A Synthesis of a $1/f$ Process Via Sobolev Spaces and Fractional Integration

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***Abstract*—We provide an almost-sure convergent expansion of a process with power law of fractional order by means of some known theorems from harmonic analysis and rather simple probability theory results.**

***Index Terms*—Fractional integration, $1/f$ process, Sobolev spaces, stochastic processes.**

I. Introduction

The family of random processes with $1/f$ spectral behavior, first introduced by Kolmogorov in the context of turbulent flows, have numerous applications in engineering, general science, and wherever strong long-range (long memory) dependence phenomena appear.

A long memory process $Z(x)$ with spectral density $\Phi_z(\omega)$ satisfies the spectral condition (see [2] and [23]): there exists $\beta > 0$ and $c_f > 0$ such that

$$\lim_{\omega \to 0} \frac{\Phi_z(\omega)}{c_f |\omega|^{-\beta}} = 1. \tag{1}$$

As pointed out by some authors ([15], [5], [1], [21]) this suggests to look for a relation between these processes and certain fractional integration operators (see (9), (10)). For example, in [1] by means of the Riesz–Bessel fractional integration operators a nonconstructive proof is given of the existence of, not necessarily Gaussian, fractional generalized random fields, namely, *Riesz–Bessel* motions; these random fields display long-range dependence and have spectral densities of the form

$$\Phi_z(\omega) = \frac{c}{|\omega|^{2\alpha}(1 + |\omega|^2)^{\beta}}, \qquad \omega \in \mathbb{R}^d \tag{2}$$

where $0 < \alpha < \frac{d}{2}, 0 \leq \beta$.

Random fields with this power spectrum are very important in the study of partial differential equations with random initial data; in particular, the Burgers equation in the study of turbulence which is extensively discussed in, for example, [25], [6]. In [12], spectral properties of the scaling limit of solutions of a multidimensional Burgers equation under Gaussian initial conditions with long-range dependence are derived. In continuous mechanics, generalized Burgers-type equations defined as fractional powers of the negative Laplacian are considerered, as for example in [3]; random fields with power spectrum as in (2) are of interest when these equations have to be solved with random initial data [16]. The power spectrum of (2) is isotropic, since it is a function only of the radial spatial frequency $|\omega|, \omega \in \mathbb{R}^d$. The motivation of this generalization not only comes from the theory of stochastic differential equations, other applications include: models of natural (fractal) landscapes [27], texture discrimination [13], [20], and other applications in image processing. In this work, we discard the intermittency term $1/(1 + |\omega|^2)^{\beta}$ in (2). We are interested in analyzing the term which characterizes long-range dependence, so in the following we will just consider the case for (2) when $\beta = 0$

$$\Phi_z(\omega) = \frac{c}{|\omega|^{2\alpha}}, \qquad \omega \in \mathbb{R}^d, \ 0 < \alpha < \frac{d}{2}. \tag{3}$$

The main goal of this work is to show that given $\{\xi_n\}_{n \in \mathbb{N}}$, a sequence of independent random variables such that $\forall n$: $\boldsymbol{E}\xi_n = 0$ and $\mathrm{Var}(\xi_n) = 1, \{\phi_n\}_{n \in \mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R}^d)$ and $I_\alpha(\cdot) = (-\Delta)^{\frac{-\alpha}{2}}(\cdot)$ a fractional integration operator (a fractional negative power of a Laplacian), then it is possible to build an almost-sure convergent sequence of elements

$$w_\alpha^N(x) = \sum_{n=0}^{N} \xi_n (I_\alpha \phi_n)(x) \tag{4}$$

such that the limit is a d-dimensional $1/f$ process (random field) with a power spectrum as in (3). Additionally, this series resembles the ordinary Karhunen–Loéve orthonormal expansion. Similar one-dimensional expansions are studied in other contexts in [17] and [10] using wavelets. Some constructions as in [17] only use second-order properties [9] and then the value of the covariance function is not changed. We will extend this construction and obtain a $1/f$-type process (field) which is stationary at the second order.

Since a power spectrum which satisfies (3) is not valid in the theory of stationary processes because it is a nonintegrable function, but it can be considered as a generalized spectrum. Through this interpretation, we will use the theory of distributions which provides a suitable frame to work with this class of spectrum. So, the limit of (4) must be understood as a distribution and not as a point process. We need the following definitions.

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II. Some Definitions

In the following, if $x \in \mathbb{C}^d (d \geq 1)$ we will denote its usual norm by $|x|$ and $\mathrm{Supp}(f) = Cl\{x : f(x) \neq 0\}$.

The Schwartz class of functions $\mathcal{S}(\mathbb{R}^d)$ is defined as the linear space of smooth functions rapidly decreasing at infinity, together with its derivatives; this means that $\phi \in \mathcal{S}(\mathbb{R}^d)$ whenever $\phi \in C^\infty(\mathbb{R}^d)$ and

$$\sup_{(x_1,\ldots x_d)\in\mathbb{R}^d} \prod_{i=1}^{d} |x_i|^{\alpha_i} \left| \frac{\partial}{\partial x_1^{\beta_1}} \cdots \frac{\partial}{\partial x_d^{\beta_d}} \phi(x_1,\ldots x_d) \right| < \infty \quad \forall \alpha_j,\ \beta_j \in \mathbb{N}.$$

We will denote $\mathcal{D}(\mathbb{R}^d)$ the space of functions which are in $C^\infty(\mathbb{R}^d)$ and have compact suport. Both spaces are topological vector spaces [29], and their duals are denoted as: $\mathcal{S}'(\mathbb{R}^d)$ (*tempered distributions*) and $\mathcal{D}'(\mathbb{R}^d)$ (*distributions*), respectively. Clearly, $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ and then $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$.

