Vector Gaussian Multiple Description with Individual and Central Receivers¹

Hua Wang and Pramod Viswanath²

Abstract

L multiple descriptions of a vector Gaussian source for individual and central receivers are investigated. The sum rate of the descriptions with covariance distortion measure constraints, in a positive semidefinite ordering, is exactly characterized. For two descriptions, the entire rate region is characterized. Jointly Gaussian descriptions are optimal in achieving the limiting rates. The key component of the solution is a novel information-theoretic inequality that is used to lower bound the achievable multiple description rates.

1 Introduction

In the multiple description problem, an information source is encoded into L packets and these packets are sent through parallel communication channels. There are several receivers, each of which can receive a subset of the packets and needs to reconstruct the information source based on the received packets. In the most general case, there are $2^{L} - 1$ receivers and the packets received in each receiver correspond to one of $2^{L} - 1$ subsets of $\{1, \ldots, L\}$. A long standing open problem in the literature [1–10] is to characterize the information-theoretic rate region subject to the specified distortion constraints. Practical multiple description codes have been discussed in [11–18] and recent work [19, 20] has considered the multiple descriptions of even the Gaussian source with quadratic distortion measures have not been fully characterized. In the special case of two descriptions of a scalar Gaussian source with quadratic distortion measures, however, the entire rate region has been characterized in [1].

Our focus is on L descriptions of a memoryless vector Gaussian source forwhere L individual and a single common receiver (cf. Figure 1). Each receiver needs to reconstruct the original source such that the empirical covariance matrix of the difference is less than, in the sense of a positive semidefinite ordering, a "distortion" matrix. In this setting, the symmetric rate multiple description problem of a scalar Gaussian source with symmetric distortion constraints has been characterized in [7,8,10], but a complete understanding of all other rate-distortion settings is open.

 $^{^1\}mathrm{This}$ research was sponsored in part by NSF CCR-0325924 and a Vodafone US Foundation Fellowship.

²The authors are with the Department of Electrical and Computer Engineering and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana IL 61801; e-mail: {huawang,pramodv}@uiuc.edu



Figure 1: MD problem with only individual reconstructions and central reconstruction

Our main result is an *exact* characterization of the sum rate for any specified L + 1 distortion matrix constraints. With L = 2, we characterize the entire rate region. Our contribution is two fold:

- First, we derive a novel information-theoretic inequality that provides a lower bound to the sum of the description rates. The key step is to avoid using the entropy power inequality, which was a central part of the proof of two descriptions of the scalar Gaussian source in [1]: the vector entropy power inequality is tight only with a certain covariance alignment condition, which arbitrary distortion matrix requirements do not necessarily allow.
- Second, we show that jointly Gaussian descriptions actually achieve the lower bound not by resorting to a direct calculation and comparison, which appears to be difficult for L > 2, but instead by arguing the equivalence of certain optimization problems.

Consider another two description problem of a pair of jointly Gaussian memoryless sources as depicted in Figure 2. There are two encoders that describe this source to three receivers: receiver i gets the description of encoder i, with i = 1, 2 and the third receiver receives both the descriptions. Suppose receiver i is interested in reconstructing the *i*th marginal of the jointly Gaussian source, with i = 1, 2. The third receiver is interested in reconstructing the entire vector source. This description problem is closely related to the vector Gaussian description problem that is the main focus of this paper. We exploit this connection and characterize the rate region where the reconstructions have a constraint on the covariance of error at each of the receivers (in the sense of a positive semidefinite order).

We have organized the results in this paper as follows. In Section 2 we give a formal description of the problem and summarize our main result. The derivation of a lower bound is in Section 3. In Section 4 we provide an upper bound and provide conditions for the achievable sum rate to meet the lower bound. We see in Section 5 that the conditions



Figure 2: Multiple Descriptions with separate distortion constraints.

are indeed satisfied in the special case of a scalar Gaussian source. The solution in the case of the more complicated vector Gaussian source is in Section 6. The solution to the multiple description problem depicted in Figure 2 is the topic of Section 7.1. Finally, while the characterization of the rate region of general multiple descriptions of the Gaussian source (with each receiver having access to some subset of the descriptions) is still open, we can use the insights derived via our sum rate characterization to solve this problem for a nontrivial set of covariance distortion constraints; this is done in Section 7.2.

