On Space–Time Trellis Codes Achieving Optimal Diversity Multiplexing Tradeoff

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***Abstract*—Multiple antennas can be used for increasing the amount of diversity (diversity gain) or increasing the data rate (the number of degrees of freedom or spatial multiplexing gain) in wireless communication. As quantified by Zheng and Tse, given a multiple-input–multiple-output (MIMO) channel, both gains can, in fact, be simultaneously obtained, but there is a fundamental tradeoff (called the Diversity-Multiplexing Gain (DM-G) tradeoff) between how much of each type of gain, any coding scheme can extract. Space–time codes (STCs) can be employed to make use of these advantages offered by multiple antennas. Space–Time Trellis Codes (STTCs) are known to have better bit error rate performance than Space–Time Block Codes (STBCs), but with a penalty in decoding complexity. Also, for STTCs, the frame length is assumed to be finite and hence zeros are forced towards the end of the frame (called the trailing zeros), inducing rate loss. In this correspondence, we derive an upper bound on the DM-G tradeoff of full-rate STTCs with nonvanishing determinant (NVD). Also, we show that the full-rate STTCs with NVD are optimal under the DM-G tradeoff for any number of transmit and receive antennas, neglecting the rate loss due to trailing zeros. Next, we give an explicit generalized full-rate STTC construction for any number of states of the trellis, which achieves the optimal DM-G tradeoff for any number of transmit and receive antennas, neglecting the rate loss due to trailing zeros.**

***Index Terms*—Diversity-multiplexing tradeoff, multiple-input–multiple-output (MIMO), space–time codes.**

I. Introduction and Preliminaries

Consider the quasi-static Rayleigh-fading space–time channel with quasi-static interval T, n_t transmit and n_r receive antennas. The $(n_r \times T)$ received matrix Y is given by

$$Y = \theta H X + W \tag{1}$$

where X is the transmitted codeword $(n_t \times T)$ drawn from a space–time code (STC) $\mathcal{X}$, H the $(n_r \times n_t)$ channel matrix and W the $(n_r \times T)$ noise matrix. The entries of H and W are assumed to be independent and identically distributed (i.i.d.), circularly symmetric complex Gaussian $\mathbb{C}\mathcal{N}(0;1)$ random variables. STCs are classified into two categories, namely: space–time block codes (STBC) and space–time trellis codes (STTC). Henceforth, we assume $\mathcal{X}$ to be always STTC. The entries of X are drawn from a constellation $\mathcal{S}$ whose size scales with signal-to-noise ratio (SNR) with θ chosen to ensure

$$\mathcal{E}(\|\theta X\|_F^2) = T\ SNR.$$

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Multiple antennas can be used for increasing the amount of diversity (diversity gain) or increasing the data rate (the number of degrees of freedom or spatial multiplexing gain) in wireless communication. As quantified by Zheng and Tse [1], given a multiple-input–multiple-output (MIMO) channel, both gains can, in fact, be simultaneously obtained, but there is a fundamental tradeoff (called the diversity-multiplexing gain (DM-G) tradeoff) between how much of each type of gain, any coding scheme can extract.

At high SNRs, the probability that the received matrix Y, is decoded to a codeword matrix $X' \neq X$, under the condition that X was transmitted, is

$$P(X \to X') = (\prod_{i=1}^{\Lambda} \lambda_i^2)^{-n_r} \mathrm{SNR}^{-n_r \Lambda}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_\Lambda$ are the Λ non-zero singular values of $\Delta X = X - X'$. Therefore, the performance of STTC, at high SNRs, is governed, by the minimum of the rank of the matrix $X - X'$ (called the diversity gain) and minimum of the product of the non-zero singular values of the matrix $X - X'$ (called the coding gain), for $X \neq X'$. A STTC is said to achieve full diversity if $\min_{X \neq X'} \mathrm{rank}(X - X') = n_t \ \forall\, X, X' \in \mathcal{X}$.

At high $SNRs$, the ergodic capacity i.e., capacity averaged over all channel realizations H of the space–time channel model in (1) is known [8] to be

$$C(n_t, n_r, \mathrm{SNR}) \approx \min\{n_t, n_r\} \log \mathrm{SNR}. \tag{2}$$

The above expression shows that the achievable data rate increases with SNR as $\min\{n_t, n_r\} \log\ \mathrm{SNR}$.

If $|\mathcal{X}|$ is the size of the STC, the STC transmits

$$R = \frac{1}{T} \log(|\mathcal{X}|)$$

bits per channel use. Let r be the normalized rate given by $R = r \log(\mathrm{SNR})$. Following [1], we will refer to r as the multiplexing gain. From (2), it is seen that the maximum achievable multiplexing gain equals $r = \min\{n_t, n_r\}$. Let the diversity gain $d(r)$ corresponding to transmission at normalized rate r be defined by

$$d(r) = -\lim_{\mathrm{SNR} \to \infty} \frac{\log(P_e)}{\log(\mathrm{SNR})}$$

where P_e denotes the average codeword error probability. A principal result in [1] is the proof that for a fixed integer multiplexing gain r, and $T \geq n_t + n_r - 1$, the maximum achievable diversity gain $d(r)$ is governed by

$$d(r) = (n_t - r)(n_r - r).$$

Therefore, from [1], both the diversity and the spatial multiplexing gain can be obtained simultaneously but with a fundamental tradeoff between them, called the diversity-multiplexing (DM-G) tradeoff.

A. *Review of Existing STTCs*

STTCs have been studied in [2], to provide improved error performance for wireless systems using multiple transmit antennas. Let $\mathcal{X}$ be a STTC and $X, X' \in \mathcal{X}$. Then, the first column of the matrix $\Delta X = X - X'$ would be the difference of the symbols transmitted by the n_t transmit antennas, when the paths in the trellis, corresponding to the codewords X and X' diverge and the last column would correspond to the symbol difference when they converge. The t^{th} column of ΔX corresponds to the symbol difference at a time $t - 1$ after divergence. Let t_d be the time (from the frame beginning), at which the paths corresponding to X and X' diverge and $t_d + l - 1$ when they converge. Then X and X' differ in at least l locations. The *convergence length* l_c of the STTC is defined as

$$l_c = \min_{X \neq X'} l.$$

For full diversity a necessary condition is that $l_c \geq n_t$.

