# A generalized skew information and uncertainty relation 

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#### Abstract

A generalized skew information is defined and a generalized uncertainty relation is established with the help of a trace inequality which was recently proven by J.I.Fujii. In addition, we prove the trace inequality conjectured by S.Luo and Z.Zhang. Finally we point out that Theorem 1 in S.Luo and Q.Zhang, IEEE Trans.IT, Vol.50, pp.1778-1782 (2004) is incorrect in general, by giving a simple counter-example.


Index Terms-Skew information, trace inequalities and uncertainty relation.

## I. Introduction

As one of the mathematical studies on entropy, the skew entropy [14], [15] and the problem of its concavity are famous. The concavity problem for the skew entropy generalized by F.J.Dyson, was solved by E.H.Lieb in [9]. It is also known that the skew entropy represents the degree of noncommutativity between a certain quantum state represented by the density matrix $\rho$ (which is a positive semidefinite matrix with unit trace) and an observable represented by the selfadjoint matrix $X$. Quite recently S.Luo and Q.Zhang studied the relation between skew information (which is equal to the opposite signed skew entropy) and the uncertainty relation in [10]. Inspired by their interesting work, we define a generalized skew information and then study the relationship between it and the uncertainty relation. In addition, we prove the trace inequality conjectured by S.Luo and Z.Zhang in [11].

## II. Preliminaries

Let $f$ and $g$ be functions on the domain $D \subset \mathrm{R} .(f, g)$ is called a monotonic pair if $(f(a)-f(b))(g(a)-g(b)) \geq 0$ for all $a, b \in D .(f, g)$ is also called an antimonotonic pair if $(f(a)-f(b))(g(a)-g(b)) \leq 0$ for all $a, b \in D$.

In what follows we consider selfadjoint matrices whose spectra are included in $D$ so that functional calculus makes sense.

Lemma II. 1 ([1], [2]) For any selfadjoint matrices $A$ and $X$, we have the following trace inequalities.

[^0](1) If $(f, g)$ is a monotonic pair, then
$$
\operatorname{Tr}(f(A) X g(A) X) \leq \operatorname{Tr}\left(f(A) g(A) X^{2}\right)
$$
(2) If $(f, g)$ is an antimonotonic pair, then
$$
\operatorname{Tr}(f(A) X g(A) X) \geq \operatorname{Tr}\left(f(A) g(A) X^{2}\right)
$$

From this lemma, we can obtain the following lemma.

Lemma II. 2 For any selfadjoint matrices $A$ and $B$, and any matrix $X$, we have the following trace inequalities.
(1) If $(f, g)$ is a monotonic pair, then

$$
\begin{aligned}
& \operatorname{Tr}\left(f(A) X^{*} g(B) X+f(B) X g(A) X^{*}\right) \\
& \leq \operatorname{Tr}\left(f(A) g(A) X^{*} X+f(B) g(B) X X^{*}\right)
\end{aligned}
$$

(2) If $(f, g)$ is an antimonotonic pair, then

$$
\begin{aligned}
& \operatorname{Tr}\left(f(A) X^{*} g(B) X+f(B) X g(A) X^{*}\right) \\
& \geq \operatorname{Tr}\left(f(A) g(A) X^{*} X+f(B) g(B) X X^{*}\right)
\end{aligned}
$$

Proof: Define on $\mathcal{H} \oplus \mathcal{H}$

$$
\widehat{A}=\left(\begin{array}{cc}
A & 0 \\
0 & B
\end{array}\right), \widehat{X}=\left(\begin{array}{cc}
0 & X^{*} \\
X & 0
\end{array}\right)
$$

where $A, B$ and $X$ act on a finite dimensional Hilbert space $\mathcal{H}$. Then $\widehat{A}$ and $\widehat{X}$ are selfadjoint. Therefore one may apply Lemma $\boxed{I I} 1$ to get

