One can now use (17) and (20) to obtain

$$H(v_2[k] \mid v_1[k]) \approx \frac{\sqrt{2b}}{N} \log\left(\frac{cN}{\sqrt{2b}}\right). \tag{21}$$

Using (16), we then obtain the desired result

$$H(v_1[k], \ldots, v_N[k]) \leq \Theta\left(\log(N)\right). \tag{22}$$

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Theorems and Fallacies in the Theory of Long-Range-Dependent Processes

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***Abstract*—It is frequently claimed in the literature that long-range dependence has equivalent formulations in the time domain and the frequency domain. Although many researchers understand that this is only "operationally true," i.e., it holds in cases of interest, many state this equivalence as a mathematical theorem. In particular, it is claimed as a theorem in the literature that if a covariance function decays like one over a fractional power of n, then the corresponding power spectral density tends to infinity at the origin. It is shown here that the power spectral density need not exist. Conversely, if the power spectral density exists and tends to infinity at the origin, it is shown here that the covariance may not have the claimed decay. To conclude, a new theorem is proved that gives sufficient conditions on the power spectral density to guarantee that a process is asymptotically second-order self-similar (ASOSS). This result is used to provide a counterexample to the claim in the literature that asymptotic second-order self-similarity implies long-range dependence.**

***Index Terms*—Asymptotic second-order self-similarity, exact second-order self-similarity, long-range dependence, slowly varying functions.**

I. Introduction

A random process X_k with constant mean $\mu := \mathsf{E}[X_n]$ and covariance $\mathsf{E}[(X_k - \mu)(X_m - \mu)]$ that depends on k and m only through their difference $k - m$ is said to be wide-sense stationary. The covariance function of the process is defined by

$$C(n) := \mathsf{E}[(X_{m+n} - \mu)(X_m - \mu)].$$

For sufficiently well-behaved sequences $C(n)$, the power spectral density of the process is defined as the Fourier series

$$S(f) = \sum_{n=-\infty}^{\infty} C(n) e^{-j2\pi fn} \tag{1}$$

and the covariance function $C(n)$ is recovered from $S(f)$ using the formula for Fourier series coefficients

$$C(n) = \int_{-1/2}^{1/2} S(f) e^{j2\pi fn} df.$$

The next two theorems are classical results about Fourier series.

Theorem 1: Let $C(n)$ be a covariance function that satisfies

$$C(n) = \frac{p(|n|)}{|n|^{\alpha}}, \qquad \text{for } n \neq 0 \tag{2}$$

where $p(t)$ is **normalized slowly varying** at infinity (see Appendix A), and $0 < \alpha < 1$. Then the power spectral density (1) exists and satisfies

$$\lim_{f \to 0} \frac{S(f)}{p\left(\frac{1}{2\pi|f|}\right)|f|^{\alpha-1}} = \frac{2}{(2\pi)^{1-\alpha}} \Gamma(1-\alpha)\sin(\pi\alpha/2) \tag{3}$$

where $\Gamma(x) := \int_0^\infty \theta^{x-1} e^{-\theta} d\theta$, $x > 0$, is the gamma function.

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If we use the fact that covariance functions are even to write

$$S(f) = \sum_{n=-\infty}^{\infty} C(n)e^{-j2\pi fn} = C(0) + 2\sum_{n=1}^{\infty} \frac{p(n)}{n^{\alpha}} \cos(n2\pi f)$$

then Theorem 1 follows immediately from [11, p. 187, Theorem 2.6] if we also use the fact that normalized slow variation implies $t^{\delta}p(t) \to \infty$ as $t \to \infty$ whenever $\delta > 0$.

Remark: Note that (3) can be rewritten as

$$\lim_{f\to\infty} 2\pi|f| \frac{S(f)}{C\left(\frac{1}{2\pi|f|}\right)} = 2\Gamma(1-\alpha)\sin(\pi\alpha/2).$$

Theorem 2: Let $C(n)$ have a power spectral density $S(f)$. Suppose that

$$S(f) = \frac{q(|f|)}{|f|^{1-\alpha}}, \qquad \text{for } f \neq 0 \tag{4}$$

where $q(f)$ is a function of bounded variation in every interval $(\varepsilon, 1/2)$, and is normalized slowly varying at zero. If $0 < \alpha < 1$, then

$$\lim_{n\to\infty} \frac{C(n)}{q\left(\frac{1}{2\pi n}\right)n^{-\alpha}} = \frac{(2\pi)^{1-\alpha}}{\pi}\Gamma(\alpha)\sin(\pi(1-\alpha)/2). \tag{5}$$

Theorem 2 is a restatement of [11, p. 190, Theorem 2.24].