The scheme used to construct STTCs in [2] is the delay diversity scheme in which the symbol transmitted from the i^{th} antenna at time t is again transmitted from the $(i+1)^{th}$ antenna at time $t+1$. It has been shown [2], that such codes constructed by the delay diversity scheme can provide full diversity gain as well as additional SNR advantage, i.e., coding gain. An example of delay diversity scheme for two transmit antennas for four-state trellis is as shown in Fig. 1.

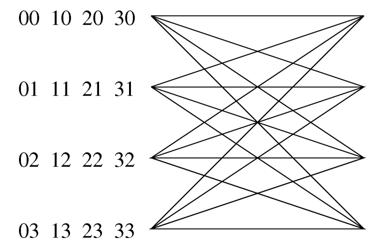

Fig. 1. 2 Tx, 4-PSK, 2 bps/Hz, four-state STTC.

There have been many efforts to improve the performance of STTCs [3]–[6], by employing exhaustive computer searches. A general method to construct STTCs by using a shift register model which ensures full-diversity and good coding gain, for any number of states, is given in [7], where coding in conjunction with delay diversity is used.

It is known that STTCs can give better bit error performance than the space–time block codes (STBCs), but with a penalty in the decoding complexity. For reducing the decoding complexity, STTC is truncated after T channel uses (called the frame length). Transmissions across different frames are independent. To truncate the STTC, we need to force the end state of the STTC to be same as the starting state. To achieve this zeros are forced at the end of each frame, called the trailing zeros, which obviously incur rate loss. Let δ be the number of channel uses for which we need to force zeros. Let us define the rate loss factor denoted by κ to be $\kappa = \frac{T-\delta}{T}$, where T is the frame length. The κ quantifies the rate loss incurred by STTC due to trailing zeros.

B. *DM-G Tradeoff of the Existing STTCs*

It is well known that any full-rank STC achieves full diversity gain for any number of transmit and receive antenna. Since all the existing STTC constructions guarantee full-rank, all of them achieve the maximum possible diversity gain for any number of transmit and receive antennas.

From [13], for a STC to achieve the maximum possible spatial multiplexing gain, a necessary condition is that, the rate of transmission should at least be $\min\{n_t, n_r\}$ complex symbols per channel use. For all the STTC constructions discussed above, the rate of transmission is limited to only one complex symbols per channel use and therefore these codes cannot meet the optimal DM-G tradeoff for more than one receive antenna. This motivates the construction of high-rate STTC's (rates more than 1 complex symbol per channel use). One such construction is given in [12], which is shown in Fig. 2, where we have proposed a STTC for 2 transmit and 2 receive antennas, whose rate of transmission is equal to $\min\{n_t, n_r\} = 2$ complex symbols per channel use. Also, only by simulation, it was shown that this STTC achieves the optimal diversity-multiplexing tradeoff for two transmit and two receive antennas. To the best of our knowledge this is the

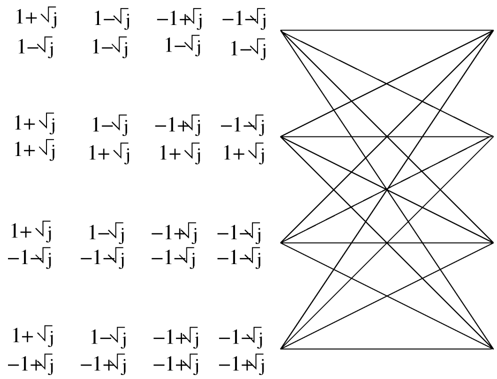

Fig. 2. 2 Tx, BPSK, 2 bps/hz, four-state high-rate STTC.

only high-rate STTC construction and no general construction exists for high-rate STTCs for arbitrary number of transmit antennas.

C. Contributions of This Correspondence

Recently [9]–[11], for STCs with $T = n_t$, it has been proved that full-rate STCs with nonvanishing determinant (NVD) property, are optimal under the DM-G tradeoff for any number of receive antennas. In this correspondence, we extend this result to the case of $T \geq n_t$ by introducing appropriate definitions and assumptions.

It is well known that by coding across larger block lengths we can get more coding and diversity gain for the slow and fast fading channels respectively. For the case of high-rate STCs, STBCs ($T = n_t$) have been studied extensively but there has been very little done for the case of $T \geq n_t$, which can offer higher coding/diversity gain at the cost of higher decoding complexity. For the case of STBCs ($T = n_t$), there have been many high-rate constructions [11], [15], [16]. All these constructions make use of cyclic division algebras (CDA) to construct high-rate STBCs.

We can construct high-rate STCs for $n_t \times T$ where $T \geq n_t$ from the known high-rate STBCs for $n_t = T$ in many ways. One way is by deleting the appropriate $T - n_t$ rows from a high rate $T \times T$ STBC of [14]. But, since $T \gg n_t$, the entries of $n_t \times T$ STC matrix will lie in a degree T extension of $\mathbb{Q}(i)$ (where $\mathbb{Q}$ is field of rational numbers) and hence the signaling complexity will be quite high. To reduce the signaling complexity one more construction is presented in [14] where $\frac{T}{n_t}$ number of $[n_t \times n_t]$ STBCs are stacked side by side to get the required $n_t \times T$ STC, if n_t divides T. Each block $n_t \times n_t$ STBC carries independent information. In this construction the $n_t \times n_t$ STBC blocks are not coded across and hence we loose on the coding gain.

In this correspondence, we give a generalized high-rate STTC construction for any number of states of trellis and any number of transmit and receive antennas using CDA for the case of $T \geq n_t$ (all the earlier CDA based construction were for $T = n_t$) by using a shift register model which does not depend on T. What we propose in this correspondence is a systematic and generalized construction for $n_t \times T$ high-rate STTC for any number of states of the trellis. Therefore by increasing the number of states of the trellis we can increase the coding gain for a fixed rate of transmission. This construction is specifically designed for n_t transmit antennas and not in a higher dimensional space and hence has the minimum signaling complexity possible. Also the information is coded through all the T channel uses and hence the scheme does not loose out on any coding gain.

