frac{2^{-\ell}}{\log n} .
$$

Hence, $P(A) \geq 1-\frac{4}{\log n}+\frac{1}{n}$. This proves the Lemma for the Haar case.
For the interval wavelet system we let

$$
\begin{array}{r}
A^{\alpha}=\left\{\omega \in \Omega^{n}:(\forall \ell \in[-1, J-\log J])\left(\forall k \in\left[0,2^{J-\log J-\ell}-1\right]\right),\right. \\
\left.\quad\left|\sum_{i=0}^{J 2^{\ell}-1} e_{k 2^{\ell} J+i+1}(\omega) \cdot \alpha_{i, J-\log J-\ell, k}\right| \leq b J 2^{\ell / 2} \sqrt{2^{-1} \ln 2}\right\},
\end{array}
$$

and

$$
\begin{array}{r}
A^{\beta}=\left\{\omega \in \Omega^{n}:(\forall \ell \in[-1, J-\log J])\left(\forall k \in\left[0,2^{J-\log J-\ell}-1\right]\right),\right. \\
\left.\quad\left|\sum_{i=0}^{J 2^{\ell}-1} e_{k 2^{\ell} J+i+1}(\omega) \cdot \beta_{i, J-\log J-\ell, k}\right| \leq b J 2^{\ell / 2} \sqrt{2^{-1} \ln 2}\right\} .
\end{array}
$$

Then $A=A^{\alpha} \cap A^{\beta}$.
We also let

$$
A_{\ell, k}^{\alpha}=\left\{\omega \in \Omega^{n}:\left|\sum_{i=0}^{J 2^{\ell}-1} e_{k 2^{\ell} J+i+1}(\omega) \cdot \alpha_{i, J-\log J-\ell, k}\right| \leq b J 2^{\ell / 2} \sqrt{2^{-1} \ln 2}\right\}
$$

and

$$
A_{\ell, k}^{\beta}=\left\{\omega \in \Omega^{n}:\left|\sum_{i=0}^{J 2^{\ell}-1} e_{k 2^{\ell} J+i+1}(\omega) \cdot \beta_{i, J-\log J-\ell, k}\right| \leq b J 2^{\ell / 2} \sqrt{2^{-1} \ln 2}\right\}
$$

Moreover, we let $A_{\ell}^{\alpha}=\bigcap_{k} A_{\ell, k}^{\alpha}$ and $A_{\ell}^{\beta}=\bigcap_{k} A_{\ell, k}^{\beta}$. Then $A_{\ell}=A_{\ell}^{\alpha} \cap A_{\ell}^{\beta}$, hence $\bar{A}_{\ell}=\overline{A_{\ell}^{\alpha}} \cup \bar{A}_{\ell}^{\beta}$.

By the same proof as for Haar wavelets above: $P\left(\bar{A}_{\ell}^{\alpha}\right)$ and $P\left(\bar{A}_{\ell}^{\beta}\right) \leq \frac{2^{-\ell}}{\log n}$.
Hence, $\quad P\left(\bar{A}_{\ell}\right) \leq \frac{2^{-\ell+1}}{\log n}$.
Since $A=\bigcap_{\ell=-1}^{J-\log J} A_{\ell}$ we obtain by a similar calculation as in the Haar case:

$$
P(A) \geq 1-\frac{8}{\log n}+\frac{2}{n}
$$

Lemma 3.3 For all $\omega \in A$, all $j \in] J_{0}, J\left[\right.$, and all $k \in\left[0,2^{j}-1\right]$, we have (for some constant $C_{\varphi} \geq 1$, depending only on the wavelet system):

$$
\left|d_{j, k}^{(e(\omega))}\right| \leq b C_{\varphi} \sqrt{\frac{\log n}{n}}
$$

and for all $k \in\left[0,2^{J_{0}}-1\right]$,

$$
\left|c_{J_{0}, k}^{(e(\omega))}\right| \leq b C_{\varphi} \sqrt{\frac{\log n}{n}}
$$