Remark: Equation (5) can be rewritten as

$$\lim_{n\to\infty} 2\pi n \frac{C(n)}{S\left(\frac{1}{2\pi n}\right)} = 2\Gamma(\alpha)\sin\left(\pi(1-\alpha)/2\right).$$

It is sometimes convenient to rewrite (3) as

$$S(f) \sim \frac{\tilde{p}(|f|)}{|f|^{1-\alpha}}, \qquad \text{for } f \text{ near } 0 \tag{6}$$

and to rewrite (5) as

$$C(n) \sim \frac{\tilde{q}(n)}{n^{\alpha}}, \qquad \text{for large } n \tag{7}$$

where $\tilde{p}$ and $\tilde{q}$ are suitably defined. Notice the similarity of (6) and (4) and of (7) and (2). If one is not careful, one might think that the conclusion of Theorem 1 is the same as the hypothesis of Theorem 2, and that the conclusion of Theorem 2 is the same as the hypothesis of Theorem 1. Combining these two misconceptions might lead one to conclude that (6) and (7) are equivalent. While Cox in his review of long-range dependence was careful to talk about "essentially equivalent" conditions [4, Sec. 3], others have asserted exact equivalence. For example, in the special case that $\tilde{p}$ and and $\tilde{q}$ are constants, say $\tilde{p}(\cdot) \equiv s$ and $\tilde{q}(\cdot) \equiv c$, it is frequently claimed in the literature, e.g., [2, p. 43, Theorem 2.1], [8, pp. 20–21], that if $C(n)$ satisfies

$$\lim_{n\to\infty} \frac{C(n)}{n^{-\alpha}} = c \tag{8}$$

where $0 < \alpha < 1$ and c is a positive finite constant, then the power spectral density $S(f)$ exists and satisfies

$$\lim_{f\to 0} \frac{S(f)}{|f|^{\alpha-1}} = s \tag{9}$$

where s is another positive finite constant. A process satisfying (8) is said to be **long-range dependent** (LRD). Conversely, the literature also frequently claims that if the power spectral density satisfies (9), then the covariance function satisfies (8).

The claimed equivalence of (8) and (9) is plausible in light of the following example. For $0 < d < 1/2$, let

$$S(f) = |1 - e^{-j2\pi f}|^{-2d} = [4\sin^2(\pi f)]^{-d}. \tag{10}$$

If we put $\alpha = 1 - 2d$, then $0 < \alpha < 1$, and

$$\lim_{f\to 0} \frac{S(f)}{|f|^{\alpha-1}} = (2\pi)^{-2d}.$$

The corresponding covariance function is [6, Theorem 1(d)]

$$C(n) = \int_{-1/2}^{1/2} S(f)e^{j2\pi fn}df = \frac{\Gamma(1-2d)\Gamma(n+d)}{\Gamma(1-d)\Gamma(d)\Gamma(n+1-d)}$$

and by Stirling's formula, $\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}$, we have

$$\lim_{n\to\infty} \frac{C(n)}{n^{-\alpha}} = \frac{\Gamma(1-2d)}{\Gamma(1-d)\Gamma(d)}. \tag{11}$$

In spite of the above example, we show in Section II that every covariance function satisfying (8) with $0 < \alpha < 1/2$ can be perturbed to obtain a new covariance function that satisfies (8) with the same values of α and c, but whose power spectral measure is singular; i.e., the power spectral density does not exist. In other words, LRD (8) does not imply (9).

If (8) holds for some $1/2 < \alpha < 1$, then the covariance sequence is square summable, and it follows that the power spectral density (1) exists as a limit in $L^2[-1/2, 1/2]$. We show in Section III that if (9) also holds, then $S(f)$ can always be perturbed to obtain a new power spectral density that satisfies (9) with the same values of α and s but whose corresponding covariance function does not satisfy (8). In other words, (9) does not imply LRD (8).

Another common claim in the literature, e.g., [8, p. 21, Sec. 1.4.1.4], is that if a process is **asymptotically second-order self-similar** (ASOSS) (defined in Section IV) with Hurst parameter $1/2 < H < 1$, then the process is LRD. In Section IV, we present a new theorem (Theorem 3) that gives sufficient conditions on the power spectral density for the process to be ASOSS. The theorem is then used to give an example of an ASOSS process (with $1/2 < H < 1$) that is not LRD. Thus, ASOSS does not imply LRD (8).

