# On the Power of Quantum Memory 

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#### Abstract

We address the question whether quantum memory is more powerful than classical memory. In particular, we consider a setting where information about a random $n$-bit string $X$ is stored in $s$ classical or quantum bits, for $s<n$, i.e., the stored information is bound to be only partial. Later, a randomly chosen predicate $F$ about $X$ has to be guessed using only the stored information. The maximum probability of correctly guessing $F(X)$ is then compared for the cases where the storage device is classical or quantum mechanical, respectively. We show that, despite the fact that the measurement of quantum bits can depend arbitrarily on the predicate $F$, the quantum advantage is negligible already for small values of the difference $n-s$. Our setting generalizes the setting of Ambainis et al. who considered the problem of guessing an arbitrary bit (i.e., one of the $n$ bits) of $X$.

An implication for cryptography is that privacy amplification by universal hashing remains essentially equally secure when the adversary's memory is allowed to be quantum rather than only classical. Since privacy amplification is a main ingredient of many quantum key distribution (QKD) protocols, our result can be used to prove the security of QKD in a generic way.


Index Terms-Cryptography, privacy amplification, quantum information theory, quantum key distribution, quantum memory, security proofs, universal hashing.

## I. Introduction

It is a well-known fact that in $s$ quantum bits one cannot reliably store more than $s$ classical bits of information. ${ }^{1}$ In other words, the raw storage capacity (like the raw transmission capacity) of a quantum bit is just one bit of information. However, since quantum memory can be read by an arbitrary measurement determined only at the time of reading the memory, quantum memory can be expected to be more powerful than classical memory in any context where a string $X$ of $n>s$ bits of information is given (and hence can be stored only partially) and it is determined only later which information about $X$ is of interest. ${ }^{2}$

The simplest setting one can consider is that one must use the stored information to guess $F(X)$ for a randomly chosen predicate $F: \mathcal{X} \rightarrow\{0,1\}$. Ambainis, Nayak, TaShma, and Vazirani [4], [5] were the first to study such a setting for the special case where $X$ is an $n$-bit string and $F(X)$ is an actual bit (i.e., one of the $n$ bits) of $X$. Because in the quantum case one can let the measurement

[^0]of the stored quantum bits depend arbitrarily on $F$, while in the classical case one can only read the stored information, quantum memory is potentially more powerful. However, we prove that having information about $X$ stored in $s$ quantum instead of $s$ classical bits is essentially useless for guessing $F(X)$, even for optimal quantum storage and measurement strategies. This is in accordance with the results in [4], [5] as well as with recent results on communication complexity (see e.g., [6]) where the power of classical and quantum communication is compared.

In a cryptographic context, our results can be applied to the security analysis of cryptographic primitives in a context where an adversary might hold quantum information. An important example is privacy amplification introduced by Bennett, Brassard, and Robert [7] (see also [8]) which is a protocol between two parties, Alice and Bob. The goal is to turn a common $n$-bit string $X$, about which an adversary Eve has some partial information, into a highly secure $k$-bit key $K$. This can be achieved as follows: Alice and Bob publicly agree on a function $G:\{0,1\}^{n} \rightarrow\{0,1\}^{k}$ chosen from a two-universal class of hash functions ${ }^{3}$ and then compute $K=G(X) .{ }^{4}$ It has been shown that, if Eve's information about $X$ consists of no more than $s$ classical bits, the final key is secure as long as $k<n-s .^{5}$

Similar to the previously described setting, it seems to be a potential advantage for the adversary to have available $s$ quantum instead of $s$ classical bits of information about $X$ because she later learns the function $G$ and can let her measurement of the $s$ quantum bits depend on $G$. This may allow her to obtain more information about the final key $K$. We prove that this is not the case, i.e., privacy amplification remains equally secure against adversaries holding quantum information.
This has interesting implications for quantum key distribution (QKD): In a QKD protocol, Alice and Bob first exchange quantum information (e.g., polarized photons) to generate a raw key $X$ which is only partially secure, i.e., Eve has some quantum information $\rho$ about $X$. In a second (purely classical) phase, Alice and Bob apply privacy amplification to generate the final secret key $K$. Our result on the security of privacy amplification thus reduces the problem of proving the security of a QKD protocol to the problem of finding a bound on the number of qubits needed to (reliably) store Eve's information $\rho$. In [10], this fact has been exploited to show the security of a generic QKD protocol which, in particular, implies the security of many known protocols such as BB84 [11]. This simplifies and generalizes ${ }^{6}$ known security proofs (see e.g.,

[^1][12]) which are based on completely different techniques. It also generalizes a proof by Ben-Or [13] which is based on a similar idea using results from communication complexity theory [14].

The paper is organized as follows. In Section III we introduce a general framework for modeling and quantifying knowledge and storage devices. The framework is then used in Section [V] to state and prove bounds on the success probability when guessing a binary predicate $F$ of $X$ given information about $X$ stored in a quantum storage device (Section [V-B). These are then compared to the situation where the information about $X$ is purely classical (Section (V-C). In Section $\nabla$ the results are extended to non-binary functions which then allows for proving the security of privacy amplification against quantum adversaries (Section $\nabla-B$ ).

## II. Preliminaries

## A. Notation

Let $\mathcal{F}(\mathcal{X} \rightarrow \mathcal{Y})$ be the set of functions with domain $\mathcal{X}$ and range $\mathcal{Y}$. The set $\mathcal{F}(\mathcal{X} \rightarrow\{0,1\})$ of binary functions with domain $\mathcal{X}$, in the following called predicates on $\mathcal{X}$, is denoted as $\mathcal{F}_{\text {bin }}^{\mathcal{X}}$. Similarly, $\mathcal{F}_{\text {bal }}^{\mathcal{X}}:=\left\{f \in \mathcal{F}_{\text {bin }}^{\mathcal{X}}:\left|f^{-1}(\{0\})\right|=\right.$ $\left.\left|f^{-1}(\{1\})\right|\right\}$ is the set of balanced predicates on $\mathcal{X}$.

Throughout this paper, random variables are denoted by capital letters (e.g., $X$ ), their range by corresponding calligraphic letters $(\mathcal{X})$, and the values they take on by lower case letters $(x)$. The event that two random variables $X$ and $Y$ take on the same value is denoted as $X=Y$. In contrast, we write $X \equiv Y$ if two random variables $X$ and $Y$ are identical (i.e., if $X=Y$ always holds). The expectation $\mathbb{E}_{x \leftarrow P_{X}}[f(x)]$ of a function $f$ on the random variable $X$ is given by $\sum_{x \in \mathcal{X}} P_{X}(x) f(x)$.

For a channel $C$ from $\mathcal{S}$ to $\mathcal{W}$ and a random variable $S$ on $\mathcal{S}$, we denote by $C_{S}$ the output of $C$ on input $S$, i.e., if the channel is defined by the conditional distributions $P_{W \mid S=s}$ for $s \in \mathcal{S}$, the joint probability distribution of $C_{S}$ and $S$ is given by $P_{C_{S} S}(w, s)=P(s) P_{C_{S} \mid S=s}(w)$ for all $(w, s) \in \mathcal{W} \times \mathcal{S}$.

A random function $G$ from $\mathcal{X}$ to $\mathcal{Y}$ is a random variable taking values from the set $\mathcal{F}(\mathcal{X} \rightarrow \mathcal{Y})$ of functions mapping elements from $\mathcal{X}$ to $\mathcal{Y}$. The set of random functions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted as $\mathcal{R}(\mathcal{X} \rightarrow \mathcal{Y})$. If $G \in \mathcal{R}(\mathcal{X} \rightarrow \mathcal{Y})$ is uniformly distributed over $\mathcal{F}(\mathcal{X} \rightarrow \mathcal{Y})$, it is called a uniform random function from $\mathcal{X}$ to $\mathcal{Y}$. Similarly, a (uniform) random predicate $F$ on $\mathcal{X}$ is a random function with (uniform) distribution over the set $\mathcal{F}_{\text {bin }}^{\mathcal{X}}$, and a (uniform) balanced random predicate is (uniformly) distributed over the set $\mathcal{F}_{\text {bal }}^{\mathcal{X}}$. In the sequel, we will only use random functions which are independent of all other (previously defined) random variables.

A random function $G$ from $\mathcal{X}$ to $\mathcal{Y}$ is called ${ }^{7}$ two-universal if $\mathbb{P}_{g \leftarrow P_{G}}\left[g(x)=g\left(x^{\prime}\right)\right] \leq 1 /|\mathcal{Y}|$ holds for any distinct $x, x^{\prime} \in$ $\mathcal{X}$. In particular, $G$ is two-universal if, for any distinct $x, x^{\prime} \in$ $\mathcal{X}$, the random variables $G(x)$ and $G\left(x^{\prime}\right)$ are independent and uniformly distributed. For instance, a uniform random function from $\mathcal{X}$ to $\mathcal{Y}$ is two-universal. Non-trivial examples where

[^2]the distribution of $G$ is over a smaller set of function (thus requiring less randomness) can, e.g., be found in [15] and [16].

## B. Distance from Uniform

The variational distance between two distributions $P$ and $P^{\prime}$ over an alphabet $\mathcal{Z}$ is defined as

$$
\delta\left(P, P^{\prime}\right):=\frac{1}{2} \sum_{z \in \mathcal{Z}}\left|P(z)-P^{\prime}(z)\right|
$$

The variational distance $\delta(P, \bar{P})$ of a distribution $P$ from the uniform distribution $\bar{P}$ (over the same alphabet $\mathcal{Z}$ ) is of particular interest in cryptographic applications. We will use the abbreviation $d(P)$ for this quantity and refer to it as the distance of $P$ from uniform. For the distance of the distribution of a random variable $Z$ from uniform, we also write $d(Z)$ instead of $d\left(P_{Z}\right)$, and, more generally, for any event $\mathcal{E}$, $d(Z \mid \mathcal{E}):=d\left(P_{Z \mid \mathcal{E}}\right)$. Note that $d$ is a convex function, i.e., for two probability distributions $P$ and $P^{\prime}$, and $q, q^{\prime} \in[0,1]$ with $q+q^{\prime}=1$, we have $d\left(q P+q^{\prime} P^{\prime}\right) \leq q d(P)+q^{\prime} d\left(P^{\prime}\right)$.

The distance $d(Z)$ of a random variable $Z$ from uniform has a natural interpretation: It equals the probability that $Z$ deviates from a uniformly distributed random variable $\bar{Z}$, in the following sense.

Lemma 1: For any probability distribution $P_{Z}$ on $\mathcal{Z}$ there exists a channel $P_{\bar{Z} \mid Z}$ such that $P_{\bar{Z}}$ is the uniform distribution on $\mathcal{Z}$ and $\mathbb{P}_{(z, \bar{z}) \leftarrow P_{Z \bar{Z}}}[z=\bar{z}]=1-d(Z)$.