One more advantage of this trellis based construction is that it does not need the *non norm* element which is so essential for the high-rate STBC constructions [11], [15], [16]. The full-rank for the high-rate STBCs is guaranteed only for a particular set of *non norm* elements depending on the signal set and n_t and finding such a set is not easy. Also the *non norm* element dictates the coding gain that the scheme achieves. For the case of high-rate STBCs, it is known that coding gain can be improved if we have freedom to choose the *non norm* element, but the full-rank requirement puts a condition on it.

In our construction we only use the structure of the trellis to guarantee full-rank and hence avoid the *non norm* element. Therefore we have more freedom to optimize the coding gain. By our shift register model and code structure we show that any codeword difference matrix is a upper triangular matrix with non zero diagonal entries and hence is full rank and the STTC so constructed achieves full diversity.

We also show that the proposed codes have the NVD property and hence achieve the optimal DM-G tradeoff (neglecting the rate loss factor).

The contributions of this correspondence can be summarized as follows:

- We quantify the effect of rate loss factor on the optimal DM-G tradeoff for STTCs.
- Using [11], we show that the full-rate STTCs with NVD property, achieve the upper bound on the optimal DM-G tradeoff for any number of receive antennas, neglecting the rate loss factor.
- The main result of this correspondence is a generalized full-rate STTC construction, for any number of transmit antennas and any number of states of the trellis with NVD property, which is shown to achieve the upper bound on the optimal DM-G tradeoff for any number of receive antennas, neglecting the rate loss factor.

The correspondence is organized as follows. In Section II, we study the DM-G tradeoff of full-rate STTCs with NVD property. Also in that section we show that full-rate STTCs with NVD property achieves the upper bound on the optimal DM-G tradeoff for any number of receive antennas, neglecting the rate loss factor. The main result of the correspondence, generalized construction of full-rate STTCs is given in Section III and proof of the optimality of full-rate STTC constructed in Section III, under the DM-G tradeoff neglecting the rate loss factor, is given in Section IV. Section V contains concluding remarks.

II. Achieving the DM-G Tradeoff

A. Signal Alphabet

Following, the definitions in [11], let $\mathcal{S} \subset \mathbb{C}$ be an alphabet (where $\mathbb{C}$ is the field of complex numbers) that can be scaled so as to approximately contain ρ elements in a circle of squared radius ρ for large ρ, for example QAM-alphabet. Let r denote the normalized rate of transmission. Then $\mathcal{S}$ is called scalably dense, if it can be scaled with SNR in such a way that

$$|\mathcal{S}| \doteq \mathrm{SNR}^{\frac{r}{n_t}} \text{ and } a \in \mathcal{S}(\mathrm{SNR}) \Longrightarrow |a|^2 \dot{\leq} \mathrm{SNR}^{\frac{r}{n_t}}$$

where we have followed the exponential equality notation used in [1]

$$f(\mathrm{SNR}) \doteq \mathrm{SNR}^b \Longrightarrow \lim_{\mathrm{SNR}\to\infty} \frac{\log f(\mathrm{SNR})}{\log \mathrm{SNR}} = b$$

and similarly with $\dot{\geq}$ and $\dot{\leq}$. Let $\mathcal{X}$ denote a STTC and let $X_{n_t \times T} \in \mathcal{X}$ and $X = [X_{i,j}], \{i = 1, 2, \ldots, n_t,\ j = 1, 2, \ldots, T\}$. We shall call $\mathcal{X}$ to be $\mathcal{S}$-linear if every entry $X_{i,j}$ of each of code matrix X is of the form

$$X_{i,j} = \sum_{k=1}^{m} c_{ijk} v_{ijk}, \quad c_{ijk} \in \mathcal{S}$$

for some $v_{ijk} \in \mathbb{C}$. As X varies over $\mathcal{X}$, the c_{ijk} vary over all of $\mathcal{S}$. We will assume that the v_{ijk} remain fixed for all SNR.

Let $X, X' \in \mathcal{X}$ and $\Delta X = X - X'$. Any $X \in \mathcal{X}$ can be written as $X = [U\ M]$ and similarly ΔX can be written as $\Delta X = [\Delta U\ \Delta M]$, where U and ΔU are of size $n_t \times n_t$ and M and ΔM are of size

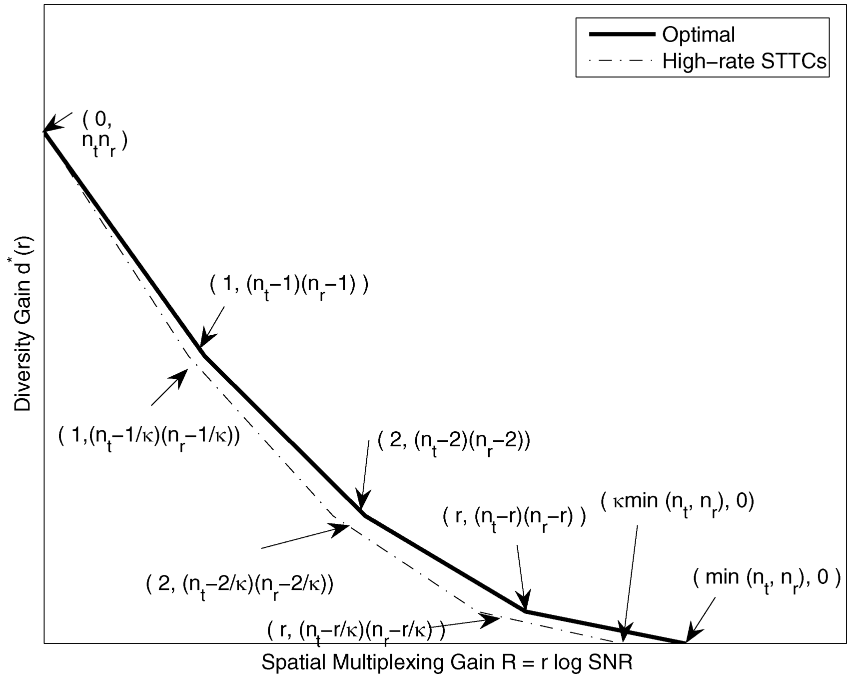

Fig. 3. Diversity-multiplexing tradeoff, $d(r)^*$ for general n_t, n_r for full-rate STTCs.