Conditions that do imply LRD are discussed in Section V.

II. LRD (8) Does Not Imply (9)

Suppose that (8) holds for some $0 < \alpha < 1/2$. Let $0 < \varepsilon < 1/2 - \alpha$. Then by [12, p. 146, Theorem 10.12], there exists a nondecreasing function $G(f)$ that is singular, and such that

$$g_n := \int_{-1/2}^{1/2} e^{j2\pi fn}dG(f)$$

satisfies, for sufficiently large n

$$|g_n| \leq \frac{K_1}{n^{1/2-\varepsilon}}$$

for some positive finite constant K_1. Consider now the covariance function[1]

$$C_{\varepsilon}(n) := C(n) + g_n$$

and write

$$\frac{C_{\varepsilon}(n)}{n^{-\alpha}} = \frac{C(n)}{n^{-\alpha}} + \frac{g_n}{n^{-\alpha}}.$$

[1]If g_n is complex valued, we can replace it by $(g_n + g_{-n})/2$ and we can replace G with the corresponding increasing function.

Observe that

$$\frac{|g_n|}{n^{-\alpha}} \leq \frac{K_1}{n^{1/2-\varepsilon-\alpha}} \to 0$$

as $n \to \infty$ since $1/2 - \varepsilon - \alpha > 0$. Hence,

$$\lim_{n\to\infty} \frac{C_\varepsilon(n)}{n^{-\alpha}} = c.$$

However, by construction, the spectral distribution corresponding to $C_\varepsilon(n)$ is singular; i.e., it does not have a density.

III. Condition (9) Does Not Imply LRD (8)

Suppose that (8) and (9) hold for some $1/2 < \alpha < 1$. Fix $0 < \varepsilon < \alpha - 1/2$, and consider the function

$$H(f) := \sum_{n=1}^{\infty} h_n e^{j2\pi fn}$$

where

$$h_n := \frac{e^{j\beta n \ln n}}{n^{1/2+\varepsilon}}$$

and $\beta > 0$. The series for $H(f)$ converges uniformly [11, p. 197]. Hence, H is continuous, and therefore bounded on $[-1/2, 1/2]$. Let $H(f)^*$ denote the complex conjugate of $H(f)$. Since $H(f) + H(f)^*$ is a bounded real-valued function, for large enough K_2

$$Q(f) := H(f) + H(f)^* + K_2 \geq 0.$$

Since $V(f) := [Q(f) + Q(-f)]/2$ is real, even, and nonnegative, $S(f) + V(f)$ is a power spectral density. Since $V(f)$ is bounded, $V(f)/|f|^{\alpha-1} \to 0$ as $f \to 0$. Hence,

$$\lim_{f\to 0} \frac{S(f) + V(f)}{|f|^{\alpha-1}} = s.$$

If we denote the covariance function corresponding to $S(f) + V(f)$ by $\widetilde{C}(n)$, then

$$\widetilde{C}(n) = C(n) + \mathrm{Re} h_n = C(n) + \frac{\cos(\beta n \ln n)}{n^{1/2+\varepsilon}}.$$

If $\beta = (\pi/2)/\ln 3$, and if we take $n = 3^k$, then

$$\widetilde{C}(n) = C(n) + \frac{\cos(\frac{\pi}{2} 3^k k)}{n^{1/2+\varepsilon}}. \tag{12}$$

If k is a multiple of 4, the above cosine is one, and

$$\frac{\widetilde{C}(n)}{n^{-\alpha}} = \frac{C(n)}{n^{-\alpha}} + n^{\alpha-(1/2+\varepsilon)}. \tag{13}$$

Since $\alpha - (1/2 + \varepsilon) > 0$, as n runs through the subsequence $n = 3^k$ with k a multiple of 4

$$\frac{\widetilde{C}(n)}{n^{-\alpha}} \to c + \infty = \infty.$$

Remark: One might wonder if

$$\frac{\widetilde{C}(n)}{n^{-\alpha'}} \to c'$$

for some $\alpha' \neq \alpha$ and some positive c' possibly different from c. However, this cannot happen. The limit (8) implies that for $0 < \alpha' < \alpha$, $C(n)/n^{-\alpha'} \to 0$, while if $1 > \alpha' > \alpha$, $C(n)/n^{-\alpha'} \to \infty$. On account of (13), the only possibility might be $\alpha' = 1/2 + \varepsilon$. If we take $n = 3^k$ with k odd in (12), we get $\widetilde{C}(n) = C(n)$, which, when divided by $n^{-\alpha'}$, goes to zero since $\alpha' = 1/2 + \varepsilon < \alpha$.