A note about the notation in this paper: we use lower case letters for scalars, lower case and bold face for vectors, upper case and bold face for matrices. The superscript tdenotes matrix transpose. We use **I** and **0** to denote the identity matrix and the all zero matrix respectively, and diag $\{p_1, \ldots, p_n\}$ to denote a diagonal matrix with the diagonal entries equal to p_1, \ldots, p_n . The partial order $\succ (\succeq)$ denotes positive definite (semidefinite) ordering: $\mathbf{A} \succ \mathbf{B}$ ($\mathbf{A} \succeq \mathbf{B}$) means that $\mathbf{A} - \mathbf{B}$ is a positive definite (semidefinite) matrix. We write $\mathcal{N}(\mu, \mathbf{Q})$ to denote a Gaussian random vector with mean μ and covariance \mathbf{Q} . All logarithms in this paper are to the natural base.

2 Problem Setting and Main Results

2.1 Problem Setting

The information source $\{\mathbf{x}[m]\}$ is an i.i.d. random process with the marginal distribution $\mathcal{N}(0, \mathbf{K}_x)$, i.e., a collection of i.i.d. Gaussian random vectors. Denoting the dimension of $\{\mathbf{x}[m]\}$ by N, we suppose that \mathbf{K}_x is an $N \times N$ positive definite matrix. There are L encoding functions at the source, encoder l encodes a source sequence, of length n, $\mathbf{x}^n = (\mathbf{x}[1], \ldots, \mathbf{x}[n])^t$ to a source code $C_l^{(n)} = f_l^{(n)}(\mathbf{x}^n)$, for $l = 1 \ldots L$. This code $C_l^{(n)}$ is sent through lth communication channel at the rate $R_l = \frac{1}{n} \log |C_l^{(n)}|$. There are L individual receivers and one central receiver.

For $l = 1, \ldots L$, the *l*th individual receiver uses its information (the output of the *l*th channel) to generate an estimate $\hat{\mathbf{x}}_l^n = g_l^{(n)} \left(f_l^{(n)}(\mathbf{x}^n) \right)$ of the source sequence \mathbf{x}^n . The central receiver uses the output of all the *L* channels to generate an estimate $\hat{\mathbf{x}}_0^n$ of the source sequence \mathbf{x}^n . Since we are interested in covariance constraints, the decoder maps can be restricted to be the minimal mean square error (MMSE) estimate of the source sequence based on the received codewords. So,

$$\hat{\mathbf{x}}_{l}^{n} = \mathbb{E}\left[\mathbf{x}^{n}|f_{l}^{(n)}(\mathbf{x}^{n})\right], \quad l = 1, \dots, L$$

$$\hat{\mathbf{x}}_{0}^{n} = \mathbb{E}\left[\mathbf{x}^{n}|f_{1}^{(n)}(\mathbf{x}^{n}), \dots, f_{L}^{(n)}(\mathbf{x}^{n})\right].$$
(1)

Suppose the reconstructed sequences satisfy the covariance constraints

$$\frac{1}{n} \sum_{m=1}^{n} \mathbb{E} \Big[(\mathbf{x}[m] - \hat{\mathbf{x}}_{l}[m])^{t} (\mathbf{x}[m] - \hat{\mathbf{x}}_{l}[m]) \Big] \preccurlyeq \mathbf{D}_{l}, \quad l = 1, \dots, L,$$

$$\frac{1}{n} \sum_{m=1}^{n} \mathbb{E} \Big[(\mathbf{x}[m] - \hat{\mathbf{x}}_{0}[m])^{t} (\mathbf{x}[m] - \hat{\mathbf{x}}_{0}[m]) \Big] \preccurlyeq \mathbf{D}_{0},$$
(2)

then we say that multiple descriptions with distortion constraints $(\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$ are achievable at the rate tuple (R_1, \ldots, R_L) .

