Asymptotically Optimal Multiple-access Communication via Distributed Rate Splitting

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Abstract—We consider the multiple-access communication problem in a distributed setting for both the additive white Gaussian noise channel and the discrete memoryless channel. We propose a scheme called Distributed Rate Splitting to achieve the optimal rates allowed by information theory in a distributed manner. In this scheme, each real user creates a number of virtual users via a power/rate splitting mechanism in the M-user Gaussian channel or via a random switching mechanism in the M-user discrete memoryless channel. At the receiver, all virtual users are successively decoded. Compared with other multiple-access techniques, Distributed Rate Splitting can be implemented with lower complexity and less coordination. Furthermore, in a symmetric setting, we show that the rate tuple achieved by this scheme converges to the maximum equal rate point allowed by the information-theoretic bound as the number of virtual users per real user tends to infinity. When the capacity regions are asymmetric, we show that a point on the dominant face can be achieved asymptotically. Finally, when there is an unequal number of virtual users per real user, we show that differential user rate requirements can be accommodated in a distributed fashion.

Index Terms—Multiple access, rate splitting, successive decoding, stripping, interference cancellation, ALOHA.

I. INTRODUCTION

We consider the basic multiple-access communication problem in a distributed setting. In Gallager's survey paper [1], it is pointed out that the multiple-access problem has been studied from a number of different perspectives, each having its own advantages and shortcomings. In the data networking community, a well-known distributed multiple access scheme is ALOHA [2], [3]. In ALOHA, it is assumed that a "collision" happens whenever more than one user transmit simultaneously. Those packets involved in a collision are discarded and retransmitted according to some retransmission probability. The collision channel model, however, does not accurately describe the underlying physical multiple-access channel. It is well known that there exist coding techniques which can decode multiple users' messages when simultaneous transmissions occur. Indeed, more sophisticated models such as signal capture [4], [5], spread ALOHA [6], and multi-packet reception [7], [8], [9] have been developed to enhance ALOHA. Even

these improved schemes, however, are not optimal from the viewpoint of information theory. In the information theory literature, the capacity regions of various multiple-access channel (MAC) models have been characterized (see [10], [11], [12], [13], [14], [15]). Rate splitting multiple-access techniques (or generalized time-sharing) are presented in [16], [17], [18], [19] to achieve every point in the Gaussian or the discrete memoryless MAC capacity region using only single-user codes. These schemes, however, require a pre-defined decoding order, which makes distributed implementation difficult. Finally, in the spread spectrum community, CDMA techniques are adopted. Here, users are decoded regarding all other users' signals as interference. This, however, is not optimal from the information theoretic viewpoint.

To address some of the shortcomings mentioned above, Medard *et al.* [20] use information-theoretic techniques to analyze different notions of capacity for time-slotted ALOHA systems. A coding/decoding scheme which combines rate splitting and superposition coding is constructed. This scheme allows some bits to be reliably received even when collision occurs, and more bits to be reliably received in the absence of collisions. Shamai [21] proposes a similar scheme to apply a broadcast strategy to multiple-access channel under static fading where the fading coefficients are not available to the transmitters or the receiver. To implement the scheme in [20], however, a pre-defined decoding order is required, as in [16], [17], [18], [19]. In [22], Cheng proposes a distributed scheme called "stripping CDMA" for the L out of K Gaussian MAC. Here, no pre-defined decoding order is required. It is shown in [22] that stripping CDMA is asymptotically optimal, although the optimal operating parameters are not specified.

In this paper, we investigate distributed multiple-access schemes based on the idea of rate splitting for both the M-user additive white Gaussian noise MAC and the M-user discrete memoryless MAC. We characterize the optimal operating parameters as well as the asymptotic optimality of these schemes from the viewpoint of information theory. Assume that every user has an infinite backlog of bits to send, and that every user knows the total number of users M. We propose a distributed scheme, called *Distributed Rate Splitting* (DRS), to achieve the optimal communication rates allowed by information theory. In this scheme, each real user creates a number of virtual users via a power/rate splitting mechanism in the M-user Gaussian channel or via a random switching mechanism in the M-user discrete memoryless channel. At the receiver, all virtual users are successively decoded. A possible advantage of the DRS scheme is that it can be implemented with lower complexity when compared with multiple-access

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schemes such as joint coding[1] and less coordination when compared with time-sharing and rate splitting.

In Sections II and III, we focus first on symmetric situations where the channel capacity regions are symmetric and every real user creates the same number of virtual users. In this case, the DRS scheme entails the following. Each user i creates L virtual users indexed by i_k, $k = 1, 2, ..., L$. The virtual user class V_k consists of users $\{1_k, ..., M_k\}$ (i.e. we have altogether L virtual user classes and there are M virtual users in each class). In the M-user Gaussian MAC, virtual users are created via a power/rate splitting mechanism. The signal transmitted by a real user is the superposition of all its virtual users' signals. The receiver receives the sum of the virtual users' signals plus noise. All virtual users are then successively decoded in increasing order of their class. That is, all virtual users in class V_k, $k = 1, ..., L$, are decoded before any virtual user in V_j, where $j > k$, is decoded. In contrast to [22], the optimal operating parameters, such as power and rate, are explicitly specified for any finite L. In the M-user discrete memoryless MAC, virtual users with the same input distribution as the real users are created, and the transmitted signal of a real user is determined by a random switch. The receiver successively decodes all virtual users in increasing order of their class given the side information of already decoded virtual users. The optimal switch is found for any finite L for the 2-user case. Finally, it is shown that for both channel models, the rate tuple achieved by the DRS scheme converges to the maximum equal rate point allowed by the information-theoretic bound as the number of virtual users per real user tends to infinity.

Next, in Section IV, we consider more general situations where the capacity regions can be asymmetric and real users may generate different numbers of virtual users. For the case of asymmetric capacity regions, new operating parameters are specified for any finite number of virtual users per real user. We show that the DRS scheme still can achieve a point on the dominant face as the number of virtual users per real user tends to infinity. For the case of unequal number of virtual users per real user, we present a variation of DRS which supports differential user rate requirements in a distributed manner. In this new scheme, each user i, independently from other users, generates L_i virtual users according to its own rate requirement. All virtual users are then decoded reliably at the receiver. Furthermore, as each real user generates more virtual users, the rate tuple achieved under this variation of DRS converges the maximum equal rate point on the dominant face.

II. M-USER GAUSSIAN MULTIPLE-ACCESS CHANNEL

We first examine a Gaussian MAC with a symmetric capacity region. Later in Section IV, we consider the asymmetric case. Consider an M-user Gaussian MAC where each transmitter has transmission power P and the receiver has
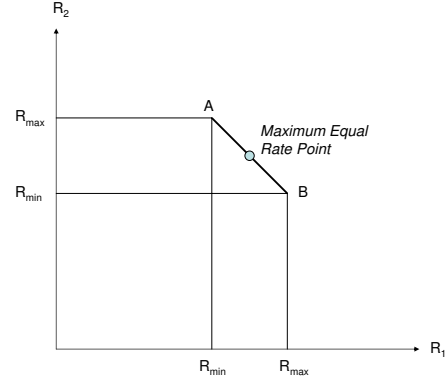

Fig. 1.   Two-user Gaussian multiple-access capacity region, where the dominant face has been highlighted.

noise variance N. The capacity region $\mathcal{C}$ is the set of $\mathbf{R} = (R_1, ..., R_M) \in \mathbb{R}_+^M$ satisfying[2]

$$\sum_{i \in S} R_i \le \frac{1}{2} \log\left(1 + \frac{|S|P}{N}\right) \ \forall S \subseteq \{1, ..., M\}, \quad (1)$$

where $|S|$ is the cardinality of the set S. The *dominant face* $\mathcal{D}$ is the subset of rate tuples which gives equality in (1) for $S = \{1, ..., M\}$. For this symmetric setting, it is easy to see that the maximum common rate that every user can achieve[3] is $R^* = \frac{1}{2M} \log(1 + \frac{MP}{N})$.

It is well-known that rate tuples on the dominant face other than the vertices cannot be achieved via standard successive decoding [11]. Note that the optimal rate tuple $\mathbf{R}^* \equiv (R^*, ..., R^*)$, called the *maximum equal rate point*, is such a point. For the two-user Gaussian MAC, the maximum equal rate point is shown in Fig. 1. Currently, three methods are known to achieve general points on the dominant face: joint encoding/decoding, time-sharing, and rate-splitting. Joint encoding/decoding is not practical because of its high complexity [1]. In time-sharing, all M users need to coordinate their transmissions. Therefore, some communication overhead is required. The rate-splitting method in [16] achieves every point in $\mathcal{C}$ via a generalized successive decoding scheme. For the two-user case, user 1 creates two virtual users, say $1a$ and $1b$, by splitting its power P into δ and $P - \delta$ and setting $r_{1a} = \frac{1}{2} \log(1 + \frac{\delta}{2P - \delta + N})$, $r_{1b} = \frac{1}{2} \log(1 + \frac{P - \delta}{N})$. User 2 does not split its power and sets its rate to $R_2 = \frac{1}{2} \log(1 + \frac{P}{P - \delta + N})$. The decoding order is $(1a, 2, 1b)$. In order to achieve the maximum equal rate point, we solve $R_2 = r_{1a} + r_{1b}$, yielding $\delta = \frac{1}{2}(N + 2P - \sqrt{N(N + 2P)})$. Thus, both time-sharing and rate splitting require some coordination among users.

In this paper, we focus on *distributed* multiple-access communication schemes. In particular, we introduce the *Distributed Rate Splitting* (DRS) scheme. The DRS scheme offers the possibility of multiple-access communication with lower complexity when compared with joint coding, and communi-

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[1]The lower complexity comes from the fact that the DRS scheme uses single-user codes instead of multi-user codes. As we show later, the DRS scheme with a reasonable number of virtual users per real user allows us to get close to the optimal operating rates.

[2]We use $\mathbf{R}$ to denote $(R_1, ..., R_M)$ throughout.

[3]$\mathbf{R}$ is achievable if for any $\varepsilon > 0$, there exists an $(n, R_1 - \varepsilon, ..., R_M - \varepsilon)$ multiple-access code with overall error probability $P_e < \varepsilon$, where n is the block length.
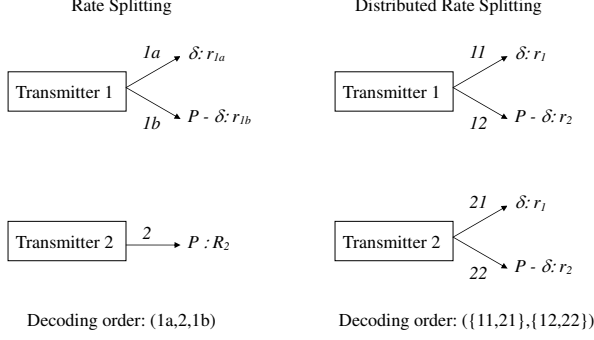
Fig. 2. Comparison of rate splitting with distributed rate splitting. In the right-hand figure, the virtual user class $V_1 = \{11, 21\}$ is decoded before virtual user class $V_2 = \{12, 22\}$.

cation with less coordination when compared with the time-sharing or rate splitting method. Moreover, we show that DRS can achieve the maximum equal rate point of the MAC capacity region asymptotically.

We now formally present the DRS scheme. In this scheme, each user creates L virtual users by splitting its power P into $(p_1, p_2, ..., p_L)$, where p_k is the power allocated to the kth virtual user and $\sum_{k=1}^{L} p_k = P$. Each user then assigns transmission rate r_k to virtual user k. Note that the proposed DRS scheme is symmetric, i.e. all M users split their powers and set their rates in the same way. The signal transmitted by a user is the superposition of its virtual users' signals. As defined in Section I, virtual user class V_k consists of all virtual users indexed by k. The receiver receives the sum of all virtual users' signals plus noise. All virtual users are then successively decoded in increasing order of their class.