$$
\begin{aligned}
& \operatorname{Tr}\left(f(A) X^{*} g(B) X+f(B) X g(A) X^{*}\right) \\
& =\operatorname{Tr}\left(\left(\begin{array}{cc}
f(A) & 0 \\
0 & f(B)
\end{array}\right)\left(\begin{array}{cc}
0 & X^{*} \\
X & 0
\end{array}\right)\right. \\
& \left.\left(\begin{array}{cc}
g(A) & 0 \\
0 & g(B)
\end{array}\right)\left(\begin{array}{cc}
0 & X^{*} \\
X & 0
\end{array}\right)\right) \\
& =\operatorname{Tr}(f(\widehat{A}) \widehat{X} g(\widehat{A}) \widehat{X}) \\
& \leq \operatorname{Tr}\left(f(\widehat{A}) g(\widehat{A}) \widehat{X}^{2}\right) \\
& =\operatorname{Tr}\left(\left(\begin{array}{cc}
f(A) & 0 \\
0 & f(B)
\end{array}\right)\left(\begin{array}{cc}
g(A) & 0 \\
0 & g(B)
\end{array}\right)\right. \\
& \left.\left(\begin{array}{cc}
0 & X^{*} \\
X & 0
\end{array}\right)\left(\begin{array}{cc}
0 & X^{*} \\
X & 0
\end{array}\right)\right) \\
& =\operatorname{Tr}\left(f(A) g(A) X^{*} X+f(B) g(B) X X^{*}\right)
\end{aligned}
$$

which is inequality (1). Inequality (2) is proven in a similar way.

## III．GENERALIZED UNCERTAINTY RELATION

For a density matrix（quantum state）$\rho$ and arbitrary matrices $X$ and $Y$ acting on $\mathcal{H}$ ，we denote $\widetilde{X} \equiv X-\operatorname{Tr}(\rho X) I$ and $\widetilde{Y} \equiv$ $Y-\operatorname{Tr}(\rho Y) I$ ，where $I$ represents the identity matrix．Then we define the covariance by $\operatorname{Cov}_{\rho}(X, Y)=\operatorname{Tr}(\rho \tilde{X} \tilde{Y})$ ．Each variance is defined by $V_{\rho}(X) \equiv \operatorname{Cov}_{\rho}(X, X)$ and $V_{\rho}(Y) \equiv$ $\operatorname{Cov}_{\rho}(Y, Y)$ ．

The famous Heisenberg＇s uncertainty relation［6］，［12］can be easily proven by the application of the Schwarz inequality and it was generalized by Schrödinger as follows：

Proposition III． 1 （Schrödinger［13］）For any density matrix $\rho$ and any two selfadjoint matrices $A$ and $B$ ，we have the uncertainty relation ：

$$
\begin{equation*}
V_{\rho}(A) V_{\rho}(B)-\left|\operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right)\right|^{2} \geq \frac{1}{4}|\operatorname{Tr}(\rho[A, B])|^{2} \tag{1}
\end{equation*}
$$

where $[X, Y] \equiv X Y-Y X$ ．
Definition III． 2 For arbitrary matrices $X$ and $Y$ ，we define

$$
I_{p}(\rho ; X, Y) \equiv \operatorname{Tr}(\rho X Y)-\operatorname{Tr}\left(\rho^{\frac{1}{p}} X \rho^{\frac{1}{p^{*}}} Y\right)
$$

where $p \in[1,+\infty]$ and with $p^{*}$ such that $\frac{1}{p}+\frac{1}{p^{*}}=1$ ．If $A$ is selfadjoint，the Wigner－Yanase－Dyson information is defined by

$$
\begin{aligned}
I_{p}(\rho ; A) & \equiv I_{p}(\rho ; A, A)=\operatorname{Tr}\left(\rho A^{2}\right)-\operatorname{Tr}\left(\rho^{\frac{1}{p}} A \rho^{\frac{1}{p^{*}}} A\right) \\
& =-\frac{1}{2} \operatorname{Tr}\left(\left[\rho^{\frac{1}{p}}, A\right]\left[\rho^{\frac{1}{p^{*}}}, A\right]\right)
\end{aligned}
$$

We use the parameters $p$ and $p^{*}$ ，since many papers［3］，［4］， ［5］，［7］in this field use such notations．The Wigner－Yanase skew information is

$$
\begin{aligned}
I(\rho ; A) & \equiv I_{2}(\rho ; A)=\operatorname{Tr}\left(\rho A^{2}\right)-\operatorname{Tr}\left(\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}} A\right) \\
& =-\frac{1}{2} \operatorname{Tr}\left(\left[\rho^{\frac{1}{2}}, A\right]^{2}\right)
\end{aligned}
$$