The closure of the set of all achievable rate tuples is called the rate region and is denoted by $\mathcal{R}_*(\mathbf{K}_x, \mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$. Throughout this paper, we suppose that $\mathbf{0} \prec \mathbf{D}_0 \prec \mathbf{D}_l \prec \mathbf{K}_x$, $\forall l = 1, \ldots, L$.³

2.2 Sum Rate

Our main result is the precise characterization of the sum rate of multiple descriptions for individual and central receivers.

Theorem 1. For distortion constraints $(\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$, the sum rate is

$$\sup_{\mathbf{K}_{z} \succ \mathbf{0}} \quad \frac{1}{2} \log \left(\frac{|\mathbf{K}_{x}| |\mathbf{K}_{x} + \mathbf{K}_{z}|^{(L-1)} |\mathbf{D}_{0} + \mathbf{K}_{z}|}{|\mathbf{D}_{0}| \prod_{l=1}^{L} |\mathbf{D}_{l} + \mathbf{K}_{z}|} \right).$$
(3)

³That $\mathbf{D}_0 \preccurlyeq \mathbf{D}_l$, is without loss of generality is seen by applying the data processing inequality for mmse estimation errors; having more access to information can only reduce the covariance of the error in a positive semidefinite sense. Similarly, $\mathbf{K}_x \preccurlyeq \mathbf{D}_0$ is also not interesting; here we simplify this condition and take $\mathbf{D}_0 \prec \mathbf{K}_x$.

This sum rate is achieved by a *jointly Gaussian random multiple description scheme*: let $\mathbf{w}_1, \dots, \mathbf{w}_L$ be zero mean jointly Gaussian random vectors independent of \mathbf{x} , with the positive definite covariance matricex $(\mathbf{w}_1, \dots, \mathbf{w}_L)$ denoted by \mathbf{K}_w . Defining

$$\mathbf{u}_l = \mathbf{x} + \mathbf{w}_l, \quad l = 1, \ \dots, \ L,$$

we consider \mathbf{K}_w such that

$$\operatorname{Cov}[\mathbf{x}|\mathbf{u}_{l}] \stackrel{\text{def}}{=} \mathbb{E}\Big[(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{u}_{l}])^{t} (\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{u}_{l}]) \Big] \preccurlyeq \mathbf{D}_{l}, \quad l = 1, \dots, L,$$

$$\operatorname{Cov}[\mathbf{x}|\mathbf{u}_{1}, \dots, \mathbf{u}_{L}] \stackrel{\text{def}}{=} \mathbb{E}\Big[(\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{u}_{1}, \dots, \mathbf{u}_{L}])^{t} (\mathbf{x} - \mathbb{E}[\mathbf{x}|\mathbf{u}_{1}, \dots, \mathbf{u}_{L}]) \Big] \preccurlyeq \mathbf{D}_{0}.$$
(4)

To construct the code book for the *l*th description, draw $e^{nR_l} \mathbf{u}_l^n$ vectors randomly according to the marginal of \mathbf{u}_l . The encoders observe the source sequence \mathbf{x}^n , look for codewords $(\mathbf{u}_1^n, \ldots, \mathbf{u}_L^n)$ that are jointly typical with \mathbf{x}^n and send the index of the resulting \mathbf{u}_l^n through the *l*th channel, respectively. The *l*th individual receiver uses this index and generates a reproduction sequence $\mathbb{E}[\mathbf{x}^n | \mathbf{u}_l^n]$ for $l = 1 \ldots L$, the central receiver uses all the *L* indices to generate a reproduction sequence $\mathbb{E}[\mathbf{x}^n | \mathbf{u}_l^n, \ldots, \mathbf{u}_L^n]$. For every \mathbf{K}_w satisfying (4), the rate tuple (R_1, \ldots, R_L) satisfying

$$\sum_{l \in S} R_l \ge \sum_{l \in S} h(\mathbf{u}_l) - h(\mathbf{u}_l, l \in S | \mathbf{x}) = \frac{1}{2} \log \frac{\prod_{l \in S} |\mathbf{K}_x + \mathbf{K}_{w_l}|}{|\mathbf{K}_{w_S}|}, \quad \forall S \subseteq \{1, \dots, L\}$$
(5)

is achievable by using this coding scheme, where \mathbf{K}_{w_S} is the covariance matrix for all $\mathbf{w}_l, l \in S$, and $\mathbf{K}_{w_l} = \mathbb{E}[\mathbf{w}_l^t \mathbf{w}_l]$. In particular, the achievable sum rate is

$$\frac{1}{2}\log\frac{\prod\limits_{l=1}^{L}|\mathbf{K}_{x}+\mathbf{K}_{w_{l}}|}{|\mathbf{K}_{w}|}.$$
(6)