To illustrate the DRS scheme, consider the case $L = 2$. Each real user splits its power P into δ and $P - \delta$. Notice there are two major differences between our scheme and the traditional rate splitting scheme in [16]. First, in our scheme, all real users split in the same way, whereas there is at least one user who does not split in the traditional rate splitting scheme. Second, virtual users in the same class, (i.e. with the same index k), are allocated the same rate in our scheme, whereas all virtual users have different rates according to the pre-defined decoding order in the traditional rate splitting scheme.[4] These differences are illustrated in Fig. 2.

Since we assume the receiver uses successive decoding method, some virtual user must be decoded first. Without loss of generality we assume one of the δ virtual users is decoded first. For the case $L = 2$, we show that there is a unique way for a real user to split its power in order to maximize its total throughput.

Lemma 1: For $L = 2$ and for a fixed δ, each real user's throughput is maximized by setting $r_1 = \frac{1}{2}\log(1 + \frac{\delta}{MP - \delta + N})$ and $r_2 = \frac{1}{2}\log(1 + \frac{P - \delta}{(M-1)(P-\delta)+N})$.

[4]In terms of achievable rate, the DRS scheme with $L = 2$ is not optimal. Later in this section, we demonstrate the asymptotic optimality of DRS by taking L to infinity.

Proof: The δ virtual user who is decoded first must have $r_1 = \frac{1}{2}\log(1 + \frac{\delta}{MP - \delta + N})$ (i.e. the virtual user regards all other virtual users as interference) in order to be decoded successfully. Due to symmetry, all other δ virtual users must have the same r_1. Then the problem of maximizing each real user's throughput reduces to $\max r_2$, subject to (i) $r_1 = \frac{1}{2}\log(1 + \frac{\delta}{MP - \delta + N})$, (ii) one of the δ virtual users is decoded first and (iii) $(r_1, r_2, r_1, r_2, ..., r_1, r_2)$ must be decodable.

Note that r_2 is maximized when the interference plus noise faced by all the $(P - \delta)$ virtual users is minimized, and the only way to minimize the interference plus noise faced by all the $(P - \delta)$ virtual users is to decode all the δ virtual users before decoding any $(P - \delta)$ virtual user.[5] Therefore, the minimum interference plus noise faced by any $(P - \delta)$ virtual user is $MP - M\delta - (P - \delta) + N = (M - 1)(P - \delta) + N$. Hence, the maximum rate associated with a $(P - \delta)$ virtual user is $r_2 = \frac{1}{2}\log(1 + \frac{P - \delta}{(M-1)(p-\delta)+N})$. $\square$

Using the DRS scheme with $L = 2$, each user can strictly increase its throughput relative to the case where users do not split their powers and decode against each other as noise. This is easily verified by observing that for any $\delta < P$,

$$\log\left(1 + \frac{\delta}{MP - \delta + N}\right) + \log\left(1 + \frac{P - \delta}{(M-1)(P-\delta)+N}\right)$$
$$> \log\left(1 + \frac{P}{(M-1)P + N}\right). \quad (2)$$

Now consider the case where each user creates more than two virtual users $(L > 2)$. Here, we show that each user's throughput increases further.

Lemma 2: Given a DRS scheme with L virtual users per real user, where $(p_1, ..., p_L)$ are the virtual users' powers, it is possible to strictly increase the throughput via an $(L + 1)$ virtual user system with powers $(p_1, ..., p_{L-1}, p'_L, p''_L)$, where $p'_L + p''_L = p_L$.

Proof: Suppose that every user splits its power into L virtual users: $(p_1, p_2, ..., p_{L-1}, p_L)$ subject to $\sum_{k=1}^{L} p_k = P$, where L is an arbitrary integer and p_k is the power of kth virtual user. Since virtual user L is decoded last, following the reasoning in the proof of Lemma 1, we have $r_L = \frac{1}{2}\log(1 + \frac{p_L}{(M-1)p_L + N})$. We now split the virtual user with power p_L into two new virtual users with powers p'_L and p''_L, where $p'_L + p''_L = p_L$. We set

$$r_{L'} = \frac{1}{2}\log\left(1 + \frac{p'_L}{Mp_L - p'_L + N}\right)$$
$$r_{L''} = \frac{1}{2}\log\left(1 + \frac{p''_L}{(M-1)p''_L + N}\right).$$

Now each real user has $L + 1$ virtual users. Notice that we do not change the power and decoding order of any of the other virtual users (i.e. virtual users $1, ..., L - 1$). From a real user's view point, the virtual user with $r_{L'}$ is decoded second

[5]We assume a genie-aided [16] decoding scheme where the previously decoded messages have been decoded correctly. In practice, errors can be made in previous decodings. However, for purposes of analyzing the overall error probability, the genie-aided model is sufficient.

to last *among all virtual users generated by this real user* and the virtual user with $r_{L''}$ is decoded last. Thus, all virtual users can be decoded and from (2), $r_{L'} + r_{L''} > r_L$. Therefore, every real user with L virtual users can strictly increase its throughput by splitting its power among $L+1$ virtual users. $\square$

Before we examine the asymptotic behavior of DRS, we solve the problem of how to split a user's power *optimally* among a fixed number of virtual users. The main difficulty here is that the objective function is not concave. In order to find the optimal splitting method, we prove the following lemma.

Lemma 3: Consider the following optimization problem:

$$\max_{p_k, p_j} \frac{1}{2} \log\left(1 + \frac{p_k}{A - p_k}\right) + \frac{1}{2} \log\left(1 + \frac{p_j}{A - Mp_k - p_j}\right) \tag{3}$$

subject to $p_k + p_j = c$ and $p_k, p_j \geq 0$, where A, M and c are positive constants and $A \geq Mc$. The unique solution to (3) is also the unique solution to $\frac{p_k}{A - p_k} = \frac{p_j}{A - Mp_k - p_j}$, subject to $p_k + p_j = c$ and $p_k, p_j \geq 0$, where $A \geq Mc$.

Proof: Substitute $p_j = c - p_k$ into the objective function, we have

$$
\begin{aligned}
f(p_k) &= \frac{1}{2} \log\left(1 + \frac{p_k}{A - p_k}\right) \\
&\quad + \frac{1}{2} \log\left(1 + \frac{c - p_k}{A - Mp_k - (c - p_k)}\right).
\end{aligned}
$$

Setting $\frac{df(p_k)}{dp_k} = 0$ subject to $0 \leq p_k \leq c$, the unique solution is $p_k^* = \frac{1}{M}(A - \sqrt{A(A - cM)})$. Thus, p_k^* is the unique stationary point of $f(p_k)$. We can also verify that $f(p_k^*) > f(0)$ and $f(p_k^*) > f(c)$. So $(p_k^*, c - p_k^*)$ is the unique solution to our maximization problem. We can directly solve $\frac{p_k}{A - p_k} = \frac{p_j}{A - Mp_k - p_j}$ subject to $p_k + p_j = c$ and $p_k, p_j \geq 0$. The unique solution is also $(p_k^*, c - p_k^*)$. $\square$

We now present the optimal splitting method. Theorem 1 states a necessary condition for the optimal splitting method, and Theorem 2 implies there is a unique optimal splitting method. In Corollary 1, we formally present the optimal splitting method and the required power levels.

Theorem 1: Let each real user split its power into L virtual users. Let p_k be the power allocated to the kth virtual users and $r_k = \frac{1}{2} \log\left(1 + \frac{p_k}{MP - M\sum_{j<k} p_j - p_k + N}\right)$. If $(p_1^*, ...p_L^*)$ maximizes $\sum_{k=1}^L r_k$ and satisfies $\sum_{k=1}^L p_k^* = P$, $p_k^* \geq 0$ for $k = 1, 2..., L$, then $r_k(p_1^*, ...p_L^*) = r^*$, for all k.

That is, the optimal power split must lead to equal transmission rates for all virtual users.

Proof: We use a perturbation argument. Suppose $(\tilde{p}_1, ..., \tilde{p}_L)$ maximizes $\sum_{k=1}^L r_k$ and satisfies $\sum_{k=1}^L \tilde{p}_k = P$, $\tilde{p}_k \geq 0 \ \forall k$ and the resulting $r_k(\tilde{p}_1, ..., \tilde{p}_L)$ is not the same for all k. Then we can find a pair of virtual users $(k, k+1)$, where virtual

user k and $k+1$ are decoded at the kth and $(k+1)$th places respectively, and $r_k(\tilde{p}_1, ..., \tilde{p}_L) \neq r_{k+1}(\tilde{p}_1, ..., \tilde{p}_L)$.

Without loss of generality, let us consider the case where $r_k(\tilde{p}_1, ..., \tilde{p}_L) > r_{k+1}(\tilde{p}_1, ..., \tilde{p}_L)$. By the definition of r_k, we have

$$\frac{\tilde{p}_k}{MP - M\sum_{j<k} \tilde{p}_j - \tilde{p}_k + N}$$
$$> \frac{\tilde{p}_{k+1}}{MP - M\sum_{j<k} \tilde{p}_j - M\tilde{p}_k - \tilde{p}_{k+1} + N}. \tag{4}$$

We can verify that if we change $\tilde{p}_k$ to $\tilde{p}_k - \varepsilon$ and $\tilde{p}_{k+1}$ to $\tilde{p}_{k+1} + \varepsilon$, where ε is a small positive number, then the first term of (4) decreases and the second term of (4) increases. Let ε^* be the solution to

$$\frac{\tilde{p}_k - \varepsilon}{MP - M\sum_{j<k} \tilde{p}_j - (\tilde{p}_k - \varepsilon) + N}$$
$$= \frac{\tilde{p}_{k+1} + \varepsilon}{MP - M\sum_{j<k} \tilde{p}_j - M(\tilde{p}_k - \varepsilon) - (\tilde{p}_{k+1} + \varepsilon) + N}.$$

(The existence of ε^* can be demonstrated). Let $\hat{p}_k = \tilde{p}_k - \varepsilon^*$ and $\hat{p}_{k+1} = \tilde{p}_{k+1} + \varepsilon^*$. Notice that $\tilde{p}_k + \tilde{p}_{k+1} = \hat{p}_k + \hat{p}_{k+1}$. Since the maximization considered in Lemma 3 has a unique solution, $r_k(\tilde{p}_1, ..., \hat{p}_k, \hat{p}_{k+1}, ..., \tilde{p}_L) + r_{k+1}(\tilde{p}_1, ..., \hat{p}_k, \hat{p}_{k+1}, ..., \tilde{p}_L) > r_k(\tilde{p}_1, ..., \tilde{p}_k, \tilde{p}_{k+1}, ..., \tilde{p}_L) + r_{k+1}(\tilde{p}_1, ..., \tilde{p}_k, \tilde{p}_{k+1}, ..., \tilde{p}_L)$. This contradicts our assumption that $(\tilde{p}_1, ..., \tilde{p}_L)$ maximizes $\sum_{k=1}^L r_k$. Therefore, the theorem follows. $\square$

By Theorem 1, if $(p_1^*, ..., p_L^*)$ maximizes $\sum_{k=1}^L r_k$ and satisfies $\sum_{k=1}^L p_k^* = P$, $p_k^* \geq 0$ for $k = 1, 2..., L$, then we must have $SIR_1(p_1^*, ..., p_L^*) = ... = SIR_L(p_1^*, ..., p_L^*)$ where $SIR_k(p_1, ..., p_L) = \frac{p_k}{MP - M\sum_{j<k} p_j - p_k + N}$. Therefore, if we show that $SIR_k(p_1, ..., p_L) = SIR_j(p_1, ..., p_L)$ for all $k \neq j$, has a unique solution, then there is *at most* one feasible solution to the maximization problem.

Theorem 2: The set of equations: $SIR_1(p_1, ..., p_L) = ... = SIR_L(p_1, ..., p_L)$, subject to $\sum_{k=1}^L p_k = P$ and $p_k \geq 0 \ \forall k$, has a unique solution.

Proof: See Appendix I.

Corollary 1: If a real user splits its power P into L virtual users, then the unique way to maximize this user's throughput is to set $p_k = \frac{N}{M}(1 + \frac{MP}{N})^{\frac{L-k}{L}}[(1 + \frac{MP}{N})^{\frac{1}{L}} - 1]$ for $k = 1, ..., L$.