$n_t \times (T - n_t)$ respectively. A $\mathcal{S}$-linear STTC will be said to have non-vanishing determinant (NVD) property, if there is a constant μ independent of SNR, such that for SNR > 0,

$$\min_{X,X' \in \mathcal{X},\, X \neq X'} |det\left(\Delta U \Delta U^\dagger\right)| \geq \mu > 0.$$

An $\mathcal{S}$-linear STTC will be said to be of full-rate, if for every SNR, the size of the code satisfies

$$|\mathcal{X}| = |\mathcal{S}|^{n_t \kappa T}$$

where κ is the rate loss factor of the STTC.

B. Signal Energy

Let R be the rate of the STTC $\mathcal{X}$ in bits per channel use and $r \triangleq \frac{R}{\log(\mathrm{SNR})}$ be the normalized rate at high SNR [1]. For full-rate STTC $\mathcal{X}$ we have $n_t\kappa$ as the average number of constellation symbols transmitted per channel use i.e., $n_t \kappa T = \log_{|S|}|\mathcal{X}|$. Thus,

$$|\mathcal{X}| = 2^{TR} = \mathrm{SNR}^{Tr} = |S|^{n_t \kappa T}$$

and

$$M^2 \triangleq |\mathcal{S}| = \mathrm{SNR}^{\frac{r}{n_t \kappa}}.$$

Let $\omega_m = exp\left(\frac{j2\pi}{m}\right)$ and let

$$J = \{-(M-1), -(M-3), \ldots, -1, +1, \ldots, M-3, M-1\}.$$

We define the signal constellations,

$$\mathcal{A}_m = \{(a + \omega_m b) | a, b \in J\}.$$

Since the scaling factor θ in (1) is required to satisfy $E[\|\theta X\|_F^2] = n_t\mathrm{SNR}$, assuming that each component X_{ij} of X is uniformly drawn from the signal constellation $\mathcal{A}_m$, it can be shown that

$$E[|X_{ij}|^2] \doteq M^2.$$

So the normalizing factor satisfies

$$\theta^2 = \frac{\mathrm{SNR}}{n_t E[|x_{ij}|^2]} \doteq \mathrm{SNR}^{1-\frac{r}{n_t\kappa}}.$$

Theorem 1: Let $\mathcal{S}$ be an alphabet that is scalably dense. Consider a STTC $\mathcal{X}$ with $T \geq n_t$, which

- is $\mathcal{S}$-linear;
- has full-rank;
- has full-rate;
- has the NVD property.

Then the DM-G tradeoff curve $d(r)$ for $\mathcal{X}$ is given by piecewise-linear function connecting the points $(r, d(r))$, $r = l\kappa$, $l = 1, 2, \ldots, \min\{n_t, n_r\}$ where

$$d(r) \geq (n_t - r/\kappa)(n_r - r/\kappa)$$

and κ is rate loss factor for STTC.

The DM-G tradeoff curve for the full-rate STTC with NVD property is plotted in Fig. 3. The DM-G tradeoff curve for the full-rate STTC intersects the r axis at $\kappa \min\{n_t, n_r\}$. This shows that the maximum achievable spatial multiplexing gain, which the high-rate STTC can achieve, is given by $\kappa \min\{n_t, n_r\}$. Since $\kappa < 1$, it follows that the full-rate STTCs, cannot achieve the maximum possible spatial multiplexing gain $\min\{n_t, n_r\}$, with n_t transmit and n_r receive antennas. On the other hand, the DM-G tradeoff curve for the full-rate STTC intersects the d axis at the maximum diversity gain $n_t n_r$, corresponding to n_t transmit and n_r receive antennas. Therefore, clearly the full-rate STTCs with NVD property, achieves the maximum possible diversity gain possible.

Since $\kappa < 1$, the upper bound on the DM-G tradeoff for the full rate STTCs is always lower than the optimal DM-G tradeoff [1], for n_t, n_r transmit and receive antennas, respectively. For non integer values of l, the DM-G tradeoff for STTCs $d(r)$ (where $r = l\kappa$) is given by straight line interpolation of $d(r)$ between $r = \lfloor l \rfloor \kappa$ and $r = \lceil l \rceil \kappa$ respectively, where $\lfloor . \rfloor$ denotes the largest integer less than $(.)$ and and $\lceil . \rceil$ denotes the smallest integer greater than $(.)$.

This result is expected, since for STTC, there is a inherent rate loss due to the trailing zeros. Also, for STTC, T is very large and since $\delta < \lambda$ where 2^{λ} is the number of states of the STTC, $\kappa = \frac{T-\delta}{T}$ is very close to 1. Therefore, Theorem 1 shows that, ignoring the trailing zeros ($\kappa = 1$), high-rate STTCs achieve the upper bound on the optimal DM-G tradeoff curve for any number of transmit and receive antennas.

Before we give the proof of Theorem 1 we establish the following lemma which will be helpful in the proof.

Lemma 1: Let $\mathcal{X}$ be a full-rank STTC and let $\Delta X = X - X'$, be the difference matrix of $X, X' \in \mathcal{X}$ of dimension $n_t \times T$. Any ΔX can be written as $[\Delta U\ \Delta M]$, where ΔU is of size $n_t \times n_t$ and ΔM is of size $n_t \times (T - n_t)$. Written in this way, ΔU is always of full-rank.

Proof: Assume to the contrary that ΔU is not always full-rank. For the case of STTC, we can choose to truncate the STTC at any $T \geq n_t$, without compromising full rank. Let us make a choice of $T = n_t$ to truncate the STTC, then $\Delta X = \Delta U$, which is a valid codeword difference matrix from $\mathcal{X}$. Since ΔU is not full rank, ΔX is not full rank and hence STTC $\mathcal{X}$ is not full rank for $T \geq n_t$, leading to a contradiction. Therefore ΔU is of full rank. □

Proof: (Theorem 1) Let $\Delta X = X - X'$, be the difference matrix of $X, X' \in \mathcal{X}$ of dimension $n_t \times T$. Any ΔX can be written as $[\Delta U\ \Delta M]$, where ΔU is of size $n_t \times n_t$ and ΔM is of size $n_t \times (T - n_t)$. Since $\mathcal{X}$ is of full-rank, ΔU is always full-rank (Lemma 1). Also, let $\mathcal{U} = \{\Delta U : X \neq X', X, X' \in \mathcal{X}\}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n_t}$, $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n_t}$ and $l_1 \geq l_2 \geq \cdots \geq l_{n_t}$ be the ordered eigen values of $H^{\dagger}H$, $\Delta U \Delta U^{\dagger}$ and $\Delta X \Delta X^{\dagger}$, respectively.

