Capacity and Random-Coding Exponents for Channel Coding with Side Information

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Abstract

Capacity formulas and random-coding exponents are derived for a generalized family of Gel'fand-Pinsker coding problems. These exponents yield asymptotic upper bounds on the achievable log probability of error. In our model, information is to be reliably transmitted through a noisy channel with finite input and output alphabets and random state sequence. and the channel is selected by a hypothetical adversary. Partial information about the state sequence is available to the encoder, adversary, and decoder. The design of the transmitter is subject to a cost constraint. Two families of channels are considered: 1) compound discrete memoryless channels (CDMC), and 2) channels with arbitrary memory, subject to an additive cost constraint, or more generally to a hard constraint on the conditional type of the channel output given the input. Both problems are closely connected. The random-coding exponent is achieved using a stacked binning scheme and a maximum *penalized* mutual information decoder, which may be thought of as an empirical generalized Maximum a Posteriori decoder. For channels with arbitrary memory, the random-coding exponents are larger than their CDMC counterparts. Applications of this study include watermarking, data hiding, communication in presence of partially known interferers, and problems such as broadcast channels, all of which involve the fundamental idea of binning.

Index terms: channel coding with side information, error exponents, arbitrarily varying channels, universal coding and decoding, randomized codes, MAP decoding, random binning, capacity, reliability function, method of types, watermarking, data hiding, broadcast channels.

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1 Introduction

In 1980, Gel'fand and Pinsker studied the problem of coding for a discrete memoryless channel (DMC) p(y|x,s) with random states S that are observed by the encoder but not by the decoder [1]. They derived the capacity of this channel and showed it is achievable by a random binning scheme and a joint-typicality decoder. Applications of their work include computer memories with defects [2], writing on dirty paper, and communication in presence of a known interference [3, 4, 5, 6]. Duality with source coding problems with side information was explored in [7, 8, 9]. In the late 1990's, it was discovered that the problems of embedding and hiding information in cover signals are closely related to the Gel'fand-Pinsker problem: the cover signal plays the role of the state sequence in the Gel'fand-Pinsker problem [10, 11, 12]. Capacity expressions were derived under expected distortion constraints for the transmitter and a memoryless adversary [12]. One difference between the basic Gel'fand-Pinsker problem and the various formulations of data-hiding and watermarking problems resides in the amount of side information available to the encoder, channel designer (adversary), and decoder. A unified framework for studying such problems is considered in this paper. The encoder, adversary and decoder have access to degraded versions s^e , s^a , s^d , respectively, of a state sequence s. Capacity is obtained as the solution to a mutual-information game:

$$C = \sup_{p_{XU|S^e}} \min_{p_{Y|XS^a}} [I(U; YS^d) - I(U; S^e)],$$

where U is an auxiliary random variable, and the sup and min are subject to appropriate constraints.

In problems such as data hiding, the assumption of a fixed channel is untenable when the channel is under partial control of an adversary. This motivated the game-theoretic approach of [12], where the worst channel in a class of memoryless channels was derived, and capacity is the solution to a maxmin mutual-information game. This game-theoretic approach was recently extended by Cohen and Lapidoth [13] and Somekh-Baruch and Merhav [14, 15], who considered a class of channels with arbitrary memory, subject to almost-sure distortion constraints. In the special case of *private data hiding*, in which the cover signal is known to both the encoder and the decoder, Somekh-Baruch and Merhav also derived random-coding and sphere-packing exponents [14]. Binning is not needed in this scenario. The channel model of [13, 14, 15] is different from but reminiscent of the classical memoryless arbitrary varying channel (AVC) [16, 17, 18] which is often used to analyze jamming problems. In the classical AVC model, no side information is available to the encoder or decoder. Error exponents for this problem were derived by Ericson [19] and Hughes and Thomas [20]. The capacity of the AVC with side information at the encoder was derived by Ahlswede [21].

The coding problems considered in this paper are motivated by data hiding applications in which the decoder has partial¹ or no knowledge of the cover signal. In all cases capacity is achievable by random-binning schemes. Roughly speaking, the encoder designs a codebook for the auxiliary U. The selected sequence **U** plays the role of input to a fictitious channel and conveys information about both the encoder's state sequence \mathbf{S}^e and the message M to the decoder. Finding the best error exponents for such schemes is challenging. Initial attempts in this direction for the Gel'fand-Pinsker DMC have been reported by Haroutunian *et al.* [22, 23], but errors were discovered later [24, 25]. Very recently, random-coding exponents have been independently obtained by Haroutunian and

¹For instance, the decoder may have access to a noisy, compressed version of the original cover signal.

Tonoyan [26] and Somekh-Baruch and Merhav [27]. Their results and ours [28] were presented at the 2004 ISIT conference.

The random-coding exponents we have derived cannot be achieved by standard binning schemes and standard maximum mutual information (MMI) decoders [16, 18]. Instead we use a *stack* of variable-size codeword-arrays indexed by the type of the encoder's state sequence \mathbf{S}^e . The appropriate decoder is a maximum penalized mutual information (MPMI) decoder, where the penalty is a function of the encoder's state sequence type. The MPMI decoder may be thought of as an empirical generalized MAP decoder, just like the conventional MMI decoder may be thought of as an empirical MAP decoder.

This paper is organized as follows. A statement of the problem is given in Sec. 2, together with basic definitions. Our main results are stated in Sec. 3 in the form of four theorems. An application to binary alphabets under Hamming cost constraints for the transmitter and adversary is given in Sec. 4. Proofs of the theorems appear in Secs. 5—8. All derivations are based on the method of types [29]. The paper concludes with a discussion in Sec. 9 and appendices.

1.1 Notation

We use uppercase letters for random variables, lowercase letters for individual values, and boldface fonts for sequences. The p.m.f. of a random variable $X \in \mathcal{X}$ is denoted by $p_X = \{p_X(x), x \in \mathcal{X}\}$, and the probability of a set Ω under p_X is denoted by $P_X(\Omega)$. Entropy of a random variable Xis denoted by H(X), and mutual information between two random variables X and Y is denoted by I(X;Y) = H(X) - H(X|Y), or by $\tilde{I}_{XY}(p_{XY})$ when the dependency on p_{XY} should be explicit; similarly we sometimes use the notation $\tilde{I}_{XY|Z}(p_{XYZ})$. The Kullback-Leibler divergence between two p.m.f.'s p and q is denoted by D(p||q). We denote by $D(p_{Y|X}||q_{Y|X}|p_X) = D(p_{Y|X}p_X||q_{Y|X}p_X)$ the conditional Kullback-Leibler divergence of $p_{Y|X}$ and $q_{Y|X}$ with respect to p_X . The base-2 logarithm of x is denoted by $\log x$, and the natural logarithm is denoted by $\ln x$.

Following the notation in Csiszár and Körner [16], let $p_{\mathbf{x}}$ denote the type of a sequence $\mathbf{x} \in \mathcal{X}^N$ ($p_{\mathbf{x}}$ is an empirical p.m.f. over \mathcal{X}) and $T_{\mathbf{x}}$ the type class associated with $p_{\mathbf{x}}$, i.e., the set of all sequences of type $p_{\mathbf{x}}$. Likewise, we define the joint type $p_{\mathbf{xy}}$ of a pair of sequences $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^N \times \mathcal{Y}^N$ (a p.m.f. over $\mathcal{X} \times \mathcal{Y}$) and $T_{\mathbf{xy}}$ the type class associated with $p_{\mathbf{xy}}$, i.e., the set of all sequences of type $p_{\mathbf{xy}}$. We define the conditional type $p_{\mathbf{y}|\mathbf{x}}$ of a pair of sequences (\mathbf{x}, \mathbf{y}) as $\frac{p_{\mathbf{xy}}(x,y)}{p_{\mathbf{x}}(x)}$ for all $x \in \mathcal{X}$ such that $p_{\mathbf{x}}(x) > 0$. The conditional type class $T_{\mathbf{y}|\mathbf{x}}$ is the set of all sequences $\tilde{\mathbf{y}}$ such that $(\mathbf{x}, \tilde{\mathbf{y}}) \in T_{\mathbf{xy}}$. We denote by $H(\mathbf{x})$ the entropy of the p.m.f. $p_{\mathbf{x}}$ and by $I(\mathbf{x}; \mathbf{y})$ the mutual information for the joint p.m.f. $p_{\mathbf{xy}}$. Recall that

$$(N+1)^{-|\mathcal{X}|} 2^{NH(\mathbf{x})} \le |T_{\mathbf{x}}| \le 2^{NH(\mathbf{x})}$$
(1.1)

and

$$(N+1)^{-|\mathcal{X}||\mathcal{Y}|} 2^{NH(\mathbf{y}|\mathbf{x})} \le |T_{\mathbf{y}|\mathbf{x}}| \le 2^{NH(\mathbf{y}|\mathbf{x})}.$$
(1.2)

We let \mathcal{P}_X and $\mathcal{P}_X^{[N]}$ represent the set of all p.m.f.'s and empirical p.m.f.'s, respectively, for a random variable X. Likewise, $\mathcal{P}_{Y|X}$ and $\mathcal{P}_{Y|X}^{[N]}$ denote the set of all conditional p.m.f.'s and all empirical conditional p.m.f.'s, respectively, for a random variable Y given X. The notations $f(N) \ll$ $g(N), f(N) = O(g(N)), \text{ and } f(N) \gg g(N) \text{ indicate that } \lim_{N \to \infty} \left| \frac{f(N)}{g(N)} \right| \text{ is zero, finite but nonzero,}$ and infinite, respectively. The shorthands $f(N) \doteq g(N), f(N) \le g(N)$ and $f(N) \ge g(N)$ denote equality and inequality on the exponential scale: $\lim_{N \to \infty} \frac{1}{N} \ln \frac{f(N)}{g(N)} = 0, \lim_{N \to \infty} \frac{1}{N} \ln \frac{f(N)}{g(N)} \le 0,$ and $\lim_{N \to \infty} \frac{1}{N} \ln \frac{f(N)}{g(N)} \ge 0$, respectively. We let $\mathbb{I}\{x \in \Omega\}$ denote the indicator function of a set Ω , and $\mathbb{U}(\Omega)$ denote the uniform p.m.f. over a finite set Ω . We define $|t|^+ \triangleq \max(0, t), \exp_2(t) \triangleq 2^t$, and $\overline{h}(t) \triangleq -t \log t - (1-t) \log(1-t)$ (the binary entropy function). We adopt the notational convention that the minimum of a function over an empty set is $+\infty$.

2 Statement of the Problem

Our generic problem of communication with side information at the encoder and decoder is diagrammed in Fig. 1. There three versions \mathbf{S}^e , \mathbf{S}^a , and \mathbf{S}^d of a state sequence are available to the encoder, adversary and decoder, respectively. We use the short hand \mathbf{S} to denote the joint state sequence ($\mathbf{S}^e, \mathbf{S}^a, \mathbf{S}^d$). This sequence consists of independent and identically distributed (i.i.d.) samples drawn from a p.m.f. $p(s^e, s^a, s^d)$. The individual sequences $\mathbf{S}^e, \mathbf{S}^a, \mathbf{S}^d$ are available noncausally to the encoder, adversary and decoder, respectively. The adversary's channel is of the form $p_{\mathbf{Y}|\mathbf{XS}^a}(\mathbf{y}|\mathbf{x}, \mathbf{s}^a)$. This includes the problems listed in Table 1 as special cases. The alphabets $\mathcal{S}^e, \mathcal{S}^a, \mathcal{S}^d, \mathcal{X}$ and \mathcal{Y} are finite.



Figure 1: Communication with side information at the encoder and decoder. Cost constraints are imposed on the encoder and channel.

Problem	\mathbf{S}^{a}	\mathbf{S}^d	Binning?
Gel'fand-Pinsker [1]	\mathbf{S}^{e}	Ø	yes
Public Watermarking [12, 15]	Ø	Ø	yes
Semiblind Watermarking [12]	Ø	$\mathbf{S}^{d} eq \mathbf{S}^{e}$	yes
Cover-Chiang [7]	$(\mathbf{S}^e, \mathbf{S}^d)$	\mathbf{S}^d	yes
Private Watermarking [12, 14]	Ø	\mathbf{S}^{e}	no

Table 1: Relation between \mathbf{S}^e , \mathbf{S}^a , and \mathbf{S}^d for various coding problems with side information.

A message M is to be transmitted to a decoder; M is uniformly distributed over the message set \mathcal{M} . The transmitter produces a sequence $\mathbf{X} = f_N(\mathbf{S}^e, M)$. The adversary passes \mathbf{X} through the channel $p_{\mathbf{Y}|\mathbf{XS}^a}(\mathbf{y}|\mathbf{x}, \mathbf{s}^a)$ to produce corrupted data \mathbf{Y} . The decoder does not know $p_{\mathbf{Y}|\mathbf{XS}^a}$ selected by the adversary and has access to \mathbf{S}^d . The decoder produces an estimate $\hat{M} = g_N(\mathbf{Y}, \mathbf{S}^d) \in \mathcal{M}$ of the transmitted message.²

We allow the encoder/decoder pair (f_N, g_N) to be randomized, i.e., the choice of (f_N, g_N) is a function of a random variable known to the encoder and decoder but not to the adversary. This random variable is independent of all other random variables and plays the role of a secret key. The randomized code will be denoted by (F_N, G_N) .

To summarize, the random variables $M, F_N, G_N, \mathbf{S}^e, \mathbf{S}^a, \mathbf{S}^d, \mathbf{X}$ and \mathbf{Y} have joint p.m.f.

$$p_M(m)p_{F_NG_N}(f_N, g_N) \left[\prod_{i=1}^N p_{S^e S^a S^d}(s_i^e, s_i^a, s_i^d)\right] \mathbb{I}\{\mathbf{x} = f_N(\mathbf{s}^e, m)\} p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^a}(\mathbf{y}|\mathbf{x}, \mathbf{s}^a).$$

2.1 Constrained Side-Information Codes

A cost function $\Gamma : S^e \times \mathcal{X} \to \mathbb{R}^+$ is defined to quantify the cost $\Gamma(s^e, x)$ of transmitting symbol x when the channel state at the encoder is s^e . This definition is extended to N-vectors using $\Gamma^N(\mathbf{s}^e, \mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \Gamma(s^e_i, x_i)$. In information embedding applications, Γ is a distortion function measuring the distortion between host signal and marked signal.

We now define a class of codes satisfying maximum-cost constraints (Def. 2.1) and a class of codes satisfying average-cost constraints (Def. 2.2). The latter class is of course larger than the former. We also define a class of randomly-modulated (RM) codes (Def. 2.3), adopting terminology from [20].

Def. 2.2 is analogous to the definition of a length-N information hiding code in [12]. The common source of randomness between encoder and decoder appears via the distribution $p_{F_NG_N}(f_N, g_N)$ whereas in [12] it appears via a cryptographic key sequence **k** with finite entropy rate.

Definition 2.1 A length-N, rate-R, randomized code with side information and maximum cost D_1 is a triple (\mathcal{M}, F_N, G_N) , where

- \mathcal{M} is the message set of cardinality $|\mathcal{M}| = \lceil 2^{NR} \rceil$;
- (F_N, G_N) has joint distribution $p_{F_NG_N}(f_N, g_N)$;
- $f_N : (S^e)^N \times \mathcal{M} \to \mathcal{X}^N$ is the encoder mapping the state sequence \mathbf{s}^e and message m to the transmitted sequence $\mathbf{x} = f_N(\mathbf{s}^e, m)$. The mapping is subject to the cost constraint

$$\Gamma^{N}(\mathbf{s}^{e}, f_{N}(\mathbf{s}^{e}, m)) \leq D_{1} \quad almost \ surely \ (p_{S^{e}}p_{F_{N}}p_{M});$$

$$(2.1)$$

² At first sight the problem setup could be simplified by eliminating the variable \mathbf{S}^{a} and considering the "average channel" $p_{\mathbf{Y}|\mathbf{XS}^{e}}(\mathbf{y}|\mathbf{x},\mathbf{s}^{e}) = \sum_{\mathbf{s}^{a}} p_{\mathbf{Y}|\mathbf{XS}^{a}}(\mathbf{y}|\mathbf{x},\mathbf{s}^{a}) p_{S^{a}|S^{e}}^{N}(\mathbf{s}^{a}|\mathbf{s}^{e})$. We do not follow this approach because $p_{S^{a}|S^{e}}$ is fixed and $p_{\mathbf{Y}|\mathbf{XS}^{a}}$ is optimized by the adversary; hence these p.m.f.'s appear separately in the problem formulation and its solution. A similar comment applies to \mathbf{S}^{d} .

• $g_N : \mathcal{Y}^N \times (\mathcal{S}^d)^N \to \mathcal{M} \cup \{e\}$ is the decoder mapping the received sequence \mathbf{y} and channel state sequence \mathbf{s}^d to a decoded message $\hat{m} = g_N(\mathbf{y}, \mathbf{s}^d)$. The decision $\hat{m} = e$ is a declaration of error.

Definition 2.2 A length-N, rate-R, randomized code with side information and expected cost D_1 is a triple (\mathcal{M}, F_N, G_N) which satisfies the same conditions as in Def. 2.1, except that (2.1) is replaced with the weaker constraint

$$\sum_{\mathbf{s}^e} p_{S^e}^N(\mathbf{s}^e) \sum_{f_N} p_{F_N}(f_N) \sum_{m \in \mathcal{M}} \frac{1}{|\mathcal{M}|} \Gamma^N(\mathbf{s}^e, f_N(\mathbf{s}^e, m)) \le D_1.$$
(2.2)

Definition 2.3 A randomly modulated (RM) code with side information is a randomized code defined via permutations of a prototype (f_N, g_N) . Such codes are of the form

$$\mathbf{x} = f_N^{\pi}(\mathbf{s}^e, m) \triangleq \pi^{-1} f_N(\pi \mathbf{s}^e, m)$$
$$g_N^{\pi}(\mathbf{y}, \mathbf{s}^d) \triangleq g_N(\pi \mathbf{y}, \pi \mathbf{s}^d)$$

where π is chosen uniformly from the set of all N! permutations and is not revealed to the adversary. The sequence $\pi \mathbf{x}$ is obtained by applying π to the elements of \mathbf{x} .

2.2 Constrained Attack Channels

Next we define a class \mathcal{A} of DMC's (Def. 2.4) and a corresponding class $\mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{A}]$ of channels with arbitrary memory (CAM) in which the conditional type of \mathbf{y} given $(\mathbf{x}, \mathbf{s}^a)$ is constrained (Def. 2.5).

Definition 2.4 A compound DMC (CDMC) class \mathcal{A} is any compact (under L_1 norm) subset of $\mathcal{P}_{Y|XS^a}$.

For CDMC's, we have $p_{\mathbf{Y}|\mathbf{XS}^a}(\mathbf{y}|\mathbf{x}, \mathbf{s}^a) = \prod_{i=1}^N p_{Y|XS^a}(y_i|x_i, s_i^a)$, where $p_{Y|XS^a} \in \mathcal{A}$. The set \mathcal{A} is defined according to the application.

- 1. In the case of a known channel [1], \mathcal{A} is a singleton.
- 2. In information hiding problems [12], \mathcal{A} is the class of DMC's that introduce expected distortion between X and Y at most equal to D_2 :

$$\sum_{s_a, x, y} p_{XS^a}(x, s^a) p_{Y|XS^a}(y|x, s^a) d(x, y) \le D_2,$$
(2.3)

where $d : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^+$ is a distortion function. \mathcal{A} can also be defined to be a subset of the above class.

3. In some applications, \mathcal{A} could be defined via multiple cost constraints.

Given a p.m.f. p_{XUS^e} , we denote by $\mathcal{P}_{YS^aS^d|XUS^e}[\mathcal{A}, p_{XUS^e}]$ the class of DMC's $p_{YS^aS^d|XUS^e}$ whose conditional marginal $p_{Y|XS^a}$ is in the CDMC class \mathcal{A} .

Definition 2.5 The CAM class $\mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{A}]$ is the set of channels such that for any channel input $(\mathbf{x}, \mathbf{s}^a)$ and output \mathbf{y} , the conditional type $p_{\mathbf{y}|\mathbf{xs}^a}$ belongs to $\mathcal{A} \bigcap \mathcal{P}_{Y|XS^a}^{[N]}$ with probability 1:

$$Pr[p_{\mathbf{y}|\mathbf{xs}^a} \in \mathcal{A}] = 1. \tag{2.4}$$

If \mathcal{A} is defined via the distortion constraint (2.3), let $d^N(\mathbf{x}, \mathbf{y}) = \frac{1}{N} \sum_{i=1}^N d(x_i, y_i)$. Condition (2.4) may then be rewritten as

$$Pr[d^N(\mathbf{x}, \mathbf{y}) \le D_2] = 1, \tag{2.5}$$

i.e., feasible channels have total distortion bounded by ND_2 and arbitrary memory.³ Comparing the CDMC class \mathcal{A} and the CAM class $\mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{A}]$, we see that 1) for $(\mathbf{X}, \mathbf{S}^a, \mathbf{Y})$ in any given type class, the conditional p.m.f. of \mathbf{Y} given $(\mathbf{X}, \mathbf{S}^a)$ is uniform in the CDMC case but not necessarily so in the CAM case, and 2) while conditional types $p_{\mathbf{y}|\mathbf{xs}^a} \notin \mathcal{A}$ may have exponentially vanishing probability under the CDMC model, such types are prohibited in the CAM case. One may expect that both factors have an effect on capacity and random-coding exponents. As we shall see, only the latter factor does have an effect on random-coding exponents.

The relation between the CAM class $\mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{A}]$ in (2.5) and the classical AVC model [16] is detailed in Appendix A. The class (2.5) is not a special case of the classical AVC model because arbitrary memory is allowed.

We also introduce the following class of attack channels, which turn out to be the worst CAM channels for the problems considered in this paper.

Definition 2.6 An attack channel $p_{\mathbf{Y}|\mathbf{XS}^a}$ uniform over single conditional types is defined via a mapping Λ : $\mathcal{P}_{XS^a}^{[N]} \to \mathcal{P}_{Y|XS^a}^{[N]}$ such that with probability 1, the channel output \mathbf{y} has conditional type $p_{\mathbf{y}|\mathbf{xs}^a} = \Lambda(p_{\mathbf{xs}^a})$. Moreover, \mathbf{Y} is uniformly distributed over the corresponding conditional type class.

Lastly, given a type $p_{\mathbf{xus}^e}$, we denote by $\mathcal{P}_{YS^aS^d|XUS^e}^{[N]}[\mathcal{A}, p_{\mathbf{xus}^e}]$ the class of conditional types $p_{\mathbf{ys}^a\mathbf{s}^d|\mathbf{xus}^e}$ such that $p_{\mathbf{y}|\mathbf{xs}^a}$ is in the CAM class $\mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{A}]$.

2.3 Probability of Error

The average probability of error for a deterministic code (f_N, g_N) when channel $p_{\mathbf{Y}|\mathbf{XS}^a}$ is in effect is given by

$$P_{e}(f_{N}, g_{N}, p_{\mathbf{Y}|\mathbf{XS}^{a}})$$

$$= \Pr(\hat{M} \neq M)$$

$$= \frac{1}{|\mathcal{M}|} \sum_{m} \sum_{\mathbf{s}^{e}, \mathbf{s}^{a}, \mathbf{x}} \sum_{(\mathbf{y}, \mathbf{s}^{d}) \notin g_{N}^{-1}(m)} p_{\mathbf{Y}|\mathbf{XS}^{a}}(\mathbf{y}|\mathbf{x}, \mathbf{s}^{a}) \mathbb{I}\{\mathbf{x} = f_{N}(\mathbf{s}^{e}, m)\} p_{S^{e}S^{a}S^{d}}^{N}(\mathbf{s}^{e}, \mathbf{s}^{a}, \mathbf{s}^{d}). \quad (2.6)$$

³ The case of channels with arbitrary memory subject to expected-distortion constraints admits a trivial solution: the adversary "obliterates" **X** with a fixed, nonzero probability that depends on D_2 but not on N, and therefore no reliable communication is possible in the sense of Def. 2.7 below.

For a randomized code the expression above is averaged with respect to $p_{F_NG_N}(f_N, g_N)$; this average is denoted by $P_e(F_N, G_N, p_{\mathbf{Y}|\mathbf{XS}^a})$. The minmax probability of error for the class of randomized codes and the class of attack channels considered is given by

$$P_{e,N}^{*} = \min_{p_{F_{N}G_{N}}} \max_{p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^{a}}} \sum_{f_{N},g_{N}} p_{F_{N}G_{N}}(f_{N},g_{N})P_{e}(f_{N},g_{N},p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^{a}}).$$
(2.7)

Definition 2.7 A rate R is said to be achievable if $P_{e,N}^* \to 0$ as $N \to \infty$.

Definition 2.8 The capacity $C(D_1, \mathcal{A})$ is the supremum of all achievable rates.

Definition 2.9 The reliability function of the class of attack channels considered is

$$E(R) = \liminf_{N \to \infty} \left[-\frac{1}{N} \log P_{e,N}^* \right].$$
(2.8)

There are four combinations of maximum/expected cost constraints for the transmitter and CDMC/CAM designs for the adversary (four flavors of the generalized Gel'fand-Pinsker problem), and a question is whether same capacity and error exponents will be obtained in all four cases. We now define *transmit channels*, which play a crucial role in deriving capacity and error-exponents.