Proof: Since the constraint region $\{\mathbf{p} : \sum_{k=1}^L p_k = P \text{ and } p_k \geq 0 \text{ for } k = 1, 2, .., L\}$ is a simplex and $\sum_{k=1}^L r_k$ is continuous, there exists at least one solution. We denote one solution by $(\hat{p}_1, ..., \hat{p}_L)$. By the necessary condition stated in Theorem 1, $(\hat{p}_1, ..., \hat{p}_L)$ must satisfy $r_1(\hat{p}_1, ..., \hat{p}_L) = ... = r_L(\hat{p}_1, ..., \hat{p}_L)$. Moreover, by the uniqueness property stated in Theorem 2, $(\hat{p}_1, ..., \hat{p}_L)$ is the unique solution to $\max_{p_1, ..., p_L} \sum_{k=1}^L r_k$ subject to $\sum_{k=1}^L p_k = P$ and $p_k \geq 0 \ \forall k$. Next, we plug p_k into the expression of $SIR_k(p_1, ..., p_L)$. Let $A = 1 + \frac{MP}{N}$, we are able to verify that $SIR_k(p_1, ..., p_L) = \frac{A^{\frac{1}{L}} - 1}{(M-1)A^{\frac{1}{L}} + 1}$, which

is independent of k. Hence, the corollary follows. $\square$

We now examine the asymptotic behavior of the DRS scheme. We first demonstrate the interesting fact that the rate tuple converges to the maximum equal rate point for a general power split as long as all virtual users' powers go to 0 as $L \to \infty$. This implies a convergence result for the optimal power split. We then analyze the rate of convergence under the optimal power split.

Theorem 3: Given any power split $(p_1, ..., p_L)$, a sufficient condition for $\lim_{L \to \infty} \sum_{k=1}^{L} r_k = \frac{1}{2M} \log(1 + \frac{MP}{N})$ is $\max_{k \in \{1, ..., L\}} p_k \to 0$ as $L \to \infty$.

Proof:

$$\lim_{L \to \infty} \sum_{k=1}^{L} r_k$$

$$= \lim_{L \to \infty} \sum_{k=1}^{L} \frac{1}{2} \log \left(1 + \frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \right)$$

$$= \lim_{L \to \infty} \sum_{k=1}^{L} \frac{1}{2} \left(\frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \right) \quad (5)$$

$$\geq \lim_{L \to \infty} \sum_{k=1}^{L} \frac{1}{2} \left(\frac{p_k}{MP - M \sum_{j<k} p_j + N} \right) \quad (6)$$

$$= \lim_{L \to \infty} \sum_{k=1}^{L} \frac{1}{2} \left(\frac{\beta_k - \beta_{k-1}}{MP + N - M\beta_{k-1}} \right) \quad (7)$$

$$= \frac{1}{2} \int_{0}^{P} \frac{1}{MP + N - Mx} dx \quad (8)$$

$$= \frac{1}{2M} \log \left(1 + \frac{MP}{N} \right) \quad (9)$$

where $\beta_0 = 0$, $\beta_k = \sum_{j=1}^{k} p_j$.

The equality in (5) is justified as follows. Note that, $\lim_{x \to 0} \frac{\log(1+x)-x}{x^2} = -\frac{1}{2}$. Hence, for sufficiently small positive δ, $|x| < \delta \Rightarrow |\frac{\log(1+x)-x}{x^2}| < 1 \Rightarrow |\log(1+x) - x| \leq x^2$. Now, we examine the error term

$$\left| \sum_{k=1}^{L} \left[\log \left(1 + \frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \right) \right. \right.$$

$$\left. \left. - \left(\frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \right) \right] \right|$$

$$\leq \sum_{k=1}^{L} \left| \log \left(1 + \frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \right) \right.$$

$$\left. - \left(\frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \right) \right|$$

$$\leq \sum_{k=1}^{L} \left(\frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \right)^2 \quad (10)$$

$$\leq \sum_{k=1}^{L} \left(\frac{1}{N} \right)^2 p_k^2 \quad (11)$$

$$\leq \left(\frac{1}{N} \right)^2 \max_k p_k \sum_{k=1}^{L} p_k \quad (12)$$

$$= P \left(\frac{1}{N} \right)^2 \max_k p_k \quad (13)$$

where inequality in (10) holds because $\frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} < \delta$ when $L \to \infty$, and the inequality in (11) follows from the fact that $\frac{p_k}{MP - M \sum_{j<k} p_j - p_k + N} \leq \frac{p_k}{N}$. Since $\max_k p_k \to 0$ as $L \to \infty$, the error term goes to zero in the limit. This justifies the equality in (5). Using the capacity bound, we also have $\lim_{L \to \infty} \sum_{k=1}^{L} r_k \leq \frac{1}{2M} \log(1 + \frac{MP}{N})$. Therefore, $\lim_{L \to \infty} \sum_{k=1}^{L} r_k = \frac{1}{2M} \log(1 + \frac{MP}{N})$. $\square$

Note that our optimal power split satisfies the sufficient condition in Theorem 3. Therefore, its convergence is assured.

Corollary 2: If each real user adopts the optimal splitting method specified in Corollary 1, then

$$\lim_{L \to \infty} \sum_{k=1}^{L} r_k = \frac{1}{2M} \log \left(1 + \frac{MP}{N} \right) \equiv R^*.$$

Next, we examine the rate of convergence to the maximum equal rate point under the optimal power split. Define the error term $e[L] \equiv R^* - \sum_{k=1}^{L} r_k$, we analyze how fast this error term tends to 0 as $L \to \infty$. We prove the following:[6]

Theorem 4: $e[L] = \Theta \left(\frac{1}{L} \right)$.

Proof: Let $A \triangleq 1 + \frac{MP}{N}$.

$$\lim_{L \to \infty} L e[L]$$

$$= \lim_{L \to \infty} L \left(\frac{1}{2M} \log(A) - \frac{L}{2} \log \left(\frac{M}{M - 1 + A^{-\frac{1}{L}}} \right) \right)$$

$$= \lim_{y \to 0} \frac{\frac{y \log(A)}{2M} - \frac{1}{2} \log \left(\frac{M}{M-1+A^{-y}} \right)}{y^2} \quad (14)$$

$$= \lim_{y \to 0} \frac{\frac{\log(A)}{2M} - \frac{1}{2} \frac{A^{-y} \log(A)}{M-1+A^{-y}}}{2y} \quad (15)$$

$$= \frac{(M-1)(\log(A))^2}{4M^2}. \quad (16)$$

Note that equalities in (15) and (16) can be verified by L'Hospital's rule.

Consequently, given any $\varepsilon > 0$, there exists a positive integer n_0 such that for all $L > n_0$, we have $\left| L e[L] - \frac{(M-1)(\log(A))^2}{4M^2} \right| < \varepsilon$. This implies $\frac{(M-1)(\log(A))^2}{4M^2} - \varepsilon < L e[L] < \frac{(M-1)(\log(A))^2}{4M^2} + \varepsilon$. Therefore, we can choose a small enough ε such that $\frac{(M-1)(\log(A))^2}{4M^2} - \varepsilon > 0$. Let $c_1 = \frac{(M-1)(\log(A))^2}{4M^2} - \varepsilon$, $c_2 = \frac{(M-1)(\log(A))^2}{4M^2} + \varepsilon$, we have $c_1 \leq L e[L] \leq c_2$. This implies that there exists a positive

---

[6]Recall that $f[n] = \Omega(g[n])$ if there exist positive constants c_1 and n_0 such that $f[n] \geq c_1 g[n]$ for all $n \geq n_0$, and $f[n] = O(g[n])$ if there are positive constants c_2 and n_0, such that $f[n] \leq c_2 g[n]$ for all $n \geq n_0$. Finally, $f[n] = \Theta(g[n])$ if $f[n] = \Omega(g[n])$ and $f[n] = O(g[n])$.
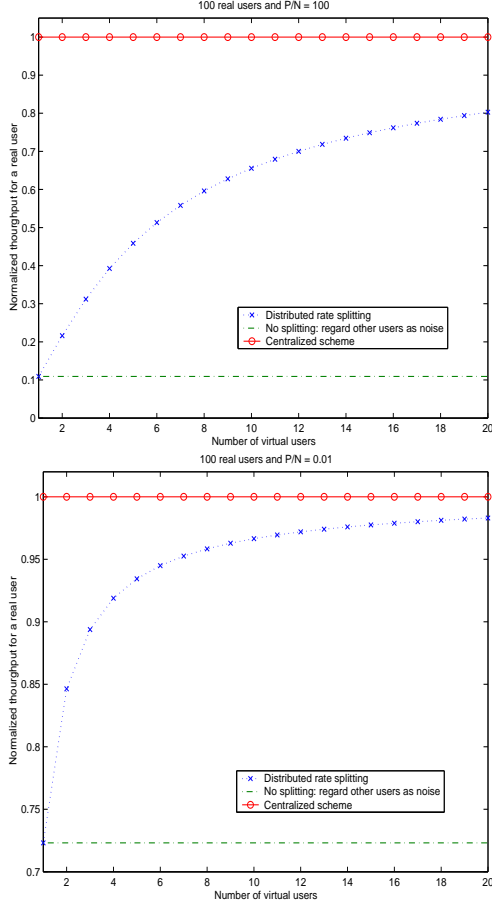
Fig. 3. Throughput per real user v.s. the number of virtual users per real user for both high and low SNR regimes. Note that the scales of the vertical axes in both figures are different.

integer n_0 and for all $L > n_0$, we have $\frac{c_1}{L} \leq e\,[L] \leq \frac{c_2}{L}$. $\square$

Finally, we note that all virtual users in one virtual user class can be decoded *in parallel*. Thus, the decoding delay of DRS is proportional to the number of virtual users L and independent of the number of real users. Since L is controlled by the designer, DRS offers a tradeoff between the throughput of a real user and the decoding delay. In Fig. 3, we present some numerical simulations illustrating the tradeoff between the number of virtual users and the throughput for both the high and low SNR regimes. A system with 100 real users is used in the simulations.

III. M-USER DISCRETE MEMORYLESS MAC

An M-user discrete memoryless MAC is defined in terms of M discrete input alphabets $\mathcal{X}_i$, $i \in \{1, ..., M\}$, an output alphabet $\mathcal{Y}$ and a stochastic matrix $W : \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_M \rightarrow \mathcal{Y}$ with entries $W\,(y \mid x_1, ..., x_M)$. For any product input distribution $P_{X_1} \cdots P_{X_M}$, let the achievable region $\mathcal{R}[W; P_{X_1} \cdots P_{X_M}]$ be the set of $\mathbf{R} \in \mathbb{R}_+^M$ satisfying

$$\sum_{i \in S} R_i \leq I\,(X_S; Y \mid X_{S^c})\,, \forall S \subseteq \{1, ..., M\}$$

where $X_S \triangleq (X_i)_{i \in S}$ and $S^c \triangleq \{1, ..., M\} \setminus S$.

The capacity region of the asynchronous MAC is [14] [15]

$$\mathcal{C} = \bigcup_{P_{X_1} \cdots P_{X_M}} \mathcal{R}[W; P_{X_1} \cdots P_{X_M}]. \qquad (17)$$

We fix the input product distribution $P_{X_1} \cdots P_{X_M}$ and focus on achieving the desired operating point in $\mathcal{R}[W; P_{X_1} \cdots P_{X_M}]$. In this section, we consider only discrete memoryless channels satisfying the following symmetry condition:

$$I\,(X_S; Y \mid X_{S^c}) = I\,(X_T; Y \mid X_{T^c})\,, \qquad (18)$$

$\forall S, T \subseteq \{1, ..., M\}$ such that $|S| = |T|$. Later, in Section IV, we consider the more general asymmetric case. We further assume that for $\forall S, T \subseteq \{1, 2, ..., M\}$, if $S \cap T = \varnothing$, then $I\,(X_S; Y) < I\,(X_S; Y | X_T)$. Under our symmetric setting, the maximum common rate that every user can achieve is $R^* = \frac{1}{M} I\,(X_1, ..., X_M; Y)$.