We denote this ensemble of descriptions, throughout this paper, as the jointly Gaussian description scheme and the time sharing between them as the jointly Gaussian description strategy. We show that jointly Gaussian description schemes are optimal in achieving the sum rate (3).

2.3 Rate Region for Two Description Problem

For two descriptions, we can characterize the entire rate region.

Theorem 2. Given distortion constraints $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$, the rate region for the two



Figure 3: Rate region for two description problem

description problem for an i.i.d. $\mathcal{N}(0, \mathbf{K}_x)$ vector Gaussian source is

$$\mathcal{R}_{*}(\mathbf{K}_{x}, \mathbf{D}_{1}, \mathbf{D}_{2}, \mathbf{D}_{0}) = \begin{cases} (R_{1}, R_{2}) : \\ R_{l} \geq \frac{1}{2} \log \frac{|\mathbf{K}_{x}|}{|\mathbf{D}_{l}|}, \quad l = 1, 2 \\ R_{1} + R_{2} \geq \sup_{\mathbf{K}_{z} \succ \mathbf{0}} \frac{1}{2} \log \frac{|\mathbf{K}_{x}||\mathbf{K}_{x} + \mathbf{K}_{z}||\mathbf{D}_{0} + \mathbf{K}_{z}|}{|\mathbf{D}_{0}||\mathbf{D}_{1} + \mathbf{K}_{z}||\mathbf{D}_{2} + \mathbf{K}_{z}|} \end{cases}$$
(7)

We show that if the distortion constraints $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$ satisfy $\mathbf{D}_0 + \mathbf{K}_x - \mathbf{D}_1 - \mathbf{D}_2 \succ \mathbf{0}$ and $\mathbf{D}_0^{-1} + \mathbf{K}_x^{-1} - \mathbf{D}_1^{-1} - \mathbf{D}_2^{-1} \succ \mathbf{0}$, we can get the optimizing \mathbf{K}_z by solving a matrix Riccati equation. An illustration of the rate region is shown in Figure 3. In this case, if we let $\mathbf{K}_{w_l} = [\mathbf{D}_l^{-1} - \mathbf{K}_x^{-1}]^{-1}$ for l = 0, 1, 2, then the optimizing \mathbf{K}_z is

$$\mathbf{K}_z = \mathbf{K}_x (\mathbf{K}_x - \mathbf{A}^*)^{-1} \mathbf{K}_x - \mathbf{K}_x,$$

where

$$\mathbf{A}^* = (\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{\frac{1}{2}} \left[(\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{-\frac{1}{2}} (\mathbf{K}_{w_2} - \mathbf{K}_{w_0}) (\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{-\frac{1}{2}} \right]^{\frac{1}{2}} (\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{\frac{1}{2}} - \mathbf{K}_{w_0}.$$

Letting R_{sum} denote the optimal sum rate, the two corner points in Figure 3 are

$$B_1 = \left(\frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_1|}, R_{sum} - \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_1|}\right), \text{ and}$$
$$B_2 = \left(R_{sum} - \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_2|}, \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_2|}\right).$$