Definition 2.10 Given alphabets \mathcal{X}, \mathcal{U} and \mathcal{S}^e , a transmit channel $p_{XU|S^e}$ is a conditional p.m.f. that satisfies the following distortion constraint on the conditional marginal $p_{X|S^e}$:

$$\sum_{u,s^e,x} p_{XU|S^e}(x,u|s^e) p_{S^e}(s^e) \Gamma(s^e,x) \le D_1.$$

Given an alphabet \mathcal{U} of cardinality L, we denote by $\mathcal{P}_{XU|S^e}(L, D_1)$ the set of feasible transmit channels. Note that transmit channels have been termed covert channels [12] and watermarking channels [14, 15] in the context of information hiding. In those papers, the channel $p_{Y|X}$ was termed attack channel; we retain this terminology for $p_{Y|XS^a}$ in this paper.

2.4 Preliminaries

Consider a sextuple of random variables (S^e, S^a, S^d, U, X, Y) with joint p.m.f. $p_{S^e S^a S^d UXY}$, where U is an auxiliary random variable taking values in $\mathcal{U} \triangleq \{1, 2, \dots, L\}$. The following difference of mutual informations plays a fundamental role in capacity analysis [1]—[15] of channels with side information. It plays a central role in the analysis of error exponents as well:

$$J_L(p_{S^e S^a S^d U X Y}) \triangleq I(U; Y S^d) - I(U; S^e).$$

$$(2.9)$$

Note that J_L depends on $p_{S^e S^a S^d UXY}$ only via the marginal $p_{US^e S^d Y}$; moreover, the cardinality L of the alphabet \mathcal{U} has been made explicit in the definition (2.9).

Channel capacity for the problems studied in [1]—[15] is given by

$$C(D_1, \mathcal{A}) = \lim_{L \to \infty} \max_{p_{XU|S^e}} \min_{p_{Y|XS^a}} J_L(p_{S^e S^a S^d} \, p_{XU|S^e} \, p_{Y|XS^a})$$
(2.10)

where restrictions are imposed on the joint distribution of (S^e, S^a, S^d) (including the absence of some of these variables, see Table 1), and the maximization over $p_{XU|S^e}$ and minimization over $p_{Y|XS^a}$ are possibly subject to cost constraints.

The cardinality of the alphabet \mathcal{U} may be unbounded [15, p. 514] ⁴, hence the infinite range for L in (2.10). To evaluate (2.10) in the case $S^a = \emptyset$, Moulin and O'Sullivan [12] claimed that one can choose $L = |\mathcal{S}^e| |\mathcal{X}| + 1$ without loss of optimality. The proof is based on Caratheodory's theorem, as suggested in [1]. However the proof in [12] applies only to the fixed-channel case ⁵.

The use of alphabets with unbounded cardinality introduces some technical subtleties. The following two lemmas are straightforward but will be useful. The proof of the first one is based on the nested nature of the feasible sets $\mathcal{P}_{XU|S^e}(L, D_1)$, $1 \leq L < \infty$.

Lemma 2.1 Let $\mathcal{U} = \{1, 2, \cdots, L\}$ and ψ_L a functional defined over $\mathcal{P}_{XU|S^e}(L, D_1)$. Then

$$\psi_L^* \triangleq \max_{p_{XU|S^e} \in \mathcal{P}_{XU|S^e}(L,D_1)} \psi_L(p_{XU|S^e})$$
(2.11)

is a nondecreasing function of L.

Proof. We need to prove that $\psi_L^* \leq \psi_{L+1}^*$ for any L. Let $p_{XU|S^e}^*$ achieve the maximum defining ψ_L^* and define the extended p.m.f. $p_{XU|S^e}^{ex}$ over $\{1, 2, \cdots, L+1\}$ as follows:

$$p_{XU|S^e}^{ex}(x, u|s^e) = \begin{cases} p_{XU|S^e}^*(x, u|s^e) & u = 1, 2, \cdots, L\\ 0 & u = L+1 \end{cases} \quad \forall x, s^e$$

Since $p_{XU|S^e}^{ex}$ and $p_{XU|S^e}^*$ have the same conditional marginal $p_{X|S^e}$, we have $p_{XU|S^e}^{ex} \in \mathcal{P}_{XU|S^e}(L+1, D_1)$, and

$$\psi_{L+1}(p_{XU|S^e}^{ex}) = \psi_L(p_{XU|S^e}^*)$$

Therefore

$$\psi_L^* = \psi_{L+1}(p_{XU|S^e}^{ex}) \le \psi_{L+1}^*.$$

Lemma 2.2 Given three compact sets \mathcal{P} , \mathcal{Q} , \mathcal{R} and a functional ϕ : $\mathcal{P} \times \mathcal{Q} \times \mathcal{R} \to \mathbb{R}$, let (p^*, q^*, r^*) achieve the min max min in

$$\min_{p \in \mathcal{P}} \max_{q \in \mathcal{Q}} \min_{r \in \mathcal{R}} \phi(p, q, r).$$
(2.12)

It is assumed that ϕ is continuous in an L_1 neighborhood of (p^*, q^*, r^*) . Then, given three sequences of subsets $\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n, 1 \leq n < \infty$ respectively dense in \mathcal{P}, \mathcal{Q} and \mathcal{R} under the L_1 norm, we have the following property:

$$\min_{p \in \mathcal{P}} \max_{q \in \mathcal{Q}} \min_{r \in \mathcal{R}} \phi(p, q, r) = \lim_{n \to \infty} \min_{p \in \mathcal{P}_n} \max_{q \in \mathcal{Q}_n} \min_{r \in \mathcal{R}_n} \phi(p, q, r).$$
(2.13)

 $^{^4}$ The capacity formula in [15, Corollary 1, p. 514] was obtained under the restriction of constant composition codes.

⁵Equation (A7) in the proof of [12, Prop. 4.1(iv)] is associated with a fixed DMC.

Proof: Denote the left side of (2.13) by ϕ^{***} and the argument of the limit in the right side by A_n . We have $\underline{A}_n \leq A_n \leq \overline{A}_n$ where

$$\underline{A}_n = \min_{p \in \mathcal{P}} \max_{q \in \mathcal{Q}_n} \min_{r \in \mathcal{R}} \phi(p, q, r),$$

$$\overline{A}_n = \min_{p \in \mathcal{P}_n} \max_{q \in \mathcal{Q}} \min_{r \in \mathcal{R}_n} \phi(p, q, r).$$

Since the maximization (resp. minimizations) defining \underline{A}_n (resp. \overline{A}_n) is over a dense subset of \mathcal{Q} (resp. $\mathcal{P} \times \mathcal{R}$), we have

$$\lim_{n \to \infty} \underline{A}_n = \lim_{n \to \infty} \overline{A}_n = \phi^{***}.$$

Hence $\lim_{n\to\infty} A_n = \phi^{***}$.

Finally, recall that the Kullback-Leibler divergence [16] and related functionals (including mutual information $\tilde{I}(p_{XY}) = D(p_{XY}||p_Xp_Y)$ and J_L functionals) are continuous with respect to L_1 norm. For instance, for any L, any p.m.f.'s p and p' with finite values of J_L , and any $\epsilon > 0$, there exists δ such that

$$||p - p'|| < \delta \quad \Rightarrow \quad |J_L(p) - J_L(p')| < \epsilon,$$

where the norm on p - p' is the L_1 norm.

3 Main Results

The main tool used to prove the coding theorems in this paper is the method of types [29]. Our random-coding schemes are binning schemes in which the auxiliary random variable U is input to a fictitious channel.

In all derivations, optimal types for sextuples $(\mathbf{s}^e, \mathbf{s}^a, \mathbf{s}^d, \mathbf{u}, \mathbf{x}, \mathbf{y})$ are obtained as solutions to maxmin problems. Two key facts used to prove the theorems are: 1) the number of conditional types is polynomial in N, and 2) in the CAM case, the worst attacks are uniform over conditional types, as in Somekh-Baruch and Merhav's watermarking capacity game [15]. Proof of the theorems appears in Secs. 5—8. Related, known results for CDMC's without side information are summarized in Appendix B.

The expression (2.10), restated below in a slightly different form, turns out to be a capacity expression for the problems considered in this paper (Theorems 3.6 and 3.7):

$$C = C(D_1, \mathcal{A}) = \lim_{L \to \infty} C_L \tag{3.1}$$

where

$$C_L \triangleq \max_{p_{XU|S^e} \in \mathcal{P}_{XU|S^e}(L,D_1)} \min_{p_{Y|XS^a} \in \mathcal{A}} J_L(p_{S^e S^a S^d} p_{XU|S^e} p_{Y|XS^a}).$$
(3.2)

By application of Lemma 2.1, the sequence C_L is nondecreasing in L.

In the special case of degenerate $p_{S^eS^d}$ (no side information at the encoder and decoder), it is known that the maximum above is achieved by U = X, and capacity reduces to the standard formula $C = \max_{p_X} \min_{p_{Y|XS^a}} \tilde{I}_{XY}(p_X p_{Y|XS^a} p_{S^a})$. If $S^e = S^d = S$ and $S^a = \emptyset$ (private watermarking), the optimal choice is again U = X, and $C = \max_{p_X|S} \min_{p_{Y|X}} \tilde{I}_{XY|S}(p_{X|S} p_{Y|X} p_S)$.

3.1 Random-Coding Exponents for CDMC Model

Lemma 3.1 The function

$$E_{r,L}^{\text{CDMC}}(R) \triangleq \min_{\tilde{p}_{S^e} \in \mathcal{P}_{S^e}} \max_{p_{XU|S^e} \in \mathcal{P}_{XU|S^e}(L,D_1)} \min_{\tilde{p}_{YS^aS^d|XUS^e} \in \mathcal{P}_{YS^aS^d|XUS^e}} \min_{p_{Y|XS^a} \in \mathcal{A}} \left[D(\tilde{p}_{S^e} p_{XU|S^e} \tilde{p}_{YS^aS^d|XUS^e} || p_{S^eS^aS^d} p_{XU|S^e} p_{Y|XS^a}) + |J_L(\tilde{p}_{S^e} p_{XU|S^e} \tilde{p}_{YS^aS^d|XUS^e}) - R|^+ \right]$$
(3.3)

satisfies the following properties:

- (i) $E_{rL}^{\text{CDMC}}(R) = 0$ if and only if $R \ge C_L$;
- (ii) $E_{r,L}^{\text{CDMC}}(R) \le |C_L R|^+;$
- (iii) $E_{r,L}^{\text{CDMC}}(R) \leq E_{r,L+1}^{\text{CDMC}}(R)$ (nondecreasing in L).

Proof. (i) Clearly $E_{r,L}^{\text{CDMC}}(R) \ge 0$, with equality if and only if the following three conditions are met:

- 1. the minimizing \tilde{p}_{S^e} in (3.3) is equal to p_{S^e} ,
- 2. the minimizing $\tilde{p}_{YS^aS^d|XUS^e}$ in (3.3) is equal to $p_{Y|XS^a} p_{S^aS^d|S^e}$, and
- 3. $R \geq C_L$.

(ii) This upper bound on (3.3) is obtained by fixing $\tilde{p}_{S^e} = p_{S^e}$ and $\tilde{p}_{YS^aS^d|XUS^e} = p_{Y|XS^a} p_{S^aS^d|S^e}$. The upper bound is achieved if the minimizing \tilde{p}_{S^e} and $\tilde{p}_{YS^aS^d|XUS^e}$ in (3.3) are equal to p_{S^e} and $p_{Y|XS^a} p_{S^aS^d|S^e}$, respectively.

(iii) This is a direct consequence of Lemma 2.1.

Theorem 3.2 For the CDMC case (Def. 2.4) with maximum-cost constraint (2.1) or expected-cost constraint (2.2) on the transmitter, the reliability function is lower-bounded by the random-coding error exponent

$$E_r^{\text{CDMC}}(R) = \lim_{L \to \infty} E_{r,L}^{\text{CDMC}}(R).$$
(3.4)

Moreover, $E_r^{\text{CDMC}}(R) = 0$ if and only if $R \ge C$.

For any value of L, the random-coding error exponent $E_{r,L}^{\text{CDMC}}(R)$ of (3.3) is achieved by a binning code with conditionally constant composition and the MPMI decoder. We now present a brief overview of this scheme and an interpretation for the MPMI decoder.



Figure 2: Representation of binning scheme as a stack of arrays indexed by the encoder's state sequence type λ . The arrays have 2^{NR} columns and $2^{N\rho(\lambda)}$ rows, and the random-coding exponent is optimized by choosing $\rho(\lambda) = I_{US}^*(\lambda) + \epsilon$.

For notational simplicity, here we use the shorthand λ to denote the type of the encoder's state sequence (recall there is a polynomial number of such types). Let $\mathcal{U} = \{1, 2, \dots, L\}$, where the value of L is arbitrary. Referring to Fig. 2, to each value of λ corresponds an array

$$\mathcal{C}(\lambda) = \{ \mathbf{u}^{lm|\lambda}, 1 \le l \le 2^{N\rho(\lambda)}, 1 \le m \le |\mathcal{M}| \}$$
(3.5)

of codewords, drawn uniformly from some optimized type class. We refer to $\rho(\lambda)$ as the *depth parameter* of the array $C(\lambda)$. The codebook C is the union of these arrays. Each array has exponential size, but the number of arrays is polynomial in N.

The array depth parameter $\rho(\lambda)$ is designed to optimally balance the probability of encoding error and the probability of decoding error, conditioned on the encoder's state sequence type λ . Upon seeing m and \mathbf{s}^e , the encoder evaluates the type λ of \mathbf{s}^e and seeks a codeword $\mathbf{u}^{lm|\lambda}$ that belongs to some optimized conditional type class $T_{\mathbf{u}|\mathbf{s}^e}$. Let $I^*_{US^e}(\lambda)$ denote the empirical mutual information associated with $T_{\mathbf{u}|\mathbf{s}^e}$. An encoding error arises when no codeword can be found in the conditional type class $T_{\mathbf{u}|\mathbf{s}^e}$. The probability of that event does not vanish when $\rho(\lambda) \leq I^*_{US^e}(\lambda)$ but vanishes at a double-exponential rate when $\rho(\lambda) > I^*_{US^e}(\lambda) + \epsilon$. The probability of decoding error increases exponentially with $\rho(\lambda)$. Therefore the optimal tradeoff is given by $\rho(\lambda) = I^*_{US^e}(\lambda) + \epsilon$.

Instead of choosing $\rho(\lambda)$ as a function of λ , a suboptimal design would be to fix the value of ρ and draw all the codewords uniformly and i.i.d. from a single type class $T_{\mathbf{u}}$. The scheme would then be more akin to the original Gel'fand-Pinsker binning scheme, which uses a single array of codewords (drawn i.i.d. from a p.m.f. p_U). When ρ is fixed, the fact that a polynomial number of equal-size arrays is used rather than a single array is inconsequential as far as error exponents are concerned.

The MPMI decoder is matched to the selected random binning scheme. Given $(\mathbf{y}, \mathbf{s}^d)$, the MPMI decoder seeks the codeword in $\mathcal{C} = \bigcup_{\lambda} \mathcal{C}(\lambda)$ that achieves the maximum of the **penalized**

empirical mutual information criterion

$$\hat{m}_{\text{MPMI}} = \operatorname{argmax}_{m} \max_{l,\lambda} [I(\mathbf{u}^{lm|\lambda}; \mathbf{ys}^{d}) - \rho(\lambda)].$$
(3.6)

As the proof of Theorem 3.2 indicates, the penalty $\rho(\lambda)$ is optimal among all functions of λ ; the optimal penalty is thus matched to the array depth parameter.

The MPMI decoder may be thought of as an *empirical generalized MAP decoder*. Indeed, all messages are equiprobable, and the encoding procedure ensures that for any given type λ , all bins are equiprobable as well. The probability of the pair (m, l) is thus equal to $1/|\mathcal{C}(\lambda)|$ for all l, m. Hence, given $\mathcal{C}(\lambda)$, the *a priori* distribution of the codewords is uniform: $p(\mathbf{u}^{lm|\lambda}) = 1/|\mathcal{C}(\lambda)| = 2^{-N[R+\rho(\lambda)]}$. Therefore

$$\rho(\lambda) = -R - \frac{1}{N} \log p(\mathbf{u}^{lm|\lambda}) \quad \forall l, m.$$
(3.7)

We may write

$$I(\mathbf{u}^{lm|\lambda}; \mathbf{ys}^d) = \frac{1}{N} \log \frac{\hat{p}(\mathbf{y}, \mathbf{s}^d | \mathbf{u}^{lm|\lambda})}{\hat{p}(\mathbf{y}, \mathbf{s}^d)}$$
(3.8)

where \hat{p} denotes an empirical p.m.f. or empirical conditional p.m.f. Substituting (3.7) and (3.8) into (3.6), we obtain

$$\hat{m}_{\text{MPMI}} = \operatorname{argmax}_{m} \max_{l,\lambda} \left[\frac{1}{N} \log \frac{\hat{p}(\mathbf{y}, \mathbf{s}^{d} | \mathbf{u}^{lm | \lambda})}{\hat{p}(\mathbf{y}, \mathbf{s}^{d})} + \frac{1}{N} \log p(\mathbf{u}^{lm | \lambda}) \right]$$
$$= \operatorname{argmax}_{m} \max_{l,\lambda} \hat{p}(\mathbf{u}^{lm | \lambda} | \mathbf{y} \mathbf{s}^{d}).$$
(3.9)

This may be thought of as an empirical version of the generalized MAP decoder

$$\hat{m}_{\text{GMAP}} \triangleq \operatorname{argmax}_{m} \max_{l,\lambda} p(\mathbf{u}^{lm|\lambda}|\mathbf{ys}^{d}) \\ = \operatorname{argmax}_{m} \max_{l,\lambda} \left[\frac{1}{N} \log \frac{p(\mathbf{y}, \mathbf{s}^{d}|\mathbf{u}^{lm|\lambda})}{p(\mathbf{y}, \mathbf{s}^{d})} + \frac{1}{N} \log p(\mathbf{u}^{lm|\lambda}) \right]$$
(3.10)

which requires knowledge of the channel from \mathbf{u} to $(\mathbf{y}, \mathbf{s}^d)$. We do not know whether the GMAP decoder is as good (on the exponential scale) as the optimal MAP decoder

$$\hat{m}_{\text{MAP}} \triangleq \operatorname{argmax}_{m} p(m|\mathbf{y}, \mathbf{s}^{d}) = \operatorname{argmax}_{m} \mathbb{E}_{l,\lambda|\mathbf{ys}^{d}} p(\mathbf{u}^{lm|\lambda}|\mathbf{y}, \mathbf{s}^{d})$$
(3.11)

which averages out the nuisance parameters (l, λ) and is more difficult to analyze.

The MPMI decoder is matched to the encoding scheme in that the same function $\rho(\lambda)$ is used as the depth parameter of the array $C(\lambda)$ and as the penalty in the decoding function. As the proof of Theorem 3.2 indicates, any other choice of the penalty function would in general result in a lower error exponent. This is not surprising in view of the above generalized MAP interpretation.

3.2 Random-Coding Exponents for CAM Model

We now turn our attention to the CAM channel model. First we state the following lemma, which is analogous to Lemma 3.1.

Lemma 3.3 The function

$$E_{r,L}^{\text{CAM}}(R) \triangleq \min_{\tilde{p}_{S^{e}} \in \mathcal{P}_{S^{e}}} \max_{p_{XU|S^{e}} \in \mathcal{P}_{XU|S^{e}}(L,D_{1})} \min_{\tilde{p}_{YS^{a}S^{d}|XUS^{e}} \in \mathcal{P}_{YS^{a}S^{d}|XUS^{e}}[\mathcal{A},p_{XU|S^{e}}\tilde{p}_{S^{e}}] \\ \left[D(\tilde{p}_{S^{e}S^{a}S^{d}}||p_{S^{e}S^{a}S^{d}}) + \tilde{I}_{Y;US^{e}S^{d}|XS^{a}}(\tilde{p}_{S^{e}} p_{XU|S^{e}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}}) + |J_{L}(\tilde{p}_{S^{e}} p_{XU|S^{e}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}}) - R|^{+} \right]$$

$$(3.12)$$

satisfies the following properties:

(i) E^{CAM}_{r,L}(R) = 0 if and only if R ≥ C_L;
(ii) E^{CAM}_{r,L}(R) ≤ |C_L − R|⁺;
(iii) E^{CAM}_{r,L}(R) ≤ E^{CAM}_{r,L+1}(R) (monotonicity in L).

Theorem 3.4 For the CAM case (Def. 2.5) with maximum-cost constraint (2.1) or expected-cost constraint (2.2) on the transmitter, the reliability function is lower-bounded by the random-coding error exponent

$$E_r^{\text{CAM}}(R) = \lim_{L \to \infty} E_{r,L}^{\text{CAM}}(R)$$
(3.13)

Moreover, $E_r^{\text{CAM}}(R) = 0$ if and only if $R \ge C$.

For any value of L, the random-coding error exponent (3.12) is achieved by a randomlymodulated code with conditionally constant composition, stacked binning, and a MPMI decoder. The worst attack channel is uniform over single conditional types (Def. 2.6).

3.3 Comparison of Random-Coding Exponents for CDMC and CAM Models

For both the CDMC and the CAM models, it should be noted that:

- 1. the worst type classes $T_{\mathbf{s}^e}$, $T_{\mathbf{ys}^a \mathbf{s}^d | \mathbf{xus}^e}$, and best type class $T_{\mathbf{xu} | \mathbf{s}^e}$ (in an appropriate min max min sense) determine the error exponents;
- 2. the order of the min, max and min is determined by the knowledge available to the encoder. The encoder knows \mathbf{s}^e and can optimize $T_{\mathbf{xu}|\mathbf{s}^e}$, but has no control over $T_{\mathbf{ys}^a\mathbf{s}^d|\mathbf{xus}^e}$;
- 3. the straight-line part of $E_r(R)$ results from the union bound;
- 4. random codes are generally suboptimal at low rates.

Theorems 3.2 and 3.4 imply the following relationship between error exponents in the CDMC and CAM cases.

Corollary 3.5 $E_r^{\text{CDMC}}(R) \leq E_r^{\text{CAM}}(R) \leq |C - R|^+$ for all R.

Proof. Fix L. Using the relation

$$I(Y; US^e S^d | XS^a) = D(p_{YXUS^e S^a S^d} | |p_{Y|XS^a} p_{XUS^e S^a S^d}),$$

we write

$$I_{Y;US^{e}S^{d}|XS^{a}}(\tilde{p}_{S^{e}} p_{XU|S^{e}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}}) = D(\tilde{p}_{S^{e}} p_{XU|S^{e}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}} ||\tilde{p}_{S} p_{XU|S^{e}} p_{Y|XS^{a}})$$

where we have defined the marginal conditional p.m.f.

$$p_{Y|XS^{a}}(y|x,s^{a}) = \frac{\sum_{us^{e}s^{d}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}}(y,s^{a},s^{d}|x,u,s^{e}) p_{XU|S^{e}}(x,u|s^{e}) \tilde{p}_{S^{e}}(s^{e})}{\sum_{yus^{e}s^{d}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}}(y,s^{a},s^{d}|x,u,s^{e}) p_{XU|S^{e}}(x,u|s^{e}) \tilde{p}_{S^{e}}(s^{e})}.$$

Since $\tilde{p}_{YS^aS^d|XUS^e}$ is an element of $\mathcal{P}_{YS^aS^d|XUS^e}[\mathcal{A}, p_{XU|S^e}\tilde{p}_{S^e}]$ in (3.12), $p_{Y|XS^a}$ defined above is an element of \mathcal{A} and may be viewed as a functional of $\tilde{p}_{YS^aS^d|XUS^e}$ (for fixed $p_{XU|S^e}$ and \tilde{p}_{S^e}). Hence the cost function in (3.12) may be written as

$$D(\tilde{p}_{S}||p_{S}) + D(\tilde{p}_{S^{e}} p_{XU|S^{e}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}}||\tilde{p}_{S} p_{XU|S^{e}} p_{Y|XS^{a}}) + |J - R|^{+}$$

= $D(\tilde{p}_{S^{e}} p_{XU|S^{e}} \tilde{p}_{YS^{a}S^{d}|XUS^{e}}||p_{S^{e}S^{a}S^{d}} p_{XU|S^{e}} p_{Y|XS^{a}}) + |J - R|^{+}$

where the equality follows from the chain rule for divergence. Thus the cost functions in (3.12) and (3.3) are identical; the only difference is the domain over which the minimizations are performed. In (3.3), the minimization over $\tilde{p}_{YS^aS^d|XUS^e}$ is unconstrained, and the minimization over $p_{Y|XS^a}$ is over \mathcal{A} . In (3.12), the minimization over $\tilde{p}_{YS^aS^d|XUS^e}$ is constrained to the set $\mathcal{P}_{YS^aS^d|XUS^e}[\mathcal{A}, p_{XU|S^e}\tilde{p}_{S^e}]$, and $p_{Y|XS^a}$ is a fixed element of \mathcal{A} once $\tilde{p}_{YS^aS^d|XUS^e}$ is fixed. In other words the minimization in (3.3) is over a larger set, and we have $E_{r,L}^{\text{CDMC}}(R) \leq E_{r,L}^{\text{CAM}}(R)$. Taking the limits of both sides of this inequality as $L \to \infty$, we obtain $E_r^{\text{CDMC}}(R) \leq E_r^{\text{CAM}}(R)$.

Similarly, from Lemma 3.3 we have $E_{r,L}^{\text{CAM}}(R) \leq |C_L - R|^+$; taking limits as $L \to \infty$, we obtain $E_r^{\text{CAM}}(R) \leq |C - R|^+$.

