the tightness of the bounds on $C_{\rm sum}(\boldsymbol{S})$ when $L \le K \equiv 2 \pmod 4$ (Cases iii) and iv)) depends on the existence of Hadamard matrices of size $K-2$ and $K+2$. Indeed, if a size $K-2$ Hadamard matrix exists and $K \ge L+2$, then the signature design method in [9] provides us with minimum-TSC sets whose $C_{\rm sum}$ achieves the upper bound in Case iii) or iv) of Proposition 2, Part b). If a size $K+2$ Hadamard matrix exists, then the minimum-TSC sets designed in [10], [11] have $C_{\rm sum}$ equal to the lower bound in Case iii) or iv) of Proposition 2, Part b). □

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A Counterexample for the Open Problem on the Minimal Delays of Orthogonal Designs With Maximal Rates

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***Abstract*—X. Liang systematically investigated orthogonal designs with maximal rates, gave the maximal rates of complex orthogonal designs and a concrete construction procedure for complex orthogonal designs with the maximal rates. He also posed an open problem on the minimal decoding delays of complex orthogonal designs with maximal rates, and proved that the problem is correct for less than or equal to six transmit antennas. In this correspondence, we give a counterexample for the open problem for $n = 8$ and prove that the minimal delay for complex orthogonal designs with eight columns is 56. Hence, we give a negative answer for the open problem.**

***Index Terms*—Complex orthogonal designs, decoding delays, full diversity, maximal rates, space–time block codes.**

I. INTRODUCTION

Recently, space–time codes have been extensively investigated for wireless communication systems with multiple transmit and receive antennas. Alamouti [1] proposed a remarkable transmission scheme using two transmit antennas, which has linear maximum-likelihood (ML) decoding complexity and full diversity. Subsequently, Tarokh, Jafarkhani, and Calderbank [9] generalized Alamouti's idea to the general case by orthogonal designs, i.e., space–time codes from orthogonal designs, and provided a systematic method to construct real orthogonal designs with code rate 1 and complex orthogonal designs with code rate $1/2$. It was proved in [8] and [9] that the code rate of real or complex orthogonal designs is not larger than 1. Hence, what are the maximal rates for complex orthogonal designs is an open problem. Lately, an upper bound of the maximal rate for space–time codes from generalized complex orthogonal designs was given by Wang and Xia in [11] by use of elegant matrix analysis. However, we do not know if the upper bound in [11] can be achieved for more than four transmit antennas. At almost the same time, Liang [3] systematically and smartly investigated the maximal rates of space–time codes from complex orthogonal designs: he not only gave the maximal rates of complex orthogonal designs for any number of transmit antennas, but also presented a concrete construction procedure for complex orthogonal designs with the maximal rates. Furthermore, Liang discussed the minimal decoding delays of complex orthogonal designs with the maximal rates. He proved that the complex orthogonal designs with the maximal rates obtained from his construction procedure have the minimal decoding delays for fewer than or equal to six transmit antennas, and posed an open problem for the minimal decoding delays.

In the correspondence, we give a counterexample for the open problem in [3], thus giving a negative answer to the open problem. In Section II, we introduce some preliminaries on orthogonal designs. A

Manuscript received February 26, 2004; revised June 1, 2004. This work was supported by the National Science Foundation of China under Grants 60003007 and 60472038 as well as by the Japan Society for Promotion of Science (JSPS) under Research Grant 14380139.

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Communicated by Ø. Ytrechus, Associate Editor for Coding Techniques.

Digital Object Identifier 10.1109/TIT.2004.839544

counterexample for the open problem is given in Section III. Furthermore, we prove that the minimal delay of complex orthogonal designs with eight columns is 56.

II. Preliminaries on Orthogonal Designs

In this section, we introduce some basic notions on space–time codes and orthogonal designs. In what follows, $\mathbb{C}$ denotes the field of all complex numbers, $\mathbb{R}$ the field of all real numbers, and $\mathbb{Z}$ the ring of all integers. All vectors are assumed to be column vectors. For any field $\mathbb{F}$, denote by $\mathbb{F}^n$ and $M_{m\times n}(\mathbb{F})$ the set of all n-dimensional vectors in $\mathbb{F}$ and the set of all $m \times n$ matrices in $\mathbb{F}$, respectively. For any vector $x \in \mathbb{F}^n$, denote by x^t the transpose of x. For any $a \in \mathbb{C}$, denote by a^* the conjugate of a. For any vector $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{C}^n$, denote by $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^t$ the conjugate of x, and denote by $x^H = (x_1^*, x_2^*, \ldots, x_n^*)$ the conjugate transpose of x. Similarly, for any matrix $A \in M_{m\times n}(\mathbb{C})$, A^t denotes the transpose of A, A^* the conjugate of A, and A^H the conjugate transpose of A. Denote by