A. Fourier Transforms

The *Fourier transform* $\hat{f}$ of $f \in \mathcal{S}(\mathbb{R}^d)$ is defined as

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \omega\cdot x}\, dx.$$

From this, $\mathcal{F}$ can be extended, as usual, as a linear map $\mathcal{F} : L^1(\mathbb{R}^d) \mapsto C(\mathbb{R}^d)$, or as an isometry on $L^2(\mathbb{R}^d)$ and by duality over the class of tempered distributions, that is, $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}^d)$.

Definition 1: The Sobolev spaces H^s ([22], [7]) are the following linear spaces defined as:

$$H^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 (1 + |\omega|^2)^s\, d\omega < \infty \right\}. \tag{5}$$

Remark: Let $s \in \mathbb{R}$, then $H^s(\mathbb{R}^d)$ is a Hilbert space with the product $(\cdot,\cdot)_{H^S} : H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \mapsto \mathbb{C}$

$$(h,g)_{H^s} = \int_{\mathbb{R}^d} \hat{h}(\omega)\bar{\hat{g}}(\omega)(1 + |\omega|^2)^s\, d\omega. \tag{6}$$

For $f, g \in \mathcal{D}(\mathbb{R}^d)$, we define the pairing

$$\langle\cdot,\cdot\rangle : \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \longrightarrow \mathbb{R}$$

as

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x) g(x)\, dx;$$

this can be extended by a density argument over $L^p \times L^q$, $\frac{1}{p} + \frac{1}{q} = 1$ (when $p = 2$ this is the usual inner product) or $H^s \times H^{-s}$.

B. Generalized Stochastic Processes

In the following, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space. A *generalized stochastic process* is a *random* functional in $\mathcal{D}'(\mathbb{R}^d)$ (or in $\mathcal{S}'(\mathbb{R}^d)$), [9]. This means that if $\varphi \in \mathcal{S}(\mathbb{R}^d)$ then a generalized stochastic process $Z(x)$ is defined by the random variable $Z(\varphi) : \Omega \mapsto \mathbb{R}$ [26]

$$Z(\varphi) = \langle Z, \varphi\rangle = \int_{\mathbb{R}^d} Z(x)\varphi(x)\, dx.$$

So in the following, for a fixed $\varpi \in \Omega$, the formula defined by (4) will be understood as a functional defined on $\mathcal{D}(\mathbb{R}^d)$. Therefore, if w_α is the limit process, we want to prove

$$\mathbb{P}\left(\omega : \forall\phi \in \mathcal{D}(\mathbb{R}^d) : \exists \lim_{N\longrightarrow\infty} w_\alpha^N(\phi) = w_\alpha(\phi)\right) = 1.$$

The *covariance functional* is defined by the bilinear form $\Gamma : \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \mapsto \mathbb{R}$

$$\Gamma(u, v) = \boldsymbol{E}[Z(u)Z(v)] = \int\int u(s)v(x)R(x - s)\, dx\, ds$$

where $R(x)$ may be a generalized function. Sometimes, we write unformally $\boldsymbol{E}[Z(x)Z(s)] = R(x - s)$. For example, if $Z(x)$ is white noise, $R(x) = \delta(x)$ in the sense of $\langle\delta, u\rangle = u(0)$, then

$$\Gamma(u, v) = \int_{\mathbb{R}^d} u(x)v(x)\, dx$$

for all u and v in $\mathcal{D}(\mathbb{R}^d)$. If $R \in \mathcal{S}'(\mathbb{R}^d)$, it is also possible to define the *spectral density* of the process as $\Phi_z = \mathcal{F}R = \hat{R}$.

III. Preliminary Results

A. Variants of Two Theorems of Kolmogorov

Several classical results for sums of random variables can be extended to the context of Hilbert (or Banach) spaces ([30] and [18] contain many examples). The following mimic two celebrated theorems by Kolmogorov (the original theorems can be found in [4]) on the convergence of sums of independent random variables. The proofs of these theorems are included in the Appendix.

Theorem 3.1: Let $\{\xi_k\}$ be a sequence of independent random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $\boldsymbol{E}\xi_k = 0$ and $\{f_k\}$ is a sequence in a Hilbert space H. If $X_k = \xi_k f_k$ and $S_n = \sum_{k=1}^n X_k$, then

$$\mathbb{P}\left(\bigvee_{k=1\ldots n} \|S_k\|_H^2 > \varepsilon^2\right) \leq \frac{\boldsymbol{E}\|S_n\|_H^2}{\varepsilon^2}. \tag{7}$$

This last result enables us to prove the following.

Theorem 3.2: Let $\{\xi_k\}$ be a sequence of independent random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $\boldsymbol{E}\xi_k = 0$ and $\{f_k\}$ is a sequence in a Hilbert space H. If

$$\sum_{n=1}^{\infty} \boldsymbol{E}|\xi_n|^2 \|f_n\|_H^2 < \infty \tag{8}$$

then S_n converges in H almost surely (a.s.), where $X_k = \xi_k f_k$ and $S_n = \sum_{k=1}^{n} X_k$.

B. Remark

The S_n and its limit are well defined *random elements* in the following sense (this can be found for example in [30, Ch. II, Definition 2.1.1.]): let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let τ be a topological space, then we will say that X is a *random element* in τ provided that $\{\omega : X(\omega) \in B\} \in \mathcal{F}$ for each $B \in B(\tau)$, where $B(\tau)$ is the Borel σ-algebra containing the open sets of τ.

C. Some Results From Harmonic Analysis

Let us consider the usual Laplacian of f [22]

$$\Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}.$$

Then, at least formally: $\widehat{\Delta f}(\omega) = -(2\pi)^2|\omega|^2 \hat{f}(\omega)$. From this we can define the operators $(-\Delta)^{-\frac{\alpha}{2}}$ as

$$\widehat{(-\Delta)^{-\frac{\alpha}{2}}} f(\omega) = (2\pi)^{-\alpha}|\omega|^{-\alpha}\hat{f}(\omega). \tag{9}$$

The formal manipulations have a precise meaning [28] as follows.