For two random variables $Z$ and $W$, the (expected) distance of $Z$ from uniform given $W$ is defined (cf. [2]) as the expectation of the distance of $Z$ from uniform conditioned on $W$, i.e., $d(Z \mid W):=\mathbb{E}_{w \leftarrow P_{W}}\left[d\left(P_{Z \mid W=w}\right)\right]$. It follows directly from the convexity of $d$ that $d(Z \mid W) \geq d(Z)$, and, more generally, for an additional random variable $V$ and an event $\mathcal{E}, d(Z \mid W V, \mathcal{E}) \geq d(Z \mid V, \mathcal{E})$.

## III. Modeling Knowledge and Storage

## A. Knowledge and Guessing

Let $Z$ be a random variable and let $\mathcal{A}$ be an entity with knowledge described by a random variable $W$ (jointly distributed with $Z$ according to some distribution $\left.P_{Z W}\right)$. Intuitively, one would say that $\mathcal{A}$ knows nothing about $Z$ if $Z$ is uniformly distributed given $\mathcal{A}$ 's knowledge $W$, i.e., $P_{Z W} \equiv P_{Z} \times P_{W}$ where $P_{Z}$ is the uniform distribution. The following straightforward generalization of Lemma suggests that the distance $d(Z \mid W)$ of $Z$ from uniform given $W$ can be interpreted as the probability of deviating from this situation.

Lemma 2: For any probability distribution $P_{W Z}$ on $\mathcal{W} \times \mathcal{Z}$ there exists a channel $P_{\bar{Z} \mid W Z}$ such that $P_{\bar{Z}}$ is the uniform distribution on $\mathcal{Z}, P_{\bar{Z} W} \equiv P_{\bar{Z}} \times P_{W}$, and $\mathbb{P}_{(z, \bar{z}) \leftarrow P_{Z \bar{Z}}}[z=$ $\bar{z}]=1-d(Z \mid W)$.

This is of particular interest in cryptography, where, for instance, $\mathcal{A}$ is an adversary with knowledge $W$ and where one wants to use $Z$ as a key. Typically, a cryptosystem based on a key $\bar{Z}$ is secure when $\bar{Z}$ is uniformly distributed and independent of $\mathcal{A}$ 's knowledge. The lemma implies that, with probability $1-d(Z \mid W), Z$ is equal to such a perfect key $\bar{Z}$. This means that any statement which is true for an ideal setting
where $\bar{Z}$ is used as a key automatically holds, with probability at least $1-d(Z \mid W)$, for a real setting where $Z$ is the key.

The distance from uniform $d(Z \mid W)$ is also a measure for the maximum success probability $P_{\text {guess }}(Z \mid W)$ of an entity $\mathcal{A}$ knowing $W$ when trying to guess $Z$,

$$
P_{\text {guess }}(Z \mid W):=\max _{C} \underset{(w, z) \leftarrow P_{W Z}}{\mathbb{P}}\left[C_{w}=z\right]
$$

where the maximum is over all channels $C$ from $\mathcal{W}$ to $\mathcal{Z} .{ }^{8}$
The following lemma is an immediate consequence of the simple fact that the best strategy for guessing $Z$ given $W=w$ is to choose a value $\hat{z}$ maximizing the probability $P_{Z \mid W}(\hat{z} \mid w)$.

Lemma 3: Let $W$ and $Z$ be random variables. Then $P_{\text {guess }}(Z \mid W) \leq \frac{1}{|\mathcal{Z}|}+d(Z \mid W)$ where equality holds if $Z$ is binary.

## B. Selectable Knowledge

The characterization of knowledge about a random variable $Z$ held by an entity $\mathcal{A}$ in terms of a random variable $W$ is sufficient whenever this knowledge is fully accessible, e.g., written down on a sheet of paper or stored in a classical storage device. However, in a more general context $\mathcal{A}$ might have an option as to which information she can obtain. For example, if her information about $Z$ is encoded into the state $\rho$ of a quantum system, she may select one arbitrary measurement to "read it out". Formally, every measurement corresponds to a channel $W$ from the state space of the quantum system to the set of possible measurement outcomes. The situation is thus completely characterized by the set of measurements (that is, channels) $\mathbf{W}$ and the joint distribution of $Z$ and $\rho$. This setting is discussed in detail in Section III-D Another (more artificial) example might be a storage unit which can hold two bits $S \equiv B_{1} B_{2}$, but which allows only to read out one of these bits, i.e., $\mathcal{A}$ can read either the value $B_{1}$ or $B_{2}$. In this case, the situation is described by the joint distribution of $Z$ and $S$ and the set of channels $\left\{p_{1}, p_{2}\right\}$, where channel $p_{i}$ maps $\left(b_{1}, b_{2}\right)$ to $b_{i}$ for $i=1,2$. To model these situations, it is useful to introduce the following notion.

Definition 4: A selectable channel $\mathbf{W}$ on $\mathcal{S}$ with range $\mathcal{W}$ is a set of channels from $\mathcal{S}$ to $\mathcal{W}$.

Consider now a setting as described above, i.e., there is a system which is in a state described by a random variable $S$ on $\mathcal{S}$, and an entity $\mathcal{A}$ has access to $S$ by means of a channel $W$ from a set $\mathbf{W}$. In the following, we say that an entity $\mathcal{A}$ has selectable knowledge $\mathbf{W}_{S}$, meaning that $\mathcal{A}$ can learn the value of exactly one arbitrarily chosen random variable $W_{S}$ with $W \in \mathbf{W}$. The knowledge of $\mathcal{A}$ about a random variable $Z$ can then be quantified by a natural generalization of the distance measure introduced above.

Definition 5: Let $S$ and $Z$ be random variables and let $\mathbf{W}$ be a selectable channel on the range of $S$. The distance of $Z$ from uniform given $\mathbf{W}_{S}$, is

$$
d\left(Z \mid \mathbf{W}_{S}\right):=\max _{W \in \mathbf{W}} d\left(Z \mid W_{S}\right)
$$

The significance of this generalized definition of distance from uniform, e.g., in cryptography, is implied by a straightforward extension of Lemma 2

[^3]Lemma 6: Let $S$ and $Z$ be random variables and let $\mathbf{W}$ be a selectable channel on the range of $S$. Then for any choice of an element $W$ of $\mathbf{W}$, there exists a random variable $\bar{Z}$ defined by a channel $P_{\bar{Z} \mid W_{S} Z}$, such that $P_{\bar{Z}}$ is the uniform distribution on $\mathcal{Z}, P_{W_{S} \bar{Z}} \equiv P_{W_{S}} \times P_{\bar{Z}}$, and $\mathbb{P}_{(z, \bar{z}) \leftarrow P_{\bar{Z} Z}}[\bar{z}=$ $z] \geq 1-d\left(Z \mid \mathbf{W}_{S}\right)$.

Similarly, Lemma 3 can be generalized to obtain a bound for the maximum success probability of an entity $\mathcal{A}$ with selectable knowledge $\mathbf{W}_{S}$ when guessing $Z$,

$$
P_{\text {guess }}\left(Z \mid \mathbf{W}_{S}\right):=\max _{W \in \mathbf{W}} P_{\text {guess }}\left(Z \mid W_{S}\right)
$$

Lemma 7: Let $S$ and $Z$ be random variables and let $\mathbf{W}$ be a selectable channel on the range of $S$. Then $P_{\text {guess }}\left(Z \mid \mathbf{W}_{S}\right) \leq$ $\frac{1}{|\mathcal{Z}|}+d\left(Z \mid \mathbf{W}_{S}\right)$, where equality holds if $Z$ is binary.

Consider now a situation where the information about $Z$ of an entity $\mathcal{A}$ is described by both some selectable knowledge $\mathbf{W}_{S}$, and, additionally, a random variable $U$ which she can use to choose an element from $\mathbf{W}$. More precisely, she applies some channel $C=P_{W \mid U}$ from $\mathcal{U}$ to $\mathbf{W}$ to the random variable $U$ and then chooses to learn $W_{S}$ for the resulting $W \equiv C_{U} \in \mathbf{W}$. We will then be interested in the maximal distance of $Z$ from uniform resulting from an optimal strategy used by $\mathcal{A}$. Such an optimal strategy consists simply of (deterministically) choosing some $W \in \mathbf{W}$ which maximizes $\mathbb{E}_{w \leftarrow P_{W_{S}}}\left[d\left(P_{Z \mid W_{S}=w, U=u}\right)\right]$, given $U=u$. We thus introduce the following quantity.

Definition 8: Let $S, U$ and $Z$ be random variables and let W be a selectable channel on the range of $S$. The distance of $Z$ from uniform given $\mathbf{W}_{S}$ and $U$ is defined as

$$
\begin{equation*}
d\left(Z \mid \mathbf{W}_{S} ; U\right):=\underset{u \leftarrow P_{U}}{\mathbb{E}}\left[\max _{W \in \mathbf{W}} d\left(Z \mid W_{S}, U=u\right)\right] \tag{1}
\end{equation*}
$$

It is easy to see that

$$
d\left(Z \mid \mathbf{W}_{S} ; U\right)=d\left(Z \mid \mathbf{V}_{(S, U)}\right)
$$

for some selectable channel $\mathbf{V}$ on $\mathcal{S} \times \mathcal{U}$ which models the fact that $\mathcal{A}$ can choose an arbitrary strategy. In particular, Lemma 6 and Lemma 7 still hold when $\mathbf{W}_{S}$ is replaced by $\mathbf{W}_{S} ; U$, where $P_{\text {guess }}\left(Z \mid \mathbf{W}_{S} ; U\right)$ is defined as the maximal probability of $\mathcal{A}$ when guessing $Z$ in the situation described above.

It is a direct consequence of the properties of the variational distance that knowledge of an additional random variable $U$ can only increase the distance from uniform given selectable knowledge.

Lemma 9: Let $S, U$ and $Z$ be random variables and let $\mathbf{W}$ be a selectable channel on the domain of $S$. Then

$$
d\left(Z \mid \mathbf{W}_{S} ; U\right) \geq d\left(Z \mid \mathbf{W}_{S}\right)
$$

## C. Storage Devices

A (physical) storage device is a physical system where the information it contains is determined by its physical state $s$. Information is stored in the device by choosing a state $s$ from its state space $\mathcal{S}$. A storage device might provide different mechanisms to read out this information, each of them resulting in some (generally only partial) information about its state $s$. However, any possible strategy of accessing the stored information can be described as a channel mapping
the memory state to a random variable $W$. We thus define a storage device with state space $\mathcal{S}$ and range $\mathcal{W}$ as a selectable channel $\mathbf{p}$ from $\mathcal{S}$ to $\mathcal{W}$.