Remark: The key to this section was the construction of a covariance function with terms that decayed at different rates. Cox suggested that an analogous decay situation could arise in the frequency domain [4, p. 58], but he did not pursue it.

IV. ASOSS Does Not Imply LRD (8)

Consider the partitioning of the sequence X_n into blocks of size m

$$\underbrace{X_1, \ldots, X_m}_{\text{1st block}} \underbrace{X_{m+1}, \ldots, X_{2m}}_{\text{2nd block}} \cdots \underbrace{X_{(n-1)m+1}, \ldots, X_{nm}}_{n\text{th block}} \cdots.$$

The average of the nth block is

$$X_n^{(m)} := \frac{1}{m} \sum_{k=(n-1)m+1}^{nm} X_k.$$

The superscript (m) indicates the block size, which is the number of terms used to compute the average. The subscript n indicates the block number. We call $\{X_n^{(m)}\}_{n=-\infty}^{\infty}$ the aggregated process. It is easy to see that the aggregated process is also wide-sense stationary. The covariance function of the aggregated process is denoted by $C^{(m)}(n)$. We say that X_n is **ASOSS** with Hurst parameter $0 < H < 1$ if

$$\lim_{m\to\infty} \frac{C^{(m)}(n)}{m^{2H-2}} = \frac{\sigma_\infty^2}{2}[|n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H}] \tag{14}$$

for some positive finite constant σ_∞^2. If we put

$$Y_n := \sum_{i=1}^{n} (X_i - \mu)$$

where $\mu := \mathsf{E}[X_n]$, then (14) is equivalent to the condition (see Appendix B)

$$\lim_{n\to\infty} \frac{\mathsf{E}\left[Y_n^2\right]}{n^{2H}} = \sigma_\infty^2. \tag{15}$$

The following result is proved in Appendix C. It gives sufficient conditions for a process to be ASOSS.

Theorem 3: Let X_n be a wide-sense stationary sequence with power spectral density $S(f)$ satisfying (9) along with the additional condition that for every $0 < \delta < 1/2$

$$\sup_{f\in[\delta,1/2]} S(f) < \infty. \tag{16}$$

Then X_n is ASOSS in that (15) holds with $H = 1 - \alpha/2$, and

$$\sigma_\infty^2 = s \cdot \frac{4\cos(\alpha\pi/2)\Gamma(\alpha)}{(2\pi)^\alpha(1-\alpha)(2-\alpha)}.$$

Hence, the $S(f)$ in (10) corresponds to a process that is asymptotically second-order self-similar by our theorem and is LRD by (11). However, if we perturb this $S(f)$ as in Section III, then $S(f) + V(f)$ also satisfies the hypotheses of our theorem, and corresponds to an ASOSS process; but this process is not LRD.

V. What Does Imply LRD?

Suppose that instead of (15), we make the stronger assumption (no limit here) [2]

$$\mathsf{E}\left[Y_n^2\right] = \sigma_\infty^2 n^{2H}, \qquad n = 1, 2, \ldots. \tag{17}$$

Then (8) holds with $\alpha = 2 - 2H$ and $c = \sigma_\infty^2 H(2H-1)$. This can be seen by substituting (17) into (22) in Appendix B to obtain

$$\begin{aligned} C(n) &= \frac{\sigma_\infty^2}{2}[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}] \\ &= \frac{\sigma_\infty^2}{2} n^{2H} \psi(1/n) \end{aligned} \tag{18}$$

where $\psi(t) := (1+t)^{2H} - 2 + (1-t)^{2H}$, and applying l'Hôpital's rule twice to $\psi(t)/t^2$ as $t \to 0$.

[2]Since this holds for all n, and not just asymptotically, taking $n = 1$ yields $\sigma_\infty^2 = C(0)$.

As just noted, (17) implies (18). The converse is also true, as can be seen by induction. Property (18) is known as **exact second-order self-similarity (ESOSS)**. Since (17) implies (15), ESOSS implies ASOSS.