Proof: We consider two cases for $j$.
Case 1: $\quad J_{0} \leq j \leq J-\log J+1$.
We write $j$ as $J-\log J-\ell$, where $-1 \leq \ell \leq J-\log J-J_{0}$. Let us first consider Haar wavelets. By (H2) (in Lemma 2.1) we have

$$
d_{j, k}^{(e(\omega))}=2^{-J+j / 2}\left(\sum_{i=0}^{2^{\ell-1} J-1} e_{k 2^{\ell} J+i+1}(\omega)-\sum_{i=0}^{2^{\ell-1} J-1} e_{(k+1 / 2) 2^{\ell} J+i+1}(\omega)\right)
$$

Since $\omega \in A$ we can apply the defining property of $A$ to

$$
\left|\sum_{i=0}^{J 2^{\ell-1}-1} e_{i+1+k 2^{\ell} J}\right|=\left|\sum_{i=0}^{J 2^{\ell-1}-1} e_{i+1+2 k 2^{\ell-1} J}\right|
$$

Since $2 k$ is in the correct range $\left[0,2^{j+1}-2\right]=\left[0, \frac{1}{J} 2^{J-(\ell-1)}-2\right]$, we have

$$
\left|\sum_{i=0}^{J 2^{\ell-1}-1} e_{i+1+k 2^{\ell} J}\right| \leq b J 2^{(\ell-1) / 2} \sqrt{2^{-1} \ln 2}
$$

Similarly,

$$
\left|\sum_{i=0}^{J 2^{\ell-1}-1} e_{i+1+\left(k+\frac{1}{2}\right) 2^{\ell} J}\right|=\left|\sum_{i=0}^{J 2^{\ell-1}-1} e_{i+1+(2 k+1) 2^{\ell-1} J}\right| \leq b J 2^{(\ell-1) / 2} \sqrt{2^{-1} \ln 2} ;
$$

we used the defining property of $A$, since the range of $2 k+1$ is

$$
\left[0,2^{j+1}-2+1\right]=\left[0, \frac{1}{J} 2^{J-(\ell-1)}-1\right] .
$$

By combining these two bounds we obtain

$$
\left|d_{j, k}^{(e(\omega))}\right| \leq 2^{-J+j / 2} \cdot 2 \cdot b J 2^{(\ell-1) / 2} \sqrt{2^{-1} \ln 2}<b \sqrt{\ln 2} \sqrt{\frac{\log n}{n}} \leq b \sqrt{\frac{\log n}{n}} .
$$

Let us now consider case 1 for the interval wavelet system. By (D2) in Lemma 2.2,

$$
d_{j, k}^{(e(\omega))}=2^{-J+j / 2} \cdot \sum_{i=0}^{2^{\ell} J-1} e_{k 2^{\ell} J+i+1}(\omega) \cdot \beta_{i, j, k} .
$$

Since $\omega \in A$,

$$
\begin{aligned}
& \left|d_{j, k}^{(e(\omega))}\right| \leq 2^{-J+j / 2} \cdot b J 2^{(\ell-1) / 2} \sqrt{2^{-1} \ln 2}=b 2^{(-J+\log J) / 2} \sqrt{2^{-1} \ln 2}=b \sqrt{\frac{\log n}{n}} \sqrt{2^{-1} \ln 2} \\
& \quad \leq b \sqrt{\frac{\log n}{n}}
\end{aligned}
$$

Case 2: $\quad J-\log J+2 \leq j<J$.
For the Haar wavelets we use the boundedness of the noise, $\left|e_{i}-e_{j}\right| \leq b$. Hence, by (H2),

$$
\left|d_{j, k}^{(e(\omega))}\right| \leq 2^{-J+j / 2} b\left(J 2^{\ell-1}-1\right) \leq b \sqrt{\frac{\log n}{n}} .
$$

For the interval wavelet system, (D2) yields

$$
\begin{aligned}
& \left|d_{j, k}^{(e(\omega))}\right| \leq 2^{-J+j / 2} \sum_{i=0}^{2^{J-j}-1}\left|e_{k 2^{\ell} J+i+1}(\omega)\right| \cdot\left|\beta_{i, j, k}\right|=2^{-J+j / 2} 2^{J-j} \frac{b}{2} C_{\varphi} \\
& \leq \frac{b}{2} C_{\varphi} 2^{-j / 2} \leq b C_{\varphi} \sqrt{\frac{\log n}{n}}
\end{aligned}
$$

by using $j \geq J-\log J+2$ for the last inequality.
By an argument similar to the above we obtain the bound for $\left|c_{J_{0}, k}^{(e(\omega))}\right|$.