Definition 2: Let $0 < \alpha < d$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we can define its *Riesz potential*

$$I_\alpha f(x) = \left((-\Delta)^{-\frac{\alpha}{2}} f\right)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy \tag{10}$$

where $\gamma(\alpha) = \frac{\pi^{\frac{d}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2} - \frac{\alpha}{2})}$.

This linear operator has the following properties [28].

Proposition 3.1: Let $0 < \alpha < d$, then

a) the Fourier transform of $|x|^{-d+\alpha}$ is $\gamma(\alpha)(2\pi)^{-\alpha}|\omega|^{-\alpha}$ in the sense that

$$\int_{\mathbb{R}^d} |x|^{-d+\alpha}\varphi(x)\, dx = \int_{\mathbb{R}^d} \gamma(\alpha)(2\pi)^{-\alpha}|\omega|^{-\alpha}\hat{\varphi}(\omega)\, d\omega$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$;

b) the Fourier transform of $I_\alpha f(x)$ is $(2\pi)^{-\alpha}|\omega|^{-\alpha}\hat{f}(\omega)$ in the sense that

$$\int_{\mathbb{R}^d} I_\alpha f(x) g(x) dx = \int_{\mathbb{R}^d} \hat{f}(\omega)(2\pi)^{-\alpha}|\omega|^{-\alpha}\hat{g}(\omega)\, d\omega$$

for all $f, g \in \mathcal{S}(\mathbb{R}^d)$.

It is easy to check the following.

Proposition 3.2: $\forall f \in \mathcal{S}(\mathbb{R}^d)$: If $\alpha + \beta < d$ then $I_\alpha(I_\beta f) = I_{\alpha+\beta}(f)$; and $\Delta(I_\alpha f) = -I_{\alpha-2}(f)$ with $\Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}$.

We recall the following bound for these operators acting in $L^p(\mathbb{R}^d)$, [8], [28].

Theorem 3.3: (Hardy, Littlewood, and Sobolev) Let $0 < \alpha < d, 1 \le p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, then

a) $\forall f \in L^p(\mathbb{R}^d)$, the integral that defines $I_\alpha f$ converges almost everywhere (a.e.);

b) if $p > 1$ then

$$\|I_\alpha f\|_{L^q} \le C_{pq}\|f\|_{L^p} \tag{11}$$

where C_{pq} is a constant depending on p and q.

We will need the following straightforward result which is a consequence of the previous theorem and Proposition 3.2.

Proposition 3.3: Let $f \in L^p(\mathbb{R}^d)$, then we have the following.

a) If $g \in L^r(\mathbb{R}^d), p \ge 1, r \ge 1$ and $0 < \alpha < d$ are such that $\frac{1}{r} + \frac{1}{p} - \frac{\alpha}{d} = 1$ then $\langle I_\alpha f, g\rangle = \langle f, I_\alpha g\rangle$.

b) If $g \in L^r(\mathbb{R}^d), p \ge 1, r \ge 1$ and $0 < \alpha + \beta < d$ are such that $\frac{1}{r} + \frac{1}{p} - \frac{(\alpha+\beta)}{d} = 1$ then $\langle I_{\alpha+\beta} f, g\rangle = \langle I_\alpha f, I_\beta g\rangle$.

Proof: Part a): From Hölder's inequality and Theorem 3.1 we have

$$\langle I_\alpha f, g\rangle = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^d} g(x) \int_{\mathbb{R}^d} f(y)|x-y|^{-d+\alpha}\, dy\, dx \tag{12}$$

$$\le C_{p\,r}\|g\|_{L^r}\|f\|_{L^p}. \tag{13}$$

Then by Fubini's theorem, (14) equals

$$\frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} g(x)|x-y|^{-d+\alpha}\, dx\, dy = \langle f, I_\alpha g\rangle. \tag{14}$$

Part b): By means of a density argument and Proposition 3.2 we have that $I_\alpha(I_\beta f) = I_{\alpha+\beta}(f)$ for $f \in L^p\mathbb{R}^d$. Now the result will follow from part a), write

$$\langle I_{\alpha+\beta} f, g\rangle = \langle I_\alpha(I_\beta f), g\rangle$$

and we get the desired result. □

Remark: These operators are the inverses of the (positive) fractional powers of the Laplacian operator. On the class $\mathcal{S}(\mathbb{R}^d), (-\Delta)^{\frac{\alpha}{2}}$ is given by

$$-(-\Delta)^{\frac{\alpha}{2}} f(x)$$
$$= c \int_{\mathbb{R}^d} f(y) - f(x) - \frac{\nabla f(x)\cdot(y-x)}{1+|y-x|^2} \frac{dy}{|y-x|^{d+\alpha}}.$$

This expression follows from [28, Sec. 6.10] and from this formula we can give a short proof of the existence of the fractional Brownian field with exponent $\alpha/2$ [5].

We will need the following result.

Theorem 3.4: (variant of Shannon's theorem) If $f \in L^2(\mathbb{R}^d)$ is such that $\mathrm{Supp}(f) \subset [-\lambda_o, \lambda_o]^d$ with $\lambda_o < 1/2$, there exists $\theta \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\hat{f}(\omega) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)\theta(\omega - k). \tag{15}$$

Proof: Let $\tilde{f}(x) = \sum_{k\in\mathbb{Z}^d} f(x+k)$ be the periodization of f. As usual, $\tilde{f}$ can be identified with a function defined on the torus, which verifies $\tilde{f} \in L^2(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$.