As an example, consider the (artificial) storage device mentioned above which allows to store two bits, but where only one of them can be read out. Formally, this storage device is a selectable channel $\mathbf{p}=\left\{p_{1}, p_{2}\right\}$ from the state space $\mathcal{S}=\{0,1\} \times\{0,1\}$ to the set $\{0,1\}$ where $p_{m}$ is the channel mapping $\left(b_{1}, b_{2}\right)$ to $b_{m}$, for $m \in\{1,2\}$.

The most trivial case is a classical storage device for storing $s$ bits and allowing to read out all $s$ bits without errors. Obviously, its state $s$ can take one of $2^{s}$ possible values. Moreover, any accessing strategy corresponds to a channel with input $s$. Formally, a classical s-bit storage device is defined as the selectable channel $\mathbf{C}^{2^{s}}$ containing all channels taking inputs from the set $\{0,1\}^{s}$. (In Section आII-D we will give an analogous definition for quantum storage devices.) Note that for a random variable $Z$ and a random variable $S$ on $\{0,1\}^{s}, d\left(Z \mid \mathbf{C}_{S}^{s^{s}}\right)=d(Z \mid S)$. Thus we omit to mention the selectable channel if it is clear from the context, e.g., we write $d(Z \mid S ; U)$ instead of $d\left(Z \mid \mathbf{C}_{S}^{2 s} ; U\right)$.

## D. Quantum Storage

An $s$-qubit storage device is a quantum system of dimension $d=2^{s}$ where information is stored by encoding it into the state of the system. This information can (partially) be read out by measuring the system's state with respect to some (arbitrarily chosen) measurement basis. Each pure state of a $d$-dimensional quantum system corresponds to a normalized vector $|\psi\rangle$ in a $d$-dimensional Hilbert space $\mathcal{H}_{d}$. Equivalently, the set of pure states can be identified with the set $\mathcal{P}\left(\mathcal{H}_{d}\right):=\{|\psi\rangle\langle\psi|:|\psi\rangle \in$ $\left.\mathcal{H}_{d},|\langle\psi \mid \psi\rangle|=1\right\}$ where $|\psi\rangle\langle\psi|$ is the projection operator in $\mathcal{H}_{d}$ along the vector $|\psi\rangle$. The set of all possible states of the quantum system is then given by the set of mixed states $\mathcal{S}\left(\mathcal{H}_{d}\right)$, which is the convex hull of $\mathcal{P}\left(\mathcal{H}_{d}\right)$.

It is well known from quantum information theory that the most general strategy to access the information contained in a quantum system is to perform a positive operator-valued measurement (POVM), which gives a classical measurement outcome $W$. Any possible measurement is specified by a family $\left\{E_{w}\right\}_{w \in \mathcal{W}}$ of nonnegative operators on $\mathcal{H}_{d}$ satisfying $\sum_{w \in \mathcal{W}} E_{w}=\operatorname{id}_{\mathcal{H}_{d}}$. If the system is in state $\rho$, the probability of obtaining the (classical) measurement outcome $w \in \mathcal{W}$ when applying measurement $\left\{E_{w}\right\}_{w \in \mathcal{W}}$ is given by $p_{\left\{E_{w}\right\}}(w \mid \rho):=\operatorname{tr}\left(E_{w} \rho\right)$.

In the framework presented in the previous section, a $d$ dimensional quantum storage device $\mathbf{Q}^{d}$ is thus defined as the set of channels $p_{\left\{E_{w}\right\}}$ describing all possible POVMs $\left\{E_{w}\right\}$ on a $d$-dimensional quantum state, i.e.,

$$
\mathbf{Q}^{d}:=\left\{p_{\left\{E_{w}\right\}}:\left\{E_{w}\right\} \in \operatorname{POVM}\left(\mathcal{H}_{d}\right)\right\}
$$

A general way of describing this setting is to define the state $S$ of the storage device by a family of quantum states $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset \mathcal{S}\left(\mathcal{H}_{d}\right)$, where $\rho_{x}$ is the conditional state of the system given $X=x$, that is $S \equiv \rho_{X}$. Similar to the notation introduced for classical storage devices $\mathbf{C}^{2^{s}}$, we will also write $\rho_{X}$ instead of $\mathbf{Q}_{\rho_{X}}^{d}$.

According to Definition 5] the distance $d\left(Z \mid \rho_{X}\right)$ of a random variable $Z$ from uniform given $\rho_{X}$ can be written as

$$
d\left(Z \mid \rho_{X}\right)=\max _{\left\{E_{w}\right\}} d(Z \mid W)
$$

where the maximum is taken over all POVMs $\left\{E_{w}\right\}$ and where $W$ is the measurement outcome of $\left\{E_{w}\right\}$ applied to the quantum state, i.e., $P_{W \mid X=x}(w)=\operatorname{tr}\left(E_{w} \rho_{x}\right)$. Similarly, for an additional random variable $U$,

$$
d\left(Z \mid \rho_{X} ; U\right)=\underset{u \leftarrow P_{U}}{\mathbb{E}}\left[\max _{\left\{E_{w}^{u}\right\}} d(Z \mid W, U=u)\right]
$$

where, for each $u,\left\{E_{w}^{u}\right\}$ is a POVM and where $W$ is defined by $P_{W \mid X=x, U=u}(w)=\operatorname{tr}\left(E_{w}^{u} \rho_{x}\right)$.

## IV. Quantum Knowledge About Predicates

## A. The Quantum Binary Decision Problem

We begin this section by stating a few known results about the so-called quantum binary decision problem, which are central to the proof of our main statements concerning quantum knowledge.

Let $\rho_{0}, \rho_{1} \in \mathcal{S}(\mathcal{H})$ be arbitrary (mixed) states of a quantum mechanical system $\mathcal{H}$, and suppose that the system is prepared either in the state $\rho=\rho_{0}$ or in $\rho=\rho_{1}$ with a priori probabilities $q$ and $1-q$, respectively. The quantum binary decision problem is the problem of deciding between these two possibilities by an appropriate measurement. Any decision strategy can be summarized by a binary valued POVM $\left\{E_{0}, E_{1}\right\}$, where the hypothesis $H_{i}: \rho=\rho_{i}$ is chosen whenever the outcome is $i \in\{0,1\}$. For a fixed strategy $\left\{E_{0}, E_{1}\right\}$, the probability of choosing $H_{i}$, when the actual state is $\rho_{j}$, is given by $\mathbb{P}\left[H_{i} \mid \rho=\rho_{j}\right]=\operatorname{tr}\left(E_{i} \rho_{j}\right), i, j \in\{0,1\}$. Thus the expected probability of success for this strategy equals

$$
\bar{P}_{q}^{\left\{E_{0}, E_{1}\right\}}\left(\rho_{0}, \rho_{1}\right):=q \operatorname{tr}\left(E_{0} \rho_{0}\right)+(1-q) \operatorname{tr}\left(E_{1} \rho_{1}\right)
$$

The maximum achievable expected success probability in the binary decision problem is the quantity

$$
\bar{P}_{q}^{\max }\left(\rho_{0}, \rho_{1}\right):=\sup _{\left\{E_{0}, E_{1}\right\} \in \mathrm{POVM}} \bar{P}_{q}^{\left\{E_{0}, E_{1}\right\}}\left(\rho_{0}, \rho_{1}\right) .
$$

The following theorem is due to Helstrom [17]. We state it using the notation of Fuchs [18] who also gave a simple proof of it.

Theorem 10: Let $\rho_{0}, \rho_{1} \in \mathcal{S}\left(\mathcal{H}_{d}\right)$ be two states, let $q \in$ $[0,1]$, and let $\left\{\mu_{i}\right\}_{i=1}^{d}$ be the eigenvalues of the Hermitian operator $\Lambda:=q \rho_{0}-(1-q) \rho_{1}$. Then the maximum achievable expected success probability in the quantum binary decision problem is

$$
\bar{P}_{q}^{\max }\left(\rho_{0}, \rho_{1}\right)=\frac{1}{2}+\frac{1}{2} \sum_{i=1}^{d}\left|\mu_{i}\right|
$$

## B. Bounds on Quantum Knowledge

Let $X$ be a random variable and let $F$ be a randomly chosen predicate on $\mathcal{X}$. The goal of this section is to derive a bound on the distance of $F(X)$ from uniform given knowledge about $X$ stored in a quantum storage device.

Such knowledge is modeled by a family of quantum states $\left\{\rho_{x}\right\}_{x \in \mathcal{X}}$, where $\rho_{x}$ is the state of the quantum system conditioned on the event that $X=x$. An explicit expression for the corresponding quantity can be obtained using a result on the quantum binary decision problem (cf. Section $V-A$.

Lemma 11: Let $X$ be a random variable with range $\mathcal{X}$ and let $F$ be a random predicate on $\mathcal{X}$. Let $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset \mathcal{S}\left(\mathcal{H}_{d}\right)$ be a family of quantum states on a $d$-dimensional Hilbert space. Then

$$
d\left(F(X) \mid \rho_{X} ; F\right)=\frac{1}{2} \underset{f \leftarrow P_{F}}{\mathbb{E}}\left[\sum_{j=1}^{d}\left|\mu_{j}^{f}\right|\right]
$$

where $\left\{\mu_{j}^{f}\right\}_{j=1}^{d}$ are the eigenvalues of the Hermitian operator $\Lambda_{f}:=\sum_{x: f(x)=0} P_{X}(x) \rho_{x}-\sum_{x: f(x)=1} P_{X}(x) \rho_{x}, \quad$ for $f \in \mathcal{F}_{\text {bin }}^{\mathcal{X}}$.

Proof: It suffices to show that

$$
\begin{equation*}
d\left(f(X) \mid \rho_{X}\right)=\frac{1}{2} \sum_{j=1}^{d}\left|\mu_{j}^{f}\right| \tag{2}
\end{equation*}
$$

for every $f \in \mathcal{F}_{\text {bin }}^{\mathcal{X}}$. Let thus $f$ be fixed and assume for simplicity that $P_{f(X)}(0)>0$ and $P_{f(X)}(1)>0$ (otherwise, (2) is trivially satisfied).