An interpretation of skew information as a measure of quantum uncertainty is given in［10］by S．Luo and Q．Zhang． They claimed the following uncertainty relation ：

$$
\begin{align*}
& I(\rho, A) I(\rho, B)-\left|\operatorname{Re}\left(\operatorname{Corr}_{\rho}(A, B)\right)\right|^{2} \\
& \geq \frac{1}{4}|\operatorname{Tr}(\rho[A, B])|^{2}, \tag{2}
\end{align*}
$$

for two selfadjoint matrices $A$ and $B$ ，and density matrix $\rho$ ， where their correlation measure was defined by

$$
\operatorname{Corr}_{\rho}(A, B) \equiv \operatorname{Tr}(\rho A B)-\operatorname{Tr}\left(\rho^{1 / 2} A \rho^{1 / 2} B\right)
$$

However，we show the inequality（2）does not hold in general． We give a counter－example for inequality（2）in the final section．

We define the generalized skew correlation and the gener－ alized skew information as follows．

Definition III． 3 For arbitrary $X$ and $Y, p \in[1,+\infty]$ with $p^{*}$ such that $\frac{1}{p}+\frac{1}{p^{*}}=1$ and $\varepsilon \geq 0$ ，set

$$
\begin{aligned}
\phi_{p, \varepsilon}(\rho ; X, Y) & \equiv \varepsilon \operatorname{Cov}_{\rho}\left(X^{*}, Y\right) \\
& +\frac{1}{2} I_{p}\left(\rho ; \widetilde{X^{*}}, \widetilde{Y}\right)+\frac{1}{2} I_{p}\left(\rho ; \widetilde{Y}, \widetilde{X^{*}}\right)
\end{aligned}
$$

If $A$ and $B$ are selfadjoint，the generalized skew correlation is defined by

$$
\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B) \equiv \phi_{p, \varepsilon}(\rho ; A, B)
$$

The generalized skew information is defined by

$$
I_{p, \varepsilon}(\rho ; A) \equiv \operatorname{Corr}_{p, \varepsilon}(\rho ; A, A)=\varepsilon V_{\rho}(A)+I_{p}(\rho ; \widetilde{A})
$$

so that

$$
I_{p, 0}(\rho ; A)=I_{p}(\rho ; \widetilde{A})=V_{\rho}(A)-\operatorname{Tr}\left(\rho^{\frac{1}{p}} \tilde{A} \rho^{\frac{1}{p^{*}}} \widetilde{A}\right)
$$

Then we have the following theorem．
Theorem III． 4 For any two selfadjoint matrices $A$ and $B$ ，any density matrix $\rho$ ，any $p \in[1,+\infty]$ with $p^{*}$ such that $\frac{1}{p}+\frac{1}{p^{*}}=1$ and $\varepsilon \geq 0$ ，we have a generalized uncertainty relation ：

$$
\begin{aligned}
& I_{p, \varepsilon}(\rho ; A) I_{p, \varepsilon}(\rho ; B)-\left|\operatorname{Re}\left(\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right)\right|^{2} \\
& \geq \frac{\varepsilon^{2}}{4}|\operatorname{Tr}(\rho[A, B])|^{2}
\end{aligned}
$$

Proof ：By Lemma 【I． $2 \phi_{p, \varepsilon}(\rho ; X, X) \geq 0$ ．Furthermore it is clear that $\phi_{p, \varepsilon}(\rho ; X, Y)$ is sesquilinear and Hermitian．Then we have

$$
\left|\phi_{p, \varepsilon}(\rho ; X, Y)\right|^{2} \leq \phi_{p, \varepsilon}(\rho ; X, X) \phi_{p, \varepsilon}(\rho ; Y, Y)
$$

by the Schwarz inequality．It follows that

$$
\left|\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right|^{2} \leq \operatorname{Corr}_{p, \varepsilon}(\rho ; A, A) \operatorname{Corr}_{p, \varepsilon}(\rho ; B, B)
$$

for any two selfadjoint matrices $A$ and $B$ ．Then

$$
\begin{equation*}
\left|\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right|^{2} \leq I_{p, \varepsilon}(\rho ; A) I_{p, \varepsilon}(\rho ; B) \tag{3}
\end{equation*}
$$