3 Lower Bound

By fairly procedural steps, we have the following lower bound to the sum rate of the multiple descriptions:

$$n\sum_{l=1}^{L} R_{l} \geq \sum_{l=1}^{L} H(C_{l}) = \sum_{l=1}^{L} H(C_{l}) - H(C_{1}, \dots, C_{L} | \mathbf{x}^{n})$$

$$= \sum_{l=1}^{L} H(C_{l}) - H(C_{1}, \dots, C_{L}) + H(C_{1}, \dots, C_{L}) - H(C_{1}, \dots, C_{L} | \mathbf{x}^{n})$$

$$= I(C_{1}; C_{2}; \dots; C_{L}) + I(C_{1}, \dots, C_{L}; \mathbf{x}^{n}),$$

(8)

where we have defined

$$I(C_1; C_2; \dots; C_L) \stackrel{\text{def}}{=} \sum_{l=1}^L H(C_l) - H(C_1, \dots, C_L) = \sum_{l=2}^L I(C_l; C_1 \dots C_{l-1}),$$

and called it the symmetric mutual information between C_1, \ldots, C_L . Note that $I(C_1; C_2; \ldots; C_L) \geq 0$ and is also well defined even when C_1, \ldots, C_L are continuous random variables. Our main result is the following information theoretic inequality which gives a lower bound to the sum of symmetric mutual information between (C_1, C_2, \ldots, C_L) and mutual information between C_1, C_2, \ldots, C_L and \mathbf{x}^n for given covariance constraints.

Lemma 1. Let $\mathbf{x}^n = (\mathbf{x}[1], \ldots, \mathbf{x}[n])$, where $\mathbf{x}[m]$'s are *i.i.d.* $\mathcal{N}(\mathbf{0}, \mathbf{K}_x)$ Gaussian random vectors for $m = 1, \ldots, n$. Let C_1, \ldots, C_L be random variables jointly distributed with \mathbf{x}^n . Let $\hat{\mathbf{x}}_0^n = \mathbb{E}[\mathbf{x}^n | C_1, \ldots, C_L]$ and $\hat{\mathbf{x}}_l^n = \mathbb{E}[\mathbf{x}^n | C_l]$ for $l = 1, \ldots, L$. Given positive definite matrices $\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0$, if

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}[(\mathbf{x}[m] - \hat{\mathbf{x}}_{l}[m])^{t}(\mathbf{x}[m] - \hat{\mathbf{x}}_{l}[m])] \preccurlyeq \mathbf{D}_{l}, \quad l = 1, \dots, L,$$

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}[(\mathbf{x}[m] - \hat{\mathbf{x}}_{0}[m])^{t}(\mathbf{x}[m] - \hat{\mathbf{x}}_{0}[m])] \preccurlyeq \mathbf{D}_{0},$$
(9)

then

$$I(C_1; C_2; \dots; C_L) + I(C_1, \dots, C_L; \mathbf{x}^n) \ge \sup_{\mathbf{K}_z \succ \mathbf{0}} \frac{n}{2} \log \frac{|\mathbf{K}_x| |\mathbf{K}_x + \mathbf{K}_z|^{(L-1)} |\mathbf{D}_0 + \mathbf{K}_z|}{|\mathbf{D}_0| \prod_{l=1}^L |\mathbf{D}_l + \mathbf{K}_z|}.$$
 (10)

Furthermore, there exists a jointly Gaussian distribution of $(C_1, \ldots, C_L, \mathbf{x}^n)$ such that the inequality in (10) is tight.

This is a fundamental information-theoretic inequality which involves only the joint distribution⁴ between C_1, C_2, \ldots, C_L and \mathbf{x}^n and bounds on mean square error estimation of \mathbf{x}^n from C_1, C_2, \ldots, C_L ; we delegate the proof of this result to Appendix B. We can now use Lemma 1 to derive a lower bound to the sum rate

$$\sum_{l=1}^{L} R_l \ge \sup_{\mathbf{K}_z \succ \mathbf{0}} \frac{1}{2} \log \frac{|\mathbf{K}_x| |\mathbf{K}_x + \mathbf{K}_z|^{(L-1)} |\mathbf{D}_0 + \mathbf{K}_z|}{|\mathbf{D}_0| \prod_{l=1}^{L} |\mathbf{D}_l + \mathbf{K}_z|}.$$
(11)

By letting L = 1 in the lemma above, we can derive a simple lower bound to the rate of the individual descriptions as well:

$$R_{l} \geq \frac{1}{n}H(C_{l}) = \frac{1}{n}\left(H(C_{l}) - H(C_{l}|\mathbf{x}^{n})\right)$$

$$= \frac{1}{n}I(\mathbf{x}^{n};C_{l})$$

$$\geq \frac{1}{2}\log\frac{|\mathbf{K}_{x}|}{|\mathbf{D}_{l}|}, \quad l = 1, \dots, L.$$
(12)

This bound is actually the point-to-point rate-distortion function for individual receivers, since each individual receiver only faces a point-to-point compression problem.