Let $z \in\{0,1\}$. Conditioned on the event that $f(X)=z$, the state $\rho$ equals $\rho_{x}$ with probability $P_{X \mid f(X)}(x \mid z)$. This situation can equivalently be described by saying that the system is in the mixed state $\sigma_{z}^{f} \in \mathcal{S}\left(\mathcal{H}_{d}\right)$, where

$$
\sigma_{z}^{f}=\sum_{x: f(x)=z} P_{X \mid f(X)}(x \mid z) \rho_{x}
$$

The problem of guessing $f(X)$ thus corresponds exactly to the quantum binary decision problem described in Section IV-A i.e.,

$$
P_{\text {guess }}(f(X) \mid \rho)=\bar{P}_{P_{f(X)}(0)}^{\max }\left(\sigma_{0}^{f}, \sigma_{1}^{f}\right)=\frac{1}{2}+\frac{1}{2} \sum_{j=1}^{d}\left|\mu_{j}^{f}\right|
$$

where the second equality follows from Theorem 10 Finally, since $f(X)$ is binary, equation (2) follows from Lemma 7

The expression for the distance of $F(X)$ from uniform provided by Lemma 11 is generally difficult to evaluate. The following theorem gives a much simpler upper bound for this quantity. ${ }^{9}$

Theorem 12: Let $X$ be a random variable with range $\mathcal{X}$ and let $F$ be a random predicate on $\mathcal{X}$. Let further $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset$

[^4]$\mathcal{S}\left(\mathcal{H}_{d}\right)$ be a family of states on a $d$-dimensional Hilbert space. Then
$d\left(F(X) \mid \rho_{X} ; F\right) \leq \frac{1}{2} d^{\frac{1}{2}} \sqrt{\sum_{x, x^{\prime} \in \mathcal{X}} P_{X}(x) P_{X}\left(x^{\prime}\right) \lambda_{x, x^{\prime}} \operatorname{tr}\left(\rho_{x} \rho_{x^{\prime}}\right)}$
where $\lambda_{x, x^{\prime}}:=2 \mathbb{P}_{f \leftarrow P_{F}}\left[f(x)=f\left(x^{\prime}\right)\right]-1$, for $x, x^{\prime} \in \mathcal{X}$.
Proof: We set out from the equation
$$
d\left(F(X) \mid \rho_{X} ; F\right)=\frac{1}{2} \underset{f \leftarrow P_{F}}{\mathbb{E}}\left[\sum_{j=1}^{d}\left|\mu_{j}^{f}\right|\right]
$$
provided by Lemma 11 Note that, for any $f \in \mathcal{F}_{\text {bin }}^{\mathcal{X}}$,
$$
\sum_{j=1}^{d}\left|\mu_{j}^{f}\right| \leq d^{\frac{1}{2}} \sqrt{\sum_{j=1}^{d}\left|\mu_{j}^{f}\right|^{2}}=d^{\frac{1}{2}} \sqrt{\operatorname{tr}\left(\Lambda_{f}^{2}\right)}
$$
where the inequality is Jensen's inequality (applied to the convex mapping $x \mapsto x^{2}$ ) and where the equality is a consequence of Schur's (in)equality (cf. Lemma 20, which can be applied because $\Lambda_{f}$ is Hermitian and thus also normal. We conclude that
\[

$$
\begin{align*}
d\left(F(X) \mid \rho_{X} ; F\right) & \leq \frac{1}{2} d^{\frac{1}{2}} \underset{f \leftarrow P_{F}}{\mathbb{E}}\left[\sqrt{\operatorname{tr}\left(\Lambda_{f}^{2}\right)}\right] \\
& \leq \frac{1}{2} d^{\frac{1}{2}} \sqrt{\underset{f \leftarrow P_{F}}{\mathbb{E}}\left[\operatorname{tr}\left(\Lambda_{f}^{2}\right)\right]} \tag{3}
\end{align*}
$$
\]

where Jensen's inequality is applied once again.
By the definition of $\Lambda_{f}$ in Lemma 11 we have

$$
\begin{aligned}
\operatorname{tr}\left(\Lambda_{f}^{2}\right)= & \sum_{\substack{x, x^{\prime} \in \mathcal{X} \\
f(x)=f\left(x^{\prime}\right)}} P_{X}(x) P_{X}\left(x^{\prime}\right) \operatorname{tr}\left(\rho_{x} \rho_{x^{\prime}}\right) \\
& -\sum_{\substack{x, x^{\prime} \in \mathcal{X} \\
f(x) \neq f\left(x^{\prime}\right)}} P_{X}(x) P_{X}\left(x^{\prime}\right) \operatorname{tr}\left(\rho_{x} \rho_{x^{\prime}}\right) \\
= & \sum_{x, x^{\prime} \in \mathcal{X}}\left(2 \delta_{f(x), f\left(x^{\prime}\right)}-1\right) P_{X}(x) P_{X}\left(x^{\prime}\right) \operatorname{tr}\left(\rho_{x} \rho_{x^{\prime}}\right)
\end{aligned}
$$

where $\delta_{y, y^{\prime}}$ is the Kronecker delta ${ }^{10}$. The assertion then follows by taking the expectation of this expression over $F$ and combining the result with (3).

If $F$ is two-universal, the quantity on the right hand side of Theorem 12 can be bounded by an expression which is independent of the particular storage function.

Corollary 13: Let $X$ be a random variable with range $\mathcal{X}$ and let $F$ be a two-universal random predicate on $\mathcal{X}$. Then for every family $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset \mathcal{S}\left(\mathcal{H}_{d}\right)$ of states on a $d$-dimensional Hilbert space

$$
d\left(F(X) \mid \rho_{X} ; F\right) \leq \frac{1}{2} d^{\frac{1}{2}} \sqrt{\sum_{x \in \mathcal{X}} P_{X}^{2}(x)}
$$

Proof: Since $F$ is two-universal, the values $\lambda_{x, x^{\prime}}$ (as defined in Theorem 12) cannot be positive for any distinct $x, x^{\prime} \in$ $\mathcal{X}$. Since $\operatorname{tr}\left(\rho_{x} \rho_{x^{\prime}}\right) \geq 0$, we conclude that $\lambda_{x, x^{\prime}} \operatorname{tr}\left(\rho_{x} \rho_{x^{\prime}}\right) \leq 0$ for $x \neq x^{\prime}$. Moreover, $\lambda_{x, x}=1$ and $\operatorname{tr}\left(\rho_{x} \rho_{x}\right) \leq 1$, for any $x \in \mathcal{X}$. Combining these facts, the assertion follows directly from the upper bound given by Theorem 12

[^5]Note that the expression under the square root is simply the collision probability $P_{C}(X)$ of $X$. Hence, with the Rényi entropy $R(X)=-\log _{2} P_{C}(X)$, the above inequality can be rewritten as

$$
\begin{equation*}
d\left(F(X) \mid \rho_{X} ; F\right) \leq \frac{1}{2} 2^{-\frac{R(X)-s}{2}} \tag{4}
\end{equation*}
$$

where $s$ is the number of qubits in which $X$ is stored, i.e., $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset \mathcal{S}\left(\mathcal{H}_{2^{s}}\right)$.

## C. Comparing Classical and Quantum Storage Devices

Since orthogonal states of a quantum system can always be perfectly distinguished, a random variable $X$ can always be stored and perfectly retrieved in a quantum storage device of dimension $d$ as long as the size of the range of $X$ does not exceed $d$. Hence, a classical $s$-bit storage device $\mathbf{C}^{2}$ cannot be more powerful than a storage device $\mathbf{Q}^{2^{s}}$ consisting of $s$ qubits. Formally, this can be stated as follows. For any random variables $X$ and $S$ on $\mathcal{X}$ and $\{0,1\}^{s}$, respectively, there is a family of states $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset \mathcal{S}\left(\mathcal{H}_{2^{s}}\right)$ such that
$d(F(X) \mid S F) \leq d\left(F(X) \mid \rho_{X} ; F\right), \quad$ for any $F \in \mathcal{R}(\mathcal{X} \rightarrow \mathcal{Y})$.
The following lemma shows that, on the other hand, a quantum storage device can indeed be more useful than a corresponding classical storage device. However, we will see later that this is only true for special cases, e.g., if the difference between the number $n$ of bits to be stored and the capacity $s$ of the storage device is small.

Lemma 14: Let $X$ be uniformly distributed over $\{0,1\}^{2}$ and let $F$ be a uniform balanced predicate on $\{0,1\}^{2}$. Then for any random variable $S$ on $\{0,1\}$ defined by a channel $P_{S \mid X}$,

$$
d(F(X) \mid S F) \leq \frac{1}{4}
$$

Similarly, for every family $\left\{\rho_{x}\right\}_{x \in\{0,1\}^{2}} \subset \mathcal{S}\left(\mathcal{H}_{2}\right)$ of quantum states on a 2 -dimensional Hilbert space

$$
d\left(F(X) \mid \rho_{X} ; F\right) \leq \frac{1}{2 \sqrt{3}} \approx 0.289
$$

and there exists families $\left\{\rho_{x}\right\}_{x \in\{0,1\}^{2}} \subset \mathcal{S}\left(\mathcal{H}_{2}\right)$ saturating this bound.

Proof: By the convexity of the variational distance, it suffices to consider random variables $S$ which depend in a deterministic way on $X$, that is, $S \equiv \varphi_{c}(X)$ for some function $\varphi_{c}:\{0,1\}^{2} \rightarrow\{0,1\}$. It can easily be verified (by an explicit calculation) that

$$
d\left(F(X) \mid \varphi_{c}(X) F\right) \leq \frac{1}{4}
$$

for any function $\varphi_{c}$ from $\{0,1\}^{2}$ to $\{0,1\}$, and that equality holds for $\varphi_{c}:\left(x_{1}, x_{2}\right) \mapsto x_{1} \cdot x_{2}$ (i.e., $\varphi_{c}\left(x_{1}, x_{2}\right)=1$ if and only if $x_{1}=x_{2}=1$ ). This proves the first (classical) statement of the lemma.

For the second (quantum) statement, for the same reason as above, it suffices to consider pure states only. Let $\left\{\left|\psi_{x}\right\rangle\left\langle\psi_{x}\right|\right\}_{x \in\{0,1\}^{2}} \subset \mathcal{S}\left(\mathcal{H}_{2}\right)$ be an arbitrary family of pure quantum states. It follows from the linearity of the trace
and Lemma 21 applied to the Hermitian operator $A:=$ $\sum_{x \in \mathcal{X}}\left|\psi_{x}\right\rangle\left\langle\psi_{x}\right|$, that

$$
\sum_{x, x^{\prime} \in \mathcal{X}}\left|\left\langle\psi_{x} \mid \psi_{x^{\prime}}\right\rangle\right|^{2} \geq|\mathcal{X}|^{2} / d
$$

The bound $\left.d\left(F(X) \| \psi_{X}\right\rangle\left\langle\psi_{X}\right| ; F\right) \leq 1 /(2 \sqrt{3})$ can then be obtained from Theorem 12 with $\mathbb{P}_{f \leftarrow P_{F}}\left[f(x)=f\left(x^{\prime}\right)\right]=\frac{1}{3}$ for distinct $x, x^{\prime}$ (implying $\lambda_{x, x^{\prime}}=-\frac{1}{3}$ ).