It is convenient to generalize the notions of ESOSS (17), ASOSS (15), and LRD (8). A process is said to be ESOSS-L if

$$\mathsf{E}\left[Y_n^2\right] = \sigma_\infty^2 n^{2H} L(n), \qquad n = 1, 2, \ldots$$

where L is **slowly varying** at infinity (see Appendix A). Similarly, a process is said to be ASOSS-L if

$$\lim_{n\to\infty} \frac{\mathsf{E}\left[Y_n^2\right]}{n^{2H} L(n)} = \sigma_\infty^2.$$

Note that ESOSS-L implies ASOSS-L. A process is said to be LRD-L if

$$\lim_{n\to\infty} \frac{C(n)}{n^{2H-2} L(n)} = \sigma_\infty^2 H(2H-1). \tag{19}$$

It was shown in [10, pp. 1721–1722, eq. (A.9) ff.] that ESOSS-L implies LRD-L. Taking $L(n) \equiv 1$ recovers the special case that ESOSS implies LRD (8). It was shown in [10, Theorem 2, g) ⇒ h)] that LRD-L implies ASOSS-L. Taking $L(n) \equiv 1$ shows that ordinary LRD (8) implies ASOSS (15) (as reported in [1, p. 261] without proof).

Remark: To prove [10, Theorem 2, g) ⇒ h)] in the special case $L(n) \equiv 1$ requires no theory of slowly varying functions. The proof in [10] simplifies and reduces to exploiting the inequality

$$\sum_{\nu=k}^{n-1} (\nu+1)^{-\alpha} \le \int_k^n t^{-\alpha} dt \le \sum_{\nu=k}^{n-1} \nu^{-\alpha}$$

and a similar one for the integral of $t^{1-\alpha}$.

After this correspondence was submitted, Taqqu [9, p. 15] reported the following result.

Proposition 4: Suppose that a wide-sense stationary process has a covariance function $C(n)$ that is ultimately monotone as $n \to \infty$ and has power spectral density $S(f)$.

i) If for some slowly varying function L, the process is LRD-L in the sense that $C(n)$ satisfies (19), then

$$\lim_{f\to 0} \frac{S(f)}{|f|^{1-2H} \tilde{L}(|f|)} = 1 \tag{20}$$

where $\tilde{L}(\cdot)$ is proportional to $L(1/\cdot)$.

ii) Conversely, if (20) holds for some $\tilde{L}$ slowly varying at zero, then the process is LRD-L in the sense that $C(n)$ satisfies (19) with $L(\cdot)$ proportional to $\tilde{L}(1/\cdot)$.

Remark: Given only the power spectral density $S(f)$ of a process, it may be difficult to use the proposition to show that the process is LRD-L. The reason is that not only do we have to establish (20), but we also have to use $S(f)$ to establish that $C(n)$ is ultimately monotone.

Appendix A
Slowly Varying Functions

Definition 5 ([3, p. 6]): A positive measurable function $p(t)$ is said to be **slowly varying** at infinity if

$$\lim_{t\to\infty} \frac{p(\lambda t)}{p(t)} = 1, \qquad \lambda > 0. \tag{21}$$

If the limit is taken as $t \downarrow 0$, the function is said to be slowly varying at zero.

Definition 6: A positive function $p(t)$ is said to be **normalized slowly varying** at infinity if for every $\delta > 0$, for sufficiently large t, $t^\delta p(t)$ is increasing and $t^{-\delta} p(t)$ is decreasing in t. Normalized slow variation at zero is defined similarly.

Remark: For *normalized* slowly varying functions, it is easy to show that (21) holds [11, p. 186]. In fact, the set of normalized slowly varying functions is a proper subclass of the slowly varying functions. See [3, pp. 15 and 24].

Appendix B
Equivalence of (14) and (15)

To show that (14) implies (15), first observe that

$$C^{(n)}(0) = \mathsf{E}\left[\left(X_1^{(n)} - \mu\right)^2\right] = \mathsf{E}\left[Y_n^2\right]/n^2.$$

Then

$$\frac{\mathsf{E}\left[Y_n^2\right]}{n^{2H}} = \frac{n^2 C^{(n)}(0)}{n^{2H}} = \frac{C^{(n)}(0)}{n^{2H-2}} \to \sigma_\infty^2 \quad \text{by (14).}$$