The inequality $E_r^{\text{CDMC}}(R) \leq E_r^{\text{CAM}}(R)$ is not as surprising as it initially seems, because the proof of Theorem 3.4 shows there is no loss in optimality in considering CAM's that are uniform over conditional types, and there are more conditional types to choose from under the CDMC model. Generally that additional flexibility is beneficial for the adversary, and the worst conditional type does not satisfy the hard constraint (2.4). See Sec. 4 for an example.

Remark 3.1 In the absence of side information (degenerate $p_{S^eS^d}$), the optimal U = X, and (3.13) becomes $E_r^{CAM}(R) = |C - R|^+$. The expression for $E_r(R)$ derived by Hughes and Thomas [20] (Eqns (9), (6), also see the observation on top of p. 96) is upper-bounded by $|C - R|^+$; they also provide a binary-Hamming example in which equality is achieved. Our result implies that the upper bound $|C - R|^+$ is in fact achieved for any problem without side information in which there exists a hard constraint on the conditional type of the channel output given the input.

3.4 Capacity

As discussed in Sec. 2.4, Gel'fand and Pinsker's proof of the converse theorem in [1] can be extended to more complex problems such as compound Gel'fand-Pinsker channels [12, 15]. The capacity for the generalized Gel'fand-Pinsker problem is given in Theorems 3.6 and 3.7, respectively. Achievability of C follows from Theorems 3.2 and 3.4. Indeed, for any $\epsilon > 0$, there exists $L(\epsilon)$ such that $C_L \geq C - \epsilon$. The proof of the converses appear in Sec. 7 and 8.

Theorem 3.6 Under the CDMC model (Def. 2.4) for the adversary, capacity for the generalized Gel'fand-Pinsker problem is given by (3.1) for both combinations of maximum-cost constraints (2.1) and expected-cost constraints (2.2) on the transmitter.

Theorem 3.7 Under the CAM model (Def. 2.5) for the adversary, capacity for the generalized Gel'fand-Pinsker problem is given by (3.1) for both combinations of maximum-cost constraints (2.1) and expected-cost constraints (2.2) on the transmitter.

The proof of the CDMC converse is similar to that in [12]; the proof in the CAM case exploits the close connection between the CAM and CDMC problems.

3.5 Remarks on Cardinality of \mathcal{U}

The sequence C_L defined in (3.2) is nondecreasing and converges to the capacity limit C, but one may ask at what rate. When the feasible set \mathcal{A} has finite cardinality, by application of Caratheodory's theorem, it suffices to select $L = |\mathcal{X}| |\mathcal{S}^e| + |\mathcal{A}| - 1$ (see [15, 30] for related problems). When \mathcal{A} is a compact set, one may construct a sequence $\epsilon_L \downarrow 0$ and a sequence $\{\mathcal{A}_L\}$ of subsets of \mathcal{A} that is dense in the L_1 norm:

$$\forall \, p_{Y|XS^a} \in \mathcal{A}, \exists \, \hat{p}_{Y|XS^a} \in \mathcal{A}_L \; : \; \max_{x,s^a} \| \hat{p}_{Y|XS^a}(\cdot|x,s^a) - p_{Y|XS^a}(\cdot|x,s^a) \| < \epsilon_L$$

This may be done, for instance, by applying a uniform quantizer to each $p_{Y|XS^a}(y|x, s^a)$ to obtain $\hat{p}_{Y|XS^a}(y|x, s^a)$. By continuity of the functional J_L , the effect of this quantization on J_L can be made arbitrarily small by letting $L \to \infty$. Finally, Caratheodory's theorem can be applied to the set of $|\mathcal{A}_L|$ attack channels so that $\max_{p_{XU|S^e}} \min_{p_{Y|XS^a} \in \mathcal{A}_L} J_L(p_{S^e} p_{XU|S^e} p_{Y|XS^a})$ is achieved using $L = |\mathcal{X}| |\mathcal{S}^e| + |\mathcal{A}_L| - 1$. Proposition 3.8 below formally states this result when the feasible set of attack channels is defined by the distortion constraint (2.3).

Proposition 3.8 Consider the subsequence $\{C_L\}$ indexed by

$$L = |\mathcal{X}| |\mathcal{S}^{e}| + (l+1)^{|\mathcal{Y}| |\mathcal{X}| |\mathcal{S}^{a}|} - 1, \quad l = 1, 2, \cdots$$
(3.14)

Then

$$C-2|\mathcal{Y}|\frac{\log l}{l} \le C_L \le C.$$

Proof: See Appendix G.

For the random-coding exponent $E_r(R)$, the idea is similar but the derivations are more involved because Kullback-Leibler divergence is not absolutely continuous with respect to its arguments; attack channels that lie on the boundary of the probability simplex require a special treatment.

In Proposition 3.10 below, the random-coding exponent is viewed as a function of D_2 and, with a little abuse of notation, written as $E_r(D_2)$. The random-coding exponent when the alphabet \mathcal{U} has size L is similarly denoted by $E_{r,L}(D_2)$. **Lemma 3.9** The function $E_r(D_2)$ is continuous and nonincreasing in D_2 .

The above statement is a consequence of the fact that the Kullback-Leibler and mutualinformation functionals are continuous in their arguments, and that the sets $\{\mathcal{A}(D_2)\}$ are continuously nested.

Proposition 3.10 Denote by $c \leq \frac{1}{|S^a| |S^d|}$ the minimum of $p_{S^aS^d|S^e}$ over its support set. Define the constants

$$l_{\min} = \max\left\{ |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}|, \exp_{2}\left[1 + \sqrt{\left|-\frac{1}{2}\log(8|\mathcal{S}^{a}|^{5}|\mathcal{S}^{d}|c)\right|^{+}}\right] \right\}$$

and $\overline{D} = \max_x \frac{1}{|\mathcal{Y}|} \sum_y d(x, y)$. Consider the subsequence $\{E_{r,L}(D_2)\}$ indexed by

$$L = |\mathcal{X}| |\mathcal{S}^{e}| + l^{|\mathcal{Y}| |\mathcal{X}| (|\mathcal{S}| + |\mathcal{S}^{a}|)} - 1, \quad l = l_{\min}, \, l_{\min} + 1, \cdots$$
(3.15)

Then

$$E_{r,L}(D_2) \le E_r(D_2) \le E_{r,L}\left(D_2 - \frac{|\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| \overline{D} + D_2 \ln l}{l}\right) + 7 |\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| \frac{\log^2 l}{l}.$$
 (3.16)

The gap between the lower and upper bounds in (3.16) is $O(\frac{\log^2 l}{l})$ as $l \to \infty$.

Proof: See Appendix H.

4 Binary-Hamming Case

In this section, we consider a problem of theoretical and practical interest where $S^e = \{0, 1\}$, \mathbf{S}^e is a Bernoulli sequence with $Pr[S^e = 1] = p^e = 1 - Pr[S^e = 0]$, transmission is subject to the cost constraint (2.1) in which Γ is Hamming distance, and the adversary is subject to the expected-distortion constraint (2.3) or to the maximum-distortion constraint (2.5), in which d is also Hamming distance. In both cases the set \mathcal{A} is given by (2.3). We study three cases:

- **Case I:** $p^e = \frac{1}{2}$, $S^a = S^d = \emptyset$. This problem is analogous to the public watermarking problem of [8, 9, 12].
- **Case II:** $p^e = \frac{1}{2}$, $S^a = \emptyset$, $S^d = S^e$. This is the private watermarking problem of [12]. The CAM version of this problem is closely related to a problem studied by Csiszár and Narayan [17] and Hughes and Thomas [20].
- **Case III:** Degenerate side information: $p^e = 0$, $S^e = S^a = S^d = \emptyset$. Unlike [17, 20], the attacker's noise may depend on X.

In all three cases, we were able to derive some analytical results and to numerically evaluate error exponents. Capacity formulas for these problems are given below and illustrated in Fig. 3.

In this section we use the notation $p \star q \triangleq p(1-q) + (1-p)q$.



Figure 3: Capacity functions for Cases I–III when $D_2 = 0.2$.

4.1 Case I: Public Watermarking

Here $p^e = \frac{1}{2}$ and $S^a = S^d = \emptyset$, so we have $S = S^e$. Capacity for a fixed-DMC problem (adversary implements a binary symmetric channel (BSC) with crossover probability D_2) is given in Barron *et al.* [8] and Pradhan *et al.* [9]:

$$C^{\text{pub}} = g^*(D_1, D_2) \triangleq \begin{cases} \frac{D_1}{\delta_2} [\overline{h}(\delta_2) - \overline{h}(D_2)], & \text{if } 0 \le D_1 < \delta_2; \\ \overline{h}(\delta_2) - \overline{h}(D_2), & \text{if } \delta_2 \le D_1 \le 1/2; \\ 1 - \overline{h}(D_2), & \text{if } D_1 > 1/2, \end{cases}$$
(4.1)

where $\delta_2 = 1 - 2^{-\overline{h}(D_2)}$ and $\overline{h}(\cdot)$ is the binary entropy function. The straight-line portion of the capacity function is achieved by time-sharing. Proposition 4.1 shows that the BSC is the worst channel for the CDMC and CAM classes considered.

Proposition 4.1 Capacity under the CDMC and CAM models defined by the distortion constraints (2.3) and (2.5), respectively, is equal to C^{pub} and is achieved for $|\mathcal{U}| = 2$.

Proof: See Appendix C.

Proposition 4.2 The random-coding error exponent is a straight line in the CAM case: $E_r^{\text{CAM,pub}}(R) = |C^{\text{pub}} - R|^+$ for all R. The minimizing \tilde{p}_S in (3.12) coincides with p_S , the maximizing $L = |\mathcal{U}| = 2$, and the minimizing $p_{Y|XUS}$ is the BSC $p_{Y|X}$ with crossover probability D_2 .

Proof: See Appendix D.

Unlike the CAM case, in the CDMC case we have no guarantee that L = 2 is optimal for random-coding exponents. The exponents $E_{r,L}^{\text{CDMC}}(R)$ and $E_{r,L}^{\text{CAM}}(R)$ are shown in Fig. 4 for the case $D_1 = 0.4$, $D_2 = 0.2$, and L = 2; see Sec. 4.4 for details of these calculations. For the CDMC case, we have found numerically that the worst attack channel $p_{Y|X}$ is the BSC with crossover probability D_2 , and that the worst-case \tilde{p}_S in (3.4) coincides with p_S .

4.2 Case II: Private Watermarking

Here $p^e = \frac{1}{2}$, $S^a = \emptyset$, $S^d = S^e = S$.

Proposition 4.3 [12]. Capacity is given by

$$C^{\text{priv}} = \overline{h}(D_1 \star D_2) - \overline{h}(D_2)$$
(4.2)

and is achieved when U = X (L = 2).

For the random-coding exponents, we have no guarantee that L = 2 is an optimal choice. The exponents $E_{r,L}^{\text{CDMC}}(R)$ and $E_{r,L}^{\text{CAM}}(R)$ in that case are shown in Fig. 5 for the case $D_1 = 0.4$, $D_2 = 0.2$. As in Case I, for both the CAM and CDMC cases, the worst-case \tilde{p}_S in (3.3) and (3.12) coincides with p_S .

The capacity expression (4.2) was also derived for the AVC problem of Csiszár and Narayan [17], albeit with different assumptions ($p^e = 0$, i.e., degenerate side information, and channel state θ selected independently of X, see Appendix A). Error exponents for the latter problem were derived by Hughes and Thomas [20]. They obtained $E_r(R) = |C - R|^+$ at all rates below capacity.

4.3 Case III: Degenerate side information

Here $p^e = 0$, $S^e = S^a = S^d = \emptyset$.

Proposition 4.4 Capacity is the same as in the public watermarking game: $C^{\text{deg}} = C^{\text{pub}}$.

Proof: See Appendix E.

Proposition 4.5 $E_r^{\text{CAM,deg}}(R) = |C^{\text{deg}} - R|^+$ for all $R < C^{\text{deg}}$.

Proof: Follows from Remark 3.1.

Unlike Case I and Case II, the worst attack is an asymmetric binary channel, favoring outputs with low Hamming weight. Error exponents in the case $D_1 = 0.4$, $D_2 = 0.2$, are given in Fig. 6.



Figure 4: Error exponents when $p^e = \frac{1}{2}$, $S^a = S^d = \emptyset$ (public watermarking).



Figure 5: Error exponents when $p^e = \frac{1}{2}$, $S^a = \emptyset$, $S^e = S^d$ (private watermarking).



Figure 6: Error exponents when $S^e = S^a = S^d = \emptyset$ ($p^e = 0$, no side information).

4.4 Discussion

Comparing Figs 4 and 5, we see that the random-coding error exponents for L = 2 are only slightly larger when side information is available to the decoder. For instance, the zero-rate exponents are 0.123 and 0.146 at rate zero in the CDMC case; and 0.249 and 0.263 in the CAM case.

Some practical comments about the optimization problems solved in this section are in order. Among these problems, the calculations of random-coding exponents for the CDMC / public watermarking scenario are the most complicated ones, both of which have four layers of minimization or maximization. The number of the parameters to be optimized is $8|\mathcal{U}| + 1$ (1 for \tilde{p}_S , $4|\mathcal{U}| - 2$ for $p_{XU|S}$, 2 for $p_{Y|X}$ and $4|\mathcal{U}|$ for $\tilde{p}_{Y|XUS}$). Other difficulties arise due to the lack of nice properties such as everywhere differentiability and convexity. There appears to be a substantial increase of computational difficulty going from $|\mathcal{U}| = 2$ to larger \mathcal{U} . Based on the analytical results derived above, it is tempting to conjecture that $|\mathcal{U}| = 2$ is an sufficient choice for optimality; unfortunately at this time we are unable to validate that conjecture analytically or numerically.

We have used a genetic algorithm [31] to numerically solve the above-mentioned optimization problems. Advantages of genetic algorithms include easy implementation, robustness with respect to selection of starting points, no need for evaluation of function derivatives, and ability to handle high-dimensional problems. The parameters of a genetic algorithm may be selected to ensure that the algorithm is globally convergent. In particular, we have used an "elitist" genetic algorithm, in which the value of the best individual in each iteration is nondecreasing for a maximization problem (or nonincreasing for a minimization problem). The sequence of the best solutions in each iteration is guaranteed to converge to the global optimum almost surely [31, 32].

5 Proof of Theorem 3.2

Owing to Lemma 3.1, for any $\epsilon > 0$, chosen independently of N, there exists $L(\epsilon)$ such that

$$E_{r,L}^{\text{CDMC}}(R) \ge E_r^{\text{CDMC}}(R) - \epsilon, \quad \forall L > L(\epsilon).$$
 (5.1)

We shall prove the existence of a sequence of codes (f_N, g_N) such that

$$\lim_{N \to \infty} \left[-\frac{1}{N} \log \max_{p_{Y|XS^a} \in \mathcal{A}} P_e(f_N, g_N, p_{Y|XS^a}) \right] = E_{r,L}^{\text{CDMC}}(R).$$

The proof is given for the maximum-cost constraint (2.1) on the transmitter. Any code that achieves the error exponent $E_{r,L}^{\text{CDMC}}(R)$ is therefore also feasible under the weaker average-cost constraint (2.2). A random ensemble of binning codes is constructed, and it is shown that the error probability averaged over this ensemble vanishes exponentially with N at rate $E_{r,L}^{\text{CDMC}}(R)$. Since the error probability functional $P_e(f_N, g_N, p_{Y|XS^a})$ is continuous in $p_{Y|XS^a}$ (by (2.6)) and the feasible set \mathcal{A} of attack channels can be approximated with arbitrary precision (in the L_1 norm) by a subset whose cardinality is polynomial in N, there exists a code (f_N, g_N) from the ensemble that achieves $E_{r,L}^{\text{CDMC}}(R)$ uniformly over \mathcal{A} . It is therefore sufficient to prove that

$$\lim_{N \to \infty} \left[-\frac{1}{N} \log \max_{p_{Y|XS^a} \in \mathcal{A}} P_e(F_N, G_N, p_{Y|XS^a}) \right] = E_{r,L}^{\text{CDMC}}(R)$$
(5.2)

for the random ensemble considered. Combining (5.2) and (5.1) then proves the claim.

The maximum-cost constraint (2.1) may be written as

$$\sum_{s^e,x} p_{\mathbf{s}^e \mathbf{x}}(s^e, x) \Gamma(s^e, x) \le D_1 \quad a.s.$$
(5.3)

Assume $R < C_L - \epsilon$. Define the function

$$E_{r,L,N}^{\text{CDMC}}(R, p_{\mathbf{s}^{e}}, p_{\mathbf{xu}|\mathbf{s}^{e}}) \triangleq \min_{\substack{p_{\mathbf{ys}^{a}s^{d}|\mathbf{xus}^{e}} \in \mathcal{P}_{YS^{a}S^{d}|XUS^{e}}^{[N]} \\ p_{\mathbf{ys}^{a}s^{d}|\mathbf{xus}^{e}} \in \mathcal{P}_{YS^{a}S^{d}|XUS^{e}}^{[N]} p_{Y|XS^{a} \in \mathcal{A}} \\ \begin{bmatrix} D(p_{\mathbf{s}^{e}} p_{\mathbf{xu}|\mathbf{s}^{e}} p_{\mathbf{ys}^{a}s^{d}|\mathbf{xus}^{e}} || p_{S^{e}S^{a}S^{d}} p_{XU|S^{e}} p_{Y|XS^{a}}) \\ + |J_{L}(p_{\mathbf{s}^{e}} p_{\mathbf{xu}|\mathbf{s}^{e}} p_{\mathbf{ys}^{a}s^{d}|\mathbf{xus}^{e}}) - \epsilon - R|^{+} \end{bmatrix}, \\ p_{\mathbf{s}^{e}} \in \mathcal{P}_{S^{e}}^{[N]}, p_{\mathbf{xu}|\mathbf{s}^{e}} \in \mathcal{P}_{XU|S^{e}}^{[N]}(L, D_{1}). \quad (5.4)$$

Let

$$E_{r,L,N}^{\text{CDMC}}(R) \triangleq \min_{p_{\mathbf{s}^e} \in \mathcal{P}_{S^e}^{[N]}} \max_{p_{\mathbf{x}\mathbf{u}|\mathbf{s}^e} \in \mathcal{P}_{XU|S^e}^{[N]}(L,D_1)} \widetilde{E}_{r,L,N}^{\text{CDMC}}(R, p_{\mathbf{s}^e}, p_{\mathbf{x}\mathbf{u}|\mathbf{s}^e}),$$
(5.5)

which differs from (3.3) in that the optimizations are performed over empirical p.m.f.'s instead of arbitrary p.m.f.'s.

Consider the maximization over the conditional type $p_{\mathbf{x}\mathbf{u}|\mathbf{s}^e}$ (viewed as a function of $p_{\mathbf{s}^e}$) in (5.5). As a result of this optimization, we may associate the following:

- to any type $p_{\mathbf{s}^e}$, a type class $T_U^*(p_{\mathbf{s}^e}) \triangleq T_{\mathbf{u}}$ and a mutual information $I_{US^e}^*(p_{\mathbf{s}^e}) \triangleq \tilde{I}_{US^e}(p_{\mathbf{u}|\mathbf{s}^e}p_{\mathbf{s}^e})$;
- to any sequence \mathbf{s}^e , a conditional type class $T^*_{U|S^e}(\mathbf{s}^e) \triangleq T_{\mathbf{u}|\mathbf{s}^e}$;
- to any sequences \mathbf{s}^e and $\mathbf{u} \in T^*_{U|S^e}(\mathbf{s}^e)$, a conditional type class $T^*_{X|US^e}(\mathbf{u}, \mathbf{s}^e) \triangleq T_{\mathbf{x}|\mathbf{us}^e}$.

A random codebook C for **U** is the union of codebooks $C(p_{\mathbf{s}^e})$ indexed by the state sequence type $p_{\mathbf{s}^e}$ (recall there is a polynomial number of types). The codebook $C(p_{\mathbf{s}^e})$ is obtained by a) drawing $2^{N(R+\rho(p_{\mathbf{s}^e}))}$ random vectors independently from the uniform distribution over $T_U^*(p_{\mathbf{s}^e})$, and b) arranging them in an array with 2^{NR} columns and $2^{N\rho(p_{\mathbf{s}^e})}$ rows. The design of the function $\rho(p_{\mathbf{s}^e})$ is arbitrary at this point but will be optimized later.

<u>Encoder</u>. The encoding (given \mathbf{s}^e and m) proceeds in two steps.

- 1. Find *l* such that $\mathbf{u}(l,m) \in \mathcal{C}(p_{\mathbf{s}^e}) \bigcap T^*_{U|S^e}(\mathbf{s}^e)$. If more than one such *l* exists, pick one of them randomly (with uniform distribution). Let $\mathbf{u} = \mathbf{u}(l,m)$. If no such *l* can be found, generate \mathbf{u} uniformly from the conditional type class $T^*_{U|S^e}(\mathbf{s}^e)$.
- 2. Generate **X** uniformly distributed over the conditional type class $T^*_{X|US^e}(\mathbf{u}, \mathbf{s}^e)$.

Clearly, the p.m.f. of $(\mathbf{S}^e, \mathbf{U}, \mathbf{X})$, conditioned on its joint type, is uniform, and the encoder's maximum-cost constraint is satisfied.

<u>Decoder</u>. Given $(\mathbf{y}, \mathbf{s}^d)$, the decoder seeks $\hat{\mathbf{u}} \in \mathcal{C} = \bigcup_{p_{\mathbf{s}^e}} \mathcal{C}(p_{\mathbf{s}^e})$ that maximizes the **penalized** empirical mutual information criterion

$$\max_{p_{\mathbf{s}^e}} \max_{\mathbf{u} \in \mathcal{C}(p_{\mathbf{s}^e})} [I(\mathbf{u}; \mathbf{ys}^d) - \psi(p_{\mathbf{s}^e})].$$
(5.6)

The decoder declares an error if maximizers with different column indices are found. Otherwise the decoder outputs the column index of $\hat{\mathbf{u}}$. The penalty function $\psi(\cdot)$ in (5.6) will soon be optimized, resulting in the "matched design" $\psi = \rho$.

We now analyze the probability of error

$$P_e \triangleq \max_{p_{Y|XS^a} \in \mathcal{A}} P_e(F_N, G_N, p_{Y|XS^a}^N)$$

of the decoder.

Step 1. An encoding error arises under the following event:

$$\mathcal{E}_m = \{ (\mathcal{C}, \mathbf{s}^e) : (\mathbf{u}(l, m) \in \mathcal{C} \text{ and } \mathbf{u}(l, m) \notin T^*_{U|S^e}(\mathbf{s}^e)) \text{ for } 1 \le l \le 2^{N\rho(p_{\mathbf{s}^e})} \}$$
(5.7)

conditioned on message *m* being selected. The probability that a vector **U** uniformly distributed over $T_U^*(p_{\mathbf{s}^e})$ also belongs to $T_{U|S^e}^*(\mathbf{s}^e)$ is equal to $\exp_2\{-NI_{US^e}^*(p_{\mathbf{s}^e})\}$ on the exponential scale. Therefore

$$Pr[\mathcal{E}_{m}|T_{\mathbf{s}^{e}}] = (1 - Pr[\mathbf{U} \in T_{U|S^{e}}^{*}(\mathbf{S}^{e}) \mid \mathbf{U} \sim \mathbb{U}(T_{U}^{*}(p_{\bar{\mathbf{s}}^{e}}))])^{2^{N\rho(p_{\mathbf{s}^{e}})}}$$

$$\stackrel{=}{=} (1 - 2^{-NI_{US^{e}}^{*}(p_{\mathbf{s}^{e}})})^{2^{N\rho(p_{\mathbf{s}^{e}})}}$$

$$\leq \exp\{-2^{N(\rho(p_{\mathbf{s}^{e}}) - I_{US^{e}}^{*}(p_{\mathbf{s}^{e}}))\}}$$
(5.8)

$$\leq \begin{cases} \exp\{-2^{N\epsilon}\} : \text{ if } \rho(p_{\mathbf{s}^e}) \ge I^*_{US^e}(p_{\mathbf{s}^e}) + \epsilon \\ 1 : \text{ else.} \end{cases}$$
(5.9)

The inequality (5.8) follows from $1 + a \le e^a$. The double-exponential term in (5.9) vanishes faster than any exponential function.