If $a_k = \int_T f(x)e^{-i2\pi k.x}\, dx$ then

$$\lim_{\lambda\to\infty} \sum_{k\in D_\lambda} a_k e^{-2\pi i x.k} \overset{L^2(\mathbb{T}^d)}{=} \tilde{f}$$

and in $L^1(\mathbb{T}^d)$ for a suitable domain $D_\lambda \in \mathbb{R}^d$. Now, we can take $\theta(x) \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\hat{\theta}(\omega) = \begin{cases} 1, & |\omega_i| < \lambda_o \\ 0, & |\omega_i| \geq 1 - \lambda_o \end{cases}$$

and define $S_\lambda(x) = \hat{\theta}(x) \sum_{k \in D_\lambda} a_k e^{-2\pi i x.k}$.

$\hat{\theta}$ is nothing else but a low-pass filter; to fix the idea assume that $d = 1$, then as f vanishes outside $[-\lambda_0, \lambda_0]$ the behavior of $\hat{\theta}$ in $[\lambda_0, 1 - \lambda_0]$ is not relevant. On the other hand, $f = \tilde{f}\hat{\theta}$. Then, it is easy to show that

$$\lim_{\lambda \to \infty} \|S_\lambda - f\|_{L^1(\mathbb{R}^d)} = 0.$$

This implies

$$\lim_{\lambda \to \infty} \mathrm{Sup}_{\omega \in \mathbb{R}^d} |\hat{S_\lambda}(\omega) - \hat{f}(\omega)| = 0$$

but (see [29]) $a_k = \hat{f}(k)$, then

$$\widehat{S_\lambda}(\omega) = \sum_{k \in D_\lambda} \hat{f}(k)\theta(\omega - k).$$

Then (15) follows immediately from this. □

Then it is possible to prove the following proposition.

Proposition 3.4: Consider $f \in L^2(\mathbb{R}^d)$ under the same hypotheses of the previous theorem then

$$\|f\|_{H^s} \leq K(s) \left(\sum_{k \in \mathbb{Z}^d} \left|\hat{f}(k)\right|^2 (1 + |k|^2)^s \right)^{1/2}.$$

Remark: This result which is a straightforward generalization of a result in [17], is a consequence of the last sampling theorem, which identifies band-limited functions with periodic functions and is related to the fact that the right-hand side of the last inequality defines a norm in the Sobolev spaces of periodic functions [11].

Proof: Recall Peetre's inequality [24]

$$(1 + (a + b)^2)^s \leq 2^{|s|}(1 + a^2)^{|s|}(1 + b^2)^s, \qquad a, b, s \in \mathbb{R}$$

and by Theorem 3.4 we can find $\theta \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\begin{aligned} &\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 (1 + |\omega|^2)^s \, d\omega \\ &\quad \leq \int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}^d} |\hat{f}(k)\theta(\omega - k)|(1 + |\omega|^2)^{s/2} \right)^2 d\omega \\ &\quad \leq \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} u_k^2(\omega) \sum_{k \in \mathbb{Z}^d} v_k^2(\omega) \, d\omega \end{aligned}$$

where $v_k(\omega) = |\theta(\omega - k)|^{1/2}$ and

$$\begin{aligned} u_k(\omega) = |\hat{f}(k)|(1 + |k|^2)^{s/2} 2^{|s|/2} \\ \cdot (1 + \|\omega| - |k\|^2)^{|s|/2} |\theta(\omega - k)|^{1/2}. \end{aligned}$$

Since $\theta(x) \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\sum_{k \in \mathbb{Z}^d} v_k^2(\omega) = \sum_{k \in \mathbb{Z}^d} |\theta(\omega - k)| \leq C < \infty.$$

We remark that C is a constant which is independent of ω: As $\theta \in S(\mathbb{R}^d)$ then $\sup_{x \in \mathbb{R}^d} |x|^{2n} |\theta(x)| < \infty$ then, there exist $R > 0$ such that $|\theta(x)| \leq \frac{1}{|x|^{2n}}$ for all $|x| \geq R$. From these facts is easy to find a radial decreasing $\phi(|x|) \in L^1\mathbb{R}^d$ such that $\theta(x) \leq \phi(|x|)$ then

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} v_k^2(\omega) &= \sum_{k \in \mathbb{Z}^d} |\theta(\omega - k)| \\ &\leq \sum_{k \in \mathbb{Z}^d} |\phi(\omega - k)| \leq k \int_{\mathbb{R}^d} |\phi| = \text{const.} < \infty. \end{aligned}$$

And then

$$\begin{aligned} &\int_{\mathbb{R}^d} (1 + \|\omega| - |k\|^2)^{|s|} |\theta(\omega - k)| \, d\omega \\ &\qquad \leq K(s) 2^{-|s|} = \int_{\mathbb{R}^d} (1 + |\omega - k|^2)^{|s|} |\theta(\omega - k)| \, d\omega < \infty. \end{aligned}$$

Finally

$$\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 (1 + |\omega|^2)^s \, d\omega \leq \mathrm{CK}(s) \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 (1 + |k|^2)^s.$$

□

IV. On the Generation of a Long Memory Process in $\mathbb{R}^d$

In the following, we construct a series which converges a.s. in the sense of distributions to a $1/f$ process.

A. Existence of the Process

First, we prove the following existence result.

Proposition 4.1: Let $\{\xi_n\} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a sequence of independent random variables such that $\boldsymbol{E}\xi_n = 0$ and $\boldsymbol{E}|\xi_n|^2 = 1$. If $\{\phi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $0 < \alpha < \frac{d}{2}$, then

$$w_\alpha^N(x) = \sum_{n=0}^{N} \xi_n (I_\alpha \phi_n)(x) \tag{16}$$

converges to a generalized process a.s.