Proving that (15) implies (14) requires a bit more work. A simple calculation, e.g., [7, eq. (8.2)], shows that

$$2C(n) = \left(\mathsf{E}\left[Y_{n+1}^2\right] - \mathsf{E}\left[Y_n^2\right]\right) - \left(\mathsf{E}\left[Y_n^2\right] - \mathsf{E}\left[Y_{n-1}^2\right]\right). \tag{22}$$

Similarly, if $\widetilde{C}^{(m)}(n)$ denotes the covariance function of $\widetilde{X}_n^{(m)} := mX_n^{(m)}$, then [7, eq. (8.3)]

$$2\widetilde{C}^{(m)}(n) = \mathsf{E}\left[Y_{(n+1)m}^2\right] - 2\mathsf{E}\left[Y_{nm}^2\right] + \mathsf{E}\left[Y_{(n-1)m}^2\right].$$

Since $C^{(m)}(n) = \widetilde{C}^{(m)}(n)/m^2$

$$\frac{C^{(m)}(n)}{m^{2H-2}} = \frac{1}{2}\left(\frac{\mathsf{E}\left[Y_{(n+1)m}^2\right]}{[(n+1)m]^{2H}}(n+1)^{2H} - 2\frac{\mathsf{E}\left[Y_{nm}^2\right]}{(nm)^{2H}} n^{2H} + \frac{\mathsf{E}\left[Y_{(n-1)m}^2\right]}{[(n-1)m]^{2H}}(n-1)^{2H}\right)$$

and we see that (15) implies (14).

Appendix C
Proof of Theorem 3

To establish (15), observe that

$$\begin{aligned}
\mathsf{E}\left[Y_n^2\right] &= \sum_{i=1}^n \sum_{k=1}^n C(i-k) \\
&= \sum_{i=1}^n \sum_{k=1}^n \int_{-1/2}^{1/2} S(f) e^{j2\pi f(i-k)} df \\
&= \int_{-1/2}^{1/2} S(f) \left|\sum_{k=1}^n e^{-j2\pi fk}\right|^2 df \\
&= \int_{-1/2}^{1/2} S(f) \left[\frac{\sin(n\pi f)}{\sin(\pi f)}\right]^2 df \\
&= n^2 \int_{-1/2}^{1/2} S(f) \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\frac{\pi f}{\sin(\pi f)}\right]^2 df.
\end{aligned}$$

We now show that

$$\lim_{n\to\infty} \frac{\mathsf{E}\left[Y_n^2\right]}{n^{2-\alpha}} = s \cdot \frac{4\cos(\alpha\pi/2)\Gamma(\alpha)}{(2\pi)^\alpha (1-\alpha)(2-\alpha)}.$$

The first step is to put

$$i_n(\alpha) := n^\alpha \int_{-1/2}^{1/2} \frac{1}{|f|^{1-\alpha}} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\frac{\pi f}{\sin(\pi f)}\right]^2 df$$

and show that

$$\frac{\mathsf{E}\left[Y_n^2\right]}{n^{2-\alpha}} - s \cdot i_n(\alpha) \to 0.$$

The second step is to put

$$j_n(\alpha) := n^\alpha \int_{-1/2}^{1/2} \frac{1}{|f|^{1-\alpha}} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 df$$

and show that $i_n(\alpha) - j_n(\alpha) \to 0$. The proof is concluded by noting that $j_n(\alpha) \to j(\alpha)$, where

$$\begin{aligned} j(\alpha) &:= \pi^{-\alpha} \int_{-\infty}^{\infty} \frac{1}{|\theta|^{1-\alpha}} \left(\frac{\sin\theta}{\theta}\right)^2 d\theta \\ &= \frac{4\cos(\alpha\pi/2)\Gamma(\alpha)}{(2\pi)^\alpha(1-\alpha)(2-\alpha)} \quad \text{by [5, p. 447].} \end{aligned}$$

Step 1

Let $\varepsilon > 0$ be given, and let $0 < \delta < 1/2$ be so small that for $0 < |f| < \delta$

$$\left|\frac{S(f)}{|f|^{\alpha-1}} - s\right| < \varepsilon$$

or

$$\left|S(f) - \frac{s}{|f|^{1-\alpha}}\right| < \frac{\varepsilon}{|f|^{1-\alpha}}.$$

Put

$$B_\delta := \sup_{|f|\in[\delta,1/2]} \left|S(f) - \frac{s}{|f|^{1-\alpha}}\right| < \infty.$$