Step 2. We have a decoding error under the following event \mathcal{E}'_m : conditioned on message m being selected, there exists \mathbf{u}' not in column m of an array $\mathcal{C}(p'_{\mathbf{s}^e})$ such that

$$I(\mathbf{u}';\mathbf{ys}^d) - \psi(p'_{\mathbf{s}^e}) \ge I(\mathbf{u};\mathbf{ys}^d) - \psi(p_{\mathbf{s}^e}).$$

Therefore

$$P_{e} = \max_{p_{Y|XS^{a}}} Pr[error \mid m = 1, p_{Y|XS^{a}}]$$

$$= \max_{p_{Y|XS^{a}}} \sum_{T_{suxy}} Pr[T_{suxy}] Pr[error \mid T_{suxy}, m = 1]$$

$$\leq \max_{p_{Y|XS^{a}}} \sum_{T_{suxy}} Pr[T_{suxy}] \left(Pr[\mathcal{E}_{1}|T_{suxy}] + Pr[\mathcal{E}_{1}' \mid T_{suxy}, \mathcal{E}_{1}^{c}] \right)$$

$$= \max_{p_{Y|XS^{a}}} \sum_{T_{suxy}} Pr[T_{suxy}] \left(Pr[\mathcal{E}_{1}|T_{s^{e}}] + Pr[\mathcal{E}_{1}' \mid T_{suxy}, \mathcal{E}_{1}^{c}] \right)$$
(5.10)

$$\leq \sum_{T_{\mathbf{suxy}}} \max_{p_{Y|XS^a}} Pr[T_{\mathbf{suxy}}] \left(Pr[\mathcal{E}_1|T_{\mathbf{s}^e}] + Pr[\mathcal{E}'_1 \mid T_{\mathbf{suxy}}, \mathcal{E}^c_1] \right).$$
(5.11)

We will see in Step 3 that $Pr[\mathcal{E}'_1 \mid T_{\mathbf{suxy}}, \mathcal{E}^c_1]$ does not depend on $p_{Y|XS^a}$ for the MPMI decoder. Using the asymptotic relations $P_Z^N(T_z) \doteq \exp_2\{-ND(p_z||p_Z)\}$ and $P_{Z|V}^N(T_z|\mathbf{v}) \doteq \exp_2\{-ND(p_z||p_Z|V||p_z)\}$ [16], we derive

$$Pr[T_{\mathbf{suxy}}] = P_S^N P_{\mathbf{XU}|\mathbf{S}^e} P_{Y|XS^a}^N(T_{\mathbf{suxy}})$$

$$\doteq \exp_2\{-ND(p_{\mathbf{suxy}}||p_S p_{\mathbf{xu}|\mathbf{s}^e} p_{Y|XS^a})\}$$

$$= \exp_2\{-ND(p_{\mathbf{s}^e} p_{\mathbf{xu}|\mathbf{s}^e} p_{\mathbf{ys}^a \mathbf{s}^d|\mathbf{xus}^e}||p_S p_{\mathbf{xu}|\mathbf{s}^e} p_{Y|XS^a})\}.$$
(5.12)

Step 3. Next we evaluate $Pr[\mathcal{E}'_1 | T_{\mathbf{suxy}}, \mathcal{E}^c_1]$, which can be written as $Pr[\mathcal{E}'_1 | T_{\mathbf{suxy}}, \mathbf{u}, \mathbf{y}, \mathbf{s}^d, \mathcal{E}^c_1]$, where $\mathbf{u}, \mathbf{y}, \mathbf{s}^d$ is an arbitrary member of the conditional type class $T_{\mathbf{uys}^d | \mathbf{xs}^e \mathbf{s}^a}$.

Denote by $p_e(\mathbf{u}, \mathbf{y}, \mathbf{s}^d, p'_{\mathbf{s}^e}, T_{\mathbf{suxy}})$ the probability that the decoder outputs the codeword in row l' and column $m' \neq 1$ of the array $\mathcal{C}(p'_{\mathbf{s}^e})$, conditioned on $\mathbf{u}, \mathbf{y}, \mathbf{s}^d$, and $T_{\mathbf{suxy}}$. This conditional error probability is independent of (l', m'). We have

$$Pr[\mathcal{E}'_{1} \mid T_{\mathbf{suxy}}, \mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, \mathcal{E}'_{1}] = 1 - \prod_{p'_{\mathbf{s}^{e}}} [1 - p_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p'_{\mathbf{s}^{e}}, T_{\mathbf{suxy}})]^{2^{N\rho(p'_{\mathbf{s}^{e}})}(2^{NR} - 1)}$$
(5.13)

where

$$p_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p_{\mathbf{s}^{e}}^{\prime}, T_{\mathbf{suxy}}) = \sum_{\mathbf{u}^{\prime} \in \mathcal{U}_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p_{\mathbf{s}^{e}}^{\prime}, p_{\mathbf{suxy}})} p(\mathbf{u}^{\prime} | p_{\mathbf{s}^{e}}^{\prime})$$
$$= \sum_{\mathbf{u}^{\prime} \in \mathcal{U}_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p_{\mathbf{s}^{e}}^{\prime}, p_{\mathbf{suxy}})} \frac{1}{|T_{U}^{*}(p_{\mathbf{s}^{e}}^{\prime})|}$$
(5.14)

and

$$\mathcal{U}_e(\mathbf{u}, \mathbf{y}, \mathbf{s}^d, p'_{\mathbf{s}^e}, p_{\mathbf{suxy}}) = \left\{ \mathbf{u}' \in T^*_U(p'_{\mathbf{s}^e}) : I(\mathbf{u}'; \mathbf{ys}^d) - \psi(p'_{\mathbf{s}^e}) \ge I(\mathbf{u}; \mathbf{ys}^d) - \psi(p_{\mathbf{s}^e}) \right\} (5.15)$$

is the set of codewords \mathbf{u}' in the array indexed by $p'_{\mathbf{s}^e}$, that cause a decoding error, conditioned on $\mathbf{u}, \mathbf{y}, \mathbf{s}^d$, and $T_{\mathbf{usxy}}$. Also define the corresponding set of conditional types

$$\mathcal{T}_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p_{\mathbf{s}^{e}}^{\prime}, p_{\mathbf{suxy}}) = \left\{ T_{\mathbf{u}^{\prime}|\mathbf{ys}^{d}} : T_{\mathbf{u}^{\prime}} = T_{U}^{*}(p_{\mathbf{s}^{e}}^{\prime}), I(\mathbf{u}^{\prime}; \mathbf{ys}^{d}) - \psi(p_{\mathbf{s}^{e}}^{\prime}) \ge I(\mathbf{u}; \mathbf{ys}^{d}) - \psi(p_{\mathbf{s}^{e}}) \right\}$$

$$\subseteq \left\{ T_{\mathbf{u}^{\prime}|\mathbf{ys}^{d}} : I(\mathbf{u}^{\prime}; \mathbf{ys}^{d}) - \psi(p_{\mathbf{s}^{e}}^{\prime}) \ge I(\mathbf{u}; \mathbf{ys}^{d}) - \psi(p_{\mathbf{s}^{e}}) \right\}.$$
(5.16)

Therefore

$$p_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p_{\mathbf{s}^{e}}', T_{\mathbf{suxy}}) = \sum_{\substack{T_{\mathbf{u}'|\mathbf{y}\mathbf{s}^{d}} \in \mathcal{T}_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p_{\mathbf{s}^{e}}', p_{\mathbf{suxy}}) \\ \vdots = \sum_{\substack{T_{\mathbf{u}'|\mathbf{y}\mathbf{s}^{d}} \in \mathcal{T}_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p_{\mathbf{s}^{e}}', p_{\mathbf{suxy}}) \\ \vdots = 2^{-N[I(\mathbf{u}; \mathbf{y}\mathbf{s}^{d}) - \psi(p_{\mathbf{s}^{e}}) + \psi(p_{\mathbf{s}^{e}}')]}$$
(5.17)

because $\frac{|T_{\mathbf{u}'|\mathbf{ys}^d}|}{|T_{\mathbf{u}'}|} \doteq 2^{-NI(\mathbf{u}';\mathbf{ys}^d)}$, and the number of conditional types $T_{\mathbf{u}'|\mathbf{ys}^d}$ is polynomial in N. Next we use the following inequality, which is proved in Appendix F.

$$1 - \prod_{i} (1 - \alpha_i)^{t_i} \leq \min\left(1, \sum_{i} \alpha_i t_i\right), \quad 0 \leq \alpha_i \leq 1, \ t_i \geq 1.$$
(5.18)

Applying (5.18) and (5.17) successively to (5.13), we obtain

$$Pr[\mathcal{E}'_{1} \mid T_{\mathbf{suxy}}, \mathcal{E}'_{1}] = Pr[\mathcal{E}'_{1} \mid T_{\mathbf{suxy}}, \mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, \mathcal{E}'_{1}]$$

$$\leq \min \left\{ 1, \sum_{p'_{\mathbf{s}^{e}}} p_{e}(\mathbf{u}, \mathbf{y}, \mathbf{s}^{d}, p'_{\mathbf{s}^{e}}, T_{\mathbf{suxy}}) 2^{N\rho(p'_{\mathbf{s}^{e}})}(2^{NR} - 1) \right\}$$

$$\leq \min \left\{ 1, \sum_{p'_{\mathbf{s}^{e}}} 2^{-N[I(\mathbf{u}; \mathbf{ys}^{d}) - \psi(p_{\mathbf{s}^{e}}) + \psi(p'_{\mathbf{s}^{e}}) - \rho(p'_{\mathbf{s}^{e}}) - R]} \right\}$$

$$\doteq \exp_{2} \left\{ -N|I(\mathbf{u}; \mathbf{ys}^{d}) - \psi(p_{\mathbf{s}^{e}}) + \min_{p'_{\mathbf{s}^{e}}} [\psi(p'_{\mathbf{s}^{e}}) - \rho(p'_{\mathbf{s}^{e}})] - R|^{+} \right\}. (5.19)$$

Step 4. Combining (5.19) and (5.9), we obtain

$$Pr[\mathcal{E}_1|T_{\mathbf{s}^e}] + Pr[\mathcal{E}'_1 \mid T_{\mathbf{suxy}}, \mathcal{E}'_1] \stackrel{!}{\leq} \exp_2\left\{-N\,\Gamma(R, \rho, \psi, p_{\mathbf{s}^e}, p_{\mathbf{xu}|\mathbf{s}^e}, p_{\mathbf{ys}^a\mathbf{s}^d|\mathbf{xus}^e})\right\}$$
(5.20)

where we have defined the function

$$\Gamma(R,\rho,\psi,p_{\mathbf{s}^{e}},p_{\mathbf{xu}|\mathbf{s}^{e}},p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}) \\ \triangleq \begin{cases} |I(\mathbf{u};\mathbf{ys}^{d})-\psi(p_{\mathbf{s}^{e}})+\min_{p_{\mathbf{s}^{e}}'}[\psi(p_{\mathbf{s}^{e}}')-\rho(p_{\mathbf{s}^{e}}')]-R|^{+} & : \ \rho(p_{\mathbf{s}^{e}}) \ge I_{US^{e}}^{*}(p_{\mathbf{s}^{e}})+\epsilon \\ 0 & : \ \text{else.} \end{cases}$$
(5.21)

Applying the inequality $\min_{p'_{\mathbf{s}^e}} F(p'_{\mathbf{s}^e}) \leq F(p_{\mathbf{s}^e})$ to the function $F = \psi - \rho$, we obtain

$$\begin{split} &\Gamma(R,\rho,\psi,p_{\mathbf{s}^e},p_{\mathbf{xu}|\mathbf{s}^e},p_{\mathbf{ys}^a\mathbf{s}^d|\mathbf{xus}^e}) \\ &\leq & \left\{ \begin{array}{l} |I(\mathbf{u};\mathbf{ys}^d) - \rho(p_{\mathbf{s}^e}) - R|^+ &: \ \rho(p_{\mathbf{s}^e}) \geq I_{US^e}^*(p_{\mathbf{s}^e}) + \epsilon \\ 0 &: \ \text{else.} \end{array} \right. \end{split}$$

and thus

$$\Gamma(R,\rho,\psi,p_{\mathbf{s}^e},p_{\mathbf{x}\mathbf{u}|\mathbf{s}^e},p_{\mathbf{y}\mathbf{s}^a\mathbf{s}^d|\mathbf{x}\mathbf{u}\mathbf{s}^e}) \le |J_L(p_{\mathbf{s}^e}p_{\mathbf{x}\mathbf{u}|\mathbf{s}^e}p_{\mathbf{y}\mathbf{s}^a\mathbf{s}^d|\mathbf{x}\mathbf{u}\mathbf{s}^e}) - \epsilon - R|^+$$
(5.22)

with equality when

$$\psi(p_{\mathbf{s}^e}) = \rho(p_{\mathbf{s}^e}) = I_{US^e}^*(p_{\mathbf{s}^e}) + \epsilon.$$
(5.23)

Combining (5.11), (5.12), (5.20), and (5.22), we obtain

$$P_{e} \leq \sum_{T_{\mathbf{suxy}}} \max_{p_{Y|XS^{a}}} Pr[T_{\mathbf{suxy}}] \left(Pr[\mathcal{E}_{1}|T_{\mathbf{s}^{e}}] + Pr[\mathcal{E}_{1}' \mid T_{\mathbf{suxy}}, \mathcal{E}_{1}^{c}] \right)$$

$$\leq \max_{p_{\mathbf{s}^{e}}} \min_{p_{\mathbf{xu}|\mathbf{s}^{e}}} \max_{\rho, \psi} \max_{p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xs}^{e}}} \max_{p_{Y|XS^{a}}} \exp_{2} \left\{ -N[D(p_{\mathbf{s}^{e}}p_{\mathbf{xu}|\mathbf{s}^{e}}p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}||p_{S}p_{\mathbf{xu}|\mathbf{s}^{e}}p_{Y|XS^{a}}) + \Gamma(R, \rho, \psi, p_{\mathbf{s}^{e}}, p_{\mathbf{xu}|\mathbf{s}^{e}}, p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}) \right\}$$
(5.24)

$$= \exp_{2} \left\{ -NE_{r,L,N}^{\text{CDMC}}(R) \right\}$$
(5.25)

where (5.24) holds because $p_{\mathbf{xu}|\mathbf{s}^e}$ and (ρ, ψ) can be optimized to achieve the exponent $E_{r,L,N}^{\text{CDMC}}(R)$ in (5.5).

Step 5. By Lemma 3.1, the function $E_{r,L}^{\text{CDMC}}(R)$ is nonnegative and upper bounded by $|C_L - R|^+$. Applying (2.13) with \tilde{p}_{S^e} , $p_{XU|S^e}$, $(\tilde{p}_{YS^aS^d|XUS^e}, p_{Y|XS^a})$, and $D + |J - R|^+$ in the roles of the variables p, q, r, and the functional ϕ , respectively, we conclude that the exponent $E_{r,L,N}^{\text{CDMC}}(R)$ in (5.25) converges to the limit $E_{r,L}^{\text{CDMC}}(R)$ in (3.3) as $N \to \infty$. Since $E_{r,L}^{\text{CDMC}}(R) > 0$ for all $R < C_L$, the probability of error vanishes if $R < C_L$. The claim follows from the fact that we can choose L such that $C_L > C - \epsilon$, for any arbitrarily small ϵ .

6 Proof of Theorem 3.4

The proof is similar to the proof of Theorem 3.2 and is again given for the maximum-cost constraint (2.1) on the transmitter. A random ensemble \mathcal{E} of binning codes (f_N, g_N) with fixed $|\mathcal{U}| = L$ is constructed. This ensemble may also be viewed as a random ensemble of RM codes (Def. 2.3). RM codes are obtained by selecting a prototype (f_N, g_N) from \mathcal{E} and generating the RM family $\{f_N^{\pi}, g_N^{\pi}\}$ according to Def. 2.3. For RM codes there is no loss of optimality in restricting the attack

channel to a class of channels that are uniform over conditional types (see Step 2 below). It is shown that the error probability averaged over the ensemble \mathcal{E} vanishes exponentially with N at the rate $E_{r,L}^{\text{CAM}}(R)$ given in (3.12). Since the class of attack channels considered in Step 2 has polynomial complexity, there exists a RM code that achieves $E_{r,L}^{\text{CAM}}(R)$ for all attack channels in $\mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{A}]$.

The codebook-generation, encoding and decoding procedures are the same as those in the CDMC case, with the difference that the types and conditional types generated/selected by the encoder are obtained by optimizing a slightly different payoff function. The probability of error analysis is similar as well.

Assume $R < C_L - \epsilon$. Define

$$E_{r,L,N}^{\text{CAM}}(R, p_{\mathbf{s}^{e}}, p_{\mathbf{xu}|\mathbf{s}^{e}}) \triangleq \min_{\substack{p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}} \in \mathcal{P}_{YS^{a}S^{d}|XUS^{e}}^{[N]}[\mathcal{A}, p_{\mathbf{xu}|\mathbf{s}^{e}}p_{\mathbf{s}^{e}}]} \\ \left[D(p_{\mathbf{s}^{e}\mathbf{s}^{a}\mathbf{s}^{d}}||p_{S^{e}S^{a}S^{d}}) + \tilde{I}_{Y;US^{e}S^{d}|XS^{a}}(p_{\mathbf{s}^{e}}p_{\mathbf{xu}|\mathbf{s}^{e}}p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}) \\ + |J_{L}(p_{\mathbf{s}^{e}}p_{\mathbf{xu}|\mathbf{s}^{e}}p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}) - \epsilon - R|^{+} \right]$$

$$(6.1)$$

for all $p_{\mathbf{s}^e} \in \mathcal{P}_{S^e}^{[N]}$ and $p_{\mathbf{xu}|\mathbf{s}^e} \in \mathcal{P}_{XU|S^e}^{[N]}(L, D_1)$. Let

$$E_{r,L,N}^{\text{CAM}}(R) \triangleq \min_{p_{\mathbf{s}^e} \in \mathcal{P}_{S^e}^{[N]}} \max_{p_{\mathbf{x}\mathbf{u}|\mathbf{s}^e} \in \mathcal{P}_{XU|S^e}^{[N]}(D_1)} \widetilde{E}_{r,N}^{\text{CAM}}(R, p_{\mathbf{s}^e}, p_{\mathbf{x}\mathbf{u}|\mathbf{s}^e}).$$
(6.2)

which differs from (3.12) in that the optimizations are performed over empirical p.m.f.'s instead of arbitrary p.m.f.'s. Consider the maximization over $p_{\mathbf{xu}|\mathbf{s}^e}$ (viewed as a function of $p_{\mathbf{s}^e}$) in (6.2). As in the proof of Theorem 3.2, to the resulting optimal $p_{\mathbf{xu}|\mathbf{s}^e}$ we can associate a type class $T^*_U(p_{\mathbf{s}^e})$, conditional type classes $T^*_{U|S^e}(\mathbf{s}^e)$ and $T^*_{X|US^e}(\mathbf{u},\mathbf{s}^e)$, and a mutual information $I^*_{US^e}(p_{\mathbf{s}^e}p_{\mathbf{u}|\mathbf{s}^e})$.

Define $\rho(p_{\mathbf{s}^e})$ and $\psi(p_{\mathbf{s}^e})$ as in (5.23). The random codebook \mathcal{C} is a stack of codebooks $\mathcal{C}(p_{\mathbf{s}^e})$, each of which is obtained by a) drawing $2^{N(R+\rho(p_{\mathbf{s}^e}))}$ independent random vectors whose components are uniformly distributed in $T^*_U(p_{\mathbf{s}^e})$, and b) arranging them in an array with 2^{NR} columns and $2^{N\rho(p_{\mathbf{s}^e})}$ rows.

Encoder. The encoding (given s^e and m) proceeds exactly as in the CDMC case:

- 1. Find l such that $\mathbf{u}(l,m) \in \mathcal{C}(p_{\mathbf{s}^e}) \bigcap T^*_{U|S^e}(\mathbf{s}^e)$. If more than one such l exists, pick one of them randomly (with uniform distribution). Let $\mathbf{u} = \mathbf{u}(l,m)$. If no such l can be found, generate \mathbf{u} uniformly from the conditional type class $T^*_{U|S^e}(\mathbf{s}^e)$.
- 2. Generate **X** uniformly distributed over the conditional type class $T^*_{X|US^e}(\mathbf{us}^e)$.

Decoder. The decoder is the MPMI decoder of (5.6). We now analyze its probability of error

$$P_e \triangleq \max_{p_{\mathbf{Y}|\mathbf{XS}^a} \in \mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{A}]} P_e(F_N, G_N, p_{\mathbf{Y}|\mathbf{XS}^a}).$$

Step 1. An encoding error arises when no codeword with the appropriate type can be found. The probability $Pr[\mathcal{E}_m|T_{\mathbf{s}^e}]$ of this event is given by (5.9). **Step 2.** We have a decoding error under the following event \mathcal{E}'_m : there exists \mathbf{u}' not in column m of an array $\mathcal{C}(p'_{\mathbf{s}^e})$ such that $I(\mathbf{u}'; \mathbf{ys}^d) - \rho(p'_{\mathbf{s}^e}) \geq I(\mathbf{u}; \mathbf{ys}^d) - \rho(p_{\mathbf{s}^e})$. Therefore

$$P_{e} = \max_{p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^{a}}} Pr[error \mid m = 1]$$

$$= \max_{p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^{a}}} \sum_{T_{\mathbf{suxy}}} Pr[T_{\mathbf{suxy}}] Pr[error \mid T_{\mathbf{suxy}}, m = 1]$$

$$= \max_{p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^{a}}} \sum_{T_{\mathbf{suxy}}} Pr[T_{\mathbf{suxy}}] \left(Pr[\mathcal{E}_{1}|T_{\mathbf{s}^{e}}] + Pr[\mathcal{E}_{1}' \mid T_{\mathbf{suxy}}, \mathcal{E}_{1}^{c}] \right).$$
(6.3)

Unlike (5.11), no dependency on a DMC $p_{Y|XS^a}$ appears here. Observe that

$$Pr[\mathcal{E}'_1 \mid T_{\mathbf{suxy}}, \mathcal{E}'_1] = \sum_{(\tilde{\mathbf{s}}\tilde{\mathbf{u}}\tilde{\mathbf{x}}\tilde{\mathbf{y}}) \in T_{\mathbf{suxy}}} p(\tilde{\mathbf{s}}|T_{\mathbf{s}}) \, p(\tilde{\mathbf{x}}\tilde{\mathbf{u}}|\tilde{\mathbf{s}}^e) p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^a}(\tilde{\mathbf{y}}|\tilde{\mathbf{x}}\tilde{\mathbf{s}}^a) Pr[\mathcal{E}'_1 \mid \tilde{\mathbf{u}}, \tilde{\mathbf{y}}, \tilde{\mathbf{s}}^d, T_{\mathbf{suxy}}, \mathcal{E}'_1].$$
(6.4)

Here we can apply the following argument from [15]. From (6.3) (6.4), we see that $P_e(F_N, G_N, p_{\mathbf{Y}|\mathbf{XS}^a})$ is an affine functional of $p_{\mathbf{Y}|\mathbf{XS}^a}$. Moreover, it can be verified that $P_e(F_N, G_N, p_{\mathbf{Y}|\mathbf{XS}^a}) = P_e(F_N, G_N, p_{\mathbf{Y}|\mathbf{XS}^a}^{\pi})$ where π is a permutation operator, and $p_{\mathbf{Y}|\mathbf{XS}^a}^{\pi}(\mathbf{y}|\mathbf{xs}^a) \triangleq p_{\mathbf{Y}|\mathbf{XS}^a}(\pi \mathbf{y}|\pi \mathbf{x}, \pi \mathbf{s}^a)$. By uniform averaging over all permutations π , we obtain an attack channel $\overline{p}_{\mathbf{Y}|\mathbf{XS}^a} = \frac{1}{N!} \sum_{\pi} p_{\mathbf{Y}|\mathbf{XS}^a}^{\pi}$ which is *strongly exchangeable*: if $(\mathbf{X}, \mathbf{S}^a)$ is uniformly distributed over a type class, then \mathbf{Y} is uniformly distributed over conditional class types. So without loss of optimality for the adversary, we can consider only strongly exchangeable channels in the analysis, for which $p_{\mathbf{Y}|\mathbf{XS}^a}(\mathbf{y}|\mathbf{x}, \mathbf{s}^a)$ is given by

$$p_{\mathbf{Y}|\mathbf{XS}^{a}}(\mathbf{y}|\mathbf{x}, \mathbf{s}^{a}) = \frac{Pr[T_{\mathbf{y}|\mathbf{xs}^{a}}]}{|T_{\mathbf{y}|\mathbf{xs}^{a}}|},$$
(6.5)

with $T_{\mathbf{y}|\mathbf{xs}^a}$ to be optimized. Using the upper bound

$$Pr[T_{\mathbf{y}|\mathbf{x}\mathbf{s}^{a}}] \le \mathbb{I}\{p_{\mathbf{y}|\mathbf{x}\mathbf{s}^{a}} \in \mathcal{A}\}$$

$$(6.6)$$

and the asymptotic relations $Pr[T_{\mathbf{s}}] \doteq \exp_2\{-ND(p_{\mathbf{s}}||p_S)\}$ and $\frac{|T_{\mathbf{y}|\mathbf{z}\mathbf{w}}|}{|T_{\mathbf{y}|\mathbf{z}}|} \doteq \exp_2\{-NI(\mathbf{y};\mathbf{w}|\mathbf{z})\}$, we obtain

$$Pr[T_{\mathbf{suxy}}] = P_S^N P_{\mathbf{XU}|\mathbf{S}^e} P_{\mathbf{Y}|\mathbf{XS}^a}(T_{\mathbf{suxy}})$$

$$= |T_{\mathbf{suxy}}| \frac{Pr[T_{\mathbf{s}}] Pr[T_{\mathbf{y}|\mathbf{xs}^a}]}{|T_{\mathbf{s}}| |T_{\mathbf{xu}|\mathbf{s}^e}| |T_{\mathbf{y}|\mathbf{xs}^a}|}$$

$$\leq Pr[T_{\mathbf{s}}] \frac{|T_{\mathbf{suxy}}|}{|T_{\mathbf{s}}| |T_{\mathbf{xu}|\mathbf{s}^e}| |T_{\mathbf{y}|\mathbf{xs}^a}|} \mathbb{I}\{p_{\mathbf{y}|\mathbf{xs}^a} \in \mathcal{A}\}$$

$$= Pr[T_{\mathbf{s}}] \frac{|T_{\mathbf{y}|\mathbf{xus}}|}{|T_{\mathbf{y}|\mathbf{xs}^a}|} \mathbb{I}\{p_{\mathbf{y}|\mathbf{xs}^a} \in \mathcal{A}\}$$

$$\doteq \exp_2\left\{-N[D(p_{\mathbf{s}}||p_S) + \tilde{I}_{Y;US^eS^d|XS^a}(p_{\mathbf{s}^e}p_{\mathbf{xu}|\mathbf{s}^e}p_{\mathbf{ys}^a\mathbf{s}^d|\mathbf{xus}^e})]\right\} \mathbb{I}\{p_{\mathbf{y}|\mathbf{xs}^a} \in \mathcal{A}\}.(6.7)$$

Step 3. This step is identical to the corresponding step in the DMC case and yields

$$Pr[\mathcal{E}'_1 \mid T_{\mathbf{suxy}}, \mathcal{E}'_1] \le \exp_2\left\{-N|J_L(p_{\mathbf{s}^e} \, p_{\mathbf{xu}|\mathbf{s}^e} \, p_{\mathbf{ys}^a \mathbf{s}^d | \mathbf{xus}^e}) - \epsilon - R|^+]\right\}$$
(6.8)

Step 4. Combining (6.3), (6.7), (5.9), and (6.8), we obtain

$$P_{e} \stackrel{\cdot}{\leq} \sum_{T_{\mathbf{suxy}}} Pr[T_{\mathbf{suxy}}] Pr[\mathcal{E}'_{1} \mid T_{\mathbf{suxy}}, \mathcal{E}'_{1}]$$

$$\stackrel{\cdot}{\leq} \max_{p_{\mathbf{s}^{e}}} \min_{p_{\mathbf{xu}|\mathbf{s}^{e}}} \max_{p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}} \exp_{2} \left\{ -N[D(p_{\mathbf{s}^{e}\mathbf{s}^{a}\mathbf{s}^{d}}||p_{S^{e}S^{a}S^{d}}) + \tilde{I}_{Y;US^{e}S^{d}|XS^{a}}(p_{\mathbf{s}^{e}}p_{\mathbf{xu}|\mathbf{s}^{e}}p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}) + |J_{L}(p_{\mathbf{s}^{e}}p_{\mathbf{xu}|\mathbf{s}^{e}}p_{\mathbf{ys}^{a}\mathbf{s}^{d}|\mathbf{xus}^{e}}) - \epsilon - R|^{+}] \right\}$$

$$= \exp_{2}\{-NE_{r,L,N}^{\mathrm{CAM}}(R)\}$$

$$(6.9)$$

because $p_{\mathbf{xu}|\mathbf{s}^e}$ was optimized to achieve the exponent $E_{r,L,N}^{\text{CAM}}(R)$ in (6.2).