Proof: Let $\{Q_p\}_p$ be a denumerable family of disjoint cubes such that by some translation τ_p equals $(-\frac{1}{2}, \frac{1}{2}]^d$. Then

$$\begin{aligned} &\left\|(I_\alpha \phi_n) \mathbf{1}_{Q_p}\right\|_{H^s} \\ &\qquad \leq K(s) \left(\sum_{k \in \mathbb{Z}^d} \left| (I_\alpha \widehat{\phi_n)\mathbf{1}}_{Q_p}(k) \right|^2 (1 + |k|^2)^s \right)^{1/2} \end{aligned}$$

with

$$|(I_\alpha \widehat{\phi_n)\mathbf{1}}_{Q_p}(k)| = |\langle (I_\alpha \phi_n) \mathbf{1}_{Q_p}, e_k \rangle|$$

and $e_k = e^{i2\pi k.x}\mathbf{1}_{\tau_p^{-1}[-\frac{1}{2},\frac{1}{2}]^d}$. By Proposition 3.4

$$\sum_n \left\|(I_\alpha\phi_n)\mathbf{1}_{Q_p}\right\|_{H^s}^2 \le \sum_n K(s)\sum_{k\in\mathbb{Z}^d}\left|(\widehat{I_\alpha\phi_n)}\mathbf{1}_{Q_p}(k)\right|^2(1+|k|^2)^s.$$

As $e_k \in L^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ and $\mathrm{Supp}((I_\alpha\phi_n)\mathbf{1}_{Q_p}) = \mathrm{Supp}(e_k)$ by Proposition 3.3, the last term equals

$$\sum_{k\in\mathbb{Z}^d} K(s)(1+|k|^2)^s\sum_n\left|\left\langle(I_\alpha\phi_n)\mathbf{1}_{Q_p}, e_k\right\rangle\right|^2 \tag{17}$$

$$= \sum_{k\in\mathbb{Z}^d} K(s)(1+|k|^2)^s\sum_n\left|\langle\phi_n, I_\alpha e_k\rangle\right|^2. \tag{18}$$

Let $s=-d$ if $p=\frac{2d}{d+2\alpha}>1$ by Theorem 3.3 then

$$\sum_{k\in\mathbb{Z}^d} K(s)(1+|k|^2)^s\sum_n\left|\langle\phi_n, I_\alpha e_k\rangle\right|^2 \le \sum_{k\in\mathbb{Z}^d}K(s)(1+|k|^2)^{-d}\|I_\alpha e_k\|_{L^2}^2 \le \sum_{k\in\mathbb{Z}^d}K(s)(1+|k|^2)^{-d}K''\|e_k\|_{L^p}^2<\infty.$$

A similar bound is obtained from the fact that

$$I_\alpha^*: L^{p'}(\mathbb{R}^d)\longrightarrow L^2(\mathbb{R}^d)$$

(where I_α^* is the adjoint of I_α and $1=\frac{1}{p}+\frac{1}{p'}$) is a bounded linear operator.

Since $\{\xi_n\}$ are independent random variables with $\mathbf{E}|\xi_n|^2=1$ then

$$\sum_n \boldsymbol{E}|\xi_n|^2\left\|(I_\alpha\phi_n)\mathbf{1}_{Q_p}\right\|_{H^{-d}}^2 = \sum_n\left\|(I_\alpha\phi_n)\mathbf{1}_{Q_p}\right\|_{H^{-d}}^2<\infty$$

By Theorem 3.2, we have $\|\sum_n\xi_n(I_\alpha\phi_n)\mathbf{1}_{Q_p}\|_{H^{-d}}<\infty$ a.s. But convergence in $H^{-d}\cong(H^d)^*$ implies convergence in $\mathcal{S}'(\mathbb{R}^d)\subset\mathcal{D}'(\mathbb{R}^d)$. Taking $\varphi\in\mathcal{S}(\mathbb{R}^d)$, $\varepsilon>0$ and calling $w_{\alpha\,p}$ the limit of $\sum_n\xi_n(I_\alpha\phi_n)\mathbf{1}_{Q_p}$ we have

$$\left|\int_{\mathbb{R}^d}\left(\widehat{w_{\alpha p}}(\omega)-\sum_{n=0}^N\xi_n(\widehat{I_\alpha\phi_n)}\mathbf{1}_{Q_p}(\omega)\varphi(\omega)\right)d\omega\right| \le \int_{\mathbb{R}^d}\left|\widehat{w_{\alpha p}}(\omega)-\sum_{n=0}^N\xi_n\widehat{I_\alpha\phi_n}\mathbf{1}_{Q_p}(\omega)\right| \cdot(1+|\omega|^2)^{-d/2}(1+|\omega|^2)^{d/2}\varphi(\omega)\,d\omega$$
$$\le\left\|w_{\alpha p}-\sum_{n=0}^N\xi_nI_\alpha\phi_n\mathbf{1}_{Q_p}\right\|_{H^{-d}}\|\varphi\|_\infty \cdot\left(\int_{\mathbb{R}^d}(1+|\omega|^2)^d|\varphi(\omega)|\,d\omega\right)^{1/2}<\varepsilon \tag{19}$$

for all $N\ge N(\varepsilon)$.

As $\mathrm{Supp}(w_{\alpha p})\cap\mathrm{Supp}(w_{\alpha p'})=\emptyset$ then $w_\alpha=\sum_p w_{\alpha\,p}$ defines an element in $\mathcal{D}'(\mathbb{R}^d)$. □

B. Remark

In the previous result, the condition that $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$ is sufficient. The completeness of the system can be avoided, but in the following it is necessary to obtain the desired result.

C. Covariance of the Limit Process

We will prove that the process we have constructed (16) has the same power spectrum as that described in (3).

Theorem 4.1: Let $\{\xi_n\}\subset L^2(\Omega,\mathcal{F},\mathbb{P})$ be a sequence of independent random variables such that $\boldsymbol{E}\xi_n=0$ and $Var(\xi_n)=1$. If $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ such that

$$w_\alpha^N(x)=\sum_{n=0}^N\xi_n(I_\alpha\phi_n)(x)$$

converges to a generalized process a.s., for $0<\alpha<\frac{d}{2}$, then

a) the covariance of $w_\alpha(x)$ is $R_{w_\alpha}(x)=\frac{1}{\gamma(2\alpha)}|x|^{-d+2\alpha}$;
b) the spectral density is $\Phi_{w_\alpha}(\omega)=(2\pi)^{-\alpha}|\omega|^{-2\alpha}$.