In the Gaussian MAC, virtual users are created via a power/rate splitting mechanism. For the discrete memoryless MAC, we adopt the random switching mechanism of [19] where virtual users with the same input distribution as the real users are created and the transmitted signal of a real user is determined by a random switch. We first consider the two-user discrete memoryless MAC $(M = 2)$, and illustrate the random switching mechanism. The optimal random switches and the asymptotic behavior of DRS under the optimal switching are presented. We then examine the M-user case $(M > 2)$, and present a sufficient condition for the random switching mechanism to converge to the information theoretic upper bound. Finally, we investigate the rate of convergence for a simple suboptimal random switch.

A. Two-user Case $(M = 2)$

Consider a two-user MAC, $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$. For a fixed input product distribution $P_{X_1} P_{X_2}$, the achievable region is given by:

$$
\begin{aligned}
R_1 &\leq I\,(X_1; Y | X_2) = I\,(X_1; Y, X_2) \\
R_2 &\leq I\,(X_2; Y | X_1) = I\,(X_2; Y, X_1) \\
R_1 + R_2 &\leq I\,(X_1, X_2; Y)\,.
\end{aligned}
$$

Under our symmetry assumption (cf (18)), we have $I(X_1; Y) = I(X_2; Y)$, $I(X_1; Y, X_2) = I(X_2; Y, X_1)$, $I(X_1; Y, X_2) > I(X_1; Y)$, and the optimal rate tuple is

$$(R^*, R^*) \equiv \left(\frac{1}{2} I\,(X_1, X_2; Y), \frac{1}{2} I\,(X_1, X_2; Y) \right). \qquad (19)$$

Let us consider the random switching mechanism for this channel. We first consider the case where each real user generates two virtual users. Later, we consider the case where the number of virtual users per real user goes to infinity. We split by means of two switches, as shown in Fig. 4. Each switch has two inputs, $X_{i1} \in \mathcal{X}_i$ and $X_{i2} \in \mathcal{X}_i$ and one output $X_i \in \mathcal{X}_i$. Switch i is controlled by a random variable
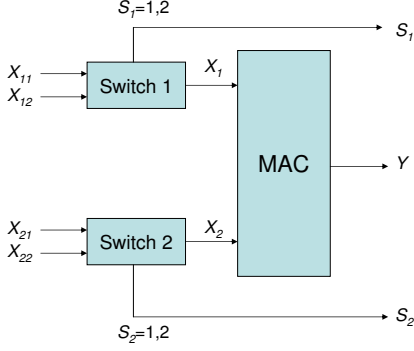
Fig. 4. Switches for two-user discrete memoryless MAC

$S_i \in \{1, 2\}$ with $P(S_i = 1) = \lambda$. The output is given by $X_i = X_{i1}$ if $S_i = 1$, and $X_i = X_{i2}$ if $S_i = 2$.

The switching random variables $\{S_1, S_2\}$ are independent of the channel inputs. We also assume that $\{S_1, S_2\}$ are available at the receiver. In practice, one would generate S_1 and S_2 at the transmitters and at the receiver, e.g. by means of a pseudorandom sequence generator. Assign to the channel inputs X_{11}, X_{12}, X_{21} and X_{22} the probability mass function $P_{X_{11}, X_{12}, X_{21}, X_{22}}(x_{11}, x_{12}, x_{21}, x_{22}) = P_{X_1}(x_{11})P_{X_1}(x_{12})P_{X_2}(x_{21})P_{X_2}(x_{22})$. Notice that X_{i1} and X_{i2} are independent and each has the same probability mass function as the random variable X_i for $i = 1, 2$.

In successive decoding for the discrete memoryless MAC, the signals of decoded virtual users are used as side information to aid the decoding process of subsequent virtual users. The first constituent decoder observes the output $\{Y, S_1, S_2\}$ and tries to decode X_{11} and X_{21}. The second constituent decoder is informed of the decision about $\{X_{11}, X_{21}\}$ made by the previous constituent decoder and tries to decode X_{12} and X_{22}. Without loss of generality, let us focus on real user 1.

$$
\begin{aligned}
r_{X_{11}} &= I(X_{11}; Y, S_1, S_2) \\
&= I(X_{11}; Y, S_2 | S_1) \\
&= \lambda I(X_{11}; Y, S_2 | S_1 = 1) \\
&\quad + (1 - \lambda) I(X_{11}; Y, S_2 | S_1 = 2) \\
&= \lambda I(X_1; Y)
\end{aligned}
$$

where the second equality follows from the independence between X_{11} and S_1, and the last equality follows from the fact that when $S_1 = 2$, X_{11} is independently of the output Y and S_2. Similarly, we have

$$
\begin{aligned}
r_{X_{12}} &= I(X_{12}; Y, S_1, S_2, X_{11}, X_{21}) \\
&= (1 - \lambda) [\lambda I(X_1; Y, X_2) + (1 - \lambda) I(X_1; Y)].
\end{aligned}
$$

It can be verified that in the two-user discrete memoryless MAC, both real users' throughput can be strictly increased by splitting their inputs via a random switch, relative to the case where they do not split, (i.e. $r_{X_{11}} + r_{X_{12}} > I(X_1; Y)$ for $\lambda \in (0, 1)$). Next, we show that by generating more virtual users, the throughput of each real user increases further.

Lemma 4: For $M = 2$, consider a distributed rate splitting scheme with L virtual users per real user. The random switch for user i is controlled by S_i, where $P(S_i = k) = \lambda_k$ for $k = 1, ..., L$. It is possible to strictly increase the throughput via an $(L + 1)$ virtual user system by splitting the Lth virtual user into two virtual users.

Proof: Without loss of generality, we consider user 1. For the kth virtual user, we have

$$
\begin{aligned}
&r_{X_{1k}} \\
&= I\left(X_{1k}; Y, S_1, S_2, X_{11}, X_{21}, ..., X_{1(k-1)}, X_{2(k-1)}\right) \\
&= I\left(X_{1k}; Y, S_2, X_{11}, X_{21}, ..., \right. \\
&\qquad\qquad \left. X_{1(k-1)}, X_{2(k-1)} \mid S_1\right) \qquad (20) \\
&= \lambda_k I\left(X_{1k}; Y, S_2, X_{11}, X_{21}, ..., \right. \\
&\qquad\qquad \left. X_{1(k-1)}, X_{2(k-1)} \mid S_1 = k\right) \\
&\quad + (1 - \lambda_k) I\left(X_{1k}; Y, S_2, X_{11}, X_{21}, ..., \right. \\
&\qquad\qquad \left. X_{1(k-1)}, X_{2(k-1)} \mid S_1 \neq k\right) \\
&= \lambda_k I\left(X_{1k}; Y, S_2, X_{11}, X_{21}, ..., \right. \\
&\qquad\qquad \left. X_{1(k-1)}, X_{2(k-1)} \mid S_1 = k\right) \qquad (21) \\
&= \lambda_k \left[\left(\sum_{j<k} \lambda_j\right) I\left(X_{1k}; Y, X_{11}, X_{21}, ..., \right.\right. \\
&\qquad\qquad \left. X_{1(k-1)}, X_{2(k-1)} \mid S_1 = k, S_2 < k\right) \\
&\quad + \left(1 - \sum_{j<k} \lambda_j\right) I\left(X_{1k}; Y, X_{11}, X_{21}, ..., \right. \\
&\qquad\qquad \left.\left. X_{1(k-1)}, X_{2(k-1)} \mid S_1 = k, S_2 \geq k\right)\right] \\
&= \lambda_k \left[\left(\sum_{j<k} \lambda_j\right) I\left(X_1; Y, X_2\right) \right. \\
&\qquad\qquad \left. + \left(1 - \sum_{j<k} \lambda_j\right) I\left(X_1; Y\right) \right], \qquad (22)
\end{aligned}
$$

where equality in (20) is due to the independence between X_{1k} and S_1, and equality in (21) follows from the fact that when $S_1 \neq k$, X_{1k} is independent of the output Y and all the other random variables. Finally, equality in (22) holds because when $S_2 < k$, one of the random variables $X_{21}, ..., X_{2(k-1)}$ is the switch output, and when $S_2 \geq k$, none of them is the switch output. Therefore,

$$
r_{X_{1L}} = \lambda_L \left[(1 - \lambda_L) I\left(X_1; Y, X_2\right) + \lambda_L I\left(X_1; Y\right)\right].
$$

Now let us split $r_{X_{1L}}$ into $r_{X_{1L}^1}$ and $r_{X_{1L}^2}$ by using a switch controlled by a binary random variable S' with $P(S' = 0) = \alpha$. We have

$$
\begin{aligned}
r_{X_{1L}^1} &= \alpha \lambda_L \left((1 - \lambda_L) I\left(X_1; Y, X_2\right) + \lambda_L I\left(X_1; Y\right)\right), \\
r_{X_{1L}^2} &= \overline{\alpha} \lambda_L \left((1 - \overline{\alpha} \lambda_L) I\left(X_1; Y, X_2\right) + \overline{\alpha} \lambda_L I\left(X_1; Y\right)\right),
\end{aligned}
$$

where $\overline{\alpha} = 1 - \alpha$. Hence,

$$
\begin{aligned}
&r_{X_{1L}^1} + r_{X_{1L}^2} - r_{X_{1L}} \\
&= \alpha \lambda_L \left((1 - \lambda_L) I\left(X_1; Y, X_2\right) + \lambda_L I\left(X_1; Y\right)\right) \\
&\quad + \overline{\alpha} \lambda_L \left((1 - \overline{\alpha} \lambda_L) I\left(X_1; Y, X_2\right) + \overline{\alpha} \lambda_L I\left(X_1; Y\right)\right)
\end{aligned}
$$

$$-\lambda_L \left[(1 - \lambda_L) I(X_1; Y, X_2) + \lambda_L I(X_1; Y) \right]$$
$$= \alpha \overline{\alpha} \lambda_L^2 \left[I(X_1; Y, X_2) - I(X_1; Y) \right]$$
$$\geq 0$$

with strict inequality if $\alpha \in (0, 1)$. $\qquad \square$

Before we examine the asymptotic behavior of the DRS scheme for $M = 2$, we first solve the problem of how to find the optimal switches for a fixed number of virtual users per real user.

Theorem 5: For $M = 2$, if a real user has L virtual users, then the optimal random variable to control the switch for user i is $S_i \in \{1, ..., L\}$ with $\mathsf{P}(S_i = k) = \frac{1}{L}$ for $k = 1, ..., L$ and $i = 1, 2$.