Note that for any positive definite \mathbf{K}_{z} ,

$$\frac{1}{2}\log\frac{|\mathbf{K}_x||\mathbf{K}_x + \mathbf{K}_z|^{(L-1)}|\mathbf{D}_0 + \mathbf{K}_z|}{|\mathbf{D}_0|\prod_{l=1}^L |\mathbf{D}_l + \mathbf{K}_z|}$$

is a lower bound to the sum rate of the multiple descriptions. Two special choices of \mathbf{K}_z are of particular interest:

• Letting $\mathbf{K}_z = \epsilon \mathbf{I}$ and $0 \epsilon \to 0^+$, we have the following lower bound:

$$\sum_{l=1}^{L} R_l \ge \frac{1}{2} \log \frac{|\mathbf{K}_x|^L}{|\mathbf{D}_1| \dots |\mathbf{D}_L|}.$$
(13)

This bound is actually the summation of the bounds on the individual rates.

• Letting some eigenvalues of \mathbf{K}_z goes to infinity, we have the following lower bound:

$$\sum_{l=1}^{L} R_l \ge \frac{1}{2} \log \frac{|\mathbf{K}_x|}{|\mathbf{D}_0|}.$$
(14)

⁴This inequality holds even when C_1, C_2, \ldots, C_L are not simply functions of \mathbf{x}^n and can also be continuous random variables.

This bound is the point-to-point rate-distortion function when we only have the central distortion constraint.

We will see later that for some distortion constraints $(\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$, (13) and (14) can be tight.

4 Upper Bound

In the previous section we gave a lower bound to the sum rate. Now we give a upper bound to the sum rate by using the jointly Gaussian description scheme described in Section 2.2.

4.1 Jointly Gaussian Multiple Description Scheme

First we give a sketch of the achievable rate region by using jointly Gaussian description scheme. Given the source sequence \mathbf{x}^n , as long as we can find a combination of codewords $(\mathbf{u}_1^n, \ldots, \mathbf{u}_L^n)$ that are jointly typical with \mathbf{x}^n , all the receivers can generate reproduction sequences that satisfy their given distortion constraints. An intuitive way to understand (5) is the following: since $(\mathbf{u}_1^n, \ldots, \mathbf{u}_L^n)$ are jointly typical with \mathbf{x}^n , then for any $S \subseteq$ $\{1, \ldots, L\}$, we have that $\mathbf{u}_l^n, l \in S$ are jointly typical with \mathbf{x}^n . Now the probability that a randomly generated combination of codewords $\mathbf{u}_l^n, l \in S$ are jointly typical with \mathbf{x}^n is roughly

$$\frac{e^{nh(\mathbf{u}_l, l \in S | \mathbf{x})}}{\prod_{l \in S} e^{nh(\mathbf{u}_l)}},$$

and the number of possible combination of codewords $\mathbf{u}_l^n, l \in S$ are $\prod_{l \in S} e^{nR_l}$. Thus, as

$$\sum_{l \in S} R_l \ge \sum_{l \in S} h(\mathbf{u}_l) - h(\mathbf{u}_l, l \in S | \mathbf{x}), \tag{15}$$

we can find a combination of codewords $\mathbf{u}_l^n, l \in S$ that are jointly typical with \mathbf{x}^n . Rigorously speaking, we need to show that as long as (15) is satisfied, then for any given source sequence \mathbf{x}^n we can find a combination of codewords $(\mathbf{u}_1^n, \ldots, \mathbf{u}_L^n)$ such that $\mathbf{u}_l^n, l \in S$ are jointly typical with \mathbf{x}^n for all $S \subseteq \{1, \ldots, L\}$. The second moment method [21] is commonly used to address this aspect, and a proof can be found in [7].