Simple calculations imply

$$
\begin{align*}
& \operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)-\operatorname{Corr}_{p, \varepsilon}(\rho ; B, A) \\
& =\varepsilon \operatorname{Tr}(\rho[\widetilde{A}, \widetilde{B}])=\varepsilon \operatorname{Tr}(\rho[A, B]),  \tag{4}\\
& \operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)+\operatorname{Corr}_{p, \varepsilon}(\rho ; B, A) \\
& =2 \operatorname{Re}\left(\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right) . \tag{5}
\end{align*}
$$

Summing both sides in the above two equalities，we have

$$
\begin{align*}
2 \operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)= & \varepsilon \operatorname{Tr}(\rho[A, B]) \\
& +2 \operatorname{Re}\left(\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right) . \tag{6}
\end{align*}
$$

Since $[A, B]$ is skew－adjoint， $\operatorname{Tr}(\rho[A, B])$ is a purely imagi－ nary number，we have

$$
\begin{align*}
\left|\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right|^{2} & =\frac{\varepsilon^{2}}{4}|\operatorname{Tr}(\rho[A, B])|^{2} \\
& +\left|\operatorname{Re}\left(\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right)\right|^{2} \tag{7}
\end{align*}
$$

Thus the proof of the theorem is completed by the use of inequality（3）and Eq．（7）．

We are interested in the relationship between the left hand sides in Proposition 【II．1 and Theorem 【II．4 The following proposition gives the relationship．

Proposition III. 5 For any two selfadjoint matrices $A$ and $B$, any density matrix $\rho$, any $p \in[1,+\infty]$ with $p^{*}$ such that $\frac{1}{p}+\frac{1}{p^{*}}=1$ and $\varepsilon \geq 0$, we have

$$
\begin{aligned}
& I_{p, \varepsilon}(\rho ; A) I_{p, \varepsilon}(\rho ; B)-\left|\operatorname{Re}\left(\operatorname{Corr}_{p, \varepsilon}(\rho ; A, B)\right)\right|^{2} \\
& \geq \varepsilon^{2} V_{\rho}(A) V_{\rho}(B)-\varepsilon^{2}\left|\operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right)\right|^{2}
\end{aligned}
$$

Proof : From Proposition III.1 we have $V_{\rho}(A) V_{\rho}(B) \geq$ $\left|\operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right)\right|^{2}$, that is,

$$
\begin{equation*}
|\operatorname{Re}(\operatorname{Tr}(\rho \widetilde{A} \widetilde{B}))|^{2} \leq \operatorname{Tr}\left(\rho \widetilde{A}^{2}\right) \operatorname{Tr}\left(\rho \widetilde{B}^{2}\right) \tag{8}
\end{equation*}
$$

By putting $\varepsilon=0$ in (3), we have

$$
\left|\operatorname{Corr}_{p, 0}(\rho ; A, B)\right|^{2} \leq I_{p, 0}(\rho ; A) I_{p, 0}(\rho ; B)
$$

It follows from (4) and (5) that

$$
\operatorname{Corr}_{p, 0}(\rho ; A, B)=\operatorname{Re}\left(\operatorname{Corr}_{p, 0}(\rho ; A, B)\right) .
$$

Thus
$\left|\operatorname{Re}\left(\operatorname{Corr}_{p, 0}(\rho ; A, B)\right)\right|^{2} \leq I_{p, 0}(\rho ; A) I_{p, 0}(\rho ; B)$.
Using Eq. (8), Eq. (9) and direct calculations, we get the following:

$$
\begin{aligned}
& \text { L.H.S. }- \text { R.H.S. } \\
& =\varepsilon \operatorname{Tr}\left(\rho \widetilde{A}^{2}\right) I_{p, 0}(\rho ; B)+\varepsilon \operatorname{Tr}\left(\rho \widetilde{B}^{2}\right) I_{p, 0}(\rho ; A) \\
& -2 \varepsilon \operatorname{Re}(\operatorname{Tr}(\rho \widetilde{A} \widetilde{B})) \operatorname{Re}\left(\operatorname{Corr}_{p, 0}(\rho ; A, B)\right) \\
& +I_{p, 0}(\rho ; A) I_{p, 0}(\rho ; B)-\left\{\operatorname{Re}\left(\operatorname{Corr}_{p, 0}(\rho ; A, B)\right)\right\}^{2} \\
& \geq \varepsilon \operatorname{Tr}\left(\rho \widetilde{A}^{2}\right) I_{p, 0}(\rho ; B)+\varepsilon \operatorname{Tr}\left(\rho \widetilde{B}^{2}\right) I_{p, 0}(\rho ; A) \\
& -2 \varepsilon \operatorname{Re}(\operatorname{Tr}(\rho \widetilde{A} \widetilde{B})) \operatorname{Re}\left(\operatorname{Corr}_{p, 0}(\rho ; A, B)\right) \\
& \geq \varepsilon \operatorname{Tr}\left(\rho \widetilde{A}^{2}\right) I_{p, 0}(\rho ; B)+\varepsilon \operatorname{Tr}\left(\rho \widetilde{B}^{2}\right) I_{p, 0}(\rho ; A) \\
& -2 \varepsilon \sqrt{\operatorname{Tr}\left(\rho \widetilde{A}^{2}\right) \operatorname{Tr}\left(\rho \widetilde{B}^{2}\right)} \sqrt{I_{p, 0}(\rho ; A) I_{p, 0}(\rho ; B)} \\
& =\varepsilon\left\{\sqrt{\operatorname{Tr}\left(\rho \widetilde{A}^{2}\right) I_{p, 0}(\rho ; B)}-\sqrt{\operatorname{Tr}\left(\rho \widetilde{B}^{2}\right) I_{p, 0}(\rho ; A)}\right\}^{2}
\end{aligned}
$$

$$
\geq 0
$$

Remark III. 6 Theorem III.4 can be also proven by Proposition III. 1 and Proposition III. 5

## IV. AN INEQUALITY RELATED TO THE UNCERTAINTY RELATION

The trace inequality

$$
\begin{aligned}
& V_{\rho}(A) V_{\rho}(B)-\left|\operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right)\right|^{2} \\
& \geq I_{2,0}(\rho ; A) I_{2,0}(\rho ; B)-\left|\operatorname{Re}\left(\operatorname{Cor}_{2,0}(\rho ; A, B)\right)\right|^{2}
\end{aligned}
$$

was conjectured in [11] and proven in [10]. As a generalization of Theorem 2 in [10], we prove a one-parameter extention of the above inequality.

Proposition IV. 1 For any two selfadjoint matrices $A$ and $B$, any density matrix $\rho$ and any $p \in[1,+\infty]$ with $p^{*}$ such that $\frac{1}{p}+\frac{1}{p^{*}}=1$, we have

$$
\begin{align*}
& V_{\rho}(A) V_{\rho}(B)-\left|\operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right)\right|^{2} \\
& \geq I_{p, 0}(\rho ; A) I_{p, 0}(\rho ; B)-\left|\operatorname{Re}\left(\operatorname{Corr}_{p, 0}(\rho ; A, B)\right)\right|^{2} \tag{10}
\end{align*}
$$

Proof: Let $\left\{\varphi_{i}\right\}$ be a complete orthonormal basis composed by eigenvectors of $\rho$. Then we calculate

$$
\operatorname{Tr}\left(\rho^{\frac{1}{p}} \widetilde{A} \rho^{\frac{1}{p^{*}}} \widetilde{A}\right)=\sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} a_{i j} a_{j i}
$$

where $a_{i j} \equiv\left\langle\tilde{A} \varphi_{i} \mid \varphi_{j}\right\rangle$ and $a_{j i} \equiv \overline{a_{i j}}$. Thus we get

$$
\begin{aligned}
& I_{p, 0}(\rho ; A)=V_{\rho}(A)-\sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} a_{i j} a_{j i} \\
& I_{p, 0}(\rho ; B)=V_{\rho}(B)-\sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} b_{i j} b_{j i}
\end{aligned}
$$