$$A(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) \text{ and } A(s_1 \sim s_2; t_1 \sim t_2)$$

the submatrix consisting of the i_1th, i_2th, $\ldots, i_k$th rows and the j_1th, j_2th, $\ldots, j_k$th columns of A, and the submatrix consisting of the s_1th, (s_1+1)th, $\ldots, s_2$th rows and the t_1th, (t_1+1)th, $\ldots, t_2$th columns of A, where $s_1 < s_2$ and $t_1 < t_2$, respectively. Sometimes, we denote by $A(i_1, i_2, \ldots, i_k; t_1 \sim t_2)$ the submatrix consisting of the i_1th, i_2th, $\ldots, i_k$th rows and the t_1th, (t_1+1)th, $\ldots, t_2$th columns of A. So $A(i;j)$ denotes the 1×1 submatrix consisting of the (i,j) element of A. We use $A(i,j)$ for the (i,j) element of the matrix A. For any $x \in \mathbb{R}$, $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the least integer larger than or equal to x and the largest integer less than or equal to x.

Definition 1: A $[p,n,k]$ complex orthogonal design O is a $p \times n$ rectangular matrix whose nonzero entries are

$$z_1, z_2, \ldots, z_k, -z_1, -z_2, \ldots, -z_k$$

or

$$z_1^*, z_2^*, \ldots, z_k^*, -z_1^*, -z_2^*, \ldots, -z_k^*$$

where $z_1, z_2, \ldots, z_k, z_1^*, z_2^*, \ldots, z_k^*$ are indeterminates over the complex number field $\mathbb{C}$, such that

$$O^H O = (|z_1|^2 + |z_2|^2 + \cdots + |z_k|^2) I_{n\times n}.$$

When $p = n$, O is called a square complex orthogonal design. k/p is called the code rate of O, and p is called the decoding delay of O.

A $[p,n,k]$ generalized complex orthogonal design O is a $p \times n$ rectangular matrix whose entries are complex linear combinations of $z_1, z_2, \ldots, z_k, z_1^*, z_2^*, \ldots, z_k^*$ such that

$$O^H O = (|z_1|^2 + |z_2|^2 + \cdots + |z_k|^2) I_{n\times n}.$$

It has been proved in [11] that the code rate of generalized complex orthogonal designs is upper-bounded by $3/4$ when $n > 2$. Here, we only consider the complex orthogonal designs defined in Definition 1.

Clearly, a $[p,n,k]$ complex orthogonal design O is still a $[p,n,k]$ complex orthogonal design under the following transformations: 1) multiplication of rows or columns with -1; 2) permutation of rows or columns of O; 3) permutation of complex variables in O; 4) multiplication of some complex variables with -1; 5) substitution of some complex variables in O with their conjugates.

From Definition 1 , it is easy to check that, for a $[p,n,k]$ complex orthogonal design O, every column of O exactly contains one of z_i, $-z_i$, z_i^*, and $-z_i^*$ for each $i = 1, 2, \ldots, k$, and every row contains at most one of $z_i, -z_i, z_i^*$ and $-z_i^*$ for each $i = 1, 2, \ldots, k$. If O includes the following submatrix:

$$\begin{pmatrix} z_i & s_1 \\ s_2 & z_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_1 & z_i \\ z_i & s_2 \end{pmatrix}$$

then $s_1 = s_2 = 0$. If O includes the following submatrix:

$$\begin{pmatrix} z_i & s_1 \\ s_2 & z_i^* \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_1 & z_i \\ z_i^* & s_2 \end{pmatrix}$$

then $s_2 = -s_1^*$.

Tarokh [9] gave a construction method for complex designs with code rate $1/2$. It is very difficult to construct complex designs with code rates larger than $1/2$. However, Liang [3] made great progress in dealing with this problem. He proved that the maximal rate of a $[p,n,k]$ complex orthogonal design is $\frac{m+1}{2m}$, where $n = 2m$ or $2m-1$, $m \geq 1$. Furthermore, he presented a procedure for constructing complex orthogonal designs with the maximal rates. For details of this procedure, the reader can refer to [3] and find many examples there.