Let n be so large that

$$\frac{1}{n^{1-\alpha}} < \frac{\varepsilon}{B_\delta}.$$

Observe that

$$\frac{\mathsf{E}\left[Y_n^2\right]}{n^{2-\alpha}} - s \cdot i_n(\alpha)$$

is equal to

$$n^\alpha \int_{-1/2}^{1/2} \left[S(f) - \frac{s}{|f|^{1-\alpha}}\right] \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\frac{\pi f}{\sin(\pi f)}\right]^2 df. \quad (23)$$

Since the integrand is even, we can restrict our attention to $[0, 1/2]$. We first consider the interval $[0, \delta]$ and later $[\delta, 1/2]$. Write

$$\begin{aligned} &\left| n^\alpha \int_0^\delta \left[S(f) - \frac{s}{f^{1-\alpha}}\right] \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\frac{\pi f}{\sin(\pi f)}\right]^2 df \right| \\ &\le \varepsilon n^\alpha \left(\frac{\pi}{2}\right)^2 \int_0^\delta \frac{1}{f^{1-\alpha}} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 df \\ &= \varepsilon n^\alpha \left(\frac{\pi}{2}\right)^2 \frac{\pi^{-\alpha}}{n^\alpha} \int_0^{n\pi\delta} \frac{1}{\theta^{1-\alpha}} \left[\frac{\sin\theta}{\theta}\right]^2 d\theta \\ &\le \varepsilon \left(\frac{\pi}{2}\right)^2 \pi^{-\alpha} \int_0^{n\pi\delta} \frac{1}{\theta^{1-\alpha}} \left[\frac{\sin\theta}{\theta}\right]^2 d\theta \\ &< \varepsilon \left(\frac{\pi}{2}\right)^2 \frac{j(\alpha)}{2}. \end{aligned}$$

We return to (23) and focus now on the range of integration $[\delta, 1/2]$. Write

$$\begin{aligned} &\left| n^\alpha \int_\delta^{1/2} \left[S(f) - \frac{s}{f^{1-\alpha}}\right] \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\frac{\pi f}{\sin(\pi f)}\right]^2 df \right| \\ &\le n^\alpha B_\delta \left(\frac{\pi}{2}\right)^2 \int_\delta^{1/2} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 df \\ &= n^\alpha B_\delta \left(\frac{\pi}{2}\right)^2 \int_{n\pi\delta}^{n\pi/2} \left[\frac{\sin\theta}{\theta}\right]^2 \frac{d\theta}{n\pi} \\ &\le \frac{B_\delta}{n^{1-\alpha}} \left(\frac{\pi}{4}\right) \int_0^\infty \left[\frac{\sin\theta}{\theta}\right]^2 d\theta \\ &< \varepsilon \left(\frac{\pi}{4}\right) \int_0^\infty \left[\frac{\sin\theta}{\theta}\right]^2 d\theta. \end{aligned}$$

Step 2

Let $\varepsilon > 0$ be given, and let $0 < \delta < 1/2$ be so small that for $0 < |f| < \delta$

$$\left|\left(\frac{\pi f}{\sin(\pi f)}\right)^2 - 1\right| < \varepsilon.$$

Let n be so large that

$$\frac{1}{n^{1-\alpha}} < \frac{\varepsilon}{\delta^{\alpha-1}}.$$

Observe that $i_n(\alpha) - j_n(\alpha)$ is equal to

$$n^\alpha \int_{-1/2}^{1/2} \frac{1}{|f|^{1-\alpha}} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\left(\frac{\pi f}{\sin(\pi f)}\right)^2 - 1\right] df$$

and the integrand is nonnegative. Write

$$\begin{aligned} &n^\alpha \int_0^\delta \frac{1}{f^{1-\alpha}} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\left(\frac{\pi f}{\sin(\pi f)}\right)^2 - 1\right] df \\ &\le \varepsilon n^\alpha \int_0^\delta \frac{1}{f^{1-\alpha}} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 df \\ &< \varepsilon j(\alpha)/2. \end{aligned}$$