Proof: We use a perturbation argument. Suppose the random variables $S_1, S_2 \in \{1, ..., L\}$ with $\mathsf{P}(S_1 = k) = \mathsf{P}(S_2 = k) = \lambda_k$ maximize R_{X_1}, where $\lambda_k \geq 0$ and $\sum_{k=1}^{L} \lambda_k = 1$. Moreover, suppose there exists λ_k such that $\lambda_k \neq \frac{1}{L}$. Let λ_k be the first element which is not equal to $\frac{1}{L}$. We consider the pair $(\lambda_k, \lambda_{k+1})$. We have

$$r_{1k} = \lambda_k \left(\frac{k-1}{L} I(X_1; Y, X_2) \right.$$
$$\left. + \left(1 - \frac{k-1}{L} \right) I(X_1; Y) \right),$$

$$r_{1(k+1)} = \lambda_{k+1} \left[\left(\frac{k-1}{L} + \lambda_k \right) I(X_1; Y, X_2) \right.$$
$$\left. + \left(1 - \frac{k-1}{L} - \lambda_k \right) I(X_1; Y) \right].$$

Therefore,

$$r_{1k} + r_{1(k+1)}$$
$$= \left[(\lambda_k + \lambda_{k+1}) \frac{k-1}{L} + \lambda_k \lambda_{k+1} \right] I(X_1; Y, X_2)$$
$$+ \left[(\lambda_k + \lambda_{k+1}) \frac{L-k+1}{L} - \lambda_k \lambda_{k+1} \right] I(X_1; Y).$$

First, we consider the case where $\lambda_k > \lambda_{k+1}$. We let $\hat{\lambda}_k = \lambda_k - \varepsilon$ and $\hat{\lambda}_{k+1} = \lambda_{k+1} + \varepsilon$ for $\varepsilon < \frac{\lambda_k - \lambda_{k+1}}{2}$. We have

$$\hat{r}_{1k} + \hat{r}_{1(k+1)}$$
$$= \left[(\lambda_k + \lambda_{k+1}) \frac{k-1}{L} + (\lambda_k - \varepsilon)(\lambda_{k+1} + \varepsilon) \right] I(X_1; Y, X_2)$$
$$+ \left[(\lambda_k + \lambda_{k+1}) \frac{L-k+1}{L} - (\lambda_k - \varepsilon)(\lambda_{k+1} + \varepsilon) \right] I(X_1; Y).$$

Thus, $\hat{r}_{1k} + \hat{r}_{1(k+1)} - (r_{1k} + r_{1(k+1)}) = [(\lambda_k + \lambda_{k+1}) \varepsilon - \varepsilon^2] (I(X_1; Y, X_2) - I(X_1; Y))$. Notice that the second term of the R.H.S. expression is positive and there exists ε such that $((\lambda_k + \lambda_{k+1}) \varepsilon - \varepsilon^2) > 0$. This, however, contradicts our assumption that the random variables $S_1, S_2 \in \{1, ..., L\}$ with $\mathsf{P}(S_1 = k) = \mathsf{P}(S_2 = k) = \lambda_k$, maximize R_{X_1}. We obtain similar contradictions for the case where $\lambda_k \leq \lambda_{k+1}$. $\square$

We now examine the asymptotic behavior of the DRS scheme in the two-user discrete memoryless MAC.

Theorem 6: For $M = 2$, if both real users adopt $\mathsf{P}(S_i = k) = \frac{1}{L}$ for $k = 1, ..., L$ and $i = 1, 2$, then $\lim_{L \to \infty} \sum_{k=1}^{L} r_{X_{ik}} = \frac{1}{2} I(X_1, X_2; Y) = R^*$ for $i = 1, 2$. Moreover, if we define the error term $e[L] \equiv R^* - \sum_{k=1}^{L} r_{X_{ik}}$, then $e[L] = \Theta(\frac{1}{L})$ for $i = 1, 2$.

Proof: Without loss of generality, we consider real user 1. Since

$$r_{X_{1k}} = \frac{1}{L} \left(\frac{k-1}{L} I(X_1; Y, X_2) + \left(1 - \frac{k-1}{L} \right) I(X_1; Y) \right),$$

we have

$$\lim_{L \to \infty} R_{X_1}$$
$$= \lim_{L \to \infty} \left[\frac{1}{L} \left(1 + \frac{1}{L} + ... + \frac{L-1}{L} \right) I(X_1; Y) \right.$$
$$\left. + \frac{1}{L} \left(\frac{L-1}{L} + ... + \frac{1}{L} \right) I(X_1; Y, X_2) \right]$$
$$= \lim_{L \to \infty} \left[\frac{L+1}{2L} I(X_1; Y) + \frac{L-1}{2L} I(X_1; YX_2) \right]$$
$$= \frac{1}{2} \left[I(X_1; Y) + I(X_1; Y, X_2) \right]$$
$$= \frac{1}{2} I(X_1, X_2; Y),$$

where the last equality follows from the fact that $I(X_1; Y, X_2) = I(X_2; Y, X_1)$ in the symmetric setting.

Next, we examine the rate of convergence. We have the error term $e[L] = \frac{1}{2} I(X_1; Y) + \frac{1}{2} I(X_1; Y, X_2) - (\frac{L+1}{2L} I(X_1; Y) + \frac{L-1}{2L} I(X_1; Y, X_2))$. Thus,

$$e[L] = \frac{1}{2L} \left(I(X_1; Y, X_2) - I(X_1; Y) \right),$$

which implies

$$\lim_{L \to \infty} L e[L] = \frac{1}{2} \left(I(X_1; Y, X_2) - I(X_1; Y) \right) \triangleq \frac{c}{2}.$$

For any $\varepsilon > 0$, there exists an n_0 such that for all $L > n_0$, $|Le[L] - \frac{c}{2}| < \varepsilon$. Hence, $\frac{c}{2} - \varepsilon < Le[L] < \frac{c}{2} + \varepsilon$. We can choose ε small enough such that $\frac{c}{2} - \varepsilon > 0$. This implies $e[L] = \Theta(\frac{1}{L})$. $\qquad \square$

B. M-user Case ($M > 2$)

There are M real users and each real user creates L virtual users. We have M switches $(S_1, ..., S_M)$ to do the splitting with probabilities $\mathsf{P}(S_i = k) = \lambda_k$ for $i = 1, ..., M$ and $k = 1, ..., L$. We assume the receiver also knows $(S_1, ..., S_M)$. This may require common randomness to exist between all transmitters and the receiver. Due to symmetry, we focus on one user, say user 1. For $k = 1, ..., L$, we set

$$r_{X_{1k}}$$
$$= I(X_{1k}; Y, \mathbf{S}_1^M, \mathbf{X}_{11}^{1(k-1)}, ..., \mathbf{X}_{M1}^{M(k-1)}) \qquad (23)$$
$$= I(X_{1k}; Y, \mathbf{S}_2^M, \mathbf{X}_{11}^{1(k-1)}, ..., \mathbf{X}_{M1}^{M(k-1)} \mid S_1) \qquad (24)$$
$$= \lambda_k I(X_{1k}; Y, \mathbf{S}_2^M, \mathbf{X}_{11}^{1(k-1)}, ..., \mathbf{X}_{M1}^{M(k-1)} \mid S_1 = k)$$
$$\quad + (1 - \lambda_k) I(X_{1k}; Y, \mathbf{S}_2^M, \mathbf{X}_{11}^{1(k-1)}, ...,$$

$$\mathbf{X}_{M1}^{M(k-1)} \mid S_1 \neq k)$$
$$= \lambda_k I(X_{1k}; Y, \mathbf{S}_2^M, \mathbf{X}_{11}^{1(k-1)}, ..., \mathbf{X}_{M1}^{M(k-1)} \mid S_1 = k) \quad (25)$$
$$= \lambda_k I(X_1; Y, \mathbf{X}_{11}^{1(k-1)}, ..., \mathbf{X}_{M1}^{M(k-1)} \mid S_1 = k, \mathbf{S}_2^M)$$
$$= \sum_{s_2,...,s_M} \mathsf{P}(S_2 = s_2, ..., S_M = s_M) \lambda_k I(X_1; Y, \mathbf{X}_{11}^{1(k-1)},$$
$$..., \mathbf{X}_{M1}^{M(k-1)} \mid S_1 = k, S_2 = s_2, ..., S_M = s_M)$$
$$= \lambda_k \left[\left(1 - \sum_{j<k} \lambda_j\right)^{M-1} I(X_1; Y) \right.$$
$$\left. + \sum_{i=1}^{M-1} \binom{M-1}{i} \left(\sum_{j<k} \lambda_j\right)^i \cdot \left(1 - \sum_{j<k} \lambda_j\right)^{M-1-i} I(X_1; Y, X_2, ..., X_{i+1}) \right], \quad (26)$$

where $\mathbf{S}_i^j \triangleq \{S_i, ..., S_j\}$ and $\mathbf{X}_{i1}^{ik} \triangleq \{X_{i1}, ..., X_{ik}\}$. Equality in (24) is due to the independence between X_{1k} and S_1. Equality in (25) holds because when $S_1 \neq k$, X_{1k} is independent of the output Y and all the other random variables. The first term in (26) follows from $\mathsf{P}(S_2 \geq k, ..., S_M \geq k) = (1 - \sum_{j<k} \lambda_j)^{M-1}$. The second summation term in (26) follows from the fact that the probability of i switching random variables among $(S_2, ..., S_M)$ having values less than k is $\binom{M-1}{i}(\sum_{j<k} \lambda_j)^i(1-\sum_{j<k} \lambda_j)^{M-1-i}$. It can be verified that real user 1 with L virtual users can strictly increase its total throughput via an $L+1$ virtual user system.

In order to maximize the total throughput of real user 1 for fixed L, we need to find the optimal $(\lambda_1^*, ..., \lambda_L^*)$ to maximize $\sum_{k=1}^L r_{X_{1k}}$. This is a non-convex optimization problem and appears to be difficult. We are able to verify that for the general M-user case (unlike the two-user case), random switches with a uniform distribution are in general suboptimal. Nevertheless, it is possible to generalize the asymptotic result of Theorem 6. We first demonstrate the fact that the convergence result holds for a general switch controlled by S_i, where $\mathsf{P}(S_i = k) = \lambda_k$ for $k = 1, ..., L$, as long as $\max_{k \in \{1,...,L\}} \lambda_k \to 0$ as $L \to \infty$. We then analyze the rate of convergence for a particular suboptimal switch, the uniform switch.

Theorem 7: For a general random switch controlled by S_i, where $\mathsf{P}(S_i = k) = \lambda_k$ for $k = 1, ..., L$, a sufficient condition for $\lim_{L\to\infty} \sum_{k=1}^L r_{X_{ik}} = \frac{1}{M} I(X_1, ..., X_M; Y)$ is $\max_{k \in \{1,...,L\}} \lambda_k \to 0$ as $L \to \infty$ for $i = 1, ..., M$.

Proof: Without loss of generality, let us examine real user 1.

$$\lim_{L\to\infty} R_{X_1}$$
$$= \lim_{L\to\infty} \sum_{k=1}^L r_{X_{1k}}$$
$$= \lim_{L\to\infty} \sum_{k=1}^L \lambda_k \left[\left(1 - \sum_{j<k} \lambda_j\right)^{M-1} I(X_1; Y) \right.$$

$$\left. + \sum_{i=1}^{M-1} \binom{M-1}{i} \left(\sum_{j<k} \lambda_j\right)^i \cdot \left(1 - \sum_{j<k} \lambda_j\right)^{M-1-i} I(X_1; Y, X_2, ..., X_{i+1}) \right]$$
$$= \lim_{L\to\infty} \sum_{i=0}^{M-1} c_i I_i,$$

where
$$I_0 = I(X_1; Y)$$
$$I_i = I(X_1; Y, X_2, ..., X_{i+1})$$
$$c_0 = \sum_{k=1}^L \lambda_k \left(1 - \sum_{j<k} \lambda_j\right)^{M-1}$$
$$c_i = \sum_{k=1}^L \lambda_k \binom{M-1}{i} \left(\sum_{j<k} \lambda_j\right)^i \left(1 - \sum_{j<k} \lambda_j\right)^{M-1-i},$$

for $i \geq 1$.

It is sufficient to prove $\lim_{L\to\infty} c_i = \frac{1}{M}$ for all i. For $i \geq 1$,

$$\lim_{L\to\infty} c_i$$
$$= \lim_{L\to\infty} \sum_{k=1}^L \lambda_k \binom{M-1}{i} \left(\sum_{j<k} \lambda_j\right)^i \left(1 - \sum_{j<k} \lambda_j\right)^{M-1-i}$$
$$= \lim_{L\to\infty} \sum_{k=1}^L (\beta_k - \beta_{k-1}) \binom{M-1}{i} \beta_{k-1}^i (1 - \beta_{k-1})^{M-1-i}$$
$$= \binom{M-1}{i} \int_0^1 x^i (1-x)^{M-1-i} dx$$
$$= \binom{M-1}{i} B(i+1, M-i)$$
$$= \frac{(M-1)!}{i!(M-i-1)!} \frac{i!(M-i-1)!}{M!}$$
$$= \frac{1}{M},$$

where $\beta_0 = 0$, $\beta_k = \sum_{j=1}^k \lambda_j$ and $B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$ is the beta function. The term c_0 can be shown to converge to $\frac{1}{M}$ as $L \to \infty$ by a similar argument. $\square$

Next, we analyze a particular suboptimal switch, the uniform switch. Since the uniform switch satisfies the sufficient condition in Theorem 7, the convergence result holds. The next lemma presents its rate of convergence.