Step 5. The last step is identical to that in the DMC case: the exponent $E_{r,L,N}^{\text{CAM}}(R)$ in (6.9) converges to the limit $E_{r,L}^{\text{CAM}}(R)$ in (3.12) as $N \to \infty$, and all rates below C_L are achievable. By choosing L large enough, C_L can be made arbitrarily close to C.

7 Proof of Converse of Theorem 3.6

The proof of the converse theorem is an extension of [12, Prop. 4.3]. To prove the claim (derive an upper bound on capacity), we only need to consider the expected-cost constraint (2.2) for the transmitter. Indeed replacing (2.2) with the stronger maximum-cost constraint (2.1) cannot increase capacity, so the same upper bound applies. Likewise, we assume as in [12] that the decoder knows the attack channel $p_{Y|XS^a}$, because the resulting upper bound on capacity applies to an uninformed decoder as well.

Step 1. Choose an arbitrary small $\eta > 0$. For any rate-*R* encoder f_N and attack channel $p_{Y|XS^a} \in \mathcal{A}$ such that

$$I(M, \mathbf{YS}^d) \le N(R - \eta), \tag{7.1}$$

we have

$$NR = H(M) = H(M|\mathbf{YS}^d) + I(M;\mathbf{YS}^d)$$

$$\leq 1 + P_e(f_N, g_N, p_{Y|XS^a}^N) NR + I(M;\mathbf{YS}^d)$$

$$\leq 1 + P_e(f_N, g_N, p_{Y|XS^a}^N) NR + N(R - \eta)$$

where the first inequality is due to Fano's inequality, and the second is due to (7.1). Hence

$$P_e(f_N, g_N, p_{Y|XS^a}^N) \ge \frac{N\eta - 1}{NR}$$

We conclude that the probability of error is bounded away from zero:

$$P_e(f_N, g_N, p_{Y|XS^a}^N) \ge \frac{\eta}{2R}$$

$$(7.2)$$

for all $N > \frac{2}{\eta}$. Therefore rate R is not achievable if

$$\min_{p_{Y|XS^a} \in \mathcal{A}} I(M; \mathbf{YS}^d) \le N(R - \eta).$$
(7.3)

Step 2. The joint p.m.f. of $(M, \mathbf{S}, \mathbf{X}, \mathbf{Y})$ is given by

$$p_{M\mathbf{S}\mathbf{X}\mathbf{Y}} = p_M p_{\mathbf{S}^e} \mathbb{I}\{\mathbf{X} = f_N(\mathbf{S}^e, M)\} \prod_{i=1}^N p_{Y|XS^a}(y_i|x_i, s_i^a) p_{S^aS^d|S^e}(s_i^a, s_i^d|s_i^e).$$
(7.4)

Define the random variables

$$W_i = (M, S_{i+1}^e, \cdots, S_N^e, S_1^d, \cdots, S_{i-1}^d, Y_1, \cdots, Y_{i-1}), \quad 1 \le i \le N.$$
(7.5)

Since $(M, \{S_j^e, S_j^d, Y_j\}_{j \neq i}) \to X_i S_i^e \to Y_i S_i^a S_i^d$ forms a Markov chain for any $1 \le i \le N$, so does

$$W_i \to X_i S_i^e \to Y_i S_i^a S_i^d. \tag{7.6}$$

Also define the quadruple of random variables (W, S, X, Y) as (W_T, S_T, X_T, Y_T) , where T is a time-sharing random variable, uniformly distributed over $\{1, \dots, N\}$ and independent of all the other random variables. The random variable W is defined over an alphabet of cardinality $\exp_2\{N[R + \log \max(|\mathcal{S}^e|, |\mathcal{Y}| |\mathcal{S}^d|)]\}$. Due to (7.4) and (7.6), $W \to XS^e \to YS^aS^d$ forms a Markov chain.

Using the same inequalities as in [1, Lemma 4] (with (Y_i, S_i^d) and S_i^e playing the roles of Y_i and S_i , respectively), we obtain

$$I(M; \mathbf{YS}^d) \le \sum_{i=1}^{N} [I(W_i; Y_i S_i^d) - I(W_i; S_i^e)].$$
(7.7)

Using the definition of (W, S, X, Y) above and the same inequalities as in [12, (C16)], we obtain

$$\sum_{i=1}^{N} [I(W_i; Y_i S_i^d) - I(W_i; S_i^e)] = N[I(W; YS^d | T) - I(W; S^e | T)]$$

$$\leq N[I(WT; YS^d) - I(WT; S^e)]$$

$$= N[I(U; YS^d) - I(U; S^e)]$$
(7.8)

where U = (W, T) is defined over an alphabet of cardinality

 $L(N) \triangleq N \exp_{2}\{N[R + \log \max(|\mathcal{S}^{e}|, |\mathcal{Y}| |\mathcal{S}^{d}|)]\}.$

Therefore

$$I(M; \mathbf{YS}^{d}) \leq N[I(U; YS^{d}) - I(U; S^{e})]$$

$$= NJ_{L(N)}(p_{USXY})$$

$$\min_{p_{Y|XS^{a} \in \mathcal{A}}} I(M; \mathbf{YS}^{d}) \leq N \min_{p_{Y|XS^{a} \in \mathcal{A}}} J_{L(N)}(p_{USXY})$$

$$\leq N \sup_{L} \max_{p_{XU|S^{e}}} \min_{p_{Y|XS^{a} \in \mathcal{A}}} J_{L}(p_{USXY})$$

$$= N \lim_{L \to \infty} \max_{p_{XU|S^{e}}} \min_{p_{Y|XS^{a} \in \mathcal{A}}} J_{L}(p_{USXY})$$
(7.9)

where the last equality follows from Lemma 2.1. Combining (7.3) and (7.9), we conclude that R is not achievable if

$$\lim_{L \to \infty} \max_{p_{XU|S^e}} \min_{p_{Y|XS^a} \in \mathcal{A}} J_L(p_{USXY}) \le R - \eta,$$

which proves the claim, because η is arbitrarily small.

8 Proof of Converse of Theorem 3.7

The proof of the converse theorem builds on the proof for the C-DMC case.

Step 1. Consider an attack channel $p_{Y|XS^a}^*$ that achieves $C(D_1, \mathcal{A})$ in (3.1). Without loss of generality, assume that $p_{S^a}(s^a) > 0$ for all $s^a \in S^a$. For any positive ϵ , consider the following L_1 neighborhood of $p_{Y|XS^a}^*$:

$$\mathcal{B}(\epsilon) = \left\{ p_{Y|XS^a} : \sum_{y,x,s^a} |p_{Y|XS^a}(y|x,s^a) - p_{Y|XS^a}^*(y|x,s^a)| \le \epsilon \right\}.$$

We have $\lim_{\epsilon \to 0} C(D_1, \mathcal{B}(\epsilon)) = C(D_1, \mathcal{A})$. For any arbitrarily small η , there exists ϵ such that

$$C(D_1, \mathcal{A}) - \eta \leq C(D_1, \mathcal{B}(\epsilon)) \leq C(D_1, \mathcal{A}).$$

In order to prove the converse theorem, it is sufficient to show that reliable communication at rates $R > C(D_1, \mathcal{B}(\epsilon)) + 2\eta \ge C(D_1, \mathcal{A}) + \eta$ is impossible for a particular attack channel $p_{\mathbf{Y}|\mathbf{XS}^a} \in \mathcal{P}_{\mathbf{Y}|\mathbf{XS}^a}[\mathcal{B}(\epsilon)]$. The channel we select is "nearly memoryless". Given any rate-R randomized code (\mathcal{M}, F_N, G_N) , we show that $\lim_{N\to\infty} P_{e,N}(F_N, G_N, p_{\mathbf{Y}|\mathbf{XS}^a}) \ge \frac{\eta}{4R}$ hence is nonzero.

Step 2: Construction of $p_{\mathbf{Y}|\mathbf{XS}^a}$. Consider any rate-*R* deterministic code (\mathcal{M}, f_N, g_N) where $R > C(D_1, \mathcal{A})$. From Theorem 3.6, we know that $\min_{f_N, g_N} P_{e,N}(f_N, g_N, (p_{Y|XS^a}^*)^N) \neq 0$ as $N \rightarrow \infty$. Define an arbitrary mapping $\Lambda : \mathcal{X}^N \times (\mathcal{S}^a)^N \to \mathcal{Y}^N$ such that $p_{\Lambda(\mathbf{x}, \mathbf{s}^a)|\mathbf{xs}^a} \in \mathcal{B}(\epsilon)$ for all $(\mathbf{x}, \mathbf{s}^a)$. Denote by $\tilde{\mathbf{Y}}$ the output of $(p_{Y|XS^a}^*)^N$. Define the following functions of $(\tilde{\mathbf{y}}, \mathbf{x}, \mathbf{s}^a)$: the binary quantity

$$B = 1 \quad \Leftrightarrow \quad p_{\tilde{\mathbf{y}}|\mathbf{xs}^a} \notin \mathcal{B}(\epsilon)$$

and the sequence

$$\mathbf{y} = \begin{cases} \tilde{\mathbf{y}} & : \text{ if } p_{\tilde{\mathbf{y}}|\mathbf{x}\mathbf{s}^a} \in \mathcal{B}(\epsilon) & (B=0) \\ \Lambda(\mathbf{x}, \mathbf{s}^a) & : \text{ else} & (B=1). \end{cases}$$
(8.1)

Therefore the p.m.f.

$$p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^{a}}(\mathbf{y}|\mathbf{x},\mathbf{s}^{a}) = \left[(p_{Y|XS^{a}}^{*})^{N}(\mathbf{y}|\mathbf{x},\mathbf{s}^{a})Pr(B=0) + \mathbb{I}\{\mathbf{y}=\Lambda(\mathbf{x},\mathbf{s}^{a})\}Pr(B=1) \right] \mathbb{I}\{p_{\mathbf{y}|\mathbf{x},\mathbf{s}^{a}} \in \mathcal{B}(\epsilon)\}$$

belongs to $\mathcal{B}(\epsilon)$.

Step 3. Now we seek an upper bound on Pr[B = 1]. Define the binary random variable A such that

$$A = 1 \quad \Leftrightarrow \quad \min_{x, s^a} p_{\mathbf{x} \mathbf{s}^a}(x, s^a) \le \epsilon^2.$$
(8.2)

The probability that A = 1 is a function of the code f_N . We assume momentarily that f_N is such that

$$Pr[A=1] < \frac{\eta}{9R}.\tag{8.3}$$

In Step 5 we show this assumption causes no loss of generality.

Define the shorthand

$$\tilde{\epsilon} = \frac{\epsilon^{\rm o}}{2\ln 2}$$

and the class of types

$$\mathcal{P}_{YXS^{a}}^{[N]}(\tilde{\epsilon}) \triangleq \left\{ p_{\tilde{\mathbf{y}}\mathbf{x}\mathbf{s}^{a}} : D(p_{\tilde{\mathbf{y}}|\mathbf{x}\mathbf{s}^{a}}||p_{Y|XS^{a}}^{*}|p_{\mathbf{x}\mathbf{s}^{a}}) > \tilde{\epsilon}, \underbrace{\min_{x,s^{a}} p_{\mathbf{x}\mathbf{s}^{a}}(x,s^{a}) > \epsilon^{2}}_{A=0} \right\}.$$
(8.4)

With this notation, we have

$$B = 1 \quad \Rightarrow \quad \sum_{y,x,s^{a}} |p_{\mathbf{y}|\mathbf{x}\mathbf{s}^{a}}(y|x,s^{a}) - p_{Y|XS^{a}}^{*}(y|x,s^{a})| > \epsilon$$

$$B = 1, A = 0 \quad \Rightarrow \quad \|p_{\tilde{\mathbf{y}}|\mathbf{x}\mathbf{s}^{a}}p_{\mathbf{x}\mathbf{s}^{a}} - p_{Y|XS^{a}}^{*}p_{\mathbf{x}\mathbf{s}^{a}}\| > \epsilon^{2} \sum_{y,x,s^{a}} |p_{\tilde{\mathbf{y}}|\mathbf{x}\mathbf{s}^{a}}(y|x,s^{a}) - p_{Y|XS^{a}}^{*}(y|x,s^{a})| > \epsilon^{3}$$

$$\Rightarrow \quad D(p_{\tilde{\mathbf{y}}|\mathbf{x}\mathbf{s}^{a}}||p_{Y|XS^{a}}^{*}|p_{\mathbf{x}\mathbf{s}^{a}}) > \tilde{\epsilon} \qquad (8.5)$$

where the first line follows from the definition of $\mathcal{B}(\epsilon)$, the second line from (8.3), and the third line from Pinsker's inequality [16, p. 58]: $D(p||p') \ge ||p - p'||^2/(2\ln 2)$.

We have

$$\Pr[B=1] \le \Pr[B=1, A=0] + \Pr[A=1].$$
(8.6)

Due to (8.4) and (8.5), the first term in the right side is bounded as follows:

$$\Pr[B = 1, A = 0] \leq \Pr\left[p_{\tilde{\mathbf{y}}\mathbf{x}\mathbf{s}^{a}} \in \mathcal{P}_{YXS^{a}}^{[N]}(\tilde{\epsilon})\right]$$

$$= \sum_{\substack{p_{\tilde{\mathbf{y}}\mathbf{x}\mathbf{s}^{a}} \in \mathcal{P}_{YXS^{a}}^{[N]}(\tilde{\epsilon})}} \Pr\left[T_{\tilde{\mathbf{y}}\mathbf{x}\mathbf{s}^{a}}\right]$$

$$\leq (N+1)^{|\mathcal{Y}| |\mathcal{X}| |\mathcal{S}^{a}|} \max_{\substack{p_{\tilde{\mathbf{y}}\mathbf{x}\mathbf{s}^{a}} \in \mathcal{P}_{YXS^{a}}^{[N]}(\tilde{\epsilon})}} \Pr\left[T_{\tilde{\mathbf{y}}\mathbf{x}\mathbf{s}^{a}}\right]$$

$$\leq (N+1)^{|\mathcal{Y}| |\mathcal{X}| |\mathcal{S}^{a}|} \max_{\substack{p_{\tilde{\mathbf{y}}\mathbf{x}\mathbf{s}^{a}} \in \mathcal{P}_{YXS^{a}}^{[N]}(\tilde{\epsilon})}} \exp_{2}\left\{-ND(p_{\tilde{\mathbf{y}}|\mathbf{x}\mathbf{s}^{a}}||p_{Y|XS^{a}}^{*}|p_{\mathbf{x}s^{a}})\right\}$$

$$\leq (N+1)^{|\mathcal{Y}| |\mathcal{X}| |\mathcal{S}^{a}|} \exp_{2}\left\{-N\tilde{\epsilon}\right\}} \tag{8.7}$$

which vanishes as $N \to \infty$. Combining (8.3), (8.6) and (8.7), we obtain

$$Pr[B=1] \leq (N+1)^{|\mathcal{Y}| |\mathcal{X}| |\mathcal{S}^a|} 2^{-N\tilde{\epsilon}} + \frac{\eta}{9R}$$
$$\leq \frac{\eta}{8R}$$

for N large enough.

Step 4. For any (f_N, g_N) , we have

$$P_{e,N}(f_N, g_N, (p_{Y|XS^a}^*)^N)$$

$$= Pr[\hat{M} \neq M | \tilde{\mathbf{Y}}, \mathbf{S}^d]$$

$$= Pr[\hat{M} \neq M | \tilde{\mathbf{Y}}, \mathbf{S}^d, B = 0] Pr(B = 0) + Pr[\hat{M} \neq M | \tilde{\mathbf{Y}}, \mathbf{S}^d, B = 1] Pr(B = 1). \quad (8.8)$$

Likewise,

$$P_{e,N}(f_N, g_N, p_{\mathbf{Y}|\mathbf{XS}^a})$$

$$= Pr[\hat{M} \neq M|\mathbf{Y}, \mathbf{S}^d]$$

$$= Pr[\hat{M} \neq M|\mathbf{Y}, \mathbf{S}^d, B = 0]Pr(B = 0) + Pr[\hat{M} \neq M|\mathbf{Y}, \mathbf{S}^d, B = 1]Pr(B = 1). \quad (8.9)$$

Noting that the terms multiplying Pr[B = 0] in (8.8) and (8.9) are identical by construction of **y** in (8.1) and that Pr[B = 1] is upper bounded by $\frac{\eta}{8R}$, we obtain

$$|P_{e,N}(f_N, g_N, p_{\mathbf{Y}|\mathbf{XS}^a}) - P_{e,N}(f_N, g_N, (p_{Y|XS^a}^*)^N)| \le 2Pr[B=1] \le \frac{\eta}{4R}, \quad \forall f_N, g_N.$$

Since $R > C(D_1, \mathcal{A}) + \eta$, Theorem 3.6 implies that

$$\lim_{N \to \infty} \min_{f_N, g_N} P_{e,N}(f_N, g_N, (p_{Y|XS^a}^*)^N) \ge \frac{\eta}{2R}$$

Hence

$$\lim_{N \to \infty} \min_{f_N, g_N} P_{e,N}(f_N, g_N, p_{\mathbf{Y}|\mathbf{XS}^a}) \ge \frac{\eta}{2R} - \frac{\eta}{4R} = \frac{\eta}{4R}$$

Therefore

$$\lim_{N \to \infty} \min_{f_N, g_N} P_{e,N}(F_N, G_N, p_{\mathbf{Y}|\mathbf{XS}^a}) \ge \frac{\eta}{4R}$$

for any randomized code (F_N, G_N) .

Step 5. It remains to prove there was no loss of generality in making the assumption (8.3). This is done as follows. Given any code f_N (that may not satisfy (8.3)), we can extend the code by appending $N\epsilon|\mathcal{X}|^{-1}$ letters x at the end of the sequence, for each $x \in \mathcal{X}$. The resulting code has length $(1 + \epsilon)N$ and will be denoted by $\tilde{f}_{(1+\epsilon)N}$. To this code we can associate a decoding function $\tilde{g}_{(1+\epsilon)N}$ that ignores the last ϵN letters of the received sequence and outputs the same decision as g_N based on the first N received letters. Hence

$$P_{e,N}(f_N, g_N, p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^a}) = P_{e,N}(f_{(1+\epsilon)N}, \tilde{g}_{(1+\epsilon)N}, p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^a}), \quad \forall p_{\mathbf{Y}|\mathbf{X}\mathbf{S}^a}.$$
(8.10)

If $\tilde{f}_{(1+\epsilon)N}$ satisfies (8.3), it follows from Step 4 that reliable communication is impossible using such codes, and from (8.10) the same conclusion applies to f_N .

We now show that $\tilde{f}_{(1+\epsilon)N}$ satisfies (8.3) for N large enough. Denote by $N'(x, s^a)$ the number of occurrences of the pair (x, s^a) in the last ϵN letters of the joint sequence $(\mathbf{x}, \mathbf{s}^a)$. Also denote by $N'(s^a)$ the number of occurrences of s^a in the last $\epsilon |\mathcal{X}|^{-1} N$ letters of the sequence \mathbf{s}^a .

If $\mathcal{S}^a \neq \emptyset$, we have

$$\forall x, s^a : Pr[N'(x, s^a) \le \epsilon^2 N] = Pr[N'(s^a) \le \epsilon^2 N].$$

Since $\mathbb{E}[N'(s^a)] = N\epsilon |\mathcal{X}|^{-1} p_{S^a}(s^a)$, the above probabilities vanish exponentially with N provided that $\epsilon < |\mathcal{X}|^{-1} \min_{s^a} p_{S^a}(s^a)$. Hence

$$\begin{aligned} \Pr[A=1] &= \Pr\left[\min_{x,s^a} p_{\mathbf{xs}^a}(x,s^a) \le \epsilon^2\right] \\ &\le |\mathcal{X}| \, |\mathcal{S}^a| \, \max_{x,s^a} \Pr[p_{\mathbf{xs}^a}(x,s^a) \le \epsilon^2] \\ &\le |\mathcal{X}| \, |\mathcal{S}^a| \, \max_{x,s^a} \Pr[N'(x,s^a) \le \epsilon^2(1+\epsilon)N] \\ &= |\mathcal{X}| \, |\mathcal{S}^a| \, \max_{s^a} \Pr[N'(s^a) \le \epsilon^2(1+\epsilon)N]. \end{aligned}$$

Thus (8.3) holds for $\epsilon(1+\epsilon) < |\mathcal{X}|^{-1} \min_{s^a} p_{S^a}(s^a)$ and N large enough.

If $S^a = \emptyset$, straightforward changes in the above derivation yield Pr[A = 1] = 0, i.e., (8.3) holds again. This concludes the proof.

9 Discussion

In their landmark paper, Gel'fand and Pinsker [1] showed that random binning achieves the capacity of a DMC with random states known to the encoder. However their encoder was not designed to provide positive error exponents at rates below capacity. In this paper we have addressed this limitation and proposed and optimized a new random-coding scheme. The codebook consists of a stack of codeword-arrays indexed by the encoder's state sequence type λ . The size of these arrays is $2^{N\rho(\lambda)} \times 2^{NR}$, i.e., the number of rows is a function of λ . The decoder is the Maximum Penalized Mutual Information decoder (3.6), where the penalty is the same function $\rho(\lambda)$ that determines the array sizes. This new MPMI decoder can be interpreted as an empirical generalized MAP decoder.

The channel models studied in this paper generalize the original Gel'fand-Pinsker setup in two ways. First, partial information about the state sequence is available to the encoder, adversary, and decoder. Second, both CDMC and CAM channel models are studied.

We have considered four combinations of maximum/expected cost constraints for the transmitter and CDMC/CAM designs for the adversary, and obtained the same capacity in all four cases. There is thus no advantage (in terms of capacity) to the transmitter in operating under expected-cost constraints instead of the stronger maximum-cost constraints.

In terms of error exponents however, there is a definite advantage to the adversary in choosing a CDMC rather than a CAM design of the channel. This is because 1) arbitrary memory does not help the adversary because randomly-modulated codes and a MMI-type decoder are used, 2) the set of conditional types the adversary can choose from is constrained in the CAM case but not in the CDMC case, and 3) the error exponents are determined by the worst types. The random-coding exponent is always upper bounded by a straight line with slope -1 at all rates below capacity. That upper bound is achieved in the CAM case, when no side information is available to the encoder.

Finally, neither the MMI nor the MPMI decoder is practical, and it remains to be seen whether good, practical encoders and decoders can be developed.

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A Relation Between CAM and AVC Models

In this appendix, we detail the relation between a channel model $p_{\mathbf{Y}|\mathbf{X}}$, with maximum distortion constraint (2.5), and the AVC model in [16]. The AVC is a family of conditional p.m.f.'s $W(y|x,\theta)$, where $\theta \in \Theta$ (finite set) is a "channel state" selected by the adversary. A cost function $l : \Theta \to \mathbb{R}^+$ for the states is also defined. The channel law is of the form

$$p(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}) = \prod_{i=1}^{N} W(y_i|x_i, \theta_i)$$
(A.1)

where the sequence $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_N\}$ is arbitrary except for a maximum-cost constraint

$$l^{N}(\boldsymbol{\theta}) \triangleq \frac{1}{N} \sum_{i=1}^{N} l(\theta_{i}) \le l_{\max}.$$
(A.2)

In some formulations of the jamming problem, $\boldsymbol{\theta}$ must be selected by the adversary before seeing \mathbf{x} ; in other formulations, θ_i is allowed to depend on x_i but not on other samples of \mathbf{x} [17, 20]; yet in other formulations (the A*VC model [16]), θ_i is allowed to depend on x_j for all $j \leq i$.