Let d_E^2 denote the squared Euclidean distance between any two codewords from $\mathcal{X}$. Then we have

$$\begin{aligned} d_E^2 &= Tr(\theta^2 H \Delta X \Delta X^{\dagger} H^{\dagger}),\ (Tr \text{ is the Trace function}) \\ &= Tr\left(\theta^2 H \left[\Delta U \Delta U^{\dagger} + \Delta M \Delta M^{\dagger}\right] H^{\dagger}\right). \end{aligned}$$

Since $\Delta M \Delta M^{\dagger}$ is Hermitian

$$d_E^2 \geq Tr\left(\theta^2 H \Delta U \Delta U^{\dagger} H^{\dagger}\right) \triangleq d_{\Delta U}^2. \tag{3}$$

Our objective is to lower bound d_E^2. Let $\lambda_i = \mathrm{SNR}^{-\alpha_i}$. Ordering of λ_i imposes $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n_t}$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{n_t})$. Also, for a particular channel realization $\boldsymbol{\alpha}$, let $P_e(\alpha)$ be the codeword error probability, and as before W is the noise matrix. Then

$$P_e(\boldsymbol{\alpha}) \leq Pr\{\|W\|^2 > \frac{d_{E,min}^2(\boldsymbol{\alpha})}{4}\} \triangleq P_{e,\Delta X}(\alpha)$$

where $d_{E,min}^2$ is the minimum squared Euclidean distance between any two distinct codewords from $\mathcal{X}$. The average codeword error probability P_e can be computed from $P_e(\alpha)$ by taking the expectation over the eigenvalues of $H^{\dagger}H$. Recall that

$$d(r) = -\lim_{\mathrm{SNR}\to\infty} \frac{\log(P_e)}{\log \mathrm{SNR}}. \tag{4}$$

From (3), $d_{E,\min}^2(\boldsymbol{\alpha}) \geq d_{\Delta U,\min}^2(\boldsymbol{\alpha})$, and hence in the worst case

$$P_e(\boldsymbol{\alpha}) \leq Pr\{\|W\|^2 > \frac{d_{\Delta U,min}^2(\boldsymbol{\alpha})}{4}\} \triangleq P_{e,\Delta U}(\alpha) \tag{5}$$

where

$$d_{\Delta U,min} = \min_{\Delta U} d_{\Delta U}.$$

This allows us to transform our original problem of lower bounding d_E^2, to lower bound $d_{\Delta U}^2$. Let us define

$$d_U(r) \triangleq \lim_{\mathrm{SNR}\to\infty} \frac{\log(P_{\Delta U})}{\log \mathrm{SNR}}$$

where $P_{\Delta U}$ is obtained by taking expectation of $P_{e,\Delta U}(\alpha)$ over the eigenvalues of $H^{\dagger}H$.

Since the STC $\mathcal{U}$ is of dimension $n_t \times n_t$, we can use the results from [11]. Also, we have $\mathcal{U}$ to be STC with dimension $n_t \times n_t$ and with NVD property and also $\theta^2 = \mathrm{SNR}^{1-\frac{r}{n_t\kappa}}$. Carrying out the analysis as similar to Theorem 1 [11] for $n_r \geq n_t$, on (5), to lower bound d_U^2, we have

$$d_U(r) \geq \inf_{\alpha\in\mathcal{B}} \sum_{i=1}^{n_t} (2i - 1 + n_r - n_t)\alpha_i$$

where the set $\mathcal{B}$ is given by

$$\mathcal{B} = \{\alpha : \alpha_i \geq 0\ \forall i, \delta_j \leq 0, j = 0, 1, \ldots, n_t - 1\}$$

and

$$\delta_j(\alpha) = 1 - \frac{r}{\kappa(j+1)} - \sum_{i=n_t-j}^{n_t} \left(\frac{\alpha_i}{j+1}\right).$$

Since $P_{e,\Delta X}(\alpha) \leq P_{e,\Delta U}(\alpha)$, clearly, $d(r) \geq d_{\Delta U}(r)$.

Evaluating the above infimum for $(l-1)\kappa \leq r \leq l\kappa$, where $l = 1, 2, \ldots, n_t$, we get,

$$d(r) \geq \sum_{i=1}^{n_t-l} (2i - 1 - n_r - n_t) + [2(n_t - l + 1) - 1 + n_r - n_t](l - \frac{r}{\kappa}).$$

The values for the non-integral r where $\{(l-1)\kappa < r < l\kappa\}$ are obtained by straight line equation above and for integral values $r = l\kappa$ we have

$$d(l\kappa) \geq (n_t - l)(n_r - l)$$

and for $r = (l-1)\kappa$

$$d((l-1)\kappa) \geq (n_t - l + 1)(n_r - l + 1).$$

Now by letting $p = l\kappa$, we have

$$\begin{aligned} d(p) &\geq (n_t - p/\kappa)(n_r - p/\kappa) \\ d(p-\kappa) &\geq (n_t - p/\kappa + 1)(n_r - p/\kappa + 1). \end{aligned}$$

Also, by making use of the results from Appendix 3 [11], the case when $n_r < n_t$ can be proved similarly. □

III. High-Rate STTCs

The shift register model for our full-rate STTCs is shown in Fig. 4. We assume that the state complexity of the STTC is 2^{λ}. Let the rate of transmission be $n_t b$ bits/s/Hz (neglecting the trailing zeros). From [2], $\lambda \geq n_t b(n_t - 1)$ for full diversity. Let $\lambda = q n_t b + k : k < n_t b$. In Fig. 4, each $B_i,\ i \in (1, 2, \ldots q+1)$ represents $n_t b$ bits. The most significant γ bits, where $\gamma = (q - n_t + 1)n_t b + k$, is denoted by B_γ in the figure. The shift register keeps shifting to the left $n_t b$ bits at a time. The past $\gamma + (n_t - 1)n_t b$ bits $(B_\gamma, B_{n_t}, B_{n_t-1}, \ldots, B_2)$ represent the state of the trellis and the current $n_t b$ bits, which have entered the shift register, represent one of the $2^{n_t b}$ branches diverging out of a state. It is easy to verify, that for the STTC so constructed, the convergence length $l_c = \lfloor \frac{\lambda}{n_t b} \rfloor + 1 = q + 1$, since $q \geq n_t - 1$, $l_c \geq n_t$.