To implement wavelet shrinkage we need two parameters: A decomposition level $J_{0}$ and a threshold $\lambda_{n, \delta}$. We define

$$
J_{1}=\left\lceil\frac{1}{1+2 \alpha}(J-\log J)\right\rceil
$$

and we choose $J_{0}$ so that $J_{0} \leq J_{1}$.
For the Haar wavelets (when $0<\alpha \leq 1$ ) we can simply pick $J_{0}=0$, but for the interval wavelet system (when $1<\alpha$ and we have $N=\lceil\alpha\rceil$ vanishing moments), we also require (see ([4) that $J_{0} \geq 1+\log (2 N-1)$. When $\alpha>1$ we choose

$$
J_{0}=1+\lceil\log (2\lceil\alpha\rceil-1)\rceil
$$

Thus, for $J_{0}$ to exist (when $\alpha>1$ ) we need $n=2^{J}$ to be such that $1+\log (2\lceil\alpha\rceil-1) \leq J_{1}$. A sufficient condition for this is that $J-\log J \geq(1+\log (2 \alpha+1))(1+2 \alpha)$, or equivalently, $\quad \frac{n}{\log n} \geq(4 \alpha+2)^{2 \alpha+1}$.
By using the fact that $\frac{n}{\log n}$ is an increasing function of $n$, and that the relation $\frac{y}{\log y} \geq x$ is implied by $y \geq x \cdot \log x \cdot \log \log x$, we have the following sufficient condition on $n$ :

When $\alpha>1$ we assume that

$$
n \geq(4 \alpha+2)^{2 \alpha+2} \cdot(\log (4 \alpha+2))^{2}
$$

We use the threshold

$$
\lambda_{n, \delta}=C_{\varphi} b(1+2 \sqrt{(1+\delta) \ln 2}) \sqrt{\frac{\log n}{n}}
$$

The first step of the wavelet shrinkage algorithm is DWT, which maps $\left(y_{1}, \ldots, y_{n}\right)$ to $\sqrt{n}\left(c_{J_{0}, 0}^{(y)}, \ldots, c_{J_{0} 2^{J_{0}-1}}^{(y)}, d_{J_{0}, 0}^{(y)}, \ldots, d_{J_{0} 2^{J_{0}-1}}^{(y)}, \ldots, \ldots, d_{J-1,0}^{(y)}, \ldots, d_{J-1,2^{J-1}-1}^{(y)}\right)$, where $n=2^{J}$. Since $y_{i}=f_{i}+e_{i}$ and the DWT is linear we have

$$
c_{J_{0}, k}^{(y)}=c_{J_{0}, k}^{(f)}+c_{J_{0}, k}^{(e)}, \quad 0 \leq k<2^{J_{0}},
$$

and

$$
d_{j, k}^{(y)}=d_{j, k}^{(f)}+d_{j, k}^{(e)}, \quad J_{0} \leq j<J, \quad 0 \leq k<2^{j},
$$

where $c_{J_{0}, k}^{(f)}, d_{j, k}^{(f)}$ and $c_{J_{0}, k}^{(e)}, d_{j, k}^{(e)}$ are the wavelet coefficients for $\left(f_{1}, \ldots, f_{n}\right)$ and $\left(e_{1}, \ldots, e_{n}\right)$, respectively.

The second step of wavelet shrinkage is thresholding. We shall prove our result for soft thresholding. But in our proofs it will be easy to see that our results will hold for hard thresholding too. For soft thresholding, we have

$$
\tilde{d}_{j, k}= \begin{cases}d_{j, k}^{(y)}-\lambda_{n, \delta} & \text { if } d_{j, k}^{(y)}>\lambda_{n, \delta} \\ 0 & \text { if }\left|d_{j, k}^{(y)}\right| \leq \lambda_{n, \delta} \\ d_{j, k}^{(y)}+\lambda_{n, \delta} & \text { if } d_{j, k}^{(y)}<-\lambda_{n, \delta}\end{cases}
$$