It remains to be proven that $\left.d\left(F(X) \| \psi_{X}\right\rangle\left\langle\psi_{X}\right| ; F\right)=$ $1 /(2 \sqrt{3})$ for a family of states $\left\{\left|\psi_{x}\right\rangle\left\langle\psi_{x}\right|\right\}_{x \in\{0,1\}^{2}} \subset \mathcal{S}\left(\mathcal{H}_{2}\right)$. Such states can be defined by setting $\left|\psi_{00}\right\rangle,\left|\psi_{01}\right\rangle\left|\psi_{10}\right\rangle$ and $\left|\psi_{11}\right\rangle$ to the vertices of a tetrahedron in $\mathcal{P}\left(\mathcal{H}_{2}\right)$ (or, more precisely, in the Bloch sphere which corresponds to $\mathcal{P}\left(\mathcal{H}_{2}\right)$ ). The assertion then follows from a straightforward calculation.

Together with Lemma 7 Lemma 14 implies that the maximum probability of correctly guessing a randomly chosen balanced predicate $F$ about a random 2-bit string $X$ is larger if information about $X$ can be stored in one qubit ( $P_{q}=$ $0.789)$ than if this information is stored in one classical bit ( $P_{c}=0.75$ ). Note that this is in accordance with earlier results showing that one individual qubit can be stronger than one classical bit (see, e.g., [4]).

Surprisingly, this advantage of a quantum storage device becomes negligible if the difference $n-s$ between the length $n$ of the bitstring $X$ and the number $s$ of bits/qubits of the storage device becomes large. To see this, let us first state a lower bound for the distance of $F(X)$ from uniform given the knowledge stored in a classical storage device.

Lemma 15: Let $X$ be uniformly distributed on $\{0,1\}^{n}$ and let $F$ be a uniform random predicate on $\{0,1\}^{n}$. Then for any $s<n$ there exists a random variable $S$ on $\{0,1\}^{s}$ defined by a channel $P_{S \mid X}$ such that

$$
\begin{equation*}
\frac{1}{2} C\left(2^{n-s}\right) \leq d(F(X) \mid S F) \tag{6}
\end{equation*}
$$

where $C(m):=\binom{m}{m / 2} 2^{-m}=\sqrt{\frac{2}{\pi m}}\left(1+O\left(\frac{1}{m}\right)\right)$. In particular,

$$
\frac{1}{\sqrt{2 \pi}} 2^{-\frac{n-s}{2}}\left(1+O\left(2^{-(n-s)}\right)\right) \leq d(F(X) \mid S F)
$$

Proof: Let $\varphi$ be a function from $\{0,1\}^{n}$ to $\{0,1\}^{s}$ such that for any $w \in\{0,1\}^{s}$, the set $\varphi^{-1}(\{w\}):=\{x \in$ $\left.\{0,1\}^{n}: \varphi(x)=w\right\}$ has size $2^{n-s}$. We claim that $S \equiv \varphi(X)$ satisfies (6).

For any fixed $w \in\{0,1\}^{s}$ and $f \in \mathcal{F}_{\text {bin }}^{\{0,1\}^{n}}$,

$$
\begin{aligned}
d(f(X) \mid \varphi(X)=w) & =\left|\underset{f \leftarrow P_{F}}{\mathbb{P}}[f(X)=0 \mid \varphi(X)=w]-\frac{1}{2}\right| \\
& =\left|\frac{k_{f}}{2^{n-s}}-\frac{1}{2}\right|
\end{aligned}
$$

where $k_{f}:=\left|f^{-1}(\{0\}) \cap \varphi^{-1}(\{w\})\right|$. Since $F$ is uniformly distributed on the set $\mathcal{F}_{\text {bin }}^{\{0,1\}^{n}}$, we have $\mathbb{P}_{f \leftarrow P_{F}}\left[k_{f}=k\right]=$ $\binom{2^{n-s}}{k} 2^{-2^{n-s}}$ for $k \in\left\{0, \ldots, 2^{n-s}\right\}$, hence

$$
\begin{aligned}
d(f(X) \mid \varphi(X)=w) & =\sum_{k=0}^{2^{n-s}}\left|\frac{k}{2^{n-s}}-\frac{1}{2}\right|\binom{2^{n-s}}{k} 2^{-2^{n-s}} \\
& =\frac{1}{2} C\left(2^{n-s}\right)
\end{aligned}
$$

where the last equality follows from equation (14) of Lemma 22 As $w \in\{0,1\}^{s}$ was arbitrary, this concludes the proof. (The approximation for $C(m)$ can be obtained from Lemma 23)

Combining Lemma 15 with inequalities (4) and (5), we conclude that the distance from uniform has the same asymptotic behavior for the classical and the quantum case: The knowledge about the predicate $F(X)$ decreases exponentially in the difference $n-s$ between the length of the bitstring $X$ and the size $s$ of the storage device.

More precisely, since, for $n-s \geq 1$,

$$
\frac{1}{2} C\left(2^{n-s}\right) \geq \frac{1}{2} 2^{-\frac{n-(s-1)}{2}}
$$

it follows from Lemma 15 and (4) that there exists a random variable $S$ on $\{0,1\}^{s}$ defined by a channel $P_{S \mid X}$ such that $d(F(X) \mid S F) \geq d\left(F(X) \mid \rho_{X} ; F\right)$ for any family of states $\left\{\rho_{x}\right\}_{x \in\{0,1\}^{n}} \subset \mathcal{S}\left(\mathcal{H}_{2^{s-1}}\right)$. This means that storing information about $X$ in $s$ classical bits instead of $s-1$ quantum bits allows to predict $F(X)$ with a lower error probability.

## V. From the Binary to the Non-Binary Case

## A. Relations Between Bounds on Knowledge

We start with a lemma bounding the distance of a random variable $X$ from uniform by the distance of a binary hash value $F(X)$ from uniform where $F$ is a randomly chosen balanced predicate. This is related to the Vazirani XOR lemma (see e.g., [19]), which gives a similar bound for the case where $F$ is chosen randomly from the set of all linear functions. ${ }^{11}$

Lemma 16 (Hashing Lemma): Let $X$ be a random variable with range $\mathcal{X}$ and let $F$ be a uniform balanced random predicate on $\mathcal{X}$. Then

$$
d(X) \leq \frac{3}{2} \sqrt{|\mathcal{X}|} d(F(X) \mid F)
$$

Proof: For any probability distribution $Q$ over $\mathcal{X}$ and any $f \in \mathcal{F}_{\text {bal }}^{\mathcal{X}}$, let $d_{f}(Q):=d\left(f\left(X^{\prime}\right)\right)$ be the distance between the uniform distribution and the distribution of $f\left(X^{\prime}\right)$ where $X^{\prime}$ is a random variable distributed according to $Q$. We have to show that

$$
\begin{equation*}
d(Q) \leq \frac{3}{2} \sqrt{|\mathcal{X}|} \underset{f \leftarrow P_{F}}{\mathbb{E}}\left[d_{f}(Q)\right] \tag{7}
\end{equation*}
$$

for any distribution $Q$ over $\mathcal{X}$. Defining the coefficients $a_{x}(Q):=Q(x)-\frac{1}{|\mathcal{X}|}$, and the sets $\mathcal{X}_{Q}^{+}:=\left\{x \in \mathcal{X}: a_{x}(Q) \geq\right.$ $0\}$ and $\mathcal{X}_{Q}^{-}:=\mathcal{X}-\mathcal{X}_{Q}^{+}$, we obtain

$$
\begin{equation*}
d(Q)=\sum_{x \in \mathcal{X}_{Q}^{+}} a_{x}(Q)=-\sum_{x \in \mathcal{X}_{Q}^{-}} a_{x}(Q) \tag{8}
\end{equation*}
$$

and, for any $f \in \mathcal{F}_{\text {bal }}^{\mathcal{X}}$ and $\mathcal{X}_{f}^{0}:=\{x \in \mathcal{X}: f(x)=0\}$,

$$
\begin{equation*}
d_{f}(Q)=\left|\sum_{x \in \mathcal{X}_{f}^{0}} a_{x}(Q)\right| \tag{9}
\end{equation*}
$$

${ }^{11}$ The following version of Vazirani's XOR lemma is proved in [20]: $d(X) \leq \sqrt{|\mathcal{X}|} \sqrt{\mathbb{E}_{\ell \leftarrow P_{L}}\left[d(\ell(X))^{2}\right]}$, where $P_{L}$ is the uniform distribution on the set of all non-zero linear functions from $\mathcal{X}$ to $\{0,1\}$.
respectively. Note that, since $d$ is convex, $d_{f}$ is convex as well and thus so is its expected value $\mathbb{E}_{f \leftarrow P_{F}}\left[d_{f}(\cdot)\right]$ (i.e., the function defined by $\left.Q \mapsto \mathbb{E}_{f \leftarrow P_{F}}\left[d_{f}(Q)\right]\right)$.

Let us first show that inequality (7) holds for distributions $\bar{Q}$ over $\mathcal{X}$ where the probabilities only take two possible values, $|\bar{Q}(\mathcal{X})| \leq 2$, i.e., there exist $a^{+} \geq 0$ and $a^{-} \leq 0$ such that $a_{x}(\bar{Q})=a^{+}$for $x \in \mathcal{X}_{\bar{Q}}^{+}$and $a_{x}(\bar{Q})=a^{-}$for $x \in \mathcal{X}_{\bar{Q}}^{-}$. Then the value $d_{f}(\bar{Q})$ in (9) only depends on the number $k(f):=\left|\mathcal{X}_{f}^{0} \cap \mathcal{X}_{\bar{Q}}^{+}\right|$of values $x \in \mathcal{X}_{\bar{Q}}^{+}$for which $f(x)=0$.

To get some intuition, consider the case where $\left|\mathcal{X}_{\bar{Q}}^{+}\right|=$ $\frac{1}{2}|\mathcal{X}|$. Since $f$ is randomly chosen, the expected deviation of $k(f)$ from its average value $\frac{1}{4}|\mathcal{X}|$ is proportional to $\sqrt{|\mathcal{X}|}$. Furthermore, $d_{f}(\bar{Q})$ is proportional to this deviation and $a^{+}$, and $a^{+}$is proportional to $d(\bar{Q})$ and inverse proportional to $|\mathcal{X}|$. Neglecting the constants, this already shows that (7) holds in this particular case.