Given a positive integer n and $n = m + l$, a $[p(m,l), n, k(m,l)]$ complex orthogonal design O can be constructed by Liang's procedure, where

$$k(m,l) = \binom{n}{m} \quad \text{and} \quad p(m,l) = \binom{n}{m+1} + \binom{n}{m-1}.$$

Furthermore, when $m = \lceil \frac{n}{2} \rceil$, $\frac{k(m,l)}{p(m,l)}$ achieves the maximal value, i.e., $\frac{k(m,l)}{p(m,l)} = \frac{m+1}{2m}$. More concretely, when $n = 2m$

$$k(m,m) = \binom{2m}{m}, \; p(m,m) = \frac{2m}{m+1}\binom{2m}{m}$$
$$\text{and} \quad \frac{k(m,m)}{p(m,m)} = \frac{m+1}{2m}.$$

When $n = 2m - 1$

$$k(m,m-1) = \frac{1}{2}k(m,m), \; p(m,m-1) = \frac{1}{2}p(m,m)$$
$$\text{and} \quad \frac{k(m,m)}{p(m,m)} = \frac{m+1}{2m}.$$

Let $\gamma_n = \frac{m+1}{2m}$, where $n = 2m$ or $2m-1$. Let $\wp_{\mathbb{C}}(n,r)$ denote the minimal positive integer p such that there exists a $[p,n,k]$ complex orthogonal design with $\frac{k}{p} \geq r$. Liang [3] conjectured that

$$\wp_{\mathbb{C}}(n,\gamma_n) = \begin{cases} \frac{2m}{m+1}\binom{n}{m}, & \text{if } n = 2m \text{ or } 2m-1, \text{ but } n \neq 4 \\ 4, & \text{if } n = 4. \end{cases} \tag{1}$$

He proved the above equation is correct for $1 \leq n \leq 6$. Is (1) correct for $n \geq 7$? This is an open problem.

III. A Counterexample for the Open Problem

In this section, we give a counterexample for (1) for $n = 8$ and prove that the minimal delay for $[p,8,k]$ complex orthogonal designs with the maximal rate $\frac{5}{8}$ is 56. We also prove that (1) is correct for $n = 7$.

According to (1), a $[p,8,k]$ complex orthogonal design O with $\frac{k}{p} = \frac{5}{8}$ should have the minimal delay

$$p = \frac{8}{5}\binom{8}{4} = 112.$$

However, we construct a $[56, 8, 35]$ complex orthogonal design G, which is given in