Lemma 5: Consider an M-user discrete memoryless MAC. Let each real user have L virtual users and each switch be controlled by an i.i.d. random variable $S_i \in \{1, 2, ..., L\}$ with $\mathsf{P}(S_i = k) = \frac{1}{L}$, $k = 1, ..., L$. Define the error term $e[L] \equiv \frac{1}{M} I(X_1, X_2, ..., X_M; Y) - \sum_{k=1}^L r_{X_{ik}}$. Then $e[L] = O(\frac{1}{L})$ for all i.

Proof: In the uniform switch setting, $\mathsf{P}(S_i = k) = \frac{1}{L}$ for $i = 1, ..., M$ and $k = 1, ..., L$. Without loss of generality, we

examine the total throughput of real user 1.

$$R_{X_1} = \sum_{k=1}^{L} r_{X_{1k}}$$

$$= \frac{1}{L} I(X_1; Y) + \sum_{k=2}^{L} \frac{1}{L} \left[\sum_{i=0}^{M-1} \binom{M-1}{i} \left(\frac{k-1}{L} \right)^i \cdot \right.$$
$$\left. \left(1 - \frac{k-1}{L} \right)^{M-1-i} I(X_1; Y, X_2, ..., X_{i+1}) \right],$$

where $I(X_1; Y, X_2, ..., X_{i+1}) = I(X_1; Y)$ for $i = 0$.

We denote $R_{X_1} \triangleq c_0 I_0 + ... + c_{M-1} I_{M-1}$, where I_i is defined in the proof of Theorem 7 for $i \geq 0$, and

$$c_0 = \frac{1}{L} \sum_{k=1}^{L} \left(1 - \frac{k-1}{L} \right)^{M-1}$$

$$c_i = \sum_{k=1}^{L} \frac{1}{L} \binom{M-1}{i} \left(\frac{k-1}{L} \right)^i \left(1 - \frac{k-1}{L} \right)^{M-1-i}.$$

for $i \geq 1$. Therefore, the error term can be calculated as follows,

$$e[L] = \left| \frac{1}{M} I(X_1, X_2, ..., X_M; Y) - \sum_{i=0}^{M-1} c_i I_i \right|$$

$$= \left| \sum_{i=0}^{M-1} \left(\frac{1}{M} - c_i \right) I_i \right|$$

$$\leq \sum_{i=0}^{M-1} \left| \frac{1}{M} - c_i \right| I_i$$

$$\leq M \left[\max_i \left(\left| \frac{1}{M} - c_i \right| I_i \right) \right].$$

Note that $x^i (1-x)^{M-1-i}$ is maximized at $x = \frac{i}{M-1}$. For $i \geq 1$, it can be verified that

$$\frac{1}{L} \sum_{l=0}^{L-1} \left(\frac{l}{L} \right)^i \left(1 - \frac{l}{L} \right)^{M-1-i}$$
$$+ \left(\frac{i}{M-1} \right)^i \left(1 - \frac{i}{M-1} \right)^{M-1-i} \frac{1}{L}$$
$$\geq \int_0^1 x^i (1-x)^{M-1-i} \, dx,$$

$$\frac{1}{L} \sum_{l=0}^{L-1} \left(\frac{l}{L} \right)^i \left(1 - \frac{l}{L} \right)^{M-1-i}$$
$$- \left(\frac{i}{M-1} \right)^i \left(1 - \frac{i}{M-1} \right)^{M-1-i} \frac{1}{L}$$
$$\leq \int_0^1 x^i (1-x)^{M-1-i} \, dx.$$

Multiplying both sides of the above two inequalities by $\binom{M-1}{i}$, we have

$$\left| \frac{1}{M} - c_i \right| \leq \binom{M-1}{i} \left(\frac{i}{M-1} \right)^i \left(1 - \frac{i}{M-1} \right)^{M-1-i} \frac{1}{L}.$$

Therefore,

$$\max_i \left(\left| \frac{1}{M} - c_i \right| I_i \right)$$
$$\leq \max_i \left[\binom{M-1}{i} \left(\frac{i}{M-1} \right)^i \left(1 - \frac{i}{M-1} \right)^{M-1-i} I_i \right] \frac{1}{L}$$
$$\equiv \alpha \frac{1}{L},$$

so $e[L] \leq M\alpha \frac{1}{L}$, where $\alpha > 0$. For the term c_0, a similar argument can be used to show that $|c_0 - \frac{1}{M}| \leq \frac{1}{L}$. Therefore, $e[L] = O\left(\frac{1}{L} \right)$. $\qquad \square$

IV. VARIATIONS OF DISTRIBUTED RATE SPLITTING

In Section II and Section III, we imposed two symmetry constraints. The first is that the capacity region for the Gaussian MAC and the achievable rate region for the discrete memoryless MAC are symmetric. The second is that users generate the same number of virtual users. In this section, we describe two variations of DRS. The first variation is presented in Section IV-A, where we relax the symmetric region constraint. In this case, we show that as the number of virtual user per real user tends to infinity, the rate tuple achieved under DRS approaches a point on the dominant face. The second variation is presented in Section IV-B, where each real user may generate a different number of virtual users. The main advantage of this variation is that it can accommodate different user rate requirements in a distributed fashion.

A. Asymmetric Capacity/Achievable Rate Region

1) M-user Gaussian MAC: In this section, we consider the case where real users in a Gaussian MAC may have different transmission powers (i.e. the capacity region may not be symmetric). We assume that user i has transmission power P_i and the power vector $(P_1, ..., P_M)$ is known to all users. We also assume that all real users split their powers into L virtual users according to the common power splitting rule defined by the vector $(\gamma_1, \gamma_2, ..., \gamma_L)$, where $\gamma_k > 0 \; \forall k$ and $\sum_{k=1}^{L} \gamma_k = 1$. The power vector for the virtual users generated by user i is $(\gamma_1 P_i, ..., \gamma_L P_i)$ for $i = 1, ..., M$.

Lemma 6: For any real user with L virtual users, the unique way to maximize this user's throughput is to set

$$\gamma_k = \frac{N}{\sum_{j=1}^{M} P_j} \left(1 + \frac{\sum_{j=1}^{M} P_j}{N} \right)^{\frac{L-k}{L}} \left[\left(1 + \frac{\sum_{j=1}^{M} P_j}{N} \right)^{\frac{1}{L}} - 1 \right]$$

for $k = 1, ..., L$. Moreover, if all real users adopt this power allocation rule, then

$$\lim_{L \to \infty} \sum_{k=1}^{L} r_{ik} = \frac{P_i}{2 \sum_{j=1}^{M} P_j} \log \left(1 + \frac{\sum_{j=1}^{M} P_j}{N} \right),$$

for $i = 1, ..., M$.

Proof: By replacing p_k by $\gamma_k P_i$ for $k = 1, ..., L$, we can use arguments similar to those in Section II to prove the following:
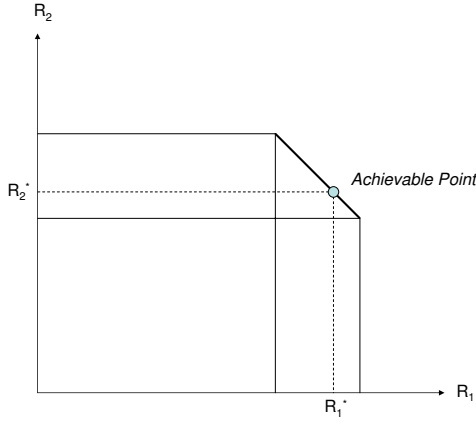
Fig. 5. Achievable point for a two-user Gaussian MAC where user 1 has higher transmission power. The transmission powers and the achievable rates satisfy $\frac{R_1^*}{R_2^*} = \frac{P_1}{P_2}$.

1) Given a DRS scheme with L virtual users per real user, it is possible to strictly increase the throughput via an $(L+1)$ virtual user system.
2) Under the optimal power split, all virtual users generated by real user i must have the same rate for $i = 1, ..., M$. (Virtual users generated by different real users may have different rates.)
3) For any real user with L virtual users, the unique way to maximize this user's throughput is to set

$$\gamma_k = \frac{N}{\sum_{i=1}^{M} P_i} \left(1 + \frac{\sum_{i=1}^{M} P_i}{N}\right)^{\frac{L-k}{L}} \left[\left(1 + \frac{\sum_{i=1}^{M} P_i}{N}\right)^{\frac{1}{L}} - 1\right]$$

for $k = 1, ..., L$.
4) If all real users adopt this power allocation rule, then

$$\lim_{L \to \infty} \sum_{k=1}^{L} r_{ik} = \frac{P_i}{2 \sum_{j=1}^{M} P_j} \log\left(1 + \frac{\sum_{i=1}^{M} P_i}{N}\right),$$

for $i = 1, ..., M$

$\square$

We illustrate this achievable point on the dominant face for a two-user Gaussian MAC in Fig. 5.

2) M-user Discrete Memoryless MAC: In Section III, we considered the symmetric setting (cf (18)):

$$I(X_S; Y \mid X_{S^c}) = I(X_T; Y \mid X_{T^c}),$$

$\forall S, T \subseteq \{1, ..., M\}$ such that $|S| = |T|$. In this section, we relax this constraint and consider an asymmetric achievable region. We require only that for $\forall S, T \subseteq \{1, 2, ..., M\}$, if $S \cap T = \varnothing$, then $I(X_S; Y) < I(X_S; Y|X_T)$. The M switches $(S_1, ..., S_M)$ have probabilities $P(S_i = k) = \lambda_k$ for $i = 1, ..., M$ and $k = 1, ..., L$.

Lemma 7: Consider a general random switch controlled by S_i, where $P(S_i = k) = \lambda_k$, $k = 1, ..., L$. Define $-i \equiv$

$\{1, ..., M\} \setminus \{i\}$. If $\max_{k \in \{1, ..., L\}} \lambda_k \to 0$ as $L \to \infty$, then

$$\lim_{L \to \infty} \sum_{k=1}^{L} r_{X_{ik}}$$
$$= \frac{1}{M} I(X_i; Y) + \sum_{j=1}^{M-1} \frac{1}{M} \frac{1}{\binom{M-1}{j}} \sum_{\substack{\varnothing \subset S \subseteq -i \\ |S| = j}} I(X_i; Y, X_S),$$

for $i = 1, ..., M$.

Proof: We can replace $\binom{M-1}{i} I(X_1; Y, X_2, ..., X_{i+1})$ by $\sum_{\varnothing \subset S \subseteq -i, |S|=j} I(X_i; Y, X_S)$ and use arguments similar to those for Theorem 7 to prove the above lemma. $\square$

B. Unequal Number of Virtual Users

1) M-user Gaussian MAC: In this section, we retain the assumption that every user has the same transmission power P, but we do not require all real users to create the same number of virtual users. That is, user i and user j create L_i and L_j virtual users independently, where L_i may not be equal to L_j. The signal transmitted by a real user is the superposition of all its virtual users' signals. We also assume in this section that user i transmits the number L_i in a header message to the receiver. The receiver receives the sum of $\sum_{i=1}^{M} L_i$ signals plus noise. We now describe a protocol which allows each user to split its power and set its rates independently, and allows the receiver to decode all virtual users one by one via a generalized successive decoding mechanism. Recall that for the Gaussian MAC, successive decoding works as follows. Users are decoded one after another regarding all other users that have not been decoded as interference, and the signals of decoded users are subtracted from the overall received signal.

PROTOCOL 1: For user i, the power split and rate allocation rule are defined as follows: for $k = 1, ..., L_i$,

$$p_{ik} = \frac{N}{M} \left(1 + \frac{MP}{N}\right)^{\frac{L_i - k}{L_i}} \left[\left(1 + \frac{MP}{N}\right)^{\frac{1}{L_i}} - 1\right],$$

$$r_{ik} = \frac{1}{2} \log\left(1 + \frac{p_{ik}}{M\left(P - \sum_{j<k} p_{ij}\right) - p_{ik} + N}\right).$$

Note that the power split and rate allocation rule in *PROTOCOL 1* are the same as that discussed in Section II.