The last step of wavelet shrinkage is the inverse of DWT which yields $\tilde{y}=\left(\tilde{y}_{1}, \ldots, \tilde{y}_{n}\right)$. If we let

$$
\begin{equation*}
\tilde{y}(x)=\sum_{k=0}^{2^{J_{0}-1}} c_{J_{0}, k}^{(y)} \varphi_{J_{0}, k}(x)+\sum_{j=J_{0}}^{J-1} \sum_{k=0}^{2^{j-1}} \tilde{d}_{j, k} \psi_{j, k}(x) \tag{2}
\end{equation*}
$$

then we obtain $\tilde{y}_{i}=\tilde{y}\left(\frac{i}{n}\right)$ for $i=1, \ldots, n$.

### 3.2 Application of Talagrand's theorem

Let $W$ be the orthogonal matrix that represents the DWT. Let $A \subseteq \Omega^{n}$ be as above. For any $\delta>0$ we define the following subset of $\Omega^{n}$ :

$$
B_{\delta}=\left\{\omega^{\prime} \in \Omega^{n}:(\forall \ell \in[1, n]), \quad \inf _{\omega \in A}\left|\sum_{i=1}^{n} W_{\ell, i}\left(e_{i}\left(\omega^{\prime}\right)-e_{i}(\omega)\right)\right|<2 b \sqrt{(1+\delta) \ln n}\right\} .
$$

Lemma 3.4 For all $\omega^{\prime} \in B_{\delta}$ and all $k \in\left[0,2^{J_{0}}-1\right]: \quad\left|c_{J_{0}, k}^{\left(e\left(\omega^{\prime}\right)\right)}\right| \leq \lambda_{n, \delta}$.
For all $j \in\left[J_{0}, J-1\right]$ and $k \in\left[0,2^{j}-1\right]: \quad\left|d_{j, k}^{\left(e\left(\omega^{\prime}\right)\right)}\right| \leq \lambda_{n, \delta}$.

Proof: By the definition of $B_{\delta}$, for every $\omega^{\prime} \in B_{\delta}$ there exists $\omega \in A$ such that

$$
\sqrt{n}\left|c_{J_{0}, k}^{(e(\omega))}-c_{J_{0}, k}^{\left(e\left(\omega^{\prime}\right)\right)}\right| \leq b 2 \sqrt{(1+\delta) \ln n}
$$

and

$$
\sqrt{n}\left|d_{j, k}^{(e(\omega))}-d_{j, k}^{\left(e\left(\omega^{\prime}\right)\right)}\right| \leq b 2 \sqrt{(1+\delta) \ln n}
$$

The Lemma then follows from Lemma 3.3.
For the following theorem we use the threshold $\lambda_{n, \delta}$ as above; we let $n_{0}=2^{9}$ when $0<\alpha \leq 1$, and $n_{0}=(4 \alpha+2)^{2 \alpha+2} \cdot(\log (4 \alpha+2))^{2}$ when $\alpha>1$.

Lemma 3.5 When $n \geq n_{0}, \quad P\left(B_{\delta}\right)>1-\frac{9}{n^{1+\delta}}$.
Proof: We first prove that

$$
\left\{\omega^{\prime} \in \Omega^{n}: d_{T}\left(\omega^{\prime}, A\right)<2 \sqrt{(1+\delta) \ln n}\right\} \subseteq B_{\delta}
$$

Recall the definition
$d_{T}\left(\omega^{\prime}, A\right)=$
$\sup \left\{\inf \left\{\sum_{i=1}^{n} \beta_{i} \cdot I\left(\omega_{i}^{\prime} \neq \omega_{i}\right):\left(\omega_{1}, \ldots, \omega_{n}\right) \in A\right\}:\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathbb{R}^{n}, \sum_{i=1}^{n} \beta_{i}^{2}=1\right\}$.
We will choose the following $n$ vectors for $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right)$ in the above formula:
$\left(\left|W_{1, \ell}\right|, \ldots,\left|W_{n, \ell}\right|\right)$, for $\ell=1, \ldots, n$.
Since $W$ is orthogonal all its row vectors have unit length. For all $\omega^{\prime} \in \Omega^{n}, \omega=\left(\omega_{1}, \ldots, \omega_{n}\right) \in A$, and $1 \leq \ell \leq n$, we have:

$$
\begin{aligned}
& \left|\sum_{i=1}^{n} W_{i, \ell}\left(e_{i}\left(\omega^{\prime}\right)-e_{i}(\omega)\right)\right| \\
& \leq \quad b \sum_{i=1}^{n}\left|W_{i, \ell}\right| \cdot I\left(e_{i}\left(\omega^{\prime}\right) \neq e_{i}(\omega)\right) \\
& \leq b \sum_{i=1}^{n}\left|W_{i, \ell}\right| \cdot I\left(\omega^{\prime} \neq \omega\right) .
\end{aligned}
$$

(The last inequality follows from the fact that $I\left(e_{i}\left(\omega^{\prime}\right) \neq e_{i}(\omega)\right) \leq I\left(\omega^{\prime} \neq \omega\right)$, because $e_{i}\left(\omega^{\prime}\right) \neq e_{i}(\omega)$ implies $\omega^{\prime} \neq \omega$.)
Hence, for all $\omega^{\prime} \in \Omega^{n}$ and $1 \leq \ell \leq n$,

$$
\begin{aligned}
& \inf \left\{\left|\sum_{i=1}^{n} W_{i, \ell}\left(e_{i}\left(\omega^{\prime}\right)-e_{i}(\omega)\right)\right|: \omega \in A\right\} \\
& \leq \inf \left\{\sum_{i=1}^{n} \mid W_{i, \ell} \cdot I\left(\omega^{\prime} \neq \omega\right) b: \omega \in A\right\} \\
& =b \inf \left\{\sum_{i=1}^{n}\left|W_{i, \ell}\right| \cdot I\left(\omega^{\prime} \neq \omega\right): \omega \in A\right\} .
\end{aligned}
$$

Therefore, if $d_{T}\left(\omega^{\prime}, A\right) \leq 2 \sqrt{(1+\delta) \ln n}$ then for all $1 \leq \ell \leq n$,

$$
\inf \left\{\left|\sum_{i=1}^{n} W_{i, \ell}\left(e_{i}\left(\omega^{\prime}\right)-e_{i}(\omega)\right)\right|: \omega \in A\right\} \leq b 2 \sqrt{(1+\delta) \ln n}
$$

This means that $\omega^{\prime} \in B_{\delta}$, and this proves that

$$
\left\{\omega^{\prime} \in \Omega^{n}: d_{T}\left(\omega^{\prime}, A\right)<2 \sqrt{(1+\delta) \ln n}\right\} \subseteq B_{\delta} .
$$

Hence, $P\left(B_{\delta}\right) \geq P\left(\left\{\omega^{\prime} \in \Omega^{n}: d_{T}\left(\omega^{\prime}, A\right)<2 \sqrt{(1+\delta) \ln n}\right\}\right)$.
By Talagrand's theorem this is $\geq 1-\exp (-(1+\delta) \ln 2) \cdot \frac{1}{P(A)}>1-\frac{9}{n^{1+\delta}}$.

Lemma 3.6 For all $\omega^{\prime} \in B_{\delta}$ we have:
(1) When $J_{1} \leq j<J, 0 \leq k<2^{j}$,

$$
\left|\tilde{d}_{j, k}\left(\omega^{\prime}\right)-d_{j, k}^{(f)}\right| \leq\left|d_{j, k}^{(f)}\right| \leq C_{\varphi} M \cdot 2^{-j\left(\frac{1}{2}+\alpha\right)}
$$

$$
\text { When } J_{0} \leq j<J_{1}, \quad 0 \leq k<2^{j}, \quad\left|\tilde{d}_{j, k}\left(\omega^{\prime}\right)-d_{j, k}^{(f)}\right| \leq 2 \lambda_{n, \delta}
$$

Proof: To prove (1), we note first that by (H3), (D3) we have $\left|d_{j, k}^{(f)}\right| \leq C_{\varphi} M 2^{-j(1 / 2+\alpha)}$. To prove the inequality $\left|d_{j, k}^{(f)}-\tilde{d}_{j, k}\right| \leq\left|d_{j, k}^{(f)}\right|$ one considers six cases, according to the possible relative positions of $0, d_{j, k}^{(f)}$, and $\tilde{d}_{j, k}$. If $0 \leq \tilde{d}_{j, k} \leq d_{j, k}^{(f)}$, or if $d_{j, k}^{(f)} \leq \tilde{d}_{j, k} \leq 0$, the inequality is obvious from the order picture. The other four cases are not possible, since they would imply that $\left|d_{j, k}^{(e(\omega))}\right|>\lambda_{n, \delta}$, contradicting what we saw a little earlier. This proves (1).