If θ is allowed to depend on the entire sequence **x** in a noncausal manner (as opposed to the above formulations of the AVC problem), the problem with maximum distortion constraint (2.5) may be formulated as (A.1) and (A.2) with state θ , channel W, and cost l defined below. Let $\theta = (\theta', \theta'')$ where $\theta' \in \mathcal{X}$ and $\theta'' \in \mathcal{Y}$, hence $\Theta = \mathcal{X} \times \mathcal{Y}$. Let

$$l(\theta) = d(\theta', \theta''), \qquad W(y|x, \theta) = \mathbb{I}\{y = \theta''\}.$$

The maximum-cost constraint (A.2) is then equivalent to the maximum-distortion constraint (2.5), with $l_{\text{max}} = D_2$. The sequence θ'' may be chosen deterministically or stochastically, using an arbitrary distribution.

B Error Exponents for Channels Without Side Information

This appendix summarizes some known results on random-coding error exponents.

Single DMC: Let $p_{Y|X}$ and p_X be the channel law and input p.m.f., respectively. Referring to [16, p. 165-166], we have

$$E_r(R, p_X, p_{Y|X}) = \min_{\tilde{p}_{Y|X}} [D(\tilde{p}_{Y|X} | | p_{Y|X} | p_X) + |\tilde{I}_{XY}(p_X, \tilde{p}_{Y|X}) - R|^+], \quad (B.1)$$

Compound DMC: Here $p_{Y|X}$ belongs to a set \mathcal{A} . We have

$$E_{r}(R, p_{X}, \mathcal{A}) = \min_{p_{Y|X} \in \mathcal{A}} E_{r}(R, p_{X}, p_{Y|X})$$

=
$$\min_{p_{Y|X} \in \mathcal{A}} \min_{\tilde{p}_{Y|X}} [D(\tilde{p}_{Y|X} || p_{Y|X} | p_{X}) + |\tilde{I}_{XY}(p_{X}, \tilde{p}_{Y|X}) - R|^{+}]$$
(B.2)

which is zero if $R \ge \min_{p_{Y|X} \in \mathcal{A}} \tilde{I}_{XY}(p_X, p_{Y|X})$.

Private Watermarking: the set \mathcal{A} is defined by the distortion constraint (2.3). Then [14]

$$E_{r}^{\text{CAM}}(R, D_{1}, D_{2}) = \max_{p_{X|S}} \min_{p_{Y|XS} \in \tilde{\mathcal{A}}} [\tilde{I}_{SY|X}(p_{S}, p_{X|S}, p_{Y|XS}) + |\tilde{I}_{XY|S}(p_{S}, p_{X|S}, p_{Y|XS}) - R|^{+}]$$
(B.3)

where $\tilde{\mathcal{A}} \triangleq \{p_{Y|XS} : \sum_{s} p_{S}(s)p_{X|S}(x|s)p_{Y|XS}(y|x,s)d(x,y) \leq D_{2}\}$. The maximization over $p_{X|S}$ is also subject to a distortion constraint.

Jamming with channel state S selected independently of input X [17, 20]. We have

$$E_{r}^{jam}(R) = \max_{p_{X}} \min_{p_{S}} \min_{\substack{\tilde{p}_{YSX} : \\ \tilde{p}_{X} = p_{X}, \tilde{p}_{S} = p_{S}}} [D(\tilde{p}_{YSX}||p_{Y|SX}p_{X}p_{S}) + |\tilde{I}_{XY}(p_{X}, \tilde{p}_{Y|X}) - R|^{+}]$$
(B.4)

C Proof of Proposition 4.1

The set \mathcal{A} , denoted here as $\mathcal{P}_{Y|X}(D_2)$, is the set of DMC's that introduce maximum Hamming distortion D_2 . Let the attack channel $p_{Y|X}^*$ be the BSC with crossover probability D_2 . Considering $p_{Y|X}^*$ may not be the worst channel, we have

$$C^{\text{pub}} = \sup_{L} \max_{p_{XU|S} \in \mathcal{P}_{XU|S}(L,D_1)} \min_{p_{Y|X} \in \mathcal{P}_{Y|X}(D_2)} J_L(p_S \, p_{XU|S} \, p_{Y|X})$$

$$\leq \sup_{L} \max_{p_{XU|S} \in \mathcal{P}_{XU|S}(L,D_1)} J_L(p_S \, p_{XU|S} \, p_{Y|X}^*)$$

$$= g^*(D_1, D_2), \qquad (C.1)$$

where the last step is derived in [8, 9]. The function g^* is defined in (4.1).

Next we prove that $C^{\text{pub}} \geq g^*(D_1, D_2)$. Consider $D_1 = D'_1 \theta$, where $D'_1 \in [0, \frac{1}{2}]$ and $\theta \in [0, 1]$. Let $p^*_{U|S}$ be the BSC with crossover probability D'_1 . Furthermore, X = U makes the distortion equal to D'_1 . (Note that $L = |\mathcal{U}| = 2$ in this case.) Clearly,

$$C^{\text{pub}}(D'_{1}) \geq \min_{\substack{p_{Y|X} \in \mathcal{P}_{Y|X}(D_{2})}} J_{L}(p_{S} p^{*}_{XU|S} p_{Y|X})$$

$$= \min_{\substack{p_{Y|X} \in \mathcal{P}_{Y|X}(D_{2})}} I(X;Y) - \underbrace{(1-h(D'_{1}))}_{I(U;S)}$$

$$= (1-h(D_{2})) - (1-h(D'_{1}))$$

$$= h(D'_{1}) - h(D_{2}), \qquad (C.2)$$

where

$$\min_{p_{Y|X} \in \mathcal{P}_{Y|X}(D_2)} I(X;Y) = 1 - h(D_2)$$

is achieved by $p_{Y|X}^*$.

Using time-sharing arguments, Barron *et al.* [8] proved that capacity is a concave function of D_1 in the case $\mathcal{A} = \{p_{Y|X}^*\}$. It can be shown that their result holds in the case $\mathcal{A} = \mathcal{P}_{Y|X}(D_2)$ considered here. Therefore we have

$$C^{\operatorname{pub}}(D_1) = C^{\operatorname{pub}}(D'_1\theta) \ge \theta C^{\operatorname{pub}}(D'_1) \ge \theta \left(h(D'_1) - h(D_2) \right), \quad \forall \theta \in [0, 1].$$

It may be verified that

$$\max_{0 \le \theta \le 1} \theta \left(h(D_1') - h(D_2) \right) = g^*(D_1, D_2).$$

Therefore

$$C^{\text{pub}} \ge g^*(D_1, D_2).$$
 (C.3)

From (C.1) and (C.3), we conclude that $C^{\text{pub}} = g^*(D_1, D_2)$; also $|\mathcal{U}| = 2$.

D Proof of Proposition 4.2

From (3.13), we have

$$E_{r}^{\text{CAM,pub}}(R) = \sup_{L} \min_{\tilde{p}_{S}} \max_{p_{XU|S} \in \mathcal{P}_{XU|S}(L,D_{1})} \min_{p_{Y|XUS} \in \mathcal{P}_{Y|XUS}(D_{2})} \left[D(\tilde{p}_{S}||p_{S}) + I_{Y;US|X}(\tilde{p}_{S} \, p_{XU|S} \, p_{Y|XUS}) + |J_{L}(\tilde{p}_{S} \, p_{XU|S} \, p_{Y|XUS}) - R|^{+} \right].$$
(D.1)

Step 1. First we prove that

$$F(D_1, D_2) \triangleq \sup_{L} \min_{\tilde{p}_S} \max_{p_{XU|S} \in \mathcal{P}_{XU|S}(L, D_1)} \min_{p_{Y|XUS} \in \mathcal{P}_{Y|XUS}(D_2)} J_L(\tilde{p}_S p_{XU|S} p_{Y|XUS})$$

= $C^{\text{pub}},$ (D.2)

with equality if $\tilde{p}_S = p_S$.

Referring to (4.1), we first consider the regime in which time sharing is not needed: $D_1 \ge \delta_2 = 1 - 2^{-\overline{h}(D_2)}$ and therefore $C^{\text{pub}} = \overline{h}(D_1) - \overline{h}(D_2)$. Letting U = X and $p_{X|S}^*$ be the BSC with crossover probability D_1 , we obtain a lower bound on $F(D_1, D_2)$:

$$F(D_{1}, D_{2}) \geq \min_{\tilde{p}_{S}} \min_{p_{Y|XS} \in \mathcal{P}_{Y|XS}(D_{2})} J_{L}(\tilde{p}_{S} p_{X|S}^{*} p_{Y|XS})$$

$$= \min_{\tilde{p}_{S}} \min_{p_{Y|XS} \in \mathcal{P}_{Y|XS}(D_{2})} \left[\tilde{I}_{X;Y}(\tilde{p}_{S} p_{X|S}^{*} p_{Y|XS}) - \tilde{I}_{X;S}(\tilde{p}_{S} p_{X|S}^{*}) \right]$$

$$= \min_{\tilde{p}_{S}} \min_{p_{Y|XS} \in \mathcal{P}_{Y|XS}(D_{2})} \left[\tilde{I}_{X;Y}(\tilde{p}_{S} p_{X|S}^{*} p_{Y|XS}) - (\overline{h}(D_{1} \star p_{0}) - \overline{h}(D_{1})) \right] \quad (D.3)$$

where we use the shorthand $p_0 = \tilde{p}_S(0)$. Next, write $p_{Y|XS}$ as

The p.m.f. of X induced by \tilde{p}_S and $p_{X|S}^*$ is given by

$$p_X = (p_{X0}, p_{X1})$$

= $(p_0(1 - D_1) + (1 - p_0)D_1, p_0D_1 + (1 - p_0)(1 - D_1)).$

We derive

$$\min_{\substack{p_{Y|XS} \in \mathcal{P}_{Y|XS}(D_{2})}} \tilde{I}_{X;Y}(\tilde{p}_{S} p_{X|S}^{*} p_{Y|XS}) \\
= \min_{\substack{e,f,g,h:\\p_{0}((1-D_{1})e+D_{1}f)+(1-p_{0})((1-D_{1})h+D_{1}g) \leq D_{2}}} \overline{h}(p_{X0}(1-\alpha) + p_{X1}\beta) - p_{X0}\overline{h}(1-\alpha) - p_{X1}\overline{h}(\beta) \\
= \overline{h}(D_{1} \star p_{0}) - \overline{h}(D_{2}) \tag{D.4}$$

where

$$\alpha = \frac{p_0(1-D_1)e + (1-p_0)D_1g}{p_{X0}} \quad \text{and} \quad \beta = \frac{p_0D_1f + (1-p_0)(1-D_1)h}{p_{X1}}.$$

The minimum is achieved by

$$\alpha^* = \frac{D_2}{p_{X0}} \frac{p_{X1} - D_2}{1 - 2D_2}, \quad \beta^* = \frac{D_2}{p_{X1}} \frac{p_{X0} - D_2}{1 - 2D_2}$$

Combining (D.3) and (D.4), we obtain

$$F(D_1, D_2) \ge \overline{h}(D_1) - \overline{h}(D_2) = C^{\text{pub}}.$$
(D.5)

In the case $D_1 < \delta_2$, capacity is achieved using time-sharing: $C^{\text{pub}} > \overline{h}(D_1) - \overline{h}(D_2)$. Similarly to [8], it can be shown that $F(D_1, D_2)$ is a nondecreasing concave function of D_1 . Hence,

$$F(D_1, D_2) = F(D'_1\theta, D_2) \ge \max_{0 \le \theta \le 1} \theta F(D'_1, D_2) \ge \max_{0 \le \theta \le 1} \theta \left(\overline{h}(D'_1) - \overline{h}(D_2)\right) = C^{\text{pub}}.$$
 (D.6)

For all values of D_1 , letting $\tilde{p}_S = p_S$ in (D.2) and further restricting the minimization over $p_{Y|XUS}$, we have

$$F(D_1, D_2) \le \sup_{L} \max_{p_{XU|S} \in \mathcal{P}_{XU|S}(L, D_1)} \min_{p_{Y|X} \in \mathcal{P}_{Y|X}(D_2)} J_L(\tilde{p}_S \, p_{XU|S} \, p_{Y|X}) = C^{\text{pub}}.$$
 (D.7)

Combining (D.5), (D.6) and (D.7), we obtain (D.2).

Step 2. The first two bracketed terms in (D.1) are nonnegative. This yields a lower bound on $E_r^{\text{CAM,pub}}(R)$:

$$E_{r}^{\text{CAM,pub}}(R) \geq \sup_{L} \min_{\tilde{p}_{S}} \max_{p_{XU|S} \in \mathcal{P}_{XU|S}(L,D_{1})} \min_{p_{Y|XUS} \in \mathcal{P}_{Y|XUS}(D_{2})} |J_{L}(\tilde{p}_{S} \, p_{XU|S} \, p_{Y|XUS}) - R|^{+}$$

= $|C^{\text{pub}} - R|^{+}$ (D.8)

where the equality is due to (D.2).

Step 3. If we fix $\tilde{p}_S = p_S = (\frac{1}{2}, \frac{1}{2})$ and restrict $p_{Y|SUX}$ to be of the form $p_{Y|X}$, we obtain an upper bound on $E_r^{\text{CAM,pub}}(R)$:

$$E_{r}^{\text{CAM,pub}}(R) \leq \sup_{L} \max_{p_{XU|S} \in \mathcal{P}_{XU|S}(L,D_{1})} \min_{p_{Y|X} \in \mathcal{P}_{Y|X}(D_{2})} |J_{L}(p_{S} \, p_{XU|S} \, p_{Y|X}) - R|^{+}$$

= $|C^{\text{pub}} - R|^{+}.$ (D.9)

Step 4. Combining (D.9) and (D.8), we obtain $E_r^{\text{CAM,pub}}(R) = |C^{\text{pub}} - R|^+$.

E Proof of Proposition 4.4

We have

$$C^{\text{deg}} = \max_{p_X \in \mathcal{P}_X(D_1)} \min_{p_Y|_X \in \mathcal{P}_Y|_{XS}(D_2)} I(X;Y).$$
(E.1)

Let $a = p_X(1)$, $e = p_{Y|X}(1|0)$, and $f = p_{Y|X}(0|1)$, which satisfy the distortion constraints

$$a \le D_1, \quad (1-a)e + af \le D_2.$$

Substituting these probabilities into (E.1), we obtain

$$C^{\text{deg}} = \max_{a \le D_1} \min_{(1-a)e + af \le D_2} \left[\overline{h}((1-a)(1-e) + af) - (1-a)\overline{h}(e) - a\overline{h}(f) \right]$$

Solving the above max-min problem in the case $D_1 \ge \delta_2 = 1 - 2^{\overline{h}(D_2)}$, we obtain the optimal p_X^* and $p_{Y|X}^*$ from

$$a = D_1, \quad e = \frac{D_2(D_1 - D_2)}{(1 - D_1)(1 - 2D_2)}, \quad f = \frac{D_2(1 - D_1 - D_2)}{D_1(1 - 2D_2)}$$

After some algebraic simplifications, we obtain $C^{\text{deg}} = \overline{h}(D_1) - \overline{h}(D_2)$. Applying the same timesharing argument as in the proof of Prop. 4.1, we obtain $C^{\text{deg}} = g^*(D_1, D_2)$, which is the same as the capacity C^{pub} for the public watermarking game.

F Proof of (5.18)

The inequality $1 - \prod_i (1 - \alpha_i)^{t_i} \leq 1$ being trivial, it remains to prove that

$$1 - \prod_{i} (1 - \alpha_i)^{t_i} \le \sum_{i} \alpha_i t_i$$

or equivalently,

$$\prod_{i} (1 - \alpha_i)^{t_i} \ge 1 - \sum_{i} \alpha_i t_i.$$

Define the K-vector 1 whose components are all equal to 1, the K-vectors $\boldsymbol{\alpha}$ and \mathbf{t} with components $\{\alpha_i\}$ and $\{t_i\}$, and $\Omega = [1, \infty)^K$, the domain of \mathbf{t} . Denote by ∇f the gradient vector of a function f defined on \mathbb{R}^K , and by $\mathbf{a} \cdot \mathbf{b}$ the dot product of two vectors in \mathbb{R}^K . Define the functions

$$f(\mathbf{t}) = \prod_{i} (1 - \alpha_i)^{t_i}$$
(F.1)

$$g(\mathbf{t}) = f(\mathbf{1}) + (\mathbf{t} - \mathbf{1}) \cdot \nabla f(\mathbf{1})$$
 (F.2)

$$h(\mathbf{t}) = 1 - \boldsymbol{\alpha} \cdot \mathbf{t} \tag{F.3}$$

We need to prove that $f(\mathbf{t}) \ge h(\mathbf{t})$ for all $\mathbf{t} \in \Omega$ and $\boldsymbol{\alpha} \in [0, 1]^K$. In Step 1 below we establish that $f(\mathbf{t}) \ge g(\mathbf{t})$. In Step 2, we prove that $g(\mathbf{t}) \ge h(\mathbf{t})$.

Step 1. The function $g(\mathbf{t})$ describes a hyperplane tangent to the graph of $f(\mathbf{t})$ at $\mathbf{t} = \mathbf{1}$. The function $f(\mathbf{t})$ may be written as

$$f(\mathbf{t}) = \exp\left\{\sum_{i} t_i \ln(1 - \alpha_i)\right\}.$$
 (F.4)

It is convex and therefore $f(\mathbf{t}) \geq g(\mathbf{t})$, owing to the hyperplane separation theorem.

Step 2. From (F.2) and (F.3), we have

$$g(\mathbf{t}) - h(\mathbf{t}) = [f(\mathbf{1}) - h(\mathbf{1})] + \sum_{i} (t_i - 1) \left[\frac{\partial f(\mathbf{t})}{\partial t_i} \Big|_{\mathbf{t} = \mathbf{1}} + \alpha_i \right].$$
(F.5)

Observe that $f(\mathbf{1}) = \prod_i (1 - \alpha_i)$ and $h(\mathbf{1}) = 1 - \sum_i \alpha_i$; therefore we have the well-known inequality $f(\mathbf{1}) \ge h(\mathbf{1})$. Next, since each term $t_i - 1$ in (F.5) is nonnegative for $\mathbf{t} \in \Omega$, it suffices to prove that

$$\left. \frac{\partial f(\mathbf{t})}{\partial t_i} \right|_{\mathbf{t}=\mathbf{1}} \ge -\alpha_i \tag{F.6}$$

to establish that $g(\mathbf{t}) - h(\mathbf{t}) \ge 0$ for all $\mathbf{t} \in \Omega$.

From (F.4), we obtain

$$\left. \frac{\partial f(\mathbf{t})}{\partial t_i} \right|_{\mathbf{t}=\mathbf{1}} = f(\mathbf{1}) \ln(1-\alpha_i), \quad 1 \le i \le K, \quad \forall \mathbf{t},$$
(F.7)

where $0 \le f(1) \le 1 - \alpha_i$. Since $\ln(1 - \alpha_i) \le 0$, this implies

$$f(\mathbf{1})\ln(1-\alpha_i) \ge (1-\alpha_i)\ln(1-\alpha_i).$$
(F.8)

We now prove that

$$(1 - \alpha_i)\ln(1 - \alpha_i) \ge -\alpha_i \tag{F.9}$$

which, combined with (F.8) and (F.7), will establish (F.6). Putting $x = 1 - \alpha_i$, we apply the inequality $\ln \frac{1}{x} \leq \frac{1}{x} - 1$ to claim that

$$x \ln x = -x \ln \frac{1}{x} \ge -x \left(\frac{1}{x} - 1\right) = x - 1$$

which proves (F.9). The proof is complete.

G Proof of Proposition 3.8

The variational distance for $p, p' \in \mathcal{P}_Y$ is defined as $||p - p'|| = \sum_y |p(y) - p'(y)|$ and extended to conditional pmf's $p, p' \in \mathcal{P}_{Y|X}$ as $||p - p'|| = \max_x \sum_y |p(y|x) - p'(y|x)|$.

Lemma G.1 [16, p. 33]. For any $p, p' \in \mathcal{P}_Y$, we have

$$||p - p'|| \le \theta \le \frac{1}{2} \quad \Rightarrow \quad |\tilde{H}_p(Y) - \tilde{H}_{p'}(Y)| \le \theta \log \frac{|\mathcal{Y}|}{\theta}$$

Lemma G.2 For any $p_X \in \mathcal{P}_X$ and $p, p' \in \mathcal{P}_{Y|X}$, we have

$$||p - p'|| \le \theta \le \frac{1}{2} \quad \Rightarrow \quad |\tilde{H}_{p_X p}(Y|X) - \tilde{H}_{p_X p'}(Y|X)| \le \theta \log \frac{|\mathcal{Y}|}{\theta}.$$

Proof:

$$\begin{aligned} |\tilde{H}_{p_X p}(Y|X) - \tilde{H}_{p_X p'}(Y|X)| &= \left| \sum_x p_X(x) [\tilde{H}_p(Y|X=x) - \tilde{H}_{p'}(Y|X=x)] \right| \\ &\leq \max_x |\tilde{H}_p(Y|X=x) - \tilde{H}_{p'}(Y|X=x)| \\ &\leq \theta \log \frac{|\mathcal{Y}|}{\theta} \end{aligned}$$

where the last inequality follows from Lemma G.1.

Proof of Proposition 3.8.

The upper bound is straightforward. We now derive the lower bound.

Step 1. Define a discretized set $A_l \subset A$ of attack channels as follows:

$$\mathcal{A}_{l} = \left\{ q \in \mathcal{A} : q(y|x, s^{a}) \in \{0, l^{-1}, 2l^{-1}, \cdots, 1\} \; \forall y, x, s^{a} \right\}.$$

Step 2. Any attack channel $p \in \mathcal{A}$ satisfies

$$\sum_{x} p_X(x) d_2(x) \le D_2$$

where

$$d_2(x,s^a) \triangleq \sum_y p(y|x,s^a)d(x,y) \tag{G.1}$$

is the distortion introduced by p when X = x and $S^a = s^a$.

We construct an approximation \hat{p} of p with the following properties:

$$\sum_{y} \hat{p}(y|x, s^a) = 1 \tag{G.2}$$

$$\sum_{y} \hat{p}(y|x, s^{a}) d(x, y) \leq d_{2}(x, s^{a}), \quad \forall x, s^{a},$$
(G.3)

hence $\hat{p} \in \mathcal{A}_l$. The construction of \hat{p} is as follows. Define $p^+(y|x, s^a)$ as the least upper bound on $p(y|x, s^a)$ in the set $\{l^{-1}, 2l^{-1}, \dots, 1\}$. Therefore $0 < p^+(y|x, s^a) - p(y|x, s^a) \le l^{-1}$. Define

$$k = \sum_{y} (l p^{+}(y|x, s^{a}) - l p(y|x, s^{a})))$$

= $l \sum_{y} p^{+}(y|x, s^{a}) - l \in \{0, 1, \cdots, |\mathcal{Y}| - 1\},$ (G.4)

 $d_k(x)$ as the sum of the k largest values of d(x, y) when y ranges over \mathcal{Y} , and $\mathcal{Y}_k(x)$ as the set of k corresponding values of y, that is, we have $d_k(x) = \sum_{y \in \mathcal{Y}_k(x)} d(x, y)$. Now let

$$\hat{p}(y|x,s^a) = p^+(y|x,s^a) - \frac{1}{l} \mathbf{1}_{\{y \in \mathcal{Y}_k\}}.$$
(G.5)

We have the following properties:

$$\sum_{y} \hat{p}(y|x, s^{a}) = \sum_{y} p^{+}(y|x, s^{a}) - \frac{k}{l} = \sum_{y} p(y|x, s^{a}) = 1;$$

therefore (G.2) holds. Also

$$\sum_{y} d(x, y)\hat{p}(y|x, s^{a}) = \sum_{y} d(x, y)p(y|x, s^{a}) - \sum_{y} d(x, y)(p^{+}(y|x, s^{a}) - \hat{p}(y|x, s^{a})) + \sum_{y} d(x, y)(p^{+}(y|x, s^{a}) - p(y|x, s^{a})).$$

From (G.1), the first sum in the right side is equal to $d_2(x, s^a)$. Owing to (G.5), the second sum is equal to

$$\sum_{y} d(x, y) \frac{1}{l} \mathbf{1}_{\{y \in \mathcal{Y}_k\}} = \frac{1}{l} d_k(x).$$

The third sum is equal to

$$\frac{1}{l}\sum_{y} d(x,y)(l\,p^+(y|x,s^a) - l\,p(y|x,s^a)) \le \frac{1}{l}d_k(x)$$

where the inequality holds because

$$l p^+(y|x, s^a) - l p(y|x, s^a) \le 1$$
 and $\sum_y (l p^+(y|x, s^a) - l p(y|x, s^a)) = k.$

Hence (G.3) holds as well, and $\hat{p} \in \mathcal{A}_l$. The cardinality of \mathcal{A}_l is at most $(l+1)^{|\mathcal{Y}||\mathcal{X}||\mathcal{S}^a|}$.