Proof:

(Part a) Given N, let us define the bilinear form $\Gamma_N:\mathcal{D}(\mathbb{R}^d)\times\mathcal{D}(\mathbb{R}^d)\mapsto\mathbb{R}$ as follows: let

$$R^N(x,s)=\boldsymbol{E}\left[w_\alpha^N(x)w_\alpha^N(s)\right]$$

and given $\varphi,\psi\in\mathcal{D}(\mathbb{R}^d)$ define

$$\Gamma_N(\varphi,\psi)=\langle\langle R^N(x,s),\varphi(s)\rangle,\psi\rangle.$$

Define the bilinear form $\Gamma:\mathcal{D}(\mathbb{R}^d)\times\mathcal{D}(\mathbb{R}^d)\mapsto\mathbb{R}$ as

$$\Gamma(\varphi,\psi)=\langle I_{2\alpha}\varphi,\psi\rangle, =\frac{1}{\gamma(2\alpha)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\varphi(s)}{|x-s|^{d-2\alpha}}\psi(x)\,ds\,dx.$$

From these facts we have

$$\langle R^N(x,.),\varphi\rangle =\int_{\mathbb{R}^d}\boldsymbol{E}\left[w_\alpha^N(x)w_\alpha^N(s)\right]\varphi(s)\,ds =\int_{\mathbb{R}^d}\boldsymbol{E}\left[\sum_{n=0}^N\xi_n(I_\alpha\phi_n)(x)\sum_{m=0}^N\xi_m(I_\alpha\phi_m)(s)\right]\varphi(s)\,ds$$
$$=\int_{\mathbb{R}^d}\left(\sum_{m=0}^N\sum_{n=0}^N\boldsymbol{E}[\xi_n\xi_m](I_\alpha\phi_n)(x)(I_\alpha\phi_m)(s)\right)\varphi(s)\,ds.$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables with $\mathrm{Var}(\xi_n)=1$ and $\boldsymbol{E}[\xi_n]=0$, then $\boldsymbol{E}[\xi_n\xi_m]=\delta_{nm}$. Then

$$\int_{\mathbb{R}^d}\left(\sum_{n=0}^N(I_\alpha\phi_n)(x)(I_\alpha\phi_m)(s)\right)\varphi(s)\,ds =\sum_{n=0}^N\int_{\mathbb{R}^d}(I_\alpha\phi_n)(s)\varphi(s)\,ds(I_\alpha\phi_n)(x)$$
$$=\left(\sum_{n=0}^N\langle\varphi,I_\alpha\phi_n\rangle I_\alpha\phi_n\right)(x)=\left(I_\alpha\sum_{n=0}^N\langle I_\alpha\varphi,\phi_n\rangle\phi_n\right)(x).$$

Taking $\alpha\in(0,d/2)$ and $\varphi,\psi\in\mathcal{S}(\mathbb{R}^d)$ then by Proposition 3.3

$$\langle I_{2\alpha}\varphi,\psi\rangle=\langle I_\alpha\varphi,I_\alpha\psi\rangle \tag{20}$$

for all $\varphi,\psi\in\mathcal{S}(\mathbb{R}^d)$, $\alpha\in(0,d/2)$.

If $\phi_n\in L^2$ and $\varphi\in\mathcal{S}(\mathbb{R}^d)$ then, by Proposition 3.3, we have $\langle I_\alpha\phi_n,\varphi\rangle=\langle\phi_n,I_\alpha\varphi\rangle$. Defining $P_Nf=\sum_{n=0}^N\langle f,\phi_n\rangle\phi_n$, if we take $\psi,\varphi\in\mathcal{S}(\mathbb{R}^d)$, we can write

$$\langle R^N(x,.),\varphi\rangle=I_\alpha P_NI_\alpha\varphi(x). \tag{21}$$

On the other hand, again by Proposition 3.3, $\langle I_\alpha P_N I_\alpha \varphi, \psi\rangle = \langle P_N I_\alpha\varphi, I_\alpha\psi\rangle$, and from these facts it follows that

$$|\langle I_\alpha P_N I_\alpha\varphi,\psi\rangle - \langle I_{2\alpha}\varphi,\psi\rangle| = |\langle P_N I_\alpha\varphi, I_\alpha\psi\rangle - \langle I_\alpha\varphi, I_\alpha\psi\rangle|. \quad (22)$$

Then

$$|\langle P_N I_\alpha\varphi - I_\alpha\varphi, I_\alpha\psi\rangle| \leq \|I_\alpha\psi\|_{L^2}\|P_N I_\alpha\varphi - I_\alpha\varphi\|_{L^2}$$

and given $\varepsilon > 0 \exists N(\varepsilon)$ such that

$$\|P_N I_\alpha\varphi - I_\alpha\varphi\|_{L^2} < \frac{\varepsilon}{\|I_\alpha\psi\|_{L^2}}, \qquad \forall N \geq N(\varepsilon).$$

Hence,

$$\Gamma_N(\varphi,\psi) = \int_{\mathbb{R}^d} (I_\alpha P_N I_\alpha\varphi)(x)\psi(x)\,dx \to \int_{\mathbb{R}^d} (I_{2\alpha}\varphi)(x)\psi(x)\,dx = \Gamma(\varphi,\psi)$$

as $N \to \infty$. Then from (21) and (22) it follows that $\Gamma_N(\varphi,\psi) \to \Gamma(\varphi,\psi)$. Hence,

$$R_{w_\alpha}(x) = \frac{1}{\gamma(2\alpha)}|x|^{-d+2\alpha}.$$

(Part b) Since $R_{w_\alpha}(x) = |x|^{-d+2\alpha} \in \mathcal{S}'(\mathbb{R}^d)$ we can calculate its Fourier transform, then the result follows immediatly by Proposition 3.1 and (Part a). □