To conclude Step 2, write

$$\begin{aligned} &n^\alpha \int_\delta^{1/2} \frac{1}{f^{1-\alpha}} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 \left[\left(\frac{\pi f}{\sin(\pi f)}\right)^2 - 1\right] df \\ &\le n^\alpha \frac{1}{\delta^{1-\alpha}} \left(\frac{\pi^2}{4} - 1\right) \int_\delta^{1/2} \left[\frac{\sin(n\pi f)}{n\pi f}\right]^2 df \\ &< \frac{\delta^{\alpha-1}}{\pi n^{1-\alpha}} \left(\frac{\pi^2}{4} - 1\right) \int_0^\infty \left[\frac{\sin\theta}{\theta}\right]^2 d\theta \\ &< \frac{\varepsilon}{\pi} \left(\frac{\pi^2}{4} - 1\right) \int_0^\infty \left[\frac{\sin\theta}{\theta}\right]^2 d\theta. \end{aligned}$$

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Survival Exponential Entropies

Konstantinos Zografos and Saralees Nadarajah

***Abstract*—The multivariate survival function of a random vector X is used to define a broad class of entropy measures. Several properties of the proposed class are studied and explicit expressions of the measures derived for specific probabilistic models. The cumulative residual entropy, introduced by Rao *et al.* and Wang *et al.*, is a particular case of the proposed class of measures.**

***Index Terms*—Exponential entropy, multivariate survival function, Shannon entropy.**

I. Introduction

The notion of entropy is of fundamental importance in different areas such as physics, probability and statistics, communication theory, and economics. It was originally developed in the field of thermodynamics and extended later to statistical mechanics. The concept of entropy is of particular importance in the field of information theory and it was introduced there by Shannon in [3]. If X is a random variable with an absolutely continuous distribution with probability density function $f(x)$ then the Shannon entropy of the density f is defined by

$$\mathcal{H}_{Sh}(X) = -\int_{-\infty}^{+\infty} f(x)\ln f(x)dx \tag{1}$$

which may be regarded as a descriptive quantity of the distribution given by f. The conditions for the existence of (1) are derived in [4].

Since Shannon's [3] pioneering work on the mathematical theory of communication, entropy (1) has been used as a major tool in information theory and in almost every branch of science and engineering. Numerous entropy and information indices have been developed and used in various disciplines and contexts. A competitor to Shannon entropy is the Rényi entropy defined by

$$\mathcal{H}_R(X) = \frac{1}{1-\alpha}\ln\int_{-\infty}^{+\infty} f^{\alpha}(x)dx \tag{2}$$

for $\alpha > 0$ and $\alpha \neq 1$ [5]. $\mathcal{H}_{Sh}(X)$ and $\mathcal{H}_R(X)$ are special cases of the exponential entropy of order α, defined by

$$\mathcal{B}_{\alpha}(X) = \left(\int_{-\infty}^{+\infty} f^{\alpha}(x)dx\right)^{1/(1-\alpha)}, \quad \alpha > 0, \ \alpha \neq 1. \tag{3}$$

In particular

$$\mathcal{H}_{Sh}(X) = \lim_{\alpha\to 1}\{\ln \mathcal{B}_{\alpha}(X)\}$$

and

$$\mathcal{H}_R(X) = \ln \mathcal{B}_{\alpha}(X).$$

Exponential entropy (3) has been defined and studied by Campbell [6] and generalized by Koski and Persson [7]. It has been successfully applied [8] in data compression or signal compression problems, in the sense of quantization.

Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields [9].

Shannon's original definition of entropy was given for discrete random variables and its continuous counterpart, defined by (1), is not a direct consequence of the definition in the discrete case. It is also well known that for continuous distributions, $\mathcal{H}_{Sh}(X)$ may be negative and infinite. Rao *et al.* [1] enumerated several drawbacks of $\mathcal{H}_{Sh}(X)$. The most important of them is that $\mathcal{H}_{Sh}(X)$ is defined only for distributions with densities. In order to overcome this drawback, Rao *et al.* [1] and Wang *et al.* [2] defined a new measure, referred to as the cumulative residual entropy, based on the probability distribution of a random variable rather than its density function. Several properties of the said measure have been stated and studied in the previously-mentioned articles.

Measures of uncertainty with origins in Shannon entropy (1) and involving, in their expression, the distribution function of a nonnegative random variable, have been defined and studied earlier by Ebrahimi and his colleagues [10]–[13]. The role of information theory in reliability analysis has been reviewed by Ebrahimi *et al.* [14].

In this article, we propose two new broad classes of measures of uncertainty of a random vector X based on the survival function of an absolute value transformation of X. We refer to them as the survival exponential and the generalized survival exponential entropies. Their definition is motivated by the work of Rao *et al.* [1] and Wang *et al.* [2],

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