For each of the $B_i,\ i \in (1, 2, \ldots q+1)$, we divide the $n_t b$ bits into n_t groups of b bits each. We represent each of these n_t groups by $b_{i,j}$ for $i = 1, 2, \ldots q+1$ and $j = 0, 1, 2, \ldots, n_t - 1$. By using some bijective map, each of these $b_{i,j}$ bits are mapped to $x_{i,j}$, where $x_{i,j} \in \mathcal{S}$ with $|\mathcal{S}| = 2^b$. At time t we represent $x_{i,j}$ by $x_{i,j}^t$. For each of $B_i,\ i = 1, 2, \ldots q+1$, we feed these constellation points for all the n_t groups to a map π, whose output is $\pi_i^t = (\sum_{j=1}^{n_t} x_{ij}^t \delta_j)$ at time t, for $i = 1, 2, \ldots, q+1$, where $\delta_j \in \mathbb{C}\ \forall j$. Each of the π_i for

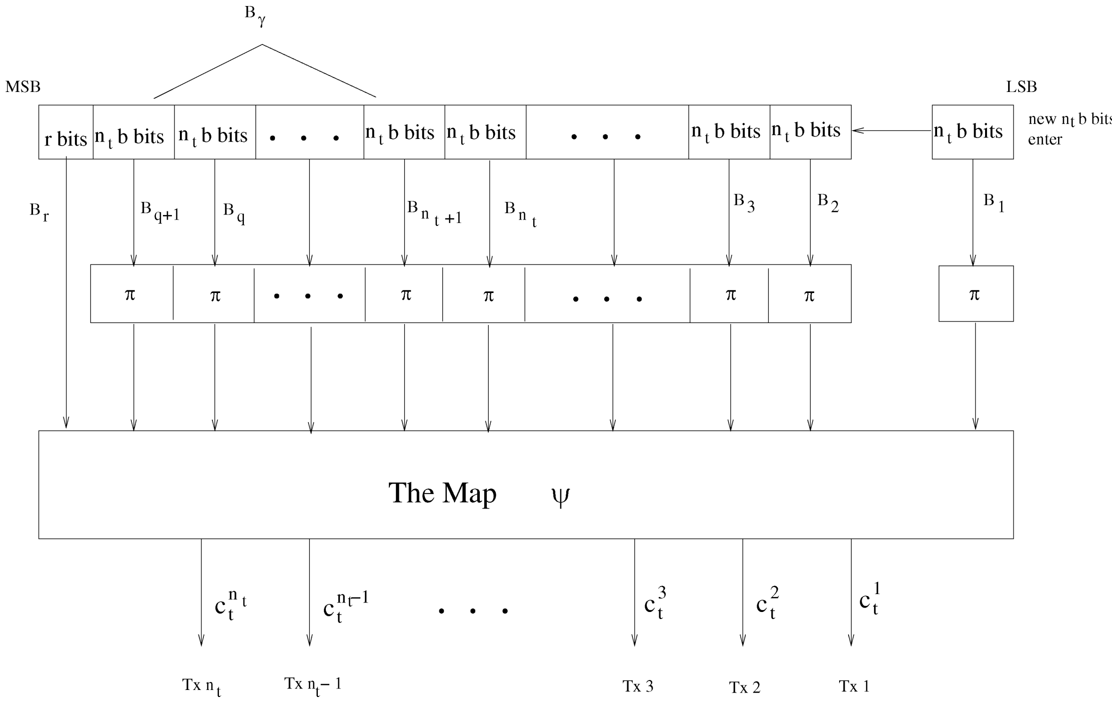

Fig. 4. Shift register model for high-rate STTC.

$i = 1, 2, \ldots q+1$, are fed to the map $\mathbf{\Psi}$, to determine what is to be transmitted from each of the transmit antenna. We here give a particular choice of $\mathbf{\Psi}$ to guarantee that the STTC so constructed is full-rank.

Lemma 2: Let $F = \mathbb{Q}(j)$, where $\mathbb{Q}$ is the field of rational numbers and $j = \sqrt{-1}$. Then from Theorem 6 and 7 [11], for any $n_t = 2^m \prod_{i=2}^{l} p_i^{m_i}$ where p_i is an arbitrary prime, there exist a cyclic Galois extension K with degree n_t over F.

Theorem 2: Let $F = \mathbb{Q}(j)$ and $\delta_1, \delta_2, \ldots \delta_{n_t}$ be the integral basis of K/F, where K is the cyclic Galois extension of F of degree n_t and σ be the generator of the cyclic Galois group denoted by $Gal(K/F)$ (Lemma 2 shows that it exists). In Fig. 4, let $x_{i,j}^t \in \mathcal{S} \in F$ and if $\mathbf{\Psi}$ is such that, $c_t^{(i)} = \pi_i^t = \sigma^{i-1}\left(\sum_{j=1}^{n_t} x_{i,j}^t \delta_i\right)$ for $i = 1, 2, \ldots n_t$, where $c_t^{(i)}$ is the transmitted signal from the i^{th} transmit antenna in the t^{th} time instant, then the STTC $\mathcal{X}$ so constructed achieves full diversity.

Proof: Consider two distinct codewords matrices C and E such that $C, E \in \mathcal{X}, C \neq E$. Let their difference matrix $B(C,E) = E - C$, be

$$\begin{bmatrix} e_1^1 - c_1^1 & \cdots & e_{t_d}^1 - c_{t_d}^1 & \cdots & e_{t_c}^1 - c_{t_c}^1 & \cdots & e_T^1 - c_T^1 \\ e_1^2 - c_1^2 & \cdots & e_{t_d}^2 - c_{t_d}^2 & \cdots & e_{t_c}^2 - c_{t_c}^2 & \cdots & e_T^2 - c_T^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ e_1^{n_t} - c_1^{n_t} & \cdots & e_{t_d}^{n_t} - c_{t_d}^{n_t} & \cdots & e_{t_c}^{n_t} - c_{t_c}^{n_t} & \cdots & e_T^{n_t} - c_T^{n_t} \end{bmatrix}$$