V. Conclusion and Some Commentaries

We constructed a series that converges a.s. in the sense of distributions to a process with a $\propto 1/|\omega|^\beta$ spectral behavior. Moreover, it converges in the norm of some Sobolev spaces over a bounded set. Just for illustration, we include some synthetic figures obtained by the simulation of approximations of these processes for several values of α. These approximations were obtained by truncation of these series. On the other hand, two-dimensional orthonormal bases are easily obtained by means of the tensor product of one-dimensional basis, taking, for example, a Shannon wavelet basis. Fractional differencing or integration can be performed in the frequency domain as proposed in other works, such as [19]. This suggests certain advantages in the use of basis with band-limited elements. Truncation errors and convergence rates will be studied elsewhere. In the two-dimensional case, it is useful to obtain textures with special spatial patterns or to construct a fractional Brownian field. As expected, the parameter α governs the long-term dependence. If α is near to $d/2$, as in the case of Fig. 3, we have a highly correlated process, as α decreases, the long-range dependence phenomena becomes weaker, see Fig. 2, finally, when α approches 0 we have a process which is near to white noise, see Fig. 1; moreover if $\alpha = 0$ this is exactly a white noise, and if we consider the one-dimensional case we obtain the same construction of generalized white noise developed in [17].

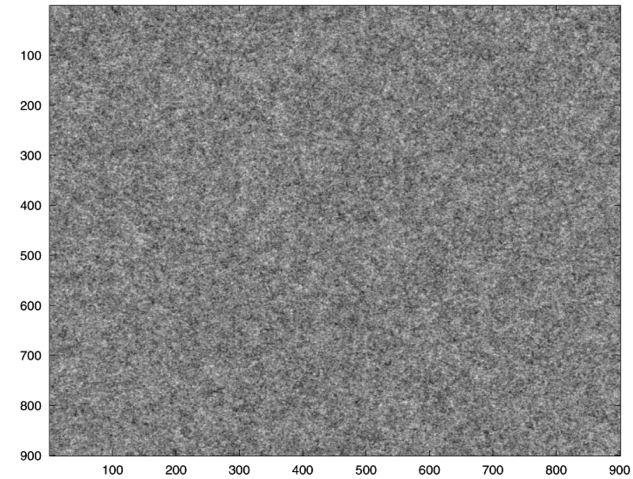

Fig. 1. A sample of a two-dimensional process ($\alpha = 0.001$).

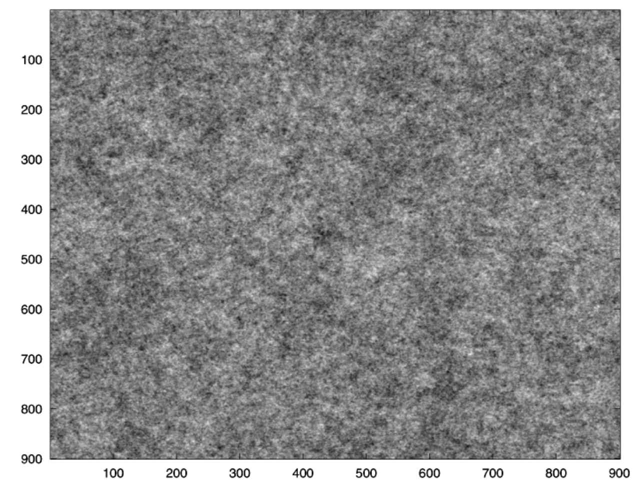

Fig. 2. A sample of a two-dimensional process ($\alpha = 0.5$).

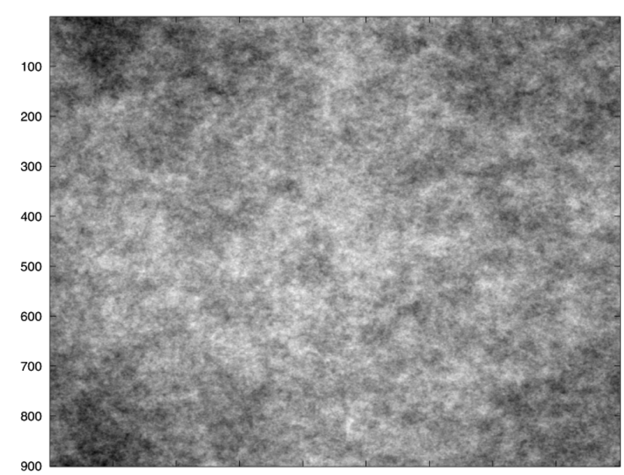

Fig. 3. A sample of a two-dimensional process ($\alpha = 0.99$).

Appendix I
Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1: Define

$$A_k = \{\varpi \in \Omega; : \|S_k\|_H \geq \varepsilon; \|S_j\|_H < \varepsilon, \forall j < k\}$$

which verify $A_{k_1} \cap A_{k_2} = \emptyset$ if $k_1 \neq k_2$, since if we assume that $\exists\varpi \in A_{k_1} \cap A_{k_2}$ and take $k_1 < k_2$, then by the definition of these sets we have $\|S_{k_1}\|_H^2 \geq \varepsilon^2$ and $\|S_{k_1}\|_H^2 < \varepsilon^2$ and then we would have a contradiction. Hence they are disjoint. Now we have

$$\begin{aligned}\boldsymbol{E}\|S_n\|_H^2 &= \int_\Omega \|S_n\|_H^2 \, d\mathbb{P} \\ &= \int_{\bigsqcup_{k=1}^n A_k} \|S_n\|_H^2 \, d\mathbb{P} + \int_{\Omega\setminus(\bigsqcup_{k=1}^n A_k)} \|S_n\|_H^2 \, d\mathbb{P} \\ &\geq \int_{\bigsqcup_{k=1}^n A_k} \|S_n\|_H^2 d\mathbb{P} = \sum_{k=1}^n \int_{A_k} \|S_n\|_H^2 \, d\mathbb{P}. \end{aligned} \tag{23}$$