Proving the exact statement (7) requires a little bit more computation. For any predicate $f \in \mathcal{F}_{\text {bal }}^{\mathcal{X}}$, expression (9) reads

$$
\begin{aligned}
d_{f}(\bar{Q}) & =\left|\sum_{x \in \mathcal{X}_{f}^{0} \cap \mathcal{X}_{\bar{Q}}^{+}} a^{+}+\sum_{x \in \mathcal{X}_{f}^{0} \cap \mathcal{X}_{\bar{Q}}^{-}} a^{-}\right| \\
& =\left|k(f) a^{+}+\left(\frac{n}{2}-k(f)\right) a^{-}\right|
\end{aligned}
$$

where $n:=|\mathcal{X}|$. With $s:=\left|\mathcal{X}_{\bar{Q}}^{+}\right|$, expression (8) implies

$$
a^{+}=\frac{d(\bar{Q})}{s} \quad \text { and } \quad a^{-}=-\frac{d(\bar{Q})}{n-s}
$$

and hence

$$
\begin{aligned}
d_{f}(\bar{Q}) & =\left|d(\bar{Q})\left(k(f)\left(\frac{1}{s}+\frac{1}{n-s}\right)-\frac{n}{2} \frac{1}{n-s}\right)\right| \\
& =d(\bar{Q})\left|k(f)-\frac{s}{2}\right| \frac{n}{s(n-s)}
\end{aligned}
$$

Consequently, for $Q=\bar{Q}$, inequality (7) is equivalent to

$$
\frac{1}{\left|\mathcal{F}_{\text {bal }}^{\mathcal{X}}\right|} \frac{n}{s(n-s)} \sum_{f \in \mathcal{F}_{\text {bal }}^{\mathcal{X}}}\left|k(f)-\frac{s}{2}\right| \geq \frac{2}{3 \sqrt{n}}
$$

Since the term in the sum over $\mathcal{F}_{\text {bal }}^{\mathcal{X}}$ only depends on $k(f)$, the sum can be replaced by a sum over $k$, i.e., we have to show that

$$
\begin{array}{r}
\frac{1}{\left(\frac{n}{2}\right)} \frac{n}{s(n-s)} \sum_{k=\max \left(0, s-\frac{n}{2}\right)}^{\min \left(s, \frac{n}{2}\right)}\binom{s}{k}\binom{n-s}{\frac{n}{2}-k}\left|k-\frac{s}{2}\right| \\
=\frac{\left(\frac{n}{2}!\right)^{2} s!(n-s)!n}{n!s(n-s)} S_{n, s} \geq \frac{2}{3 \sqrt{n}} \tag{10}
\end{array}
$$

with

$$
S_{n, s}=\sum_{k=\max \left(0,-\frac{n}{2}+s\right)}^{\min \left(s, \frac{n}{2}\right)} \frac{\left|k-\frac{s}{2}\right|}{k!(s-k)!\left(\frac{n}{2}-s+k\right)!\left(\frac{n}{2}-k\right)!}
$$

The term $S_{n, s}$ has different analytic solutions depending on whether $s$ is even or odd. Let us first assume that $s$ is even. Replacing the summation index $k$ by $\bar{k}=k-\frac{s}{2}$ and making
use of the symmetry of the resulting terms with respect to the sign of $\bar{k}$, we get

$$
\begin{aligned}
S_{n, s} & =2 \sum_{\bar{k}=0}^{\min \left(\frac{s}{2}, \frac{n-s}{2}\right)} \frac{\bar{k}}{\left(\frac{s}{2}+\bar{k}\right)!\left(\frac{s}{2}-\bar{k}\right)!\left(\frac{n-s}{2}+\bar{k}\right)!\left(\frac{n-s}{2}-\bar{k}\right)!} \\
& =\frac{s(n-s)}{2 n\left(\frac{s}{2}!\right)^{2}\left(\frac{n-s}{2}!\right)^{2}}
\end{aligned}
$$

where the second equality follows from equation (15) of Lemma 22 with $a=\frac{s}{2}$ and $b=\frac{n-s}{2}$. A straightforward calculation then shows that for fixed $n$ the minimum of the left hand side of the inequality in (10) is taken for $s$ as close as possible to $\frac{n}{2}$, i.e., $s=2\left\lfloor\frac{n}{4}\right\rfloor$ and $n-s=2\left\lceil\frac{n}{4}\right\rceil$, that is

$$
\begin{aligned}
\frac{\left(\frac{n}{2}!\right)^{2} s!(n-s)!n}{n!s(n-s)} S_{n, s} & \geq \frac{\frac{n}{2}!^{2} s!(n-s)!}{2 n!\left(\frac{s}{2}!\right)^{2}\left(\frac{n-s}{2}!\right)^{2}} \\
& \geq \frac{\frac{n}{2}!^{2}\left(2\left\lfloor\frac{n}{4}\right\rfloor\right)!\left(2\left\lceil\frac{n}{4}\right\rceil\right)!}{2 n!\left(\left\lfloor\frac{n}{4}\right\rfloor!\right)^{2}\left(\left\lceil\frac{n}{4}\right\rceil!\right)^{2}}
\end{aligned}
$$

Lemma 23 is then used to derive a lower bound for the term on the right hand side of this inequality, leading to

$$
\begin{aligned}
& \frac{\left(\frac{n}{2}!\right)^{2} s!(n-s)!n}{n!s(n-s)} S_{n, s} \\
\geq & \sqrt{\frac{2}{\pi n}}^{\frac{2}{6 n+1}+\frac{1}{24\left\lfloor\frac{n}{4}\right\rfloor+1}+\frac{1}{24\left\lceil\frac{n}{4}\right]+1}-\frac{1}{12 n}-\frac{1}{6\left\lfloor\frac{n}{4}\right\rfloor}-\frac{1}{6\left\lceil\frac{n}{4}\right\rceil}} \geq \frac{2}{3 \sqrt{n}}
\end{aligned}
$$

where the last inequality holds for $n \geq 6$.
Similarly, for $s$ odd, applying equation (16) of Lemma 22 with $a=\frac{s-1}{2}$ and $b=\frac{n-s-1}{2}$ leads to

$$
\begin{array}{r}
S_{n, s}=2 \sum_{\bar{k}=0}^{\min (a, b)} \frac{\left|\bar{k}+\frac{1}{2}\right|}{(a+\bar{k}+1)!(a-\bar{k})!(b+\bar{k}+1)!(b-\bar{k})!} \\
=\frac{2}{n\left(\frac{s-1}{2}!\right)^{2}\left(\frac{n-s-1}{2}!\right)^{2}}
\end{array}
$$

resulting in the same lower bound $\frac{2}{3 \sqrt{n}}$ for the left hand side of the inequality in 10 for $n \geq 8$. Moreover, an explicit calculation shows that (10) also holds for $n=2, n=4$, and $n=6$ which concludes the proof of inequality (7) for $Q=\bar{Q}$ with $|\bar{Q}(\mathcal{X})| \leq 2$.

Let now $Q$ be an arbitrary distribution on $\mathcal{X}$ and let $\Gamma$ be the set of permutations on $\mathcal{X}$ with invariant sets $\mathcal{X}_{Q}^{+}$and $\mathcal{X}_{Q}^{-}$, i.e., $\gamma\left(\mathcal{X}_{Q}^{+}\right)=\mathcal{X}_{Q}^{+}$and $\gamma\left(\mathcal{X}_{Q}^{-}\right)=\mathcal{X}_{Q}^{-}$, for $\gamma \in \Gamma$. Since $d(Q)=d(Q \circ \gamma)$ for $\gamma \in \Gamma$, we find that

$$
\bar{Q}:=\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} Q \circ \gamma
$$

is a probability distribution satisfying $d(\bar{Q})=d(Q)$ and taking identical probabilities for all elements in $\mathcal{X}_{Q}^{+}$as well as for all elements in $\mathcal{X}_{Q}^{-}$, i.e., $|\bar{Q}(\mathcal{X})| \leq 2$. Since inequality $\mathbb{7}$ is already proven for distributions of this form, we conclude

$$
\begin{aligned}
d(Q)=d(\bar{Q}) & \leq \frac{3}{2} \sqrt{|\mathcal{X}|} \underset{f \leftarrow P_{F}}{\mathbb{E}}\left[d_{f}(\bar{Q})\right] \\
& \leq \frac{3}{2} \sqrt{|\mathcal{X}|} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \underset{f \leftarrow P_{F}}{\mathbb{E}}\left[d_{f}(Q \circ \gamma)\right]
\end{aligned}
$$

where the second inequality is a consequence of the convexity of $\mathbb{E}_{f \leftarrow P_{F}}\left[d_{f}(\cdot)\right]$. Assertion (7) then follows from $d_{f}(Q \circ$ $\gamma)=d_{f \circ \gamma^{-1}}(Q)$, for all $f \in \mathcal{F}_{\text {bal }}^{\mathcal{X}}, \gamma \in \Gamma$, and the fact that $F \circ \gamma^{-1}$ is a uniform balanced random predicate, i.e., $\mathbb{E}_{f \leftarrow P_{F}}\left[d_{f \circ \gamma^{-1}}(Q)\right]=\mathbb{E}_{f \leftarrow P_{F}}\left[d_{f}(Q)\right]$.

In order to apply the hashing lemma to generalize the results of the previous section to the non-binary case, we need a relation between binary random functions (i.e., random predicates) and non-binary random functions.

Lemma 17: Let $G$ be a two-universal random function from $\mathcal{X}$ to $\mathcal{Y}$ and let $F$ be a uniform balanced random predicate on $\mathcal{Y}$. Then the random predicate $H:=F \circ G$ is two-universal.