where T, is the frame length of the STTC. Since at the beginning of the frame and the end of the frame, the state of the encoder is zero, the trellis path corresponding to the two codewords considered, should diverge and converge at a later time, assume that t_d is the time when the paths corresponding to the two codewords diverge, and t_c is the time when they converge. Then the first $t_d - 1$ columns and the last $T - t_c$ columns of the above matrix are zero. Thus we need to consider only the columns from t_d to t_c. Note that the convergence length $l_c = q + 1$ and hence $l_c \geq n_t$. We have $t_d + t_c - 1$ columns to consider. We see that $t_c - t_d + 1 \geq n_t$, since the convergence length is at least $n_t(\gamma \geq 0)$ and the number of non-zero columns here is at least the convergence length. Let us call the matrix that contains the non-zero columns of $B(C,E)$ as $B^{\text{eff}}(C,E)$. Let the total number of columns of matrix $B^{\text{eff}}(C,E)$ be l, $(l = t_c - t_d + 1 \geq l_c = n_t)$. We take the worst case of l being equal to l_c. Therefore the matrix $B^{\text{eff}}(C,E)$ has dimensions $n_t \times l_c$.

Our aim is to lower bound the rank of the matrix $B^{\text{eff}}(C,E)$. We can write $B^{\text{eff}}(C,E) = [U\ M]$, where U and M are of dimensions $n_t \times n_t$ and $n_t \times (l_c - n_t)$, respectively. Since $\operatorname{rank}(B^{eff}(C,E)) = \operatorname{rank}(U)$, for STTC to achieve full diversity, it is sufficient to show that the matrix U is of full-rank $(\operatorname{rank}(U) = n_t)$.

We will prove the theorem in two parts, in the first part, we show that the matrix U is upper triangular (i.e., all the elements below the main diagonal are zero) and in the second part we show that all the diagonal elements of the matrix U are non-zero.

Part 1) By definition

$$U = \begin{bmatrix} e_{t_d}^1 - c_{t_d}^1 & e_{t_d+1}^1 - c_{t_d+1}^1 & \cdots & e_{t_d+n_t-1}^1 - c_{t_d+n_t-1}^1 \\ e_{t_d}^2 - c_{t_d}^2 & e_{t_d+1}^2 - c_{t_d+1}^2 & \cdots & e_{t_d+n_t-1}^2 - c_{t_d+n_t-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ e_{t_d}^{n_t} - c_{t_d}^{n_t} & e_{t_d+1}^{n_t} - c_{t_d+1}^{n_t} & \cdots & e_{t_d+n_t-1}^{n_t} - c_{t_d+n_t-1}^{n_t} \end{bmatrix}.$$

The i,jth entry of U is

$$e_{t_d+j-1}^{(i)} - c_{t_d+j-1}^{(i)} = \sigma^{i-1}\left(\sum_{k=0}^{n_t-1}(x_{i,k}^{j+i-1} - x_{i,k}^{\prime\, j+i-1})\delta_k\right)$$

where at least for one $k, x_{i,k} \neq x'_{i,k}$ and $i = 1, 2, \ldots, n_t$, $j = t_d, t_d+1, t_d+n_t-1$. Since at time t_d, when there is divergence for codewords C and E, all the bits in $B_i^{t_d}$ for $i = 2, 3, \ldots, \gamma$ are same. Also from Fig. 4, it is clear that, $x_{i,j}^{t_d}$ depends only on $B_i^{t_d}$. Therefore, $c_{t_d+t}^{(i)} = e_{t_d+t}^{(i)}$ for $i > t+2, i = 1, 2, \ldots n_t$; $t = 0, 1, \ldots n_t - 2$. Hence all the elements of matrix U below the main diagonal are zero.

Part 2) Now, consider the diagonal elements of U. These are of the form,

$$e_t^{(i)} - c_t^{(i)} = \sigma^{i-1}\left(\sum_{k=0}^{n_t-1}\left(x_{i,k}^t - x_{i,k}^{\prime t}\right)\delta_j\right)$$

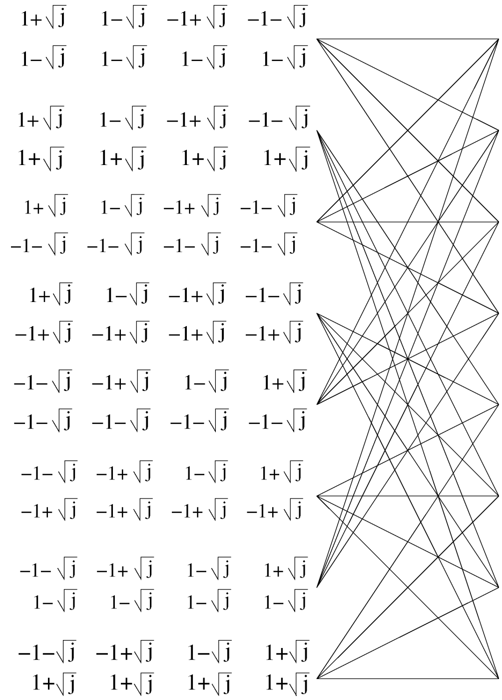

Fig. 5. New code for 2 Tx, BPSK, 2 bps/Hz, eight-state STTC.

for $i = 1, 2, \ldots n_t$ and $t = t_d, t_d + 1, \ldots t_d + n_t - 1$. Since $\sigma \in Gal(K/F)$, σ is an automorphism from $K \to K$

$$\sigma^{i-1}\left(\sum_{k=0}^{n_t-1}(x_{i,k}^t - x_{i,k}'^t)\delta_j\right) = 0$$

if and only if

$$\left(\sum_{k=0}^{n_t-1}(x_{i,k}^t - x_{i,k}'^t)\delta_j\right) = 0.$$

Since $C \neq E$, at least for one value of k, $(x_{i,k}^t \neq x_{i,k}'^t)$, hence

$$\left(\sum_{j=0}^{n_t-1}(x_{i,j}^t - x_{i,j}'^t)\delta_j\right) \neq 0$$

for $i = 1, 2, \ldots n_t$ and $t = t_d, t_d + 1, \ldots t_d + n_t - 1$. Therefore, all the diagonal elements of the matrix U, are of the form

$$e_t^{(i)} - c_t^{(i)} = \sigma^{i-1}\left(\sum_{j=0}^{n_t-1}(x_{i,j}^t - x_{i,j}'^t)\delta_j\right) \neq 0.$$