where $b_{i j} \equiv\left\langle\widetilde{B} \varphi_{i} \mid \varphi_{j}\right\rangle$ and $b_{j i} \equiv \overline{b_{i j}}$. In a similar way, we obtain

$$
\begin{aligned}
& \operatorname{Re}\left(\operatorname{Corr}_{p, 0}(\rho ; A, B)\right)=\operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right) \\
& -\frac{1}{2} \sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \operatorname{Re}\left(a_{i j} b_{j i}\right)-\frac{1}{2} \sum_{j, i} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \operatorname{Re}\left(b_{i j} a_{j i}\right)
\end{aligned}
$$

In order to prove the present proposition, we have only to show the inequality $\xi \geq \eta$, where,

$$
\begin{aligned}
& \xi \equiv V_{\rho}(A) \sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} b_{i j} b_{j i}+V_{\rho}(B) \sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} a_{i j} a_{j i} \\
& -\left(\sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} a_{i j} a_{j i}\right)\left(\sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} b_{i j} b_{j i}\right) \\
& \eta \equiv \operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right) \sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \operatorname{Re}\left(a_{i j} b_{j i}\right) \\
& +\operatorname{Re}\left(\operatorname{Cov}_{\rho}(A, B)\right) \sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \operatorname{Re}\left(b_{i j} a_{j i}\right) \\
& -\frac{1}{4}\left(\sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \operatorname{Re}\left(a_{i j} b_{j i}\right)+\sum_{i, j} \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \operatorname{Re}\left(b_{i j} a_{j i}\right)\right)^{2}
\end{aligned}
$$

Since $V_{\rho}(A)=\operatorname{Tr}\left(\rho \widetilde{A}^{2}\right)=\frac{1}{2} \sum_{i, j}\left(\lambda_{i}+\lambda_{j}\right) a_{i j} a_{j i}, V_{\rho}(B)=$ $\operatorname{Tr}\left(\rho \widetilde{B}^{2}\right)=\frac{1}{2} \sum_{i, j}\left(\lambda_{i}+\lambda_{j}\right) b_{i j} b_{j i}$, and $\left(\lambda_{i}+\lambda_{j}\right) \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}}+$ $\left(\lambda_{k}+\lambda_{l}\right) \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}}-2 \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}} \geq 0$, we calculate

$$
\begin{align*}
& \xi=\frac{1}{4} \sum_{i, j, k, l}\left\{\left(\lambda_{i}+\lambda_{j}\right) \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}}+\left(\lambda_{k}+\lambda_{l}\right) \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}}\right. \\
& \left.-2 \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}}\right\}\left(a_{i j} a_{j i} b_{k l} b_{l k}+b_{i j} b_{j i} a_{k l} a_{l k}\right) \\
& \geq \frac{1}{2} \sum_{i, j, k, l}\left\{\left(\lambda_{i}+\lambda_{j}\right) \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}}+\left(\lambda_{k}+\lambda_{l}\right) \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}}\right. \\
& \left.-2 \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}}\right\}\left|a_{i j} b_{j i}\right|\left|a_{k l} b_{l k}\right| \tag{11}
\end{align*}
$$

Since $\operatorname{Re}\left(b_{k l} a_{l k}\right)=\operatorname{Re}\left(\overline{b_{l k}} \overline{a_{k l}}\right)=\operatorname{Re}\left(b_{l k} a_{k l}\right)=$ $\operatorname{Re}\left(a_{k l} b_{l k}\right), \operatorname{Re}\left(b_{i j} a_{j i}\right)=\operatorname{Re}\left(a_{i j} b_{j i}\right)$, we calculate

$$
\begin{aligned}
& \eta=\frac{1}{2} \sum_{i, j, k, l}\left\{\left(\lambda_{i}+\lambda_{j}\right) \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}}+\left(\lambda_{k}+\lambda_{l}\right) \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}}\right. \\
& \left.-2 \lambda_{i}^{\frac{1}{p}} \lambda_{j}^{\frac{1}{p^{*}}} \lambda_{k}^{\frac{1}{p}} \lambda_{l}^{\frac{1}{p^{*}}}\right\} \operatorname{Re}\left(a_{i j} b_{j i}\right) \operatorname{Re}\left(a_{k l} b_{l k}\right) .
\end{aligned}
$$

Thus we conclude $\xi \geq \eta$, since $\left|a_{i j} b_{j i}\right|\left|a_{k l} b_{l k}\right| \geq$ $\left|\operatorname{Re}\left(a_{i j} b_{j i}\right) \operatorname{Re}\left(a_{k l} b_{l k}\right)\right|$.