Proof: For any distinct $x, x^{\prime} \in \mathcal{X}$,

$$
\begin{aligned}
& \underset{h \leftarrow P_{H}}{\mathbb{P}_{P}}\left[h(x)=h\left(x^{\prime}\right)\right]=\underset{g \leftarrow P_{G}}{\mathbb{P}}\left[g(x)=g\left(x^{\prime}\right)\right] \\
& \quad+\left(1-\underset{g \leftarrow P_{G}}{\mathbb{P}}\left[g(x)=g\left(x^{\prime}\right)\right]\right) \underset{\substack{\prime \\
g \leftarrow P_{G \mid G(x) \neq G\left(x^{\prime}\right)}^{\rightleftarrows}}}{\mathbb{P}}\left[f(g(x))=f\left(g\left(x^{\prime}\right)\right)\right] .
\end{aligned}
$$

Note that $\mathbb{P}_{f \leftarrow P_{F}, g \leftarrow P_{G \mid G(x) \neq G\left(x^{\prime}\right)}}\left[f(g(x))=f\left(g\left(x^{\prime}\right)\right)\right]$ is the collision probability of the uniform balanced random predicate $F, \mathbb{P}_{f \leftarrow P_{F}}\left[f(y)=f\left(y^{\prime}\right)\right]$ (for distinct $y, y^{\prime} \in \mathcal{Y}$ ), which can easily be computed,

$$
\underset{f \leftarrow P_{F}}{\mathbb{P}}\left[f(y)=f\left(y^{\prime}\right)\right]=\frac{|\mathcal{Y}|-2}{2(|\mathcal{Y}|-1)} .
$$

Since $G$ is two-universal, i.e., $\mathbb{P}_{g \leftarrow P_{G}}\left[g(x)=g\left(x^{\prime}\right)\right] \leq \frac{1}{|\mathcal{Y}|}$, we have

$$
\begin{aligned}
& \underset{h \leftarrow P_{H}}{\mathbb{P}}\left[h(x)=h\left(x^{\prime}\right)\right] \\
& \quad \underset{g \leftarrow P_{G}}{\mathbb{P}}\left[g(x)=g\left(x^{\prime}\right)\right]\left(1-\underset{f \leftarrow P_{F}}{\mathbb{P}}\left[f(y)=f\left(y^{\prime}\right)\right]\right) \\
& \quad+\underset{f \leftarrow P_{F}}{\mathbb{P}}\left[f(y)=f\left(y^{\prime}\right)\right] \\
& \quad \leq \frac{1}{|\mathcal{Y}|}+\left(1-\frac{1}{|\mathcal{Y}|}\right)_{f}^{\mathbb{P}} \underset{f P_{F}}{\mathbb{P}}\left[f(y)=f\left(y^{\prime}\right)\right] \\
& \quad=\frac{1}{|\mathcal{Y}|}+\left(1-\frac{1}{|\mathcal{Y}|}\right) \frac{|\mathcal{Y}|-2}{2(|\mathcal{Y}|-1)}=\frac{1}{2}
\end{aligned}
$$

i.e., the random predicate $H$ is two-universal.

Combining Lemma 16 and Lemma 17 leads to a relation between the distance from uniform of the outcomes of binary and general (non-binary) two-universal functions on a random variable $X$, given some knowledge $\mathbf{W}_{S} .{ }^{12}$

Theorem 18: Let $X$ and $S$ be random variables on $\mathcal{X}$ and $\mathcal{S}$, respectively and let $\mathbf{W}$ be a selectable channel on $\mathcal{S}$. If, for all two-universal random predicates $H$ on $\mathcal{X}$,

$$
\begin{equation*}
d\left(H(X) \mid \mathbf{W}_{S} ; H\right) \leq \varepsilon \tag{11}
\end{equation*}
$$

then, for all two-universal random functions $G$ from $\mathcal{X}$ to $\mathcal{Y}$,

$$
\begin{equation*}
d\left(G(X) \mid \mathbf{W}_{S} ; G\right) \leq \frac{3}{2} \sqrt{|\mathcal{Y}|} \varepsilon \tag{12}
\end{equation*}
$$

Proof: From Definition (1), we have

$$
d\left(G(X) \mid \mathbf{W}_{S} ; G\right)=\underset{g \leftarrow P_{G}}{\mathbb{E}}\left[\max _{W \in \mathbf{W}} d\left(g(X) \mid W_{S}\right)\right]
$$

[^6]The expression in the maximum can then be bounded using Lemma 16, that is

$$
d\left(g(X) \mid W_{S}\right) \leq \frac{3}{2} \sqrt{|\mathcal{Y}|} d\left(F(g(X)) \mid W_{S} F\right)
$$

This leads to

$$
\begin{aligned}
d\left(G(X) \mid \mathbf{W}_{S} ; G\right) & \leq \frac{3}{2} \sqrt{|\mathcal{Y}|} \underset{\substack{g \leftarrow P_{G}}}{\mathbb{E}}\left[\max _{W \in \mathbf{W}} d\left(F(g(X)) \mid W_{S} F\right)\right] \\
& \leq \frac{3}{2} \sqrt{|\mathcal{Y}|} \underset{\substack{f \leftarrow P_{F} \\
g \leftarrow P_{G}}}{\mathbb{E}}\left[\max _{W \in \mathbf{W}} d\left(f(g(X)) \mid W_{S}\right)\right]
\end{aligned}
$$

Defining $H:=F \circ G$, we obtain

$$
\begin{aligned}
d\left(G(X) \mid \mathbf{W}_{S} ; G\right) & \leq \frac{3}{2} \sqrt{|\mathcal{Y}|} \underset{h \leftarrow P_{H}}{\mathbb{E}}\left[\max _{W \in \mathbf{W}} d\left(h(X) \mid W_{S}\right)\right] \\
& =\frac{3}{2} \sqrt{|\mathcal{Y}|} d\left(H(X) \mid \mathbf{W}_{S} ; H\right) .
\end{aligned}
$$

Finally, Lemma 17 states that $H$ is a two-universal random predicate on $\mathcal{X}$, hence the assertion of the theorem follows.

## B. Application: Privacy Amplification with a Quantum Adversary

Consider two parties, Alice and Bob, being connected by an authentic but otherwise completely insecure communication channel. Assume that they initially share a uniformly distributed $n$-bit key $X$ about which an adversary Eve has some partial information, where the only bound known on Eve's information is that it consists of no more than $s$ bits. Privacy amplification, introduced by Bennett, Brassard, and Robert [7], is a method to transform $X$ into an almost perfectly secure key $K$. It has been shown that if Alice and Bob publicly (by communication over the insecure channel) choose a twouniversal random function $G$ mapping the $n$-bit string to an $k$-bit string $K=G(X)$, for $k$ smaller than $n-s$, then the resulting string $K$ is secure (i.e., Eve has virtually no information about $K$ ). Note that $n-s$ is roughly Eve's entropy about the initial string $X$, i.e., privacy amplification with two-universal random functions is asymptotically optimal with respect to the number of extractable key bits. In our formalism, the possibility of privacy amplification by applying a (twouniversal) random function $G$, as proved in [7] (a simplified proof has been given in [8]), reads

$$
\begin{equation*}
d(G(X) \mid S G)=O\left(2^{-\frac{n-s-k}{2}}\right) \tag{13}
\end{equation*}
$$

for any random variable $S$ on $\{0,1\}^{s}$ defined by a channel $P_{S \mid X}$.

Combining the results from the previous section, we obtain a similar statement for the situation where Eve's knowledge about $X$ is stored in $s$ quantum instead of $s$ classical bits. More precisely, we can derive a bound on the distance of the final key $K \equiv G(X)$ from uniform, from an adversary's point of view, where $G$ is a two-universal random function applied
to an initial string $X$, assuming only that the adversary's knowledge about $X$ is stored in a limited number $s$ of qubits. ${ }^{13}$

Corollary 19: Let $X$ be a random variable with range $\mathcal{X}$ and Rényi entropy $R(X)=n$ and let $G$ be a two-universal random function from $\mathcal{X}$ to $\{0,1\}^{k}$. Then, for any family of states $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset \mathcal{S}\left(\mathcal{H}_{2^{s}}\right)$

$$
d\left(G(X) \mid \rho_{X} ; G\right) \leq \frac{3}{4} 2^{-\frac{n-s-k}{2}}
$$

Proof: Theorem 18 together with Corollary 13 implies

$$
d\left(G(X) \mid \rho_{X} ; G\right) \leq \frac{3}{4} \sqrt{2^{k} \cdot 2^{s} \sum_{x \in \mathcal{X}} P_{X}^{2}(x)}
$$

for any family of states $\left\{\rho_{x}\right\}_{x \in \mathcal{X}} \subset \mathcal{S}\left(\mathcal{H}_{2^{s}}\right)$. The corollary then follows from the definition of the Renyi entropy (cf. remark after the proof of Corollary 13).

We thus have a quantum analogue to (13), implying that privacy amplification remains equally secure (with the same parameters) if an adversary has quantum rather than only classical bits to store her information. Note that a similar bound follows from [13] together with a result of [5], for the case where $G$ is the inner product with a randomly chosen string.

This generalization of the security proof of privacy amplification immediately extends a result by Csiszár and Körner [21] (see also [22]) to the quantum case. Consider a situation where Alice and Bob share information described by $N$ independent realizations of random variables $X$ and $Y$, respectively, and where Eve has information described by realizations of a classical random variable $Z$. The result of [21] says that the number of secret key bits that can be generated by one-way communication (from Alice to Bob) over a public channel is at least (roughly) $N(I(X ; Y)-I(X ; Z)$ ), for large $N$. The protocol that Alice and Bob have to apply consists of an error correction step followed by a privacy amplification step using a two-universal random function. If we now consider a situation where Eve holds $s$ qubits of quantum information about $X$, it follows immediately from Corollary 19 that the same protocol can be used to generate a secret key of length roughly $N(I(X ; Y)-s)$.
In most QKD protocols, Alice encodes some classical information $X$ into the state of a quantum system and sends it to Bob. Upon receiving this state, Bob applies a measurement, resulting in classical information $Y$. After this step, the adversary might hold some quantum information about $X$ and $Y$. The situation is thus characterized by classical random variables $X$ and $Y$ together with the quantum system of Eve, where the size of her system depends on the error rate tolerated by the protocol (see [10]). Hence, the generalization of the Csiszár-Körner bound described above directly gives an

[^7]expression for the amount of key that can be generated by the protocol. In particular, it proves that the security holds against any type of attack (including coherent measurements on Eve's whole quantum system).

## VI. Conclusions and Open Problems

It is a fundamental question whether $s$ quantum bits are more powerful than $s$ classical bits in order to store information about an $n$-bit value $X$ (for $n>s$ ). We considered the problem of answering a randomly chosen question $F$ about $X$, given only the stored information about $X$. The uncertainty about the answer $F(X)$ is then a measure for the usefulness of the stored information. It can be quantified in terms of the distance of $F(X)$ from uniform conditioned on the stored information, which, for binary questions $F$, corresponds to the advantage over $1 / 2$ of the success probability when guessing $F(X)$. It turns out that when storing a bitstring $X$ of length $n=2$ bits, one quantum bit can indeed be more useful than one classical bit (cf. Lemma 14). However, for larger values of $n-s$, the difference between classical and quantum memory becomes inessential. ${ }^{14}$

We have shown that this has interesting implications for cryptography. In particular, privacy amplification by twouniversal hashing remains secure even against adversaries holding quantum information (cf. Corollary 19. This also leads to conceptually simpler and more general security proofs for quantum key distribution, where privacy amplification is used for the classical post-processing of the raw key (cf. [10], [13]).

It is well-known that so-called strong extractors [9] can be used to do privacy amplification in the classical case. While two-universal hashing can be seen as special case of this, the converse generally does not hold. It is an open problem whether strong extractors are sufficient to generate a key which is secure against a quantum adversary in general.