The generalized successive decoding algorithm is given by the following pseudo-program. Note that after a virtual user is decoded, its signal is subtracted from the overall received signal.

decode virtual users $(11, 21, ..., M1)$ in any order or in parallel.
set $\mathbf{p} = (p_{11}, p_{21}, ..., p_{M1})$
while (some virtual users are not decoded),
      Find the minimal element in $\mathbf{p}$, say the ith entry;
      Decode the subsequent virtual user of user i;
      Update the ith entry of $\mathbf{p}$: $\mathbf{p}(i) = \mathbf{p}(i) + \tilde{p}$,
         where $\tilde{p}$ is the power of the virtual user
         being decoded in the previous step;

end

Lemma 8: If all M users adopt the power split and the rate allocation rule described in *PROTOCOL* 1, then for any $L_i \in \mathbb{Z}^+$, $i = 1, 2, ..., M$, the decoder can decode all virtual users one by one following the decoding algorithm.

Proof: By the rate allocation rule, $r_{i1} = \frac{1}{2}\log(1 + \frac{p_{i1}}{MP - p_{i1} + N})$ for $i = 1, ..., M$. Thus, each of them can tolerate the maximum amount of interference plus noise, $MP - p_{i1} + N$. It is then easy to see that virtual users $(11, ..., M1)$ can be decoded reliably in any order or in parallel. Now, we set $\mathbf{p} = (p_{11}, ..., p_{M1})$.

In the first run of the **while** loop, if p_{i1} is the minimum entry in $\mathbf{p}$, the receiver decodes virtual user $i2$. By the rate allocation rule, $r_{i2} = \frac{1}{2}\log(1 + \frac{p_{i2}}{M(P - p_{i1}) - p_{i2} + N})$. This implies that the maximum amount of interference plus noise that virtual user $i2$ can tolerate is $M(P - p_{i1}) - p_{i2} + N$. However, the real amount of interference plus noise it faces is $MP - \sum_{j=1}^{M} p_{j1} - p_{i2} + N$, which is smaller than or equal to what it can tolerate because $Mp_{i1} \le \sum_{j=1}^{M} p_{j1}$. Therefore, virtual user $i2$ can be decoded reliably at the receiver.

Suppose the decoding process succeeds in the tth run of the **while** loop. Now, $\mathbf{p} \triangleq (p_1^t, ..., p_M^t)$. Let us consider the $(t+1)$th run of the **while** loop. Suppose the ith entry, p_i^t, is the minimum entry in $\mathbf{p}$. The receiver decodes the subsequent virtual user of user i, denoted by il. By the rate allocation rule, $r_{il} = \frac{1}{2}\log(1 + \frac{p_{il}}{M(P - \sum_{j<l} p_{ij}) - p_{il} + N})$. This implies the maximum amount of interference plus noise that virtual user il can tolerate is $M(P - \sum_{j<l} p_{ij}) - p_{il} + N = M(P - p_i^t) - p_{il} + N$. However, the real amount of interference plus noise it faces is $MP - \sum_{j=1}^{M} p_j^t - p_{il} + N$, which is smaller than or equal to what virtual user il can tolerate because $Mp_i^t \le \sum_{j=1}^{M} p_j^t$. Therefore, virtual user il can be decoded reliably. Hence, the lemma follows by induction. $\square$

To illustrate the decoding algorithm, let us carefully examine a three-user example shown in Fig. 6. The shaded regions correspond to the virtual users $(11, 21, 31)$, which are decoded in any order. Suppose we decode "11" first. By the rate allocation rule, $r_{11} = \frac{1}{2}\log(1 + \frac{p_{11}}{3P - p_{11} + N})$, which means the maximum amount interference plus noise that virtual user "11" can tolerate is $3P - p_{11} + N$. This is exactly the amount of interference plus noise it faces. Therefore, "11" can be decoded reliably and we can subtract the signal of virtual user "11" from the overall received signal. Similarly, $(21, 31)$ can be decoded reliably and subtracted from the overall received signal. Now $\mathbf{p} = (p_{11}, p_{21}, p_{31})$. The subsequent decoding order is illustrated by the numbers in Fig. 6. In the first run of the **while** loop, since p_{31} is the minimum in $\mathbf{p}$, the receiver decodes virtual user "32". By the rate allocation rule, $r_{32} = \frac{1}{2}\log(1 + \frac{p_{32}}{3(P - p_{31}) - p_{32} + N})$, which implies the maximum amount interference plus noise it can tolerate is $3(P - p_{31}) - p_{32} + N$. However, the real interference plus noise it faces is $(P - p_{11}) + (P - p_{21}) + (P - p_{31} - p_{32}) + N$, which is smaller than what it can tolerate since $2p_{31} < p_{11} + p_{21}$. So virtual user "32" can be decoded reliably and subtracted from the received signal. By searching for the minimum entry in $\mathbf{p}$
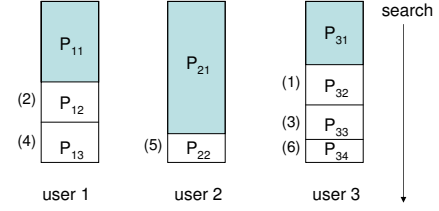

Fig. 6. Three-user example to illustrate the decoding algorithm. The numbers correspond to the decoding order inside the **while** loop.

in each run, we always decode a virtual user that can tolerate more interference than what it really faces. This assures the validity of our decoding algorithm. All of the other virtual users can be decoded in a similar fashion.

By Lemma 2 and Lemma 8, user i can choose any $L_i \in \mathbb{Z}^+$, independently from other users, and have all virtual users decoded reliably at the receiver. Therefore, user i can choose L_i according to its own service requirement. For example, if user i wants to send low rate voice communication packets, it can set $L_i = 1$, which corresponds to the basic CDMA scheme. If user i wants to send high rate stream video, it can set L_i equal to a large value in order to get higher throughput at the expense of higher coding complexity. Thus, this variation of DRS provides an explicit way for end users to trade off throughput and coding complexity, making differential rate requirements achievable in a distributed manner. Finally, Corollary 2 and Theorem 4 demonstrate the asymptotic optimality of this scheme and its rate of convergence.

2) M-user Discrete Memoryless MAC: In this section, we describe a variation of the DRS scheme for the discrete memoryless MAC which supports differential rate requirements to end users in a distributed manner. In this scheme, we adopt the uniform switch, but we do not require every user to have the same number of virtual users. We split by means of M independent switches. Without loss of generality, let us consider user i. If user i has L_i virtual users, then switch i has L_i inputs, $X_{ik} \in \mathcal{X}_i$ for $k = 1, ..., L_i$, and one output $X_i \in \mathcal{X}_i$. Switch i is controlled by a uniform random variable $S_i \in \{1, ..., L_i\}$ with $P(S_i = k) = \frac{1}{L_i}$ for $k = 1, ..., L_i$. The output is given by: $X_i = X_{ik}$, if $S_i = k$.

We now describe the protocol for the discrete memoryless MAC which allows user i to choose L_i independently. We show the asymptotic optimality of this variation of DRS under the protocol.

PROTOCOL 2: For user i with L_i virtual users, the switch i is controlled by a uniform random variable S_i where $P(S_i = k) = \frac{1}{L_i}$ for $k = 1, ..., L_i$. The rate allocation rule is defined as follows:

$$r_{X_{ik}} = \frac{1}{L_i}\left[\left(1 - \frac{k-1}{L_i}\right)^{M-1} I(X_1; Y)\right.$$
$$\left. + \sum_{l=1}^{M-1}\binom{M-1}{l}\left(\frac{k-1}{L_i}\right)^l\right].$$

$$\left(1 - \frac{k-1}{L_i}\right)^{M-1-l} I(X_1; Y, X_2, .., X_{l+1})\right].$$

The decoding algorithm is given by the following pseudo-program. Note that after a virtual user is decoded, its signal is used as side information to aid the decoding process of subsequent virtual users.

decode virtual users $(11, 21, ..., M1)$ in any order or in parallel.

  set $\mathbf{s} = (\frac{1}{L_1}, \frac{1}{L_2}, ..., \frac{1}{L_M})$

  while (some virtual users are not decoded),

      Find the minimal element in $\mathbf{s}$, say the ith entry;

      Decode the subsequent virtual user of user i;

      Update the ith entry of $\mathbf{s}$: $\mathbf{s}(i) = \mathbf{s}(i) + \frac{1}{L_i}$;

  end

Lemma 9: If all M users adopt the rate allocation rule in *PROTOCOL 2*, then for any $L_i \in \mathbb{Z}^+ \; \forall i$, the decoder can decode all virtual users one by one following the decoding algorithm.

Proof: Virtual users $(11, 21, ..., M1)$ can be decoded in any order if and only if $r_{X_{i1}} \le I(X_{i1}; Y, \mathbf{S}_1^M)$ for all i. This is true because under our rate allocation rule, $r_{X_{i1}} = \frac{1}{L_i} I(X_1; Y) = \frac{1}{L_i} I(X_i; Y) = I(X_{i1}; Y, \mathbf{S}_1^M)$. We set $\mathbf{s} = (\frac{1}{L_1}, ..., \frac{1}{L_M})$.

In the first run of the **while** loop, if $\frac{1}{L_i}$ is the minimum entry of $\mathbf{s}$, the receiver decodes virtual user $i2$. By the rate allocation rule

$$
\begin{aligned}
r_{X_{i2}} = \; & \frac{1}{L_i}\left[\left(1 - \frac{1}{L_i}\right)^{M-1} I(X_1; Y) \right. \\
& + \sum_{l=1}^{M-1} \binom{M-1}{l}\left(\frac{1}{L_i}\right)^l \cdot \\
& \left.\left(1 - \frac{1}{L_i}\right)^{M-1-l} I(X_1; Y, X_2, .., X_{l+1})\right] \quad (27)
\end{aligned}
$$

Virtual user $i2$ can be decoded reliably if $r_{X_{i2}} \le I(X_{i2}; Y, \mathbf{S}_1^M, X_{11}, ..., X_{M1})$. This mutual information can be simplified in the same way as described in equations (23) - (26). Recall the definition $-i \equiv \{1, ..., M\} \setminus \{i\}$.

$$
\begin{aligned}
& I\left(X_{i2}; Y, \mathbf{S}_1^M, X_{11}, ..., X_{M1}\right) \\
= \; & \frac{1}{L_i} I\left(X_{i2}; Y, X_{11}, ... X_{M1} | \mathbf{S}_{-i}\right) \quad (28) \\
= \; & \frac{1}{L_i} \sum_{\mathbf{s}_{-i}} \mathsf{P}\left(\mathbf{S}_{-i} = \mathbf{s}_{-i}\right) I\left(X_1; Y, X_{11}, ..., \right. \\
& \left. X_{M1} | \mathbf{S}_{-i} = \mathbf{s}_{-i}\right). \quad (29)
\end{aligned}
$$

We can verify that the $I(X_1; Y)$ term in (27) is less than or equal to the corresponding $I(X_1; Y)$ term in Equation (29). This follows from

$$
\begin{aligned}
& \mathsf{P}\left(S_1 > 1, ..., S_{i-1} > 1, S_{i+1} > 1, ... S_M > 1\right) \\
= \; & \prod_{j \ne i}\left(1 - \frac{1}{L_j}\right) \\
\ge \; & \left(1 - \frac{1}{L_i}\right)^{M-1},
\end{aligned}
$$

since $\frac{1}{L_i} \le \frac{1}{L_j}$ for all $j \ne i$. Similarly, it can be verified that the second term in (27) is less than or equal to the corresponding term in Equation (29). Therefore, virtual user $i2$ can be decoded reliably.