$$G = \begin{pmatrix}
z_1 & 0 & 0 & 0 & z_2 & z_3 & z_4 & z_5 \\
0 & z_1 & 0 & 0 & z_6 & z_7 & z_8 & z_9 \\
0 & 0 & z_1 & 0 & z_{10} & z_{11} & z_{12} & z_{13} \\
0 & 0 & 0 & z_1 & z_{14} & z_{15} & z_{16} & z_{17} \\
-z_2^* & -z_6^* & -z_{10}^* & -z_{14}^* & z_1^* & 0 & 0 & 0 \\
-z_3^* & -z_7^* & -z_{11}^* & -z_{15}^* & 0 & z_1^* & 0 & 0 \\
-z_4^* & -z_8^* & -z_{12}^* & -z_{16}^* & 0 & 0 & z_1^* & 0 \\
-z_5^* & -z_9^* & -z_{13}^* & -z_{17}^* & 0 & 0 & 0 & z_1^* \\
-z_6 & z_2 & 0 & 0 & 0 & z_{18} & z_{19} & z_{20} \\
-z_{10} & 0 & z_2 & 0 & 0 & z_{21} & z_{22} & z_{23} \\
-z_{14} & 0 & 0 & z_2 & 0 & z_{24} & z_{25} & z_{26} \\
0 & -z_{18}^* & -z_{21}^* & -z_{24}^* & -z_3^* & z_2^* & 0 & 0 \\
0 & -z_{19}^* & -z_{22}^* & -z_{25}^* & -z_4^* & 0 & z_2^* & 0 \\
0 & -z_{20}^* & -z_{23}^* & -z_{26}^* & -z_5^* & 0 & 0 & z_2^* \\
-z_7 & z_3 & 0 & 0 & -z_{18} & 0 & z_{27} & z_{28} \\
-z_{11} & 0 & z_3 & 0 & -z_{21} & 0 & z_{29} & z_{30} \\
-z_{15} & 0 & 0 & z_3 & -z_{24} & 0 & z_{31} & z_{32} \\
0 & -z_{27}^* & -z_{29}^* & -z_{31}^* & 0 & -z_4^* & z_3^* & 0 \\
0 & -z_{28}^* & -z_{30}^* & -z_{32}^* & 0 & -z_5^* & 0 & z_3^* \\
-z_8 & z_4 & 0 & 0 & -z_{19} & -z_{27} & 0 & z_{33} \\
-z_{12} & 0 & z_4 & 0 & -z_{22} & -z_{29} & 0 & z_{34} \\
-z_{16} & 0 & 0 & z_4 & -z_{25} & -z_{31} & 0 & z_{35} \\
0 & -z_{33}^* & -z_{34}^* & -z_{35}^* & 0 & 0 & -z_5^* & z_4^* \\
-z_9 & z_5 & 0 & 0 & -z_{20} & -z_{28} & -z_{33} & 0 \\
-z_{13} & 0 & z_5 & 0 & -z_{23} & -z_{30} & -z_{34} & 0 \\
-z_{17} & 0 & 0 & z_5 & -z_{26} & -z_{32} & -z_{35} & 0 \\
0 & -z_{10} & z_6 & 0 & 0 & z_{35}^* & -z_{32}^* & z_{31}^* \\
0 & -z_{14} & 0 & z_6 & 0 & -z_{34}^* & z_{30}^* & -z_{29}^* \\
z_{18}^* & 0 & -z_{35} & z_{34} & -z_7^* & z_6^* & 0 & 0 \\
z_{19}^* & 0 & z_{32} & -z_{30} & -z_8^* & 0 & z_6^* & 0 \\
z_{20}^* & 0 & -z_{31} & z_{29} & -z_9^* & 0 & 0 & z_6^* \\
0 & -z_{11} & z_7 & 0 & -z_{35}^* & 0 & z_{26}^* & -z_{25}^* \\
0 & -z_{15} & 0 & z_7 & z_{34}^* & 0 & -z_{23}^* & z_{22}^* \\
z_{27}^* & 0 & -z_{26} & z_{23} & 0 & -z_8^* & z_7^* & 0 \\
z_{28}^* & 0 & z_{25} & -z_{22} & 0 & -z_9^* & 0 & z_7^* \\
0 & -z_{12} & z_8 & 0 & z_{32}^* & -z_{26}^* & 0 & z_{24}^* \\
0 & -z_{16} & 0 & z_8 & -z_{30}^* & z_{23}^* & 0 & -z_{21}^* \\
z_{33}^* & 0 & -z_{24} & z_{21} & 0 & 0 & -z_9^* & z_8^* \\
0 & -z_{13} & z_9 & 0 & -z_{31}^* & z_{25}^* & -z_{24}^* & 0 \\
0 & -z_{17} & 0 & z_9 & z_{29}^* & -z_{22}^* & z_{21}^* & 0 \\
0 & 0 & -z_{14} & z_{10} & 0 & z_{33}^* & -z_{28}^* & z_{27}^* \\
z_{21}^* & z_{35} & 0 & -z_{33} & -z_{11}^* & z_{10}^* & 0 & 0 \\
z_{22}^* & -z_{32} & 0 & z_{28} & -z_{12}^* & 0 & z_{10}^* & 0 \\
z_{23}^* & z_{31} & 0 & -z_{27} & -z_{13}^* & 0 & 0 & z_{10}^* \\
0 & 0 & -z_{15} & z_{11} & -z_{33}^* & 0 & z_{20}^* & -z_{19}^* \\
z_{29}^* & z_{26} & 0 & -z_{20} & 0 & -z_{12}^* & z_{11}^* & 0 \\
z_{30}^* & -z_{25} & 0 & z_{19} & 0 & -z_{13}^* & 0 & z_{11}^* \\
0 & 0 & -z_{16} & z_{12} & z_{28}^* & -z_{20}^* & 0 & z_{18}^* \\
z_{34}^* & z_{24} & 0 & -z_{18} & 0 & 0 & -z_{13}^* & z_{12}^* \\
0 & 0 & -z_{17} & z_{13} & -z_{27}^* & z_{19}^* & -z_{18}^* & 0 \\
z_{24}^* & -z_{34} & z_{33} & 0 & -z_{15}^* & z_{14}^* & 0 & 0 \\
z_{25}^* & z_{30} & -z_{28} & 0 & -z_{16}^* & 0 & z_{14}^* & 0 \\
z_{26}^* & -z_{29} & z_{27} & 0 & -z_{17}^* & 0 & 0 & z_{14}^* \\
z_{31}^* & -z_{23} & z_{20} & 0 & 0 & -z_{16}^* & z_{15}^* & 0 \\
z_{32}^* & z_{22} & -z_{19} & 0 & 0 & -z_{17}^* & 0 & z_{15}^* \\
z_{35}^* & -z_{21} & z_{18} & 0 & 0 & 0 & -z_{17}^* & z_{16}^*
\end{pmatrix}. \quad (2)$$

It is very easy, but tedious, to verify

$$G^H G = \left(\sum_{1 \le i \le 35} |z_i|^2 \right) I_8.$$

In the above example, the numbers of variables and rows of G are 35 and 56, respectively. So G has the maximal code rate $\frac{5}{8}$ and the delay 56. Hence, the open problem, as shown in (1), is wrong.