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## REFERENCES

[1] A. S. Holevo, "Statistical problems in quantum physics," in Proceedings of the Second Japan-USSR Symposium on Probability Theory, ser. Lecture Notes in Mathematics, vol. 330. Springer, 1973, pp. 104-119.
[2] S. Dziembowski and U. Maurer, "Optimal randomizer efficiency in the bounded-storage model," Journal of Cryptology, vol. 17, no. 1, pp. 5-26, 2004, conference version appeared in Proc. of STOC '02.
[3] S. Vadhan, "On constructing locally computable extractors and cryptosystems in the bounded storage model," in Advances in Cryptology CRYPTO 2003, 2003, pp. 61-77.
[4] A. Ambainis, A. Nayak, A. Ta-Shma, and U. Vazirani, "Dense quantum coding and a lower bound for 1-way quantum automata," in Proceedings of the 31th ACM Symposium on Theory of Computing, 1999, quant-ph/9804043

[^8][5] A. Nayak, "Optimal lower bounds for quantum automata and random access codes," in Proceedings of the 40th Annual Symposium on Foundations of Computer Science, 1999, pp. 369-377, quant-ph/9904093
[6] R. Jain, J. Radhakrishnan, and P. Sen, "Privacy and interaction in quantum communication complexity and a theorem about the relative entropy of quantum states," in The 43rd Annual IEEE Symposium on Foundations of Computer Science (FOCS '02), 2002.
[7] C. H. Bennett, G. Brassard, and J.-M. Robert, "Privacy amplification by public discussion," SIAM Journal on Computing, vol. 17, no. 2, pp. 210-229, 1988
[8] C. H. Bennett, G. Brassard, C. Crépeau, and U. Maurer, "Generalized privacy amplification," IEEE Transaction on Information Theory, vol. 41, no. 6, pp. 1915-1923, 1995.
[9] N. Nisan and D. Zuckerman, "Randomness is linear in space," Journal of Computer and System Sciences, vol. 52, pp. 43-52, 1996, a preliminary version appeared at STOC '93.
[10] M. Christandl, R. Renner, and A. Ekert, "A generic security proof for quantum key distribution," February 2004, available at http://arxiv.org/abs/quant-ph/0402131
[11] C. H. Bennett and G. Brassard, "Quantum cryptography: Public-key distribution and coin tossing," in Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, 1984, pp. 175-179.
[12] D. Mayers, "Unconditional security in quantum cryptography," Journal of the ACM, vol. 48, no. 3, pp. 351-406, 2001, quant-ph/9802025
[13] M. Ben-Or, "Security of BB84 QKD protocol," 2002, slides available at http://www.msri.org/publications/ln/msri/2002/quantumintro/ben-or/2/
[14] A. Ambainis, L. J. Schulman, A. Ta-Shma, U. Vazirani, and A. Wigderson, "The quantum communication complexity of sampling," in Proceedings of the 39th Annual Symposium on Foundations of Computer Science, 1998, pp. 342-351.
[15] J. L. Carter and M. N. Wegman, "Universal classes of hash functions," Journal of Computer and System Sciences, vol. 18, pp. 143-154, 1979.
[16] M. N. Wegman and J. L. Carter, "New hash functions and their use in authentication and set equality," Journal of Computer and System Sciences, vol. 22, pp. 265-279, 1981.
[17] C. W. Helstrom, Quantum Detection and Estimation Theory. Academic Press, New York, 1976.
[18] C. A. Fuchs, "Distinguishability and accessible information in quantum theory," Ph.D. dissertation, University of New Mexico, 1995, quant-ph/9601020
[19] O. Goldreich, "Three XOR-lemmas - an exposition," Electronic Colloquium on Computational Complexity, Tech. Rep. TR95-056, 1995, available at http://eccc.uni-trier.de/eccc//
[20] A. Elbaz, "Improved constructions for extracting quasi-random bits from sources of weak randomness," Master's thesis, Weizmann Institute of Science, 2003, available at http://www1.cs.columbia.edu/~ arielbaz/
[21] I. Csiszár and J. Körner, "Broadcast channels with confidential messages," IEEE Transactions on Information Theory, vol. 24, pp. 339-348, 1978.
[22] U. M. Maurer, "Secret key agreement by public discussion from common information," IEEE Transactions on Information Theory, vol. 39, no. 3, pp. 733-742, 1993.
[23] R. A. Horn and C. R. Johnson, Matrix analysis. Cambridge University Press, 1985.
[24] D. Zeilberger, "A fast algorithm for proving terminating hypergeometric series identities," Discrete Math, vol. 80, pp. 207-211, 1990.
[25] W. Feller, An Introduction to Probability Theory and Its Applications, 3rd ed., ser. Wiley Series in Probability and Mathematical Statistics. New York: Wiley, 1968, vol. 1.

## ApPENDIX

Lemma 20 (Schur's inequality): Let $A$ be a linear operator on a $d$-dimensional Hilbert space $\mathcal{H}_{d}$ and let $\left\{\mu_{i}\right\}_{i=1}^{d}$ be its eigenvalues. Then

$$
\sum_{i=1}^{d}\left|\mu_{i}\right|^{2} \leq \operatorname{tr}\left(A A^{\dagger}\right)
$$

with equality if and only if $A$ is normal (i.e., $A A^{\dagger}=A^{\dagger} A$ ).
Proof: See, e.g., [23].

Lemma 21: Let $A$ be a normal operator on a $d$-dimensional Hilbert space $\mathcal{H}_{d}$. Then

$$
|\operatorname{tr}(A)|^{2} \leq d \cdot \operatorname{tr}\left(A A^{\dagger}\right)
$$

Proof: Since $A$ is normal, we have

$$
\operatorname{tr}(A)=\sum_{i=1}^{d} \mu_{i} \quad \text { and } \quad \operatorname{tr}\left(A A^{\dagger}\right)=\sum_{i=1}^{d}\left|\mu_{i}\right|^{2}
$$

where $\left\{\mu_{i}\right\}_{i=1}^{d}$ are the eigenvalues of $A$. The assertion then follows from Jensen's inequality stating that

$$
\left|\sum_{i=1}^{d} \mu_{i}\right|^{2} \leq d \cdot \sum_{i=1}^{d}\left|\mu_{i}\right|^{2}
$$

Lemma 22: Let $a, b \in \mathbb{N}$. Then the following equalities hold:

$$
\begin{equation*}
\sum_{z=0}^{2 a}\binom{2 a}{z} \cdot\left|\frac{1}{2}-\frac{z}{2 a}\right|=\frac{1}{2}\binom{2 a}{a} \tag{14}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{z=0}^{\min (a, b)} \frac{z}{(a+z)!(a-z)!(b+z)!(b-z)!}=\frac{a b}{2(a+b)(a!)^{2}(b!)^{2}} \tag{15}
\end{equation*}
$$

$$
\begin{array}{r}
\sum_{z=0}^{\min (a, b)} \frac{z+\frac{1}{2}}{(a+z+1)!(a-z)!(b+z+1)!(b-z)!}  \tag{16}\\
=\frac{1}{2(a+b+1)(a!)^{2}(b!)^{2}}
\end{array}
$$

Proof: The first equality follows from a straightforward calculation, using the identity $\binom{a}{z} \cdot \frac{z}{a}=\binom{a-1}{z-1}$. The second and the third equality can be obtained with Zeilberger's algorithm [24] which is implemented in many standard computer algebra systems (e.g., Mathematica or Maple).

Lemma 23 (Stirling's approximation): For $n \in \mathbb{N}$,
$\sqrt{2 \pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12 n+1}}<n!<\sqrt{2 \pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12 n}}$.
Proof: A proof of this extension of Stirling's approximation can be found in [25].


[^0]:    The material in this paper was presented at the Seventh Workshop on Quantum Information Processing, Waterloo, Canada, January 2004.

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    ${ }^{1}$ This is a direct consequence of the Holevo bound [1] stating that the accessible information contained in a quantum state cannot be larger than its von Neumann entropy. This assertion is also a consequence of the general results proven in this paper (cf. Section IV-C.
    ${ }^{2}$ A typical example of such a setting is the bounded-storage model [2], [3].

[^1]:    ${ }^{3}$ See Section 历I-A for a definition of two-universality.
    ${ }^{4}$ Equivalently, they can use an extractor [9].
    ${ }^{5}$ More precisely, her information is exponentially small in $n-s-k$.
    ${ }^{6}$ Most known security proofs are restricted to one specific QKD protocol.

[^2]:    ${ }^{7}$ In the literature, two-universality is usually defined for families $\mathcal{G}$ of functions: A family $\mathcal{G}$ is called two-universal if the random function $G$ with uniform distribution over $\mathcal{G}$ is two-universal. For our purposes, however, our more general definition is more convenient.

[^3]:    ${ }^{8}$ Recall that $C_{W}$ denotes the output of the channel $C$ on input $W$.

[^4]:    ${ }^{9}$ The main idea in the proof of Theorem 12 is to replace occurrences of density operators by their squares. The resulting expressions correspond to classical collision probabilities, as used in the well-known classical analysis of privacy amplification. The application of Jensen's inequality corresponds to the transition from the variational to the Euclidean distance. In this sense, this proof can be seen as a generalization of the classical derivation.

[^5]:    ${ }^{10} \delta_{y, y^{\prime}}$ equals 1 if $y=y^{\prime}$ and 0 otherwise.

[^6]:    ${ }^{12}$ Using the version of Vazirani's XOR-Lemma stated in Footnote 11 the constant $\frac{3}{2}$ in the bound [12] of Theorem 18 can be eliminated by replacing condition 11] by the stronger requirement $\sqrt{\mathbb{E}_{h \leftarrow P_{H}}\left[d\left(h(X) \mid \mathbf{W}_{S}\right)^{2}\right]} \leq \varepsilon$.

[^7]:    ${ }^{13}$ Note that this is an example illustrating the fact that a bound on the expected distance of a single bit $H(X)$ from uniform $d\left(H(X) \mid \mathbf{W}_{S} ; H\right)$ suffices to derive bounds on the expected distance from uniform $d\left(G(X) \mid \mathbf{W}_{S} ; G\right)$ of a long key $G(X)$ obtained by privacy amplification. In the case of quantum knowledge, however, it is possible to prove even stronger statements for the single-bit case, resulting in a strengthened version of Corollary 13 which gives a bound on a quantity similar to $d\left(H(X) \mid \mathbf{W}_{S} ; H\right)$ Using this and Footnote 12 the constant $\frac{3}{4}$ in Corollary 19 can be replaced by $\frac{1}{2}$.

[^8]:    ${ }^{14}$ As shown in Section IV-C $s$ classical bits can be more useful than $s-1$ quantum bits.