For the proof of (2) we consider two cases. If $\tilde{d}_{j, k}=0,\left|d_{j, k}^{(y)}\right| \leq \lambda_{n, \delta}$, hence $\left|d_{j, k}^{(f)}-\tilde{d}_{j, k}\right|=$ $\left|d_{j, k}^{(f)}\right|=\left|d_{j, k}^{(y)}-d_{j, k}^{(e)}\right| \leq\left|d_{j, k}^{(y)}\right|+\left|d_{j, k}^{(e)}\right| \leq \lambda_{n, \delta}+\lambda_{n, \delta}$. In the second case, $\left|d_{j, k}^{(y)}\right|>\lambda_{n, \delta}$, and $\left|d_{j, k}^{(f)}-\tilde{d}_{j, k}\right|=\left|d_{j, k}^{(e)}-\lambda_{n, \delta}\right| \leq \lambda_{n, \delta}+\lambda_{n, \delta}$. This proves the inequality.

Theorem 3.7 (Deviation bound for max square error) For wavelet shrinkage with threshold $\lambda_{n, \delta}$ we have for all $n \geq n_{0}$ :

$$
P\left(\max _{0 \leq i \leq n}\left(f_{i}-\tilde{y}_{i}\right)^{2} \leq\left(c_{1}+c_{2} \delta\right)\left(\frac{\log n}{n}\right)^{\frac{2 \alpha}{1+2 \alpha}}\right) \geq 1-\frac{9}{n^{1+\delta}}
$$

where $c_{1}$ and $c_{2}$ depend only on $b, M$, and $\alpha$.
As a consequence (deviation bound for mean square error),

$$
P\left(\frac{1}{n} \sum_{i=0}^{n}\left(f_{i}-\tilde{y_{i}}\right)^{2} \leq\left(c_{1}+c_{2} \delta\right)\left(\frac{\log n}{n}\right)^{\frac{2 \alpha}{1+2 \alpha}}\right) \geq 1-\frac{9}{n^{1+\delta}}
$$

Proof: At the beginning of subsection 2.1 we defined the function $\bar{f}$, and its wavelet coefficients. We have

$$
\bar{f}(x)=\sum_{k=0}^{2^{J_{0}-1}} c_{J_{0}, k}^{(f)} \varphi_{J_{0}, k}(x)+\sum_{j=J_{0}}^{J_{1}-1} \sum_{k=0}^{2^{j}-1} d_{j, k}^{(f)} \psi_{j, k}(x)+\sum_{j=J_{1}}^{J-1} \sum_{k=0}^{2^{j}-1} d_{j, k}^{(f)} \psi_{j, k}(x),
$$

and $f_{i}=\bar{f}\left(\frac{i}{n}\right)$ for $1 \leq i \leq n$.
In connection with the thresholding of $y$ we define the function

$$
\tilde{y}(x)=\sum_{k=0}^{2^{J_{0}}-1} c_{J_{0}, k}^{(y)} \varphi_{J_{0}, k}(x)+\sum_{j=J_{0}}^{J_{1}-1} \sum_{k=0}^{2^{j}-1} \tilde{d}_{j, k} \psi_{j, k}(x)+\sum_{j=J_{1}}^{J-1} \sum_{k=0}^{2^{j}-1} \tilde{d}_{j, k} \psi_{j, k}(x) .
$$

By Lemma 3.4 we have for all $\omega^{\prime} \in B_{\delta}$ :

$$
\begin{equation*}
\left|c_{J_{0}, k}^{(y)}-c_{J_{0}, k}^{(f)}\right|=\left|c_{J_{0}, k}^{\left(e\left(\omega^{\prime}\right)\right)}\right| \leq \lambda_{n, \delta} \tag{0}
\end{equation*}
$$