By construction of \hat{p} , we have

$$||p_{Y|XS^a} - \hat{p}_{Y|XS^a}|| \le \frac{|\mathcal{Y}|}{l} \triangleq \theta.$$

Step 3. Consider an alphabet \mathcal{U} of arbitrarily large cardinality. We have

$$J_{|\mathcal{U}|}(\cdot) = I(U; YS^d) - I(U; S^e) = H(Y|S^d) - H(Y|US^d) + I(U; S^d) - I(U; S^e),$$

hence

$$\begin{aligned} |J_{|\mathcal{U}|}(p_{S} \, p_{XU|S^{e}} \, p_{Y|XS^{a}}) &- J_{|\mathcal{U}|}(p_{S} \, p_{XU|S^{e}} \, \hat{p}_{Y|XS^{a}})| \\ &= |\tilde{H}_{p_{Y|XS^{a}}}(Y|S^{d}) - \tilde{H}_{p_{Y|XS^{a}}}(Y|US^{d}) - \tilde{H}_{\hat{p}_{Y|XS^{a}}}(Y|S^{d}) + \tilde{H}_{\hat{p}_{Y|XS^{a}}}(Y|US^{d})| \\ &\leq |\tilde{H}_{p_{Y|XS^{a}}}(Y|S^{d}) - \tilde{H}_{\hat{p}_{Y|XS^{a}}}(Y|S^{d})| + |\tilde{H}_{p_{Y|XS^{a}}}(Y|US^{d}) - \tilde{H}_{\hat{p}_{Y|XS^{a}}}(Y|US^{d})| \\ &\leq 2\theta \log \frac{|\mathcal{Y}|}{\theta} = 2|\mathcal{Y}| \frac{\log l}{l} \end{aligned}$$
(G.6)

where the second inequality is obtained by application of Lemmas G.1 and G.2.

Step 4. By application of Caratheodory's theorem, given a pmf p_{XUS^e} where \mathcal{U} has arbitrarily large cardinality, and given L real-valued functionals $f_i, 1 \leq i \leq L$ defined over the set \mathcal{P}_{XS^e} , there exist L elements u_1, \dots, u_L of \mathcal{U} and L nonnegative numbers $\alpha_1, \dots, \alpha_L$ summing to 1 such that

$$\sum_{u \in \mathcal{U}} p_U(u) f_i(p_{XS^e | U=u}) = \sum_{u=1}^L \alpha_u f_i(p_{XS^e | U=u}), \quad i = 1, 2, \cdots, L$$

The payoff function in the mutual-information game takes the form

$$\begin{aligned} J_{|\mathcal{U}|}(p_{S}p_{XU|S^{e}}p_{Y|XS^{a}}) &= I(U;YS^{d}) - I(U;S^{e}) \\ &= \sum_{u \in \mathcal{U}} p_{U}(u)[-H(YS^{d}|U=u) + H(S^{e}|U=u)] + H(YS^{d}) - H(S^{e}). \end{aligned}$$

We apply Caratheodory's theorem to our problem by letting

$$\begin{aligned} f_i(p_{XS^e|U=u}) &= p_{XS^e}(x, s^e), & 1 \le i(x, s^e) \le |\mathcal{X}| \, |\mathcal{S}^e| - 1, \\ f_i(p_{XS^e|U=u}) &= H(YS^d|U=u) - H(S^e|U=u), \quad |\mathcal{X}| \, |\mathcal{S}^e| \le i(p_{Y|XS^a}) \le |\mathcal{X}| \, |\mathcal{S}^e| + |\mathcal{A}_l| - 1. \end{aligned}$$

The first $|\mathcal{X}||\mathcal{S}^e| - 1$ functions correspond to the marginals of p_{XS^e} except one, and the next $|\mathcal{A}_l| = l^{|\mathcal{Y}||\mathcal{X}||\mathcal{S}^a|}$ functions are indexed by the attack channels $p_{Y|XS^a} \in \mathcal{A}_l$. Hence, defining $\mathcal{U}' = \{1, \dots, L\}$, there exist L nonnegative numbers $\alpha_1, \dots, \alpha_L$ summing to 1 and a random variable $U' \in \mathcal{U}'$ such that

$$p_{XU'S^e}(x, u', s^e) = p_{XS^e|U}(x, s^e|u_{u'}) \alpha_{u'} \quad \forall x, s^e$$
$$J_L(p_S p_{XU'|S^e} p_{Y|XS^a}) = J_{|\mathcal{U}|}(p_S p_{XU|S^e} p_{Y|XS^a}) \quad \forall p_{Y|XS^a} \in \mathcal{A}_l.$$

Hence it suffices to consider

$$|\mathcal{U}| = L = |\mathcal{X}| |\mathcal{S}^e| + l^{|\mathcal{Y}| |\mathcal{X}| |\mathcal{S}^a|} - 1$$

as stated in (3.14), to achieve the maximum in

$$\max_{p_{XU|S^e} \in \mathcal{P}_{XU|S^e}(|\mathcal{U}|, D_1)} \min_{p_{Y|XS^a} \in \mathcal{A}_l} J_L(p_S p_{XU|S^e} p_{Y|XS^a}).$$

Step 5. For any choice of \mathcal{U} , we have

$$C_{L} \stackrel{(a)}{=} \max_{\substack{p_{XU|S^{e}} \in \mathcal{P}_{XU|S^{e}}(L,D_{1}) p_{Y|XS^{a}} \in \mathcal{A} \\ p_{XU|S^{e}} \in \mathcal{P}_{XU|S^{e}}(L,D_{1}) p_{Y|XS^{a}} \in \mathcal{A} } J_{L}(p_{S}p_{XU|S^{e}}p_{Y|XS^{a}}) - 2|\mathcal{Y}| \frac{\log l}{l}}$$

$$\stackrel{(c)}{=} \max_{\substack{p_{XU|S^{e}} \in \mathcal{P}_{XU|S^{e}}(|\mathcal{U}|,D_{1}) p_{Y|XS^{a}} \in \mathcal{A} \\ p_{XU|S^{e}} \in \mathcal{P}_{XU|S^{e}}(|\mathcal{U}|,D_{1}) p_{Y|XS^{a}} \in \mathcal{A} } J_{|\mathcal{U}|}(p_{S}p_{XU|S^{e}}p_{Y|XS^{a}}) - 2|\mathcal{Y}| \frac{\log l}{l}}$$

$$\stackrel{(d)}{=} \max_{\substack{p_{XU|S^{e}} \in \mathcal{P}_{XU|S^{e}}(|\mathcal{U}|,D_{1}) p_{Y|XS^{a}} \in \mathcal{A} \\ p_{XU|S^{e}} \in \mathcal{P}_{XU|S^{e}}(|\mathcal{U}|,D_{1}) p_{Y|XS^{a}} \in \mathcal{A} } J_{|\mathcal{U}|}(p_{S}p_{XU|S^{e}}p_{Y|XS^{a}}) - 2|\mathcal{Y}| \frac{\log l}{l}}$$

$$(G.7)$$

where (a) is the definition of C_L ; (b) is because (G.6) holds uniformly for all $p_{XU|S^e}$ and $p_{Y|XS^a} \in \mathcal{A}$; and (c) is a consequence of Caratheodory's theorem in Step 4; and (d) holds because $\mathcal{A}_l \subset \mathcal{A}$.

The inequality (G.7) holds in the limit as $|\mathcal{U}| \to \infty$, hence

$$C_L \ge C - 2|\mathcal{Y}| \frac{\log l}{l}$$

which proves the claim.

H Proof of Proposition 3.10

Define sets $\mathcal{P}_Y(\epsilon) = \{p_Y \in \mathcal{P}_Y : \min_y p(y) \geq \epsilon\}$ and similarly, $\mathcal{P}_{Y|X}(\epsilon) = \{p_{Y|X} \in \mathcal{P}_{Y|X} : \min_{x,y} p(y|x) \geq \epsilon\}$, for any $\epsilon \in [0, 1/|\mathcal{Y}|]$. In preparation for the proof of the proposition, we define a log-uniform quantizer Φ_l and present three lemmas.

Pmf quantization. Given $l \geq |\mathcal{Y}|$, we define $\epsilon = l^{-1}$ and a pmf quantization mapping $\Phi_l : \mathcal{P}_Y(\epsilon) \to \mathcal{P}_Y(\epsilon)$ as follows. Define the log-uniform quantizer $Q_l : [\epsilon, 1] \to Q_l$ with l reproduction levels

$$\mathcal{Q}_l = \{ \epsilon = \epsilon^{l\epsilon}, \epsilon^{(l-1)\epsilon}, \cdots, \epsilon^{2\epsilon}, \epsilon^{\epsilon} \}.$$
(H.1)

and quantization function

$$Q_l(z) = \begin{cases} \epsilon & : z = \epsilon \\ \epsilon^{i\epsilon} & : \epsilon^{i\epsilon} < z \le \epsilon^{(i-1)\epsilon}, \quad i = 1, 2, \cdots, l. \end{cases}$$
(H.2)

Observe that the *ratio* between adjacent reproduction levels, $\epsilon^{\epsilon} \uparrow 1$ as $l \to \infty$. Moreover the *difference* between adjacent reproduction levels is upper-bounded by $1 - \epsilon^{\epsilon} \leq \epsilon \ln \epsilon^{-1} = \frac{\ln l}{l}$. Both notions of precision will be useful in the proof.

For any $p \in \mathcal{P}_Y(\epsilon)$, define

$$q(y) = Q_l(p(y)), \tag{H.3}$$

the sum $\sigma = \sum_{y} q(y)$, and the pmf $\hat{p}(y) = \frac{1}{\sigma}q(y)$. Hence

$$\hat{p}(y) = \Phi_l \, p(y) \triangleq \frac{Q_l(p(y))}{\sum_y Q_l(p(y))}, \quad y \in \mathcal{Y}.$$
(H.4)

Lemma H.1 For any integer $l \ge |\mathcal{Y}| \ge 2$ and $p, \tilde{p} \in \mathcal{P}_Y(l^{-1})$, we have

$$|D(\tilde{p}||p) - D(\Phi_l \tilde{p}||\Phi_l p)| \le 2(|\mathcal{Y}| + 1)\frac{\log^2 l}{l}.$$

Proof. Let $\epsilon = l^{-1}$. Since $p(y) > \epsilon$, it follows from (H.2) and (H.3) that

$$\epsilon^{\epsilon} p(y) \le q(y) < p(y). \tag{H.5}$$

Summing (H.5) over $y \in \mathcal{Y}$, we obtain

$$\epsilon^{\epsilon} \le \sigma = \sum_{y} q(y) \le 1.$$
 (H.6)

We have

$$\epsilon^{\epsilon} \leq \frac{q(y)}{p(y)} \leq 1 \quad \text{and} \quad 1 \leq \frac{1}{\sigma} = \frac{\Phi_l \, p(y)}{q(y)} \leq \epsilon^{-\epsilon}$$
(H.7)

For each $y \in \mathcal{Y}$, we have

$$|p(y) - \Phi_l p(y)| \le |p(y) - q(y)| + |q(y) - \Phi_l p(y)| \le \frac{\ln l}{l} + (1 - \sigma)\Phi_l p(y) \le [1 + \Phi_l p(y)]\frac{\ln l}{l},$$

hence

$$||p - \Phi_l p|| \le (|\mathcal{Y}| + 1) \frac{\ln l}{l}.$$
 (H.8)

Multiplying the inequalities in (H.7) and taking logarithms, we obtain

$$\left|\log\frac{\Phi_l p(y)}{p(y)}\right| \le -\epsilon \log \epsilon = \frac{\log l}{l}.$$
(H.9)

Similarly,

$$\log \frac{\Phi_l \, p(y)}{p(y)} \bigg| \le \frac{\log l}{l}, \quad \forall y.$$
(H.10)

Hence

$$D(\tilde{p}||p) - D(\Phi_l \tilde{p}||\Phi_l p) = \sum_y \tilde{p}(y) \log \frac{\tilde{p}(y)}{p(y)} - \sum_y \Phi_l \tilde{p}(y) \log \frac{\Phi_l \tilde{p}(y)}{\Phi_l p(y)}$$

$$= \sum_y \tilde{p}(y) \left(\log \frac{\tilde{p}(y)}{p(y)} - \log \frac{\Phi_l \tilde{p}(y)}{\Phi_l p(y)} \right) + \sum_y (\tilde{p}(y) - \Phi_l \tilde{p}(y)) \log \frac{\Phi_l \tilde{p}(y)}{\Phi_l p(y)}$$

$$= \sum_y \tilde{p}(y) \left(\log \frac{\Phi_l p(y)}{p(y)} - \log \frac{\Phi_l \tilde{p}(y)}{\tilde{p}(y)} \right) + \sum_y (\tilde{p}(y) - \Phi_l \tilde{p}(y)) \log \frac{\Phi_l \tilde{p}(y)}{\Phi_l p(y)}.$$

Hence

$$\begin{aligned} |D(\tilde{p}||p) - D(\Phi_l \, \tilde{p}||\Phi_l \, p)| &\leq \max_y \left| \log \frac{\Phi_l \, p(y)}{p(y)} \right| + \max_y \left| \log \frac{\Phi_l \, \tilde{p}(y)}{\tilde{p}(y)} \right| + \|\tilde{p} - \Phi_l \, \tilde{p}\| \max_y \left| \log \frac{\Phi_l \, \tilde{p}(y)}{\Phi_l \, p(y)} \right| \\ &\leq 2 \frac{\log l}{l} + (|\mathcal{Y}| + 1) \frac{\ln l}{l} \log l \\ &= 2 \frac{\log l}{l} \left[1 + (|\mathcal{Y}| + 1) \frac{\ln 2}{2} \log l \right] \\ &\leq 2 (|\mathcal{Y}| + 1) \frac{\log^2 l}{l} \end{aligned}$$

where the second inequality follows from (H.8), (H.9), and (H.10), and the third inequality from the fact that $l \ge |\mathcal{Y}| \ge 2$.

The following lemma establishes a bound on the variation in conditional Kullback-Leibler divergence $D(\tilde{p}_{Y|X}||p_{Y|X}||\tilde{p}_X)$ when the mapping Φ_l is applied to each pmf $p_{Y|X}(\cdot|x) \in \mathcal{P}_Y(\epsilon)$. The resulting pmf is denoted by $\Phi_l p_{Y|X} = \{\Phi_l p_{Y|X}(\cdot|x), x \in \mathcal{X}\}.$

Lemma H.2 For any integer $l \geq |\mathcal{Y}| \geq 2$, $\tilde{p}_{XY} = \tilde{p}_X \tilde{p}_{Y|X} \in \mathcal{P}_{XY}$, and $p_{Y|X}, \tilde{p}_{Y|X} \in \mathcal{P}_{Y|X}(\epsilon)$, we have

$$|D(\tilde{p}_{Y|X}||p_{Y|X}||\tilde{p}_X) - D(\Phi_l \,\tilde{p}_{Y|X}||\Phi_l \, p_{Y|X}||\tilde{p}_X)| \le 2(|\mathcal{Y}|+1)\frac{\log^2 l}{l}.$$

Proof.

$$\begin{aligned} &|D(\tilde{p}_{Y|X} \| p_{Y|X} | \tilde{p}_X) - D(\Phi_l \, \tilde{p}_{Y|X} \| \Phi_l \, p_{Y|X} | \tilde{p}_X)| \\ &= \left| \sum_x \tilde{p}_X(x) [D(\tilde{p}_{Y|X=x} \| p_{Y|X=x}) - D(\Phi_l \, \tilde{p}_{Y|X=x} \| \Phi_l \, p_{Y|X=x})] \right| \\ &\leq \max_x |D(\tilde{p}_{Y|X=x} \| p_{Y|X=x}) - D(\Phi_l \, \tilde{p}_{Y|X=x} \| \Phi_l \, p_{Y|X=x})| \\ &\leq 2(|\mathcal{Y}| + 1) \frac{\log^2 l}{l} \end{aligned}$$

where the last inequality follows from Lemma H.1.

Proof of Proposition 3.10.

The lower bound is straightforward. We now derive the upper bound. The class of attack channels under expected distortion constraint D_2 is denoted by $\mathcal{A}(D_2)$; the dependency on p_X is not explicitly indicated.

Define $\mathcal{U}_L = \{1, \dots, L\}$, where L is given in (3.15). Let \mathcal{U} have arbitrarily cardinality, possibly larger than L. Define the shorthands

$$\tilde{\mathcal{A}} \triangleq \mathcal{P}_{YS^aS^d|XS^e}$$
 and $\tilde{\mathcal{A}}_{\mathcal{U}} \triangleq \mathcal{P}_{YS^aS^d|XUS^e}$

and the functionals (with a little abuse of notation)

$$\begin{split} &E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{p}_{YS^aS^d|XUS^e}, p_{Y|XS^a}) \\ &\triangleq \quad D(\tilde{p}_{S^e}p_{XU|S^e}\tilde{p}_{YS^aS^d|XUS^e} \| p_S p_{XU|S^e} p_{Y|XS^a}) + |J_{|\mathcal{U}|}(\tilde{p}_{S^e}p_{XU|S^e}\tilde{p}_{YS^aS^d|XUS^e}) - R|^+ \end{split}$$

and

$$E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}},\tilde{\mathcal{B}}_{\mathcal{U}},\mathcal{B}) \triangleq \max_{\substack{p_{XU|S^{e}}\in\mathcal{P}_{XU|S^{e}}(|\mathcal{U}|,D_{1}) \\ \tilde{p}_{YS^{a}S^{d}|XUS^{e}}\in\tilde{\mathcal{B}}_{\mathcal{U}}} \min_{p_{Y|XS^{a}}\in\mathcal{B}}} E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}},p_{XU|S^{e}},\tilde{p}_{YS^{a}S^{d}|XUS^{e}},p_{Y|XS^{a}}), \quad \forall \tilde{\mathcal{B}}_{\mathcal{U}} \subseteq \tilde{\mathcal{A}}_{\mathcal{U}}, \mathcal{B} \subseteq \mathcal{A}(\mathrm{H.11})$$

Hence

$$E_{r,|\mathcal{U}|}(D_2) = \min_{\tilde{p}_{S^e}} E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, \tilde{\mathcal{A}}_{\mathcal{U}}, \mathcal{A}(D_2))$$
$$E_r(D_2) = \lim_{|\mathcal{U}| \to \infty} E_{r,|\mathcal{U}|}(D_2).$$

Let $\epsilon = 1/l$ and

$$D_2' = D_2(1 + \epsilon \ln \epsilon) < D_2 \tag{H.12}$$
$$D_2' = D_2' = \epsilon^{1/2} ||\cdot| |S_1^a| + |S_2^d| |\overline{D}|$$

$$D_2'' = D_2' - \epsilon |\mathcal{Y}|| |\mathcal{S}^a| |\mathcal{S}^d| \overline{D}$$

= $D_2 - \epsilon (|\mathcal{Y}|| |\mathcal{S}^a| |\mathcal{S}^d| \overline{D} + D_2 \ln \epsilon^{-1}).$ (H.13)

The proof consists of the following steps:

1. (*Pmf lifting step*). Define the subsets

$$\mathcal{A}(D_2;\epsilon) \triangleq \left\{ p_{Y|XS^a} \in \mathcal{A}(D_2) : \min_{y,x,s^a} p_{Y|XS^a}(y|x,s^a) \ge \epsilon \right\}$$
$$\tilde{\mathcal{A}}(\epsilon) \triangleq \left\{ \tilde{p}_{YS^aS^d|XS^e} \in \tilde{\mathcal{A}} : \min_{y,x,s} \tilde{p}_{YS^aS^d|XS^e}(y,s^a,s^d|x,s^e) \ge \epsilon \right\}$$

of $\mathcal{A}(D_2)$ and $\tilde{\mathcal{A}}$ in which the conditional pmf's $p_{Y|XS^a}$ and $\tilde{p}_{YS^aS^d|XS^e}$ are lower-bounded by ϵ . Also define $\tilde{\mathcal{A}}_{\mathcal{U}}(\epsilon)$ as the set of all pmf's $\tilde{p}_{YS^aS^d|XUS^e}$ whose conditional marginals $\tilde{p}_{YS^aS^d|XS^e,U=u}$ are in $\tilde{\mathcal{A}}(\epsilon)$ for all $u \in \mathcal{U}$.

For any \tilde{p}_{S^e} , $p_{XU|S^e} \in \mathcal{P}_{XU|S^e}(|\mathcal{U}|, D_1)$, $\tilde{p}_{YS^aS^d|XUS^e} \in \tilde{\mathcal{A}}_{\mathcal{U}}$ and $p_{Y|XS^a} \in \mathcal{A}(D''_2)$, we show there exist $\tilde{p}'_{YS^aS^d|XUS^e} \in \tilde{\mathcal{A}}_{\mathcal{U}}(\epsilon)$ and $p'_{Y|XS^a} \in \mathcal{A}(D'_2; \epsilon)$ such that

$$E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{XU|S^{e}}, \tilde{p}_{YS^{a}S^{d}|XUS^{e}}, p_{Y|XS^{a}}) \\ \geq E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{XU|S^{e}}, \tilde{p}'_{YS^{a}S^{d}|XUS^{e}}, p'_{Y|XS^{a}})| - \frac{4}{l}|\mathcal{Y}||\mathcal{S}^{a}||\mathcal{S}^{d}| \log \frac{l}{2|\mathcal{S}^{a}|^{5/4}|\mathcal{S}^{d}|^{1/4}c^{1/4}}$$
(H.14)

where the constant c was defined in the statement of the proposition.

2. (*Pmf quantization step*). We define finite nets $\tilde{\mathcal{A}}_{l}(\epsilon) \subset \tilde{\mathcal{A}}(\epsilon)$ and $\mathcal{A}_{l}(D_{2};\epsilon) \subset \mathcal{A}(D_{2};\epsilon)$ whose cardinalities are at most $l^{|\mathcal{Y}||\mathcal{X}||S|}$ and $l^{|\mathcal{Y}||\mathcal{X}||S^{a}|}$, respectively. Also define $\tilde{\mathcal{A}}_{l,\mathcal{U}}(\epsilon)$ as the set of all pmf's $\tilde{p}_{YS^{a}S^{d}|XUS^{e}}$ whose conditional marginals $\tilde{p}_{YS^{a}S^{d}|XS^{e},U=u}$ are in $\tilde{\mathcal{A}}_{l}(\epsilon)$ for all $u \in \mathcal{U}$.

For any \tilde{p}_{S^e} , $p_{XU|S^e}$, $\tilde{p}'_{YS^aS^d|XUS^e} \in \tilde{\mathcal{A}}_{\mathcal{U}}(\epsilon)$ and $p'_{Y|XS^a} \in \mathcal{A}(D'_2;\epsilon)$, we show there exist $\hat{p}_{YS^aS^d|XUS^e} \in \tilde{\mathcal{A}}_{l,\mathcal{U}}(\epsilon)$ and $\hat{p}_{Y|XS^a} \in \mathcal{A}_l(D_2;\epsilon)$ such that

$$E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{XU|S^{e}}, \tilde{p}'_{YS^{a}S^{d}|XUS^{e}}, p'_{Y|XS^{a}})$$

$$\geq E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{XU|S^{e}}, \hat{p}_{YS^{a}S^{d}|XUS^{e}}, \hat{p}_{Y|XS^{a}}) - 5|\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \frac{\log^{2} l}{l}.$$
(H.15)

From (H.14) and (H.15), for

$$l \ge \exp_2\left[1 + \sqrt{\left|-\frac{1}{2}\log(8|\mathcal{S}^a|^5|\mathcal{S}^d|c)\right|^+}\right]$$
(H.16)

we obtain

$$E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{p}_{YS^aS^d|XUS^e}, p_{Y|XS^a})$$

$$\geq E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \hat{p}_{YS^aS^d|XUS^e}, \hat{p}_{Y|XS^a}) - \Delta(l)$$
(H.17)

where

$$\Delta(l) \triangleq 7|\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| \frac{\log^2 l}{l}.$$
 (H.18)

- 3. By application of Caratheodory's theorem, we show that for each \tilde{p}_{S^e} , the supremum of the function $E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{A}_{l,\mathcal{U}}(\epsilon), \mathcal{A}_l(D_2; \epsilon))$ over $p_{XU|S^e}$ is achieved for $|\mathcal{U}| = L$ given in (3.15).
- 4. Combining the results above, we show that $E_{r,L}(D_2'') \ge E_{r,|\mathcal{U}|}(D_2) \Delta(l)$. Taking the limit as $|\mathcal{U}| \to \infty$ proves the claim.

Step 1. (*Pmf lifting*). Denote by

$$\mu_Y(y) = \frac{1}{|\mathcal{Y}|}$$
 and $\mu_{S^aS^d}(s^a, s^d) = \frac{1}{|\mathcal{S}^a| |\mathcal{S}^d|}$

the uniform distributions over \mathcal{Y} and $\mathcal{S}^a \times \mathcal{S}^d$, respectively. The average distortion for the hypothetical attack channel $p_{Y|XS^a} = \mu_Y$ satisfies

$$\mathbb{E}d(X,Y) \le \max_{x} \mathbb{E}d(x,Y) = \max_{x} \frac{1}{|\mathcal{Y}|} \sum_{y} d(x,y) = \overline{D}.$$
 (H.19)

To each $p_{Y|XS^a} \in \mathcal{A}(D_2'')$ and $\tilde{p}_{YS^aS^d|XUS^e} \in \tilde{\mathcal{A}}$, associate the conditional pmf's

$$p'_{Y|XS^{a}}(y|x,s^{a}) = (1 - \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}|) p_{Y|XS^{a}}(y|x,s^{a}) + \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \mu_{Y}(y)$$

$$\tilde{p}'_{YS^{a}S^{d}|XUS^{e}}(y,s^{a},s^{d}|x,u,s^{e}) = (1 - \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}|) \tilde{p}_{YS^{a}S^{d}|XUS^{e}}(y,s^{a},s^{d}|x,u,s^{e})$$

$$+\epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \mu_{Y}(y) \mu_{S^{a}S^{d}}(s^{a},s^{d}), \quad \forall y,x,u,s \quad (H.20)$$

which are slight modifications of $p_{Y|XS^a}$ and $\tilde{p}_{YS^aS^d|XUS^e}$ and are lower-bounded by ϵ . We refer to the mappings in (H.20) as "pmf lifting".