We have shown that, $rank(U) = n_t$, implying that $rank\left(B^{eff}(C, E)\right) = n_t$ and hence the STTC $\mathcal{X}$ achieves full diversity. □

Example 1: Suppose we need to construct a STTC for two transmit antennas with rate 2 bits/s/Hz. We choose the BPSK signal set and for the case of 4 states trellis, we have parameters $\lambda = 2$, $b = 1$ and $\gamma = \eta = 0$. For $F = \mathbb{Q}(j)$ and $n_t = 2$, from [11] we have $K = \mathbb{Q}(\omega_8)$ where $\omega_n = exp(\frac{-j2\pi}{n})$. The basis of $\mathbb{Q}(\omega_8)$ over $\mathbb{Q}(j)$ is $\{1, \sqrt{j}\}$ and $\sigma \in Gal(K/F)$ is such that $\sigma : \sqrt{j} \to -\sqrt{j}$. Then from Theorem 2 we have a full rank STTC, which is same as given in [12] and also as shown in Fig. 2.

Example 1 shows that the code construction in [12], is a special case of the generalized full-rate STTC construction of this correspondence. With $\lambda = 2$, $b = 1$ and changing $\gamma = \eta = 1$, we have a 8 state STTC for two transmit antennas with rate 2 bits/s/Hz as shown in Fig. 5,

Example 2: For the case of three transmit antennas, rate 3 bits/s/Hz, we have $b = 1$ for BPSK signal set and $\lambda \geq 6$. If we need to construct a 64–state STTC for three transmit antennas with 3 bits/s/Hz, we have $\lambda = 6$, $\gamma = \eta = 0$. So, we take $F = \mathbb{Q}(\omega_3, \omega_{12})$. Then the polynomial $x^3 - \omega_{12}$ is irreducible over F. Hence ω_{36} is a root of the polynomial $x^3 - \omega_{12}$ and the automorphism is given by $\sigma : \omega_{36} \to \omega_{36}w_3$. For the sake of brevity we do not draw the trellis, but it can be easily constructed with all these parameters from Theorem 2.

IV. DM-G Tradeoff of Full-Rate STTCs

In this section we study the DM-G tradeoff of the proposed full-rate STTC construction. From the previous section, $B^{\text{eff}}(C, E)$ can be written as $B^{\text{eff}}(C, E) = [U\ M]$ where U is an upper triangular matrix of dimension $n_t \times n_t$ and M is a matrix of dimension $n_t \times (l_c - n_t)$. As proved in Theorem 2 for our proposed full-rate STTCs, the matrix U is upper triangular and full rank. Hence U is of the form

$$U = \begin{bmatrix} e_{t_d}^1 - c_{t_d}^1 & e_{t_d+1}^1 - c_{t_d+1}^1 & \cdots & e_{t_d+n_t-1}^1 - c_{t_d+n_t-1}^1 \\ 0 & e_{t_d+1}^2 - c_{t_d+1}^2 & \cdots & e_{t_d+n_t-1}^2 - c_{t_d+n_t-1}^2 \\ \vdots & 0 & \ddots & e_{t_d+n_t-1}^{n_t-2} - c_{t_d+n_t-1}^{n_t-2} \\ \vdots & \vdots & \vdots & e_{t_d+n_t-1}^{n_t-1} - c_{t_d+n_t-1}^{n_t-1} \\ 0 & 0 & 0 & e_{t_d+n_t-1}^{n_t} - c_{t_d+n_t-1}^{n_t} \end{bmatrix}.$$

It is easy to verify that

$$\begin{aligned} det(U) &= \prod_{i=1}^{n_t}\left(e_{t_d+i-1}^i - c_{t_d+i-1}^i\right) \\ &= \prod_{i=1}^{n_t}\sigma^{i-1}\left(\sum_{j=0}^{n_t-1}(x_{i,j}^{t_d+i-1} - x_{i,j}'^{t_d+i-1})\delta_j\right). \end{aligned}$$

Since σ is an automorphism of degree n_t, we have $\sigma(det(U)) = det(U)$. Thus, $det(U)$ belongs to $F = \mathbb{Q}(j)$. Moreover, with the signal set $\mathcal{S} \subset \mathbb{Z}[j]$ ($\mathbb{Z}[j] = a + jb : a, b \in \mathbb{Z}$) and the basis $\{\delta_1, \delta_2, \ldots \delta_{n_t}\}$ an integral basis, we have $det(U) \in \mathbb{Z}[j]$ [17]. Thus, $|det(U)| \geq 1$, and thus the STTC $\mathcal{X}$ has the NVD property. Since, $\mathcal{X}$ is $\mathcal{S}$-linear, has full-rate and has the *NVD* property, from Theorem 1, STTC $\mathcal{X}$ achieves the upper bound on the optimal DM-G tradeoff neglecting the rate loss factor, which is very small.

As discussed in the last section, STTC constructed for two transmit antennas [12] is a special case of our proposed full-rate STTC construction. It is shown in [12] only by simulation, that the high-rate STTC for 2 transmit antennas achieves the DM-G tradeoff, where as Theorem 1 proves this analytically, neglecting the rate loss factor.

V. Conclusion

In this correspondence, we derive the bound on the the DM-G tradeoff of full-rate STTCs with NVD property. We show that the full-rate STTCs with NVD property are optimal under the DM-G tradeoff for any number n_t of transmit antennas and any number n_r of receive antennas, neglecting the rate loss factor. In the process of bounding the DM-G tradeoff of full-rate STTCs with NVD property, we generalize the result that for $T = n_t$, STCs with full-rate and NVD property achieves the upper bound on optimal DM-G tradeoff [9]–[11], to the case $T \geq n_t$ where it is sufficient that the matrix formed by the first n_t columns of the full-rate STC has the NVD property to achieve the upper bound on the DM-G tradeoff.

Also, we show that the existing schemes to construct STTCs, do not achieve the optimal diversity-multiplexing tradeoff, for n transmit and n receive antennas ($n \geq 2$), except for the STTC [12]. We then propose a full-rate STTC construction, which is shown to achieve the upper bound of the optimal DM-G tradeoff for any number of transmit and receive antennas, neglecting the rate loss factor. We also show that the STTC [12] is a special case of our proposed high-rate STTC.

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