By Lemma 3.6 we have for all $\omega^{\prime} \in B_{\delta}$ :

$$
\begin{align*}
& \left|\tilde{d}_{j, k}-d_{j, k}^{(f)}\right| \leq\left|d_{j, k}^{(f)}\right| \leq C_{\varphi} M \cdot 2^{-j\left(\frac{1}{2}+\alpha\right)} \quad \text { for } J_{1} \leq j<J, 0 \leq k<2^{j}  \tag{1}\\
& \left|\tilde{d}_{j, k}-d_{j, k}^{(f)}\right| \leq 2 \lambda_{n, \delta}^{(f)} \quad \text { for } J_{0} \leq j<J_{1}, \quad 0 \leq k<2^{j} \tag{2}
\end{align*}
$$

Let us first deal with the case of Haar wavelets (when $\alpha \leq 1$ ). For a given $j$, the supports of different Haar wavelets do not overlap. Therefore, for all $x \in] 0,1]$ there exist $K_{1}$ and $K(j)$ such that

$$
\begin{aligned}
& |\tilde{f}(x)-\tilde{y}(x)| \leq \\
& \quad\left|c_{J_{0}, K_{1}}^{(y)}-c_{J_{0}, K_{1}}^{(f)}\right| \cdot 2^{J_{0} / 2}+\sum_{j=J_{0}}^{J_{1}-1}\left|\tilde{d}_{j, K(j)}-d_{j, K(j)}^{(f)}\right| \cdot 2^{j / 2}+\sum_{j=J_{1}}^{J-1}\left|\tilde{d}_{j, K(j)}-d_{j, K(j)}^{(f)}\right| \cdot 2^{j / 2}
\end{aligned}
$$

This and (0), (1), (2) imply for all $x \in] 0,1]$ :

$$
\begin{aligned}
& |\tilde{f}(x)-\tilde{y}(x)| \leq C_{1} \cdot\left(\frac{\log n}{n}\right)^{\frac{\alpha}{1+2 \alpha}}+C_{2} \cdot\left(\frac{\log n}{n}\right)^{\frac{\alpha}{1+2 \alpha}}+C_{3} \cdot\left(\frac{\log n}{n}\right)^{\frac{\alpha}{1+2 \alpha}} \\
& =\left(c_{1}^{\prime}+c_{2}^{\prime} \sqrt{1+\delta}\right) \cdot\left(\frac{\log n}{n}\right)^{\frac{\alpha}{1+2 \alpha}}
\end{aligned}
$$

Letting $x=\frac{i}{n}(1 \leq i \leq n)$ we obtain for all $\omega^{\prime} \in B_{\delta}$ :

$$
\left|f_{i}-\tilde{y}_{i}\left(\omega^{\prime}\right)\right|=\left|\tilde{f}\left(\frac{i}{n}\right)-\tilde{y}\left(\frac{i}{n}\right)\right| \leq\left(c_{1}^{\prime}+c_{2}^{\prime} \sqrt{1+\delta}\right) \cdot\left(\frac{\log n}{n}\right)^{\frac{\alpha}{1+2 \alpha}}
$$

In the Haar case the theorem follows from this and the fact that $P\left(B_{\delta}\right)>1-\frac{9}{n^{1+\delta}}$ (when $n \geq n_{0}$ ).

For wavelets on the interval (when $\alpha>1$, and the number of vanishing moments is $N=\lceil\alpha\rceil$ ), there are never more than $2 N$ wavelets that overlap (for a given $j$ ). Indeed, in the above sums we have for each $j$ and each $x: 0 \leq 2^{j} x-k \leq 2 N-1$. (Other values of $k$ would place the argument $2^{j} x-k$ of the wavelet functions outside of the support and would ence only produce zero-terms in the sums.) Hence $k$ only needs to range from $\left\lceil 2^{j} x\right\rceil-2 N+1$ through $\left\lceil 2^{j} x\right\rceil$, which corresponds to $2 N$ values of $k$.

Hence, the same calculation as for Haar wavelets applies, except that the constants $C_{1}, C_{2}$, $C_{3}, c_{1}^{\prime}, c_{2}^{\prime}$ need to be multiplied by $2 N$.