Since average distortion is a linear functional of the attack channel, the average distortion for $p'_{Y|XS^a}$ is upper bounded by

$$(1 - \epsilon |\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d|) D_2'' + \epsilon |\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| \overline{D} \le D_2'.$$

Hence $p'_{Y|XS^a} \in \mathcal{A}(D'_2; \epsilon)$.

Since $E_{r,|\mathcal{U}|}(\cdot)$ takes the form $D(\cdot) + |J_{|\mathcal{U}|}(\cdot) - R|^+$, the variation in the error exponent due to the above pmf lifting operations satisfies the upper bound:

$$E_{r,L}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{p}'_{YS^aS^d|XUS^e}, p'_{Y|XS^a}) - E_{r,L}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{p}_{YS^aS^d|XUS^e}, p_{Y|XS^a})$$

$$\leq \Delta D + \Delta J_{|\mathcal{U}|} \qquad (\text{H.21})$$

where

$$\Delta J_{|\mathcal{U}|} = J_{|\mathcal{U}|}(\tilde{p}_{S^e} \, p_{XU|S^e} \, \tilde{p}'_{YS^aS^d|XUS^e}) - J_{|\mathcal{U}|}(\tilde{p}_{S^e} \, p_{XU|S^e} \, \tilde{p}_{YS^aS^d|XUS^e})$$

and

$$\Delta D = D(\tilde{p}_{S^e} p_{XU|S^e} \tilde{p}'_{YS^a S^d | XUS^e} \| p_S p_{XU|S^e} p'_{Y|XS^a}) - D(\tilde{p}_{S^e} p_{XU|S^e} \tilde{p}_{YS^a S^d | XUS^e} \| p_S p_{XU|S^e} p_{Y|XS^a})$$

$$= D(\tilde{p}'_{YS^a S^d | XUS^e} \| p_{S^a S^d | S^e} p'_{Y|XS^a} | \tilde{p}_{S^e} p_{XU|S^e}) - D(\tilde{p}_{YS^a S^d | XUS^e} \| p_{S^a S^d | S^e} p_{Y|XS^a} | \tilde{p}_{S^e} p_{XU|S^e})$$

The last equality follows from the chain rule for Kullback-Leibler divergence,

$$D(\tilde{p}_{S^{e}}p_{XU|S^{e}}\tilde{p}_{YS^{a}S^{d}|XUS^{e}}||p_{S}p_{XU|S^{e}}p'_{Y|XS^{a}}) = D(\tilde{p}_{S^{e}}||p_{S^{e}}) + D(\tilde{p}_{YS^{a}S^{d}|XUS^{e}}||p_{S^{a}S^{d}|S^{e}}p'_{Y|XS^{a}}||\tilde{p}_{S^{e}}p_{XU|S^{e}}).$$

The effect of pmf lifting on ΔD is as follows. By convexity of conditional Kullback-Leibler divergence [16], from (H.20) we have

$$D(\tilde{p}'_{YS^{a}S^{d}|XUS^{e}} \| p_{S^{a}S^{d}|S^{e}} p'_{Y|XS^{a}} | \tilde{p}_{S^{e}} p_{XU|S^{e}}) \\ \leq (1 - \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}|) D(\tilde{p}_{YS^{a}S^{d}|XUS^{e}} \| p_{S^{a}S^{d}|S^{e}} p_{Y|XS^{a}} | \tilde{p}_{S^{e}} p_{XU|S^{e}}) \\ + \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| D(\mu_{Y} \mu_{S^{a}S^{d}} \| p_{S^{a}S^{d}|S^{e}} \mu_{Y} | \tilde{p}_{S^{e}} p_{XU|S^{e}})$$

$$\leq D(\tilde{p}_{YS^{a}S^{d}|XUS^{e}} \| p_{S^{a}S^{d}|S^{e}} p_{Y|XS^{a}} | \tilde{p}_{S^{e}} p_{XU|S^{e}}) + \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \max_{s} \log \frac{\mu_{S^{a}S^{d}}(s^{a}, s^{d})}{p_{S^{a}S^{d}|S^{e}}(s^{a}, s^{d}|s^{e})} \\ \leq D(\tilde{p}_{YS^{a}S^{d}|XUS^{e}} \| p_{S^{a}S^{d}|S^{e}} p_{Y|XS^{a}} | \tilde{p}_{S^{e}} p_{XU|S^{e}}) + \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \log \frac{1}{|\mathcal{S}^{a}| |\mathcal{S}^{d}| c}$$

where the constant c was defined in the statement of the proposition. Hence

$$\Delta D \le \epsilon \left| \mathcal{Y} \right| \left| \mathcal{S}^{a} \right| \left| \mathcal{S}^{d} \right| \log \frac{1}{\left| \mathcal{S}^{a} \right| \left| \mathcal{S}^{d} \right| c}.$$
(H.22)

The effect of pmf lifting on $J_{|\mathcal{U}|}(\cdot) = H(YS^d) - H(YS^d|U) - I(U;S^e)$ is as follows. From (H.20), we have

$$\begin{aligned} \|\tilde{p}_{YS^{a}S^{d}|XUS^{e}} - \tilde{p}'_{YS^{a}S^{d}|XUS^{e}}\| &= \|\epsilon|\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| p_{YS^{a}S^{d}|XUS^{e}} - \epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \mu_{Y} \mu_{S^{a}S^{d}}\| \\ &\leq 2\epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \triangleq \theta. \end{aligned} \tag{H.23}$$

Analogously to (G.6), we have

$$\begin{aligned} |\Delta J_{|\mathcal{U}|}| &= \left| \tilde{H}_{\tilde{p}_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}) - \tilde{H}_{\tilde{p}_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}|U) - \tilde{H}_{\tilde{p}'_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}) + \tilde{H}_{\tilde{p}'_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}|U) \right| \\ &\leq \left| \tilde{H}_{\tilde{p}_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}) - \tilde{H}_{\tilde{p}'_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}) \right| \\ &+ \left| \tilde{H}_{\tilde{p}_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}|U) + \tilde{H}_{\tilde{p}'_{YS^{a}S^{d}|XUS^{e}}}(YS^{d}|U) \right| \\ &\leq 2\theta \log \frac{|\mathcal{Y}| |\mathcal{S}^{d}|}{\theta} \\ &= 4\epsilon |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \log \frac{1}{2\epsilon |\mathcal{S}^{a}|} \end{aligned}$$
(H.24)

where the last inequality follows from (H.23) and Lemmas G.1 and G.2.

Combining (H.21), (H.22), and (H.24), we obtain (H.14).

Step 2. (*Pmf quantization*). Consider $p'_{Y|XS^a} \in \mathcal{A}(D'_2; \epsilon)$. For each value of (x, s^a) , apply the quantization mapping $\Phi_l : \mathcal{P}_Y(\epsilon) \to \mathcal{P}_Y(\epsilon)$ defined in (H.4) to the pmf $p'_{Y|XS^a}(\cdot|x, s^a)$ and denote by $\Phi_l p'_{Y|XS^a} \in \mathcal{P}_{Y|XS^a}(\epsilon)$ the resulting conditional pmf. Also let

$$\sigma(x, s^a) = \sum_{y} Q_l(p'_{Y|XS^a}(y|x, s^a)) \ge \epsilon^{\epsilon}$$

where the inequality is obtained as in (H.6). Similarly, given $\tilde{p}'_{YS^aS^d|XUS^e} \in \tilde{\mathcal{A}}_{\mathcal{U}}(\epsilon)$, define the quantized conditional pmf $\Phi_l \tilde{p}'_{YS^aS^d|XUS^e}$ which belongs to the finite set $\tilde{\mathcal{A}}_{l,\mathcal{U}}(\epsilon)$.

The average distortion associated with $\Phi_l \, p'_{Y|XS^a}$ is

$$\sum_{y,x,s^a} d(x,y) \Phi_l p'_{Y|XS^a}(y|x,s^a) p_{XS^a}(x,s^a)$$

$$= \frac{1}{\sigma(x,s^a)} \sum_{y,x,s^a} d(x,y) Q_l(p'_{Y|XS^a}(y|x,s^a)) p_{XS^a}(x,s^a)$$

$$\leq \frac{1}{\sigma(x,s^a)} \sum_{y,x,s^a} d(x,y) p'_{Y|XS^a}(y|x,s^a) p_{XS^a}(x,s^a)$$

$$\leq \epsilon^{-\epsilon} D'_2$$

$$= \epsilon^{-\epsilon} (1+\epsilon \ln \epsilon) D_2$$

$$\leq \epsilon^{-\epsilon} e^{\epsilon \ln \epsilon} D_2$$

$$= D_2.$$

Therefore $\Phi_l p'_{Y|XS^a} \in \mathcal{A}_l(D_2; \epsilon).$

For any choice of \tilde{p}_{S^e} , $p_{XU|S^e}$, $\tilde{p}'_{YS^aS^d|XUS^e} \in \tilde{\mathcal{A}}_{\mathcal{U}}(\epsilon)$ and $p'_{Y|XS^a} \in \mathcal{A}(D'_2;\epsilon)$, we now bound the effect of Φ_l on the error exponent as follows:

$$E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{p}'_{YS^aS^d|XUS^e}, p'_{Y|XS^a}) - E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \Phi_l \, \tilde{p}'_{YS^aS^d|XUS^e}, \Phi_l \, p'_{Y|XS^a}) \\ \leq \Delta D + \Delta J_{|\mathcal{U}|}.$$
(H.25)

Here

$$\Delta J_{|\mathcal{U}|} = J_{|\mathcal{U}|}(\tilde{p}_{S^e} \, p_{XU|S^e} \, \tilde{p}'_{YS^a S^d | XUS^e}) - J_{|\mathcal{U}|}(\tilde{p}_{S^e} \, p_{XU|S^e} \, \Phi_l \, \tilde{p}'_{YS^a S^d | XUS^e})$$

and

$$\begin{split} \Delta D | &= \left| D(\tilde{p}_{S^e} p_{XU|S^e} \tilde{p}'_{YS^a S^d | XUS^e} \| p_S p_{XU|S^e} p'_{Y|XS^a}) \right. \\ &\quad - D(\tilde{p}_{S^e} p_{XU|S^e} \Phi_l \tilde{p}'_{YS^a S^d | XUS^e} \| p_S p_{XU|S^e} \Phi_l p'_{Y|XS^a}) \right| \\ &= \left| D(\tilde{p}'_{YS^a S^d | XUS^e} \| p'_{Y|XS^a} p_{S^a S^d | S^e} | \tilde{p}_{S^e} p_{XU|S^e}) \right. \\ &\quad - D(\Phi_l \tilde{p}'_{YS^a S^d | XUS^e} \| \Phi_l p'_{Y|XS^a} p_{S^a S^d | S^e} | \tilde{p}_{S^e} p_{XU|S^e}) \right| \\ &\leq 2(|\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| + 1) \frac{\log^2 l}{l} \\ &\leq 3|\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| \frac{\log^2 l}{l} \end{split}$$
(H.26)

where the first inequality is obtained by application of Lemma H.2, and the second because $|\mathcal{Y}| \geq 2$. Next, we have

$$\|\tilde{p}'_{YS^aS^d|XUS^e}) - \Phi_l \, \tilde{p}'_{YS^aS^d|XUS^e} \| \le (|\mathcal{Y}| \, |\mathcal{S}^a| \, |\mathcal{S}^d| + 1) \, \frac{\ln l}{l} \triangleq \theta.$$

where the inequality is a straightforward generalization of (H.8). Similarly to (H.24), we obtain

$$\begin{aligned} |\Delta J_{|\mathcal{U}|}| &\leq 2\theta \log \frac{|\mathcal{Y}| |\mathcal{S}^d|}{\theta} &= 2(|\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| + 1) \frac{\ln l}{l} \log \frac{l}{|\mathcal{S}^a| \ln l} \\ &\leq 2|\mathcal{Y}| |\mathcal{S}^a| |\mathcal{S}^d| \frac{\log^2 l}{l} \end{aligned} \tag{H.27}$$

where the last inequality holds because $\log e < \frac{2}{3}$ and $|S^a| \ln l > 1$.

Combining (H.25), (H.26) and (H.27), we obtain

$$\begin{aligned} |E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{p}_{YS^aS^d|XUS^e}, p'_{Y|XS^a}) - E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \Phi_l \, \tilde{p}_{YS^aS^d|XUS^e}, \Phi_l \, p'_{Y|XS^a})| \\ \leq |\Delta D| + |\Delta J_{|\mathcal{U}|}| \leq 5|\mathcal{Y}| \, |\mathcal{S}^a| \, |\mathcal{S}^d| \, \frac{\log^2 l}{l}, \end{aligned}$$

which establishes (H.15).

Combining (H.11), (H.14) and (H.15), for any choice of \tilde{p}_{S^e} we obtain

$$E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}}, \tilde{\mathcal{A}}_{\mathcal{U}}, \mathcal{A}(D_{2}'')) - E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}}, \tilde{\mathcal{A}}_{l,\mathcal{U}}, \mathcal{A}_{l}(D_{2}; \epsilon))$$

$$\geq -5|\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \frac{\log^{2} l}{l} - \frac{4}{l} |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \log \frac{l}{2|\mathcal{S}^{a}|^{5/4}|\mathcal{S}^{d}|^{1/4}c^{1/4}}$$

$$= -\Delta(l) + \frac{2}{l} |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \left[\log^{2} l - 2\log \frac{l}{2|\mathcal{S}^{a}|^{5/4}|\mathcal{S}^{d}|^{1/4}c^{1/4}}\right]$$

$$= -\Delta(l) + \frac{2}{l} |\mathcal{Y}| |\mathcal{S}^{a}| |\mathcal{S}^{d}| \left[\log^{2} l - 2\log l + \frac{1}{2}\log(16|\mathcal{S}^{a}|^{5}|\mathcal{S}^{d}|c)\right]$$

$$\geq -\Delta(l) \qquad (H.28)$$

where $\Delta(l)$ was defined in (H.18). The term in brackets is positive when (H.16) is satisfied.

Step 3. (Caratheodory). Define

$$\begin{split} \tilde{J}_{|\mathcal{U}|}(\tilde{p}_{S^e}, p_{X|S^e, U=u}, \tilde{p}_{YS^aS^d|XS^e, U=u}) \\ &\triangleq \quad H(YS^d) - H(YS^d|U=u) - H(S^e) + H(S^e|U=u), \quad \forall u \in \mathcal{U} \end{split}$$

where the quintuple $(S^e S^a S^d X Y)$, conditioned on U = u, is distributed as $\tilde{p}_{S^e} p_{X|S^e, U=u} \tilde{p}_{YS^a S^d|XS^e, U=u}$. Likewise, we view the conditional divergence

$$D(\tilde{p}_{YS^{a}S^{d}|XS^{e},U=u} \| p_{Y|XS^{a}} p_{S^{a}S^{d}|S^{e}} \| p_{X|S^{e},U=u} \tilde{p}_{S^{e}})$$

as a function of \tilde{p}_{S^e} , $p_{X|S^e,U=u}$, $\tilde{p}_{YS^aS^d|XS^e,U=u}$, and $p_{Y|XS^a}$. We may thus write

$$\begin{split} D(\tilde{p}_{YS^{a}S^{d}|XUS^{e}} \| p_{Y|XS^{a}} p_{S^{a}S^{d}|S^{e}} | p_{X|US^{e}} \tilde{p}_{S^{e}}) \\ &= \sum_{u \in \mathcal{U}} p_{U}(u) D(\tilde{p}_{YS^{a}S^{d}|XS^{e},U=u} \| p_{Y|XS^{a}} p_{S^{a}S^{d}|S^{e}} | p_{X|S^{e},U=u} \tilde{p}_{S^{e}}) \\ J_{|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{XU|S^{e}}, \tilde{p}_{YS^{a}S^{d}|XUS^{e}}) &= \sum_{u \in \mathcal{U}} p_{U}(u) \tilde{J}_{|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{X|S^{e},U=u}, \tilde{p}_{YS^{a}S^{d}|XS^{e},U=u}) \end{split}$$

 $E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, p_{XU|S^e}, \tilde{p}_{YS^aS^d|XUS^e}, p_{Y|XS^a})$

$$= D(\tilde{p}_{S^{e}} || p_{S^{e}}) + \sum_{u \in \mathcal{U}} p_{U}(u) D(\tilde{p}_{YS^{a}S^{d}|XS^{e}, U=u} || p_{Y|XS^{a}} p_{S^{a}S^{d}|S^{e}} || p_{X|S^{e}, U=u} \tilde{p}_{S^{e}}) \\ + \left| \sum_{u \in \mathcal{U}} p_{U}(u) \tilde{J}_{|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{X|S^{e}, U=u}, \tilde{p}_{YS^{a}S^{d}|XS^{e}, U=u}) - R \right|^{+}.$$

The cardinality of the discretized set $\mathcal{A}_l(D_2;\epsilon)$ of attack channels $\hat{p}_{Y|XS^a}$ is less than $l^{|\mathcal{Y}||\mathcal{X}||\mathcal{S}^a|}$. Likewise the cardinality of the discretized set $\tilde{\mathcal{A}}_l(\epsilon)$ of channels $\hat{p}_{YS^aS^d|XS^e}$ is less than $l^{|\mathcal{Y}||\mathcal{X}||\mathcal{S}|}$.

We now define the following L functionals over \mathcal{P}_{SXY} (recall that $S = (S^e, S^a, S^d)$):

$$\begin{aligned} f_{i}(p_{SXY|U=u}) &= \tilde{p}_{S^{e}}(s^{e}) \, p_{X|S^{e}}(x|s^{e}), & 1 \leq i(x,s^{e}) \leq |\mathcal{X}| \, |\mathcal{S}^{e}| - 1, \\ f_{i}(p_{SXY|U=u}) &= D(\hat{p}_{YS^{a}S^{d}|XS^{e},U=u}\| \, \hat{p}_{Y|XS^{a}} \, p_{S^{a}S^{d}|S^{e}} \, |p_{X|S^{e},U=u} \, \tilde{p}_{S^{e}}) \\ &\quad |\mathcal{X}| \, |\mathcal{S}^{e}| \leq i(\hat{p}_{YS^{a}S^{d}|XS^{e}}, \hat{p}_{Y|XS^{a}}) \leq |\mathcal{X}| \, |\mathcal{S}^{e}| + |\tilde{\mathcal{A}}_{l}(\epsilon)| \, |\mathcal{A}_{l}(D_{2};\epsilon)| - 1 \\ f_{i}(p_{SXY|U=u}) &= \tilde{J}_{L}(\tilde{p}_{S^{e}}, p_{X|S^{e},U=u}, \hat{p}_{YS^{a}S^{d}|XS^{e},U=u}) \\ &\quad |\mathcal{X}| \, |\mathcal{S}^{e}| + |\tilde{\mathcal{A}}_{l}| \, |\mathcal{A}_{l}| \leq i(\hat{p}_{YS^{a}S^{d}|XS^{e}}) \\ &\leq |\mathcal{X}| \, |\mathcal{S}^{e}| + |\tilde{\mathcal{A}}_{l}(\epsilon)| \, (1 + |\mathcal{A}_{l}(D_{2};\epsilon)|) - 1. \end{aligned}$$

The first $|\mathcal{X}| |\mathcal{S}^e| - 1$ functions correspond to the marginals of p_{XS^e} except one, and the next

$$|\tilde{\mathcal{A}}_{l}(\epsilon)| \left(1 + |\mathcal{A}_{l}(D_{2};\epsilon)|\right) \leq l^{|\mathcal{Y}||\mathcal{X}|(|\mathcal{S}| + |\mathcal{S}^{a}|)}$$

functions are indexed by the channels $\hat{p}_{Y|XS^a} \in \mathcal{A}_l(D_2, \epsilon)$ and $\hat{p}_{YS^aS^d|XS^e} \in \tilde{\mathcal{A}}_l(\epsilon)$. Hence, applying Caratheodory's theorem, we conclude there exist L nonnegative numbers $\alpha_1, \cdots, \alpha_L$ summing to 1 and a random variable $U' \in \mathcal{U}_L$ such that

$$p_{SU'XY}(s, u', x, y) = p_{XS^{e}|U}(x, s^{e}|u_{u'}) \alpha_{u'} \hat{p}_{YS^{a}S^{d}|XUS^{e}}(y, s^{a}, s^{d}|x, u_{u'}, s^{e}) \\ \forall s, u', x, y, \\ E_{r,L}(\tilde{p}_{S^{e}}, p_{XU'|S^{e}}, \hat{p}_{YS^{a}S^{d}|XU'S^{e}}, \hat{p}_{Y|XS^{a}}) = E_{r,|\mathcal{U}|}(\tilde{p}_{S^{e}}, p_{XU|S^{e}}, \hat{p}_{YS^{a}S^{d}|XUS^{e}}, \hat{p}_{Y|XS^{a}}), \\ \forall \hat{p}_{Y|XS^{a}} \in \mathcal{A}_{l}(D_{2}; \epsilon), \ \hat{p}_{YS^{a}S^{d}|XUS^{e}} \in \tilde{\mathcal{A}}_{l,\mathcal{U}}(\epsilon).$$

Hence, given any \tilde{p}_{S^e} , it suffices to consider

$$|\mathcal{U}| = L = |\mathcal{X}| |\mathcal{S}^e| + l^{|\mathcal{Y}| |\mathcal{X}| (|\mathcal{S}| + |\mathcal{S}^a|)} - 1$$

to achieve

$$\max_{p_{XU|S^e}\in\mathcal{P}_{XU|S^e}(|\mathcal{U}|,D_1)}\min_{\hat{p}_{Y|XS^a}\in\mathcal{A}_l(D_2;\epsilon)}\min_{\hat{p}_{YS^aS^d|XUS^e}\in\tilde{\mathcal{A}}_l(\epsilon)}E_{r,|\mathcal{U}|}(\tilde{p}_{S^e},p_{XU'|S^e},\hat{p}_{YS^aS^d|XUS^e},\hat{p}_{Y|XS^a}).$$

Hence

$$E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, \tilde{\mathcal{A}}_{l,\mathcal{U}}(\epsilon), \mathcal{A}_{l}(D_2; \epsilon)) = E_{r,L}(\tilde{p}_{S^e}, \tilde{\mathcal{A}}_{l,\mathcal{U}_L}(\epsilon), \mathcal{A}_{l}(D_2; \epsilon)).$$
(H.29)

and

Step 4. Let $\tilde{p}^*_{S^e}$ achieve the minimum in

$$E_{r,L}(D_2'') = \min_{\tilde{p}_{S^e}} E_{r,L}(\tilde{p}_{S^e}, \tilde{\mathcal{A}}_{\mathcal{U}_L}, \mathcal{A}(D_2'')).$$

It follows from the previous steps that

$$E_{r,L}(D_2'') \stackrel{(a)}{=} E_{r,L}(\tilde{p}_{S^e}^*, \tilde{\mathcal{A}}_{\mathcal{U}_L}, \mathcal{A}(D_2''))$$

$$\stackrel{(b)}{\geq} E_{r,L}(\tilde{p}_{S^e}^*, \tilde{\mathcal{A}}_{l,\mathcal{U}_L}(\epsilon), \mathcal{A}_l(D_2; \epsilon)) - \Delta(l)$$

$$\stackrel{(c)}{=} E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}^*, \tilde{\mathcal{A}}_{l,\mathcal{U}}(\epsilon), \mathcal{A}_l(D_2; \epsilon)) - \Delta(l)$$

$$\stackrel{(d)}{\geq} E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}^*, \tilde{\mathcal{A}}_{\mathcal{U}}, \mathcal{A}(D_2)) - \Delta(l)$$

$$\stackrel{(d)}{\geq} \min_{\tilde{p}_{S^e}} E_{r,|\mathcal{U}|}(\tilde{p}_{S^e}, \tilde{\mathcal{A}}_{\mathcal{U}}, \mathcal{A}(D_2)) - \Delta(l)$$

$$= E_{r,|\mathcal{U}|}(D_2) - \Delta(l) \qquad (H.30)$$

where (a) follows from the definition of $\tilde{p}_{S^e}^*$, (b) follows from (H.28) with $\mathcal{U} = \mathcal{U}_L$, (c) follows from (H.29), and (d) holds because $\tilde{\mathcal{A}}_{l,\mathcal{U}}(\epsilon) \subset \tilde{\mathcal{A}}_{\mathcal{U}}$ and $\mathcal{A}_l(D_2;\epsilon) \subset \mathcal{A}(D_2)$.

Equation (H.30) holds for any \mathcal{U} , hence

$$E_{r,L}(D_2'') \geq \lim_{|\mathcal{U}| \to \infty} E_{r,|\mathcal{U}|}(D_2) - \Delta(l)$$

= $E_r(D_2) - \Delta(l),$

which concludes the proof.

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