But $\|S_n\|_H^2 = \|S_k\|_H^2 + 2\langle S_k, S_n - S_k\rangle + \|S_n - S_k\|_H^2$, then

$$\sum_{k=1}^n \int_{A_k} \|S_n\|_H^2 \, d\mathbb{P} \geq \sum_{k=1}^n \int_{A_k} \|S_k\|_H^2 + 2\langle S_k, S_n - S_k\rangle \, d\mathbb{P}$$

and using the independence of the random variables we have

$$\begin{aligned}\int_{A_k} \langle S_k, S_n - S_k\rangle \, d\mathbb{P} &= \int_{A_k} \left\langle \sum_{j=1}^k \xi_j f_j, \sum_{i=k+1}^n \xi_i f_i \right\rangle d\mathbb{P} \\ &= \sum_{j=1}^k \sum_{i=k+1}^n \langle f_j, f_i\rangle \int_{A_k} \xi_j \xi_i \, d\mathbb{P} \\ &= \sum_{j=1}^k \sum_{i=k+1}^n \langle f_j, f_i\rangle \int_\Omega \xi_j \xi_i \mathbf{1}_{A_k} \, d\mathbb{P} = 0. \end{aligned} \tag{24}$$

This is so, since for $j \leq k \leq i$

$$\int_\Omega \xi_j \xi_i \mathbf{1}_{A_k} \, d\mathbb{P} = \int_\Omega \xi_j \mathbf{1}_{A_k} \, d\mathbb{P} \int_\Omega \xi_i \, d\mathbb{P} = 0$$

because $\mathbf{1}_{A_k}$ is a random variable that only depends on ξ_l for $1 \leq l \leq k$ and ξ_i is independent of all these variables. Finally, from (23) and (24) and the definition of the A_k's

$$\begin{aligned}&\sum_{k=1}^n \int_{A_k} \|S_k\|_H^2 \, d\mathbb{P} \\ &\quad \geq \varepsilon^2 \sum_{k=1}^n \mathbb{P}(A_k) = \varepsilon^2 \mathbb{P}\left(\bigvee_{k=1\ldots n} \|S_k\|_H^2 > \varepsilon^2 \right). \end{aligned} \tag{25}$$

□

Proof of Theorem 3.2: We need to find a bound for

$$\mathbb{P}\left(\bigvee_{k=1\ldots r} \|S_{n+k} - S_n\|_H^2 > \varepsilon^2 \right). \tag{26}$$

Since $S_{n+k} - S_n = \sum_{j=1}^k X_{j+n}$ then

$$\begin{aligned}\boldsymbol{E}\|S_{n+r} - S_n\|_H^2 &= \boldsymbol{E}\left\langle \sum_{j=1}^r X_{j+n}, \sum_{i=1}^r X_{i+n} \right\rangle \\ &= \sum_{j=1}^r \sum_{i=1}^r \boldsymbol{E}\langle X_{j+n}, X_{i+n}\rangle \\ &= \sum_{j=1}^r \sum_{i=1}^r \boldsymbol{E}\xi_{j+n}\xi_{i+n}\langle f_{j+n}, f_{i+n}\rangle \end{aligned} \tag{27}$$

and since $\boldsymbol{E}\xi_k = 0$, and from the independence of the sequence, (27) equals

$$\sum_{k=1}^r \boldsymbol{E}\xi_{k+n}^2 \|f_{k+n}\|_H^2. \tag{28}$$

Then by (26), (28), and Theorem 3.1

$$\mathbb{P}\left(\bigvee_{k=1\ldots r} \|S_{n+k} - S_n\|_H^2 > \varepsilon^2 \right) \leq \frac{1}{\varepsilon^2} \sum_{k=1}^r \boldsymbol{E}\xi_{k+n}^2 \|f_{k+n}\|_H^2 \tag{29}$$

so that

$$\mathbb{P}\left(\mathrm{Sup}_{k>1} \|S_{n+k} - S_n\|_H^2 > \varepsilon^2 \right) \leq \frac{1}{\varepsilon^2} \sum_{k=1}^\infty \boldsymbol{E}\xi_{k+n}^2 \|f_{k+n}\|_H^2 \tag{30}$$

and from the condition $\sum_{n=1}^\infty \mathrm{Var}(\xi_n)\|f_n\|_H^2 < \infty$ we get

$$\lim_{n\to\infty} \sum_{k=1}^\infty \boldsymbol{E}\xi_{k+n}^2 \|f_{k+n}\|_H^2 = 0.$$

Taking $\varepsilon = \frac{2}{N}$ we obtain

$$\lim_{n\to\infty} \mathbb{P}\left(\mathop{\mathrm{Sup}}_{k\geq 1} \|S_{n+k} - S_n\|_H^2 > \left(\frac{2}{N}\right)^2 \right) = 0.$$

If

$$E_{n,N} \triangleq \left\{ \varpi \in \Omega : \mathop{\mathrm{Sup}}_{j,k\geq n} \|S_j - S_k\|_H > \frac{2}{N} \right\}$$

then we have $E_{n,N} \searrow E_N$ and $\mathbb{P}(E_N) = 0$, and then

$$\mathbb{P}\left(\bigcup_{N\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \left\{ \varpi \in \Omega : \mathop{\mathrm{Sup}}_{j,k\geq n} \|S_j - S_k\|_H > \frac{2}{N} \right\} \right) = 0.$$

□

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