## Appendix

## Proof of Lemma 2.1

Properties (H1) and (H2) follow from a direct calculation based on the exact formulas for the Haar wavelets $\varphi_{j, k}$ and $\psi_{j, k}$.

$$
\begin{aligned}
& c_{j, k}^{(e)}=\int_{0}^{1} \bar{e}(x) \varphi_{j, k}(x) d x=2^{j / 2} \int_{k 2^{-j}}^{(k+1) 2^{-j}} \bar{e}(x) d x= \\
& \sum_{i=k 2^{J-j}}^{\left(k+12^{J-j}-1\right.} e_{i+1} 2^{-J}=2^{-J+j / 2} \sum_{i=0}^{2^{J-j}-1} e_{i+1+k 2^{J-j}} .
\end{aligned}
$$

The calculation for (H2) is similar. The same calculation as for (H2) will give for $\bar{f}$ :

$$
d_{j, k}^{(f)}=2^{-J-1+j / 2} \sum_{i=0}^{2^{J-j-1}}\left(f\left(i+1+k 2^{J-j}\right)-f\left(i+1+\left(k+\frac{1}{2}\right) 2^{J-j}\right)\right) .
$$

Then we use the Hölder condition $\left|f\left(i+1+k 2^{J-j}\right)-f\left(i+1+\left(k+\frac{1}{2}\right) 2^{J-j}\right)\right| \leq M\left(\frac{1}{2} 2^{J-j}\right)^{\alpha}$.

## Proof of Lemma 2.2

Property (D1) follows from a direct calculation:

$$
c_{j, k}^{(e)}=\int_{0}^{1} \bar{e}(x) \varphi_{j, k}(x) d x=\sum_{i=0}^{n-1} e_{i} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \varphi_{j k}(x) d x
$$

where we denote the functions $\varphi_{j k}^{\text {left }}$ by $\varphi_{j, 2^{j}-2 N+k}$, and $\varphi_{j k}^{\text {right }}$ by $\varphi_{j, 2^{j}-N+k}$.
For the $\varphi_{j k}$ "in the middle" of the interval we have

$$
\int_{\frac{i}{n}}^{\frac{i+1}{n}} \varphi_{j k}(x) d x=2^{j / 2} \int_{i 2^{-J+j}-k}^{(i+1) 2^{-J+j}-k} \varphi(t) 2^{-j} d t=2^{j / 2} 2^{-J} \alpha_{i j k}
$$

by the the mean-value theorem, for some numbers $\alpha_{i j k}$ with $\left|\alpha_{i j k}\right| \leq \sup _{[0,1]}|\varphi|$.
For the $\varphi_{j, 2^{j}-2 N+k}$ "at the left end" of the interval,

$$
\begin{aligned}
& \int_{\frac{i}{n}}^{\frac{i+1}{n}} \varphi_{j k}^{\mathrm{left}}(x) d x=\int_{\frac{i}{n}}^{\frac{i+1}{n}} \sum_{s=0}^{2 N-1}(-s)^{k} \varphi\left(2^{j} x+s\right) d x=\sum_{s=0}^{2 N-1}(-s)^{k} \int_{i 2^{-J+j}+s}^{(i+1) 2^{-J+j}+s} \varphi(y) 2^{-j} d y \\
& =\sum_{s=0}^{2 N-1}(-s)^{k} 2^{-j} 2^{-J+j} \gamma_{i j s}
\end{aligned}
$$

by the mean-value theorem, for some numbers $\gamma_{i j s}$ with $\left|\gamma_{i j s}\right| \leq \sup _{[0,1]}|\varphi|$. By taking

$$
\alpha_{i j k}=2^{-j / 2} \sum_{s=0}^{2 N-1}(-s)^{k} \gamma_{i j s}
$$

we obtain (D1). At the left end, $k \leq N$, so $\left|\alpha_{i j k}\right| \leq 2 N(2 N-1)^{N} \cdot \sup |\varphi|$.
The scaling functions "at the right end" of the interval are handled in a similar way. The calculation for (D2) is similar. (D3) follows from the wavelet characterization of Hölder classes (5], page 299, and [10]).

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