Suppose the decoding process succeeds in the tth run of the **while** loop. Now, $\mathbf{s} \triangleq (s_1^t, ..., s_M^t)$. Let us consider the $(t+1)$th run of the **while** loop. Suppose the ith entry, s_i^t, is the minimum entry in $\mathbf{s}$. The receiver decodes the subsequent virtual user of user i, denoted by ij. By the rate allocation rule

$$
\begin{aligned}
r_{X_{ij}} = \; & \frac{1}{L_i}\left[\left(1 - \frac{j-1}{L_i}\right)^{M-1} I(X_1; Y) \right. \\
& + \sum_{l=1}^{M-1} \binom{M-1}{l}\left(\frac{j-1}{L_i}\right)^l \cdot \\
& \left.\left(1 - \frac{j-1}{L_i}\right)^{M-1-l} I(X_1; Y, X_2, .., X_{l+1})\right].
\end{aligned}
$$

Again, we can simplify the mutual information $I(X_{ij}; Y, \mathbf{S}_1^M, \mathbf{X}_1^{j-1}, ..., \mathbf{X}_M^{j-1})$ and show that it is great than or equal to $r_{X_{ij}}$, which implies that virtual user ij can be decoded reliably. Hence, the lemma follows by induction. $\square$

Let us illustrate the decoding algorithm by the following example. We consider a two-user discrete memoryless MAC where user 1 creates 2 virtual users and user 2 creates 3 virtual users. Random switch 1 is controlled by S_1 where $\mathsf{P}(S_1 = 1) = \mathsf{P}(S_1 = 2) = \frac{1}{2}$, and random switch 2 is controlled by S_2 where $\mathsf{P}(S_2 = k) = \frac{1}{3}$ for $k = 1, 2, 3$. By the rate allocation rule, the virtual users' rates can be simplified as follows

$$
\begin{aligned}
r_{X_{11}} = \; & \frac{1}{2} I(X_1; Y) \\
r_{X_{12}} = \; & \frac{1}{2}\left[\frac{1}{2} I(X_1; Y) + \frac{1}{2} I(X_1; Y, X_2)\right] \\
r_{X_{21}} = \; & \frac{1}{3} I(X_1; Y) \\
r_{X_{22}} = \; & \frac{1}{3}\left[\frac{2}{3} I(X_1; Y) + \frac{1}{3} I(X_1; Y, X_2)\right] \\
r_{X_{23}} = \; & \frac{1}{3}\left[\frac{1}{3} I(X_1; Y) + \frac{2}{3} I(X_1; Y, X_2)\right].
\end{aligned}
$$

We first decode $(11, 21)$ in any order. Suppose we decode "21" first. Virtual user "21" can be decoded reliably if $r_{X_{21}} \le I(X_{21}; Y, S_1, S_2)$. The condition holds because

$$
\begin{aligned}
I(X_{21}; Y, S_1, S_2) = \; & I(X_{21}; Y, S_1 | S_2) \\
= \; & \frac{1}{3} I(X_{21}; Y | S_2 = 1) \\
= \; & \frac{1}{3} I(X_2; Y) \\
= \; & \frac{1}{3} I(X_1; Y) \\
= \; & r_{X_{21}}
\end{aligned}
$$

Virtual user "11" can be decoded similarly. Now, we set $\mathbf{s} = (\frac{1}{2}, \frac{1}{3})$.

In the first run of the **while** loop, the receiver decodes "22" since $\frac{1}{3} < \frac{1}{2}$. Let us calculate the mutual information between X_{22} and Y given S_1, S_2 and previously decoded X_{11}, X_{21}.

$$
\begin{aligned}
& I\left(X_{22}; Y, S_1, S_2, X_{11}, X_{21}\right) \\
=\ & \frac{1}{3} I\left(X_{22}; Y | S_1, X_{11}, X_{21}, S_2 = 2\right) \\
=\ & \frac{1}{3}\left(\frac{1}{2} I(X_2; Y, X_1) + \frac{1}{2} I(X_2; Y)\right) \\
=\ & \frac{1}{3}\left(\frac{1}{2} I(X_1; Y, X_2) + \frac{1}{2} I(X_1; Y)\right) \\
\geq\ & \frac{1}{3}\left(\frac{1}{3} I(X_1; Y, X_2) + \frac{2}{3} I(X_1; Y)\right) \\
=\ & r_{X_{22}}.
\end{aligned}
$$

The third equality is due to our symmetric assumption. Therefore, virtual user "22" can be decoded. By searching for the minimum entry in **s** in each run, we always decode a virtual user whose rate is smaller than or equal to the corresponding mutual information. This guarantees the correctness of our decoding algorithm. Virtual users $(12, 23)$ can be decoded reliably at the receiver in a similar fashion. The asymptotic optimality of this scheme in the discrete memoryless MAC can be demonstrated by Theorem 7.

V. Concluding Remarks

In this paper, we take an information-theoretic approach to the problem of distributed multiple-access communication. We present a Distributed Rate Splitting scheme whereby each real user creates a number of virtual users and all virtual users are successively decoded at the receiver. One possible advantage of Distributed Rate Splitting is that it can be implemented with lower complexity when compared with joint coding schemes, and less coordination among users when compared with either time-sharing or rate splitting. For the symmetric M-user Gaussian MAC, each real user creates the same number of virtual users via a power/rate splitting mechanism. The transmitted signal of a real user is the superposition of all its virtual users' signals. For the symmetric M-user discrete memoryless MAC, each real user creates the same number of virtual users via a random switching mechanism, and the transmitted signal of a real user is determined by a random switch. All virtual users are successively decoded at the receiver. It is shown that DRS can achieve the maximum equal rate point for both channel models as the number of virtual users per real user tends to infinity. Finally, we present two variations of the DRS scheme. For the case of asymmetric capacity regions, we show that a point on the dominant face can be achieved asymptotically. For the case of an unequal number of virtual users, we show that different user rates requirements can be accommodated independently in a distributed manner.

Appendix I
Proof of Theorem 2

Proof: We use induction on the number of virtual users. For $L = 2$, the original problem reduces to: $SIR_1(p_1, p_2) =$ $SIR_2(p_1, p_2)$ subject to $p_1 + p_2 = P$ and $p_1, p_2 \geq 0$. The unique solution is

$$
\begin{aligned}
(p_1^*(P), p_2^*(P)) = & \left(\frac{1}{M}\left(N + MP - \sqrt{N(N + MP)}\right),\right. \\
& \left.\frac{1}{M}\left(-N + \sqrt{N(N + MP)}\right)\right).
\end{aligned}
$$

For $L = j - 1$, suppose $(p_1^*(P), ..., p_{j-1}^*(P))$ uniquely solves $SIR_1(p_1, ..., p_{j-1}) = ... = SIR_{j-1}(p_1, ..., p_{j-1})$ for any $P > 0$, subject to $\sum_{k=1}^{j-1} p_k = P$ and $p_k \geq 0$ for $k = 1, 2, .., j-1$. Let us consider the j virtual users case. Given any tuple $(p_1, ..., p_j)$ such that $\sum_{k=1}^{j} p_k = P$ and $p_k \geq 0$ for $k = 1, 2, .., j$, we can fix p_j, so $p_1 + ... + p_{j-1} = P - p_j$. We now solve

$$
SIR_1(p_1, ..., p_{j-1}) = ... = SIR_{j-1}(p_1, ..., p_{j-1}) \quad (30)
$$

subject to $\sum_{k=1}^{j-1} p_k = P - p_j$, $p_k \geq 0$ for $k = 1, 2, .., j$. For fixed p_j, by the induction hypothesis, we have a unique solution $(p_1^*(P - p_j), ..., p_{j-1}^*(P - p_j))$ which solves equation (30). Let $s^*(p_j) \triangleq SIR_{j-1}\left(p_1^*(P - p_j), ..., p_{j-1}^*(P - p_j)\right) = \frac{p_{j-1}^*(P - p_j)}{(M-1)p_{j-1}^*(P - p_j) + N}$. We are able to verify that $s^*(p_j)$ is a strictly decreasing function of p_j, and $SIR_j(p_j) = \frac{p_j}{(M-1)p_j + N}$ is a strictly increasing function of p_j. Moreover, the function values at boundary points satisfy $s^*(0) > SIR_j(0)$ and $s^*(P) < SIR_j(P)$. So there exists a unique p_j^* such that $SIR_j(p_j^*) = s^*(p_j^*)$. Hence, we conclude $\left(p_1^*(P - p_j^*), ..., p_{j-1}^*(P - p_j^*), p_j^*\right)$ is the unique solution to the j virtual users case. The theorem follows by induction. $\square$

References

[1] R. Gallager, "A perspective on multiacces channels," *IEEE Trans. Inform. Theory*, vol. 31, pp. 124–142, Mar. 1985.

[2] N. Abramson, "The aloha system-another alternative for computer communications," in *Fall Joint Computer Conference*, 1970.

[3] D. Bertsekas and R. Gallager, *Data Networks*. Prentice Hall, 1992.

[4] N. Abramson, "The throughput of packet broadcasting channels," *IEEE Trans. Commun.*, vol. 25, pp. 117–128, 1977.

[5] C. Namislo, "Analysis of mobile radio slotted aloha networks," *IEEE J. Select. Areas Commun.*, vol. 2, pp. 583 – 588, July 1984.

[6] D. Makrakis and K. Murthy, "Spread slotted aloha techniques for mobile and personal satellite communication systems," *IEEE J. Select. Areas Commun.*, vol. 10, pp. 985 – 1002, Aug. 1992.

[7] S. Ghez, S. Verdu, and S. Schwartz, "Optimal decentralized control in the random access multipacket channel," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1153–1163, Nov. 1989.

[8] V. Naware, G. Mergen, and L. Tong, "Stability and delay of slotted aloha with multipacket reception," *IEEE Trans. Inform. Theory*, submitted for publication.

[9] D. Raychaudhuri, "Performance analysis of random access packet-switched code division multiple access systems," *IEEE Trans. Commun.*, vol. 29, pp. 895–901, 1981.

[10] A. D. Wyner, "Recent results in Shannon Theory," *IEEE Transactions on Information Theory*, vol. 20, pp. 2–10, 1974.

[11] T. Cover and J. Thomas, *Elements of Information Theory*. Wiley-Interscience, 1991.

[12] R. Ahlswede, "Multi-way communication channels," in *Proc. IEEE International Symposium on Information Theory*, 1971, pp. 23–52.

[13] H. Liao, "Multiple access channels," Ph.D. dissertation, University of Hawaii, 1972.

[14] G. Poltyrev, "Coding for channel with asynchronous multiple access," *Probl. Pered. Inform.*, vol. 19, 1983.

[15] J. Hui and P. Humblet, "The capacity region of the totally asynchronous multiple-access channel," *IEEE Trans. Inform. Theory*, vol. 31, Mar. 1985.

[16] B. Rimoldi and R. Urbanke, "A rate-splitting approach to the gaussian multiple-access channel," *IEEE Trans. Inform. Theory*, vol. 42, pp. 364–375, Mar. 1996.

[17] E. Yeh and R. Gallager, "Achieving the multiple access capacity region via projective time sharing," in *Proc. IEEE International Symposium on Information Theory*, Cambridge, MA, Aug. 1998, p. 213.

[18] A. Grant, B. Rimoldi, R. Urbanke, and P. Whiting, "Rate-splitting multiple-access for discrete memoryless channels," *IEEE Trans. Inform. Theory*, vol. 47, pp. 873 – 890, Mar. 2001.

[19] B. Rimoldi, "Generalized time sharing: a low-complexity capacity achieving multiple-access technique," *IEEE Trans. Inform. Theory*, vol. 47, pp. 2432 – 2442, Sept. 2001.

[20] M. Medard, J. Huang, A. Goldsmith, S. Meyn, and T. Coleman, "Capacity of time-slotted aloha packetized multiple-access systems over the awgn channel," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 486–499, Mar. 2004.

[21] S. Shamai, "A broadcast approach for the multiple-access slow fading channel," in *Proc. IEEE International Symposium on Information Theory*, Sorrento, Italy, June 2000, p. 128.

[22] R. Cheng, "Stripping cdma – an asymptotically optimal coding scheme for l-out-of-k white gaussian channels," in *Proc. IEEE Global Telecommunications Conference*, Sorrento, Italy, Nov. 1996, pp. 142–146.

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