In fact, Liang gave a $[112, 8, 70]$ complex orthogonal design, i.e., matrices (167) and (168) in [3, p. 2497]. Since all complex variables z_i, $1 \le i \le 53$, in Liang's matrix (167) are arbitrary, our $[56, 8, 35]$ complex orthogonal design G is actually a special case of Liang's matrix (167) when we make the following replacements for complex variables z_i, $36 \le i \le 53$:

$$z_{36} = z_{35}^*, z_{37} = -z_{32}^*, \ldots, z_{53} = z_{18}^*.$$

However, finding these replacements is not straightforward, but somewhat clever. To construct the $[56, 8, 35]$ complex orthogonal design G, we begin from the submatrix $G(1 \sim 8; 1 \sim 8)$, then gradually extend $G(1 \sim 8; 1 \sim 8)$. Finally, we get the $[56, 8, 35]$ complex orthogonal design G.

Furthermore, we can prove that the minimal delay of $[p, 8, k]$ complex orthogonal designs is 56. Let O be any $[p, 8, k]$ complex orthogonal design with the maximal rate $\frac{m+1}{2m} = \frac{5}{8}$, where $m = 4$. Since O has the maximal rate, according to [3, proof of Proposition 6], we can assume that O contains the following submatrix:

$$O_1 = \begin{pmatrix}
z_1 & 0 & 0 & 0 & z_2 & z_3 & z_4 & z_5 \\
0 & z_1 & 0 & 0 & z_6 & z_7 & z_8 & z_9 \\
0 & 0 & z_1 & 0 & z_{10} & z_{11} & z_{12} & z_{13} \\
0 & 0 & 0 & z_1 & z_{14} & z_{15} & z_{16} & z_{17} \\
-z_2^* & -z_6^* & -z_{10}^* & -z_{14}^* & z_1^* & 0 & 0 & 0 \\
-z_3^* & -z_7^* & -z_{11}^* & -z_{15}^* & 0 & z_1^* & 0 & 0 \\
-z_4^* & -z_8^* & -z_{12}^* & -z_{16}^* & 0 & 0 & z_1^* & 0 \\
-z_5^* & -z_9^* & -z_{13}^* & -z_{17}^* & 0 & 0 & 0 & z_1^*
\end{pmatrix}.$$

Extending O_1 so that each column of O includes $\pm z_j$ or $\pm z_j^*$, $2 \le j \le 5$, we conclude, under the suitable transformations mentioned early, O must contain the submatrix $\begin{pmatrix} O_1 \\ O_2 \end{pmatrix}$, where

$$O_2 = \begin{pmatrix}
-z_6 & z_2 & 0 & 0 & 0 & \spadesuit & \spadesuit & \spadesuit \\
-z_{10} & 0 & z_2 & 0 & 0 & \spadesuit & \spadesuit & \spadesuit \\
-z_{14} & 0 & 0 & z_2 & 0 & \spadesuit & \spadesuit & \spadesuit \\
0 & \spadesuit & \spadesuit & \spadesuit & -z_3^* & z_2^* & 0 & 0 \\
0 & \spadesuit & \spadesuit & \spadesuit & -z_4^* & 0 & z_2^* & 0 \\
0 & \spadesuit & \spadesuit & \spadesuit & -z_5^* & 0 & 0 & z_2^* \\
-z_7 & z_3 & 0 & 0 & \spadesuit & 0 & \spadesuit & \spadesuit \\
-z_{11} & 0 & z_3 & 0 & \spadesuit & 0 & \spadesuit & \spadesuit \\
-z_{15} & 0 & 0 & z_3 & \spadesuit & 0 & \spadesuit & \spadesuit \\
0 & \spadesuit & \spadesuit & \spadesuit & 0 & -z_4^* & z_3^* & 0 \\
0 & \spadesuit & \spadesuit & \spadesuit & 0 & -z_5^* & 0 & z_3^* \\
-z_8 & z_4 & 0 & 0 & \spadesuit & \spadesuit & 0 & \spadesuit \\
-z_{12} & 0 & z_4 & 0 & \spadesuit & \spadesuit & 0 & \spadesuit \\
-z_{16} & 0 & 0 & z_4 & \spadesuit & \spadesuit & 0 & \spadesuit \\
0 & \spadesuit & \spadesuit & \spadesuit & 0 & 0 & -z_5^* & z_4^* \\
-z_9 & z_5 & 0 & 0 & \spadesuit & \spadesuit & \spadesuit & 0 \\
-z_{13} & 0 & z_5 & 0 & \spadesuit & \spadesuit & \spadesuit & 0 \\
-z_{17} & 0 & 0 & z_5 & \spadesuit & \spadesuit & \spadesuit & 0
\end{pmatrix}$$

where $\spadesuit$ means an unoccupied or unfilled position. Now we prove that the submatrices $O_2(4 \sim 6; 2 \sim 4)$, $O_2(10 \sim 11; 2 \sim 4)$, and $O_2(15; 2 \sim 4)$ cannot include $\pm z_i$ or $\pm z_i^*$, $6 \le i \le 17$. Clearly, $O_2(4 \sim 6; 2 \sim 4) = O(12 \sim 14; 2 \sim 4)$. First, we note that $-z_i$,