The inequality (10) was independently proven by H.Kosaki in [8]. Our proof is simpler than Kosaki's one.

As a concluding remark, we point out that Theorem 1 in [10] is incorrect in general.

Remark IV. 2 Theorem 1 in [10] is not true in general. A counter-example is given as follows. Let

$$
\rho=\frac{1}{4}\left(\begin{array}{ll}
3 & 0 \\
0 & 1
\end{array}\right), A=\left(\begin{array}{cc}
0 & i \\
-i & 0
\end{array}\right), B=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) .
$$

Then we have, $I(\rho, A) I(\rho, B)-\left|\operatorname{Re}\left(\operatorname{Corr}_{\rho}(A, B)\right)\right|^{2}=$ $\frac{7-4 \sqrt{3}}{4}$ and $|\operatorname{Tr}(\rho[A, B])|^{2}=1$. These imply

$$
I(\rho, A) I(\rho, B)-\left|\operatorname{Re}\left(\operatorname{Corr}_{\rho}(A, B)\right)\right|^{2}<\frac{1}{4}|\operatorname{Tr}(\rho[A, B])|^{2} .
$$

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## REFERENCES

[1] J.-C.Bourin, Some inequalities for norms on matrices and operators, Linear Alg.Appl., Vol.292, pp.139-154 (1999).
[2] J.I.Fujii, A trace inequality arising from quantum information theory, Linear Alg.Appl.,Vol.400, pp.141-146(2005).
[3] P.Gibilisco and T.Isola, On the characterization of paired monotone metrics, Ann.Inst.Stat.Math.,Vol.56,pp.369-381 (2004).
[4] P.Gibilisco and T.Isola, On monotonicity of scalar curvature in classical and quantum information geometry, J.Math.Phys.,Vol.46,pp.1-14(2005).
[5] M.R.Grasselli, Duality, monotonicity and Wigner-Yanase-Dyson metrics, Inf.Dimens.Anal.Quantum Prob.Relat.Top.,Vol.7,pp.215-232(2004).
[6] W.Heisenberg, Über den anschaulichen Inhalt der quantummechanischen Kinematik und Mechanik, Zeitschrift für Physik, Vol.43,pp.172198(1927).
[7] A.Jencova, Flat connections and Wigner-Yanase-Dyson metrics, Pep.Math.Phys.,Vol.52,pp.331-351(2003).
[8] H.Kosaki, Matrix trace inequality related to uncertainty principle, International Journal of Mathematics, Vol.16,pp.629-646(2005).
[9] E.H.Lieb, Convex trace functions and the Wigner-Yanase-Dyson conjecture, Adv. Math.,Vol.11,pp.267-288(1973).
[10] S.Luo and Q.Zhang, On skew information, IEEE Trans.IT, Vol.50,No.8,pp.1778-1782 (2004).
[11] S.Luo and Z.Zhang, An informational characterization of Schrödinger's uncertainty relations,J.Stat.Phys.,Vol.114,pp.1557-1576(2004).
[12] H.P.Robertson, The uncertainty principle, Phys.Rev.,Vol.34,pp.163164(1929).
[13] E.Schrödinger, About Heisenberg uncertainty relation, Proc.Prussian Acad.Sci.,Phys.Math. Section,Vol.XIX,pp.293(1930).
[14] E.P.Wigner and M.M.Yanase, Information content of distribution, Proc.Nat.Acad.Sci. U.S.A., Vol.49,pp.910-918(1963).
[15] E.P.Wigner and M.M.Yanase, On the positive semidefinite nature of certain matrix expression, Canad. J. Math.,Vol.16,pp.397-406(1964).


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