$6 \le i \le 17$, have already appeared in the first column of O_2. Obviously, $\pm z_i^*, 6 \le i \le 17$, cannot appear in $O_2(4 \sim 6; 2 \sim 4)$. It is easy to see that $\pm z_6$, $\pm z_{10}$, and $\pm z_{14}$ cannot appear in $O_2(4 \sim 6; 2 \sim 4)$. For example, we verify $O(14,3) \ne z_7$. If $O(14,3) = z_7$, then

$$O(14,2) = -z_{11},\ O(10,8) = -z_7^* \text{ and } O(10,6) = z_9^*.$$

Since

$$O(10,6) = z_9^* \text{ and } O(24,1) = -z_9 \text{ and } O(10,1) = -z_{10},$$

$O(24,6) = -z_{10}^*$.

Since

$$O(14,2) = -z_{11} \text{ and } O(14,5) = -z_5^* \text{ and } O(24,2) = z_5$$

$O(24,5) = -z_{11}^*$. Then, we have the following submatrix:

$$O(3,24;5,6) = \begin{pmatrix} z_{10} & z_{11} \\ -z_{11}^* & -z_{10}^* \end{pmatrix}$$

which is a contradiction. Hence, $O(14,3) \ne z_7$. If $O(14,3) = -z_7$, then, according to the above procedure, we get the submatrix

$$O(3,24;5,6) = \begin{pmatrix} z_{10} & z_{11} \\ z_{11}^* & z_{10}^* \end{pmatrix}$$

which is also a contradiction. It is similar to verify that $\pm z_8, \pm z_9, \pm z_{11}$, $\pm z_{12}$, $\pm z_{13}$, $\pm z_{15}$, $\pm z_{16}$, and $\pm z_{17}$ cannot appear in $O_2(4 \sim 6; 2 \sim 4)$ according to the steps that we just verify $O(14,3) \ne z_7$. So $O_2(4 \sim 6; 2 \sim 4)$ cannot include $\pm z_i$ or $\pm z_i^*, 6 \le i \le 17$. Similarly, $O_2(10 \sim 11; 2 \sim 4)$ and $O_2(15; 2 \sim 4)$ do not include $\pm z_i$ or $\pm z_i^*, 6 \le i \le 17$. For example, let us prove $O(19,4) \ne z_8$. If $O(19,4) = z_8$, then

$$O(19,2) = -z_{16}, O(17,8) = -z_8^* \text{ and } O(17,7) = z_9^*.$$

Since

$$O(17,7) = z_9^* \text{ and } O(24,1) = -z_9 \text{ and } O(17,1) = -z_{15}$$

$O(24,7) = -z_{15}^*$.

Since

$$O(19,2) = -z_{16} \text{ and } O(19,5) = -z_5^* \text{ and } O(24,2) = z_4$$

$O(24,6) = -z_{16}^*$. So we get a contradiction

$$O(4,24;6,7) = \begin{pmatrix} z_{15} & z_{16} \\ -z_{16}^* & -z_{15}^* \end{pmatrix}.$$

Hence, $O(19,4) \ne z_8$. Since $\pm z_i$ or $\pm z_i^*, 6 \le i \le 17$, do not appear in $O_2(4 \sim 6; 2 \sim 4)$, $O_2(10 \sim 11; 2 \sim 4)$, and $O_2(15; 2 \sim 4)$, they do not appear in other unoccupied positions of O_2 either. Because the unoccupied positions in O_2 do not include $\pm z_i$ or $\pm z_i^*, 6 \le i \le 9$, we can imply that O has the following submatrix:

$$\begin{pmatrix} O_1 \\ O_2 \\ O_3 \end{pmatrix}$$

where

$$O_3 = \begin{pmatrix}
0 & -z_{10} & z_6 & 0 & 0 & \spadesuit & \spadesuit & \spadesuit \\
0 & -z_{14} & 0 & z_6 & 0 & \spadesuit & \spadesuit & \spadesuit \\
\spadesuit & 0 & \spadesuit & \spadesuit & -z_7^* & z_6^* & 0 & 0 \\
\spadesuit & 0 & \spadesuit & \spadesuit & -z_8^* & 0 & z_6^* & 0 \\
\spadesuit & 0 & \spadesuit & \spadesuit & -z_9^* & 0 & 0 & z_6^* \\
0 & -z_{11} & z_7 & 0 & \spadesuit & 0 & \spadesuit & \spadesuit \\
0 & -z_{15} & 0 & z_7 & \spadesuit & 0 & \spadesuit & \spadesuit \\
\spadesuit & 0 & \spadesuit & \spadesuit & 0 & -z_8^* & z_7^* & 0 \\
\spadesuit & 0 & \spadesuit & \spadesuit & 0 & -z_9^* & 0 & z_7^* \\
0 & -z_{12} & z_8 & 0 & \spadesuit & \spadesuit & 0 & \spadesuit \\
0 & -z_{16} & 0 & z_8 & \spadesuit & \spadesuit & 0 & \spadesuit \\
\spadesuit & 0 & \spadesuit & \spadesuit & 0 & 0 & -z_9^* & z_8^* \\
0 & -z_{13} & z_9 & 0 & \spadesuit & \spadesuit & \spadesuit & 0 \\
0 & -z_{17} & 0 & z_9 & \spadesuit & \spadesuit & \spadesuit & 0
\end{pmatrix}.$$

Noting that $-z_i$, $10 \le i \le 17$, have already appeared in the second column of O_3, we can similarly verify that $\pm z_i$ or $\pm z_i^*$, $10 \le i \le 17$, do not appear in $O_3(3 \sim 5; 3,4)$, $O_3(8,9;3,4)$, and $O_3(12;3,4)$. For instance, we verify $O(31,4) \ne z_{11}$. If $O(31,4) = z_{11}$, then

$$O(31,3) = -z_{15}, O(28,8) = -z_{11}^* \text{ and } O(28,6) = z_{13}^*.$$

Since

$$O(28,6) = z_{13}^* \text{ and } O(39,2) = -z_{13} \text{ and } O(28,2) = -z_{14}$$

$O(39,6) = -z_{14}^*$.

Since

$$O(31,3) = -z_{15} \text{ and } O(31,5) = -z_9^* \text{ and } O(39,3) = z_9$$

$O(39,5) = -z_{15}^*$. Hence, we get the folowing contradiction:

$$O(4,39;5,6) = \begin{pmatrix} z_{14} & z_{15} \\ -z_{15}^* & -z_{14}^* \end{pmatrix}.$$

Consequently, $O(31,4) \ne z_{11}$. Because other unoccupied positions in O_3 are determined by the elements in $O_2(4 \sim 6; 2 \sim 4)$, $O_2(10 \sim 11; 2 \sim 4)$, $O_2(15; 2 \sim 4)$, $O_3(3 \sim 5; 3,4)$, $O_3(8,9;3,4)$, and $O_3(12;3,4)$, so $\pm z_i$ or $\pm z_i^*$, $10 \le i \le 17$, do not appear in O_2 and O_3. Thus, to make each column of O contain $\pm z_i$ or $\pm z_i^*$, $10 \le i \le 17$, we can similarly imply that O has the following submatrix:

$$\begin{pmatrix} O_1 \\ O_2 \\ O_3 \\ O_4 \\ O_5 \end{pmatrix}$$

where

$$O_4 = \begin{pmatrix}
0 & 0 & -z_{14} & z_{10} & 0 & \spadesuit & \spadesuit & \spadesuit \\
\spadesuit & \spadesuit & 0 & \spadesuit & -z_{11}^* & z_{10}^* & 0 & 0 \\
\spadesuit & \spadesuit & 0 & \spadesuit & -z_{12}^* & 0 & z_{10}^* & 0 \\
\spadesuit & \spadesuit & 0 & \spadesuit & -z_{13}^* & 0 & 0 & z_{10}^* \\
0 & 0 & -z_{15} & z_{11} & \spadesuit & 0 & \spadesuit & \spadesuit \\
\spadesuit & \spadesuit & 0 & \spadesuit & 0 & -z_{12}^* & z_{11}^* & 0 \\
\spadesuit & \spadesuit & 0 & \spadesuit & 0 & -z_{13}^* & 0 & z_{11}^* \\
0 & 0 & -z_{16} & z_{12} & \spadesuit & \spadesuit & 0 & \spadesuit \\
\spadesuit & \spadesuit & 0 & \spadesuit & 0 & 0 & -z_{13}^* & z_{12}^* \\
0 & 0 & -z_{17} & z_{13} & \spadesuit & \spadesuit & \spadesuit & \spadesuit
\end{pmatrix}$$

and

$$O_5 = \begin{pmatrix} \spadesuit & \spadesuit & \spadesuit & 0 & -z_{15}^* & z_{14}^* & 0 & 0 \\ \spadesuit & \spadesuit & \spadesuit & 0 & -z_{16}^* & 0 & z_{14}^* & 0 \\ \spadesuit & \spadesuit & \spadesuit & 0 & -z_{17}^* & 0 & 0 & z_{14}^* \\ \spadesuit & \spadesuit & \spadesuit & 0 & 0 & -z_{16}^* & z_{15}^* & 0 \\ \spadesuit & \spadesuit & \spadesuit & 0 & 0 & -z_{17}^* & 0 & z_{15}^* \\ \spadesuit & \spadesuit & \spadesuit & 0 & 0 & 0 & -z_{17}^* & z_{16}^* \end{pmatrix}.$$

Consequently, the number of rows of O is not less than 56. Therefore, 56 is the minimal delay for $[p, 8, k]$ complex orthogonal designs. We summarize the above results by the following theorem.

Theorem 1: The minimal delay of $[p, 8, k]$ complex orthogonal designs with the maximal rate $\frac{5}{8}$ is 56.

Finally, we claim that the minimal delay of $[p, 7, k]$ complex orthogonal designs with the maximal rate $\frac{5}{8}$ is also 56, which can be easily verified by the above procedure. This shows that (1) is correct for $n = 7$. Furthermore, we conjecture that (1) is correct when $n \neq 4t$, and $\wp_{\mathbb{C}}(n, \gamma_n) = \frac{m}{m+1}\binom{n}{m}$ when $n = 4t$, i.e., the factor "2" in (1) is removed for $n = 4k$, where t is a natural number.

ACKNOWLEDGMENT

The authors would like to thank two referees for their careful reading and comments which have improved the clarity of this correspondence. They also acknowledge Prof. Ø. Ytrehus for his help.

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Distance-Increasing Mappings From Binary Vectors to Permutations

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***Abstract*—Mappings from the set of binary vectors of a fixed length to the set of permutations of the same length that strictly increase Hamming distances except when that is obviously not possible are useful for the construction of permutation codes. In this correspondence, we propose recursive and explicit constructions of such mappings. Some comparisons show that the new mappings have better distance expansion distributions than other known distance-preserving mappings (DPMs). We also give some examples to illustrate the applications of these mappings to permutation arrays (PAs).**

***Index Terms*—Code constructions, distance-preserving mappings (DPMs), Hamming distance, mapping, permutation arrays (PAs).**

I. INTRODUCTION

A distance-preserving mapping, shortly DPM, is a mapping from the set of all binary vectors of length n to the set of all permutations of $Z_n = \{1, 2, \dots, n\}$ that preserves or increases the Hamming distance. Recently, Chang and others [1] proposed several nice constructions of DPMs and used their DPMs to improve some lower bounds on the size of permutation arrays. Lee [2] also devised a construction of DPMs of odd length. DPMs for vectors of length n are called n-DPMs.

The main objects studied in this correspondence are special n-DPMs that strictly increase Hamming distances except when that is obviously not possible. We call these special distance-preserving mappings n-DIMs (distance-increasing mappings for vectors of length n). From the point of view of DIMs, for $n = 4$ or $n > 4$ and $n \mod 4 = 2$, Chang's n-DPMs are in fact n-DIMs. Unfortunately, Lee's n-DPMs are not n-DIMs.

In this correspondence, we devise recursive and explicit constructions of n-DIMs for all n greater than or equal to 4. Some comparisons of the distance expansion distribution of the newly constructed DIMs and other known DPMs are then given. In the last section, we also give some examples to illustrate the applications of these mappings to permutation arrays (PAs).

II. BASIC NOTATIONS

Let S_n be the set of all $n!$ permutations of $Z_n = \{1, 2, \dots, n\}$. A permutation $\pi : Z_n \rightarrow Z_n$ is represented by an n-tuple $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ where $\pi_i = \pi(i)$. Let Z_2^n denote the set of all binary vectors of length n. A binary vector $x \in Z_2^n$ is denoted by an n-tuple: $x = (x_1, x_2, \dots, x_n)$ where x_i is the ith bit of x.

The Hamming distance between two n-tuples $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$, denoted $d(\boldsymbol{a}, \boldsymbol{b})$, is defined to be the number of positions where they differ, that is,

$$d(\boldsymbol{a}, \boldsymbol{b}) = |\{j \in Z_n \mid a_j \neq b_j\}|.$$

A distance-increasing mapping of length n, an n-DIM for short, is a mapping $f : Z_2^n \rightarrow S_n$ such that for any pair of distinct binary

Manuscript received June 1, 2004; revised September 20, 2004. This work was supported in part by the Taiwan National Science Council under Contract NSC 93-2213-E-305-003.

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Communicated by V. A. Vaishampayan, Associate Editor At Large.
Digital Object Identifier 10.1109/TIT.2004.839527