Evaluating (15) based on the jointly Gaussian distribution of \mathbf{x} and $\mathbf{u}_1, \ldots, \mathbf{u}_L$, we get that all the rate tuples (R_1, \ldots, R_L) satisfying

$$\sum_{l \in S} R_l \ge \sum_{l \in S} h(\mathbf{u}_l) - h(\mathbf{u}_l, l \in S | \mathbf{x}) = \frac{1}{2} \log \frac{\prod_{l \in S} |\mathbf{K}_x + \mathbf{K}_{w_l}|}{|\mathbf{K}_{w_S}|}, \quad \forall S \subseteq \{1, \dots, L\}$$
(16)

are achievable by the jointly Gaussian description scheme. In particular, we have that the achievable sum rate is

$$\sum_{l=1}^{L} h(\mathbf{u}_{l}) - h(\mathbf{u}_{1}, \dots, \mathbf{u}_{L} | \mathbf{x}) = \frac{1}{2} \log \frac{\prod_{l=1}^{L} |\mathbf{K}_{x} + \mathbf{K}_{w_{l}}|}{|\mathbf{K}_{w}|}.$$
 (17)

The resulting distortions $(\mathbf{D}_1^*, \ldots, \mathbf{D}_L^*, \mathbf{D}_0^*)$ by using jointly Gaussian description scheme can be calculated as

$$\mathbf{D}_{l}^{*} = \operatorname{Cov}[\mathbf{x}|\mathbf{u}_{l}] = [\mathbf{K}_{x}^{-1} + \mathbf{K}_{w_{l}}^{-1}]^{-1}, \quad l = 1, \dots, L, \mathbf{D}_{0}^{*} = \operatorname{Cov}[\mathbf{x}|\mathbf{u}_{1}, \dots, \mathbf{u}_{L}] = [\mathbf{K}_{x}^{-1} + (\mathbf{I}, \dots, \mathbf{I})\mathbf{K}_{w}^{-1}(\mathbf{I}, \dots, \mathbf{I})^{t}]^{-1}.$$
(18)

4.2 Combinatorial Property of the Achievable Region

The achievable region given in (15) has useful combinatorial properties; in particular it belongs to the class of *contra-polymatroids* [22]. Certain rate regions of the multiple access channel [23] and distributed source coding problems [24] are also known to have this specific combinatorial property. To see this, let

$$\phi(S) \stackrel{\text{def}}{=} \sum_{l \in S} h(\mathbf{u}_l) - h(\mathbf{u}_l, l \in S | \mathbf{x}), \quad S \subseteq \{1, \ldots, L\}.$$

We can readily verify that

$$\phi(S \cup \{t\}) \ge \phi(S), \quad \forall t \in \{1, \dots, L\},$$

$$\phi(S \cup T) + \phi(S \cap T) \ge \phi(S) + \phi(T).$$
(19)

By definition, we conclude that the achievable rate region of a jointly Gaussian multiple description scheme is a contra-polymatroid. The key advantage of this combinatorial property is that we can exactly characterize the vertices of the achievable rate region (15). Letting π to be a permutation on $\{1, \ldots, L\}$, define

$$b_i^{(\pi)} \stackrel{\text{def}}{=} \phi(\{\pi_1, \pi_2, \dots, \pi_i\}) - \phi(\{\pi_1, \pi_2, \dots, \pi_{i-1}\}), \quad i = 1, \dots, L,$$

and $\mathbf{b}^{(\pi)} = \left(b_1^{(\pi)}, \ldots, b_L^{(\pi)}\right)$. Then the *L*! points $\{\mathbf{b}^{(\pi)}, \pi \text{ a permutation}\}\$ are the vertices of the contra-polymetroid (15).

4.3 Comparison of Upper Bound and the Lower Bound

Our goal is to show that the jointly Gaussian description scheme achieves the lower bound to the sum rate. In general it does not seem facile to do a direct calculation and comparison. We forgo this strategy and, instead, provide an alternative characterization of the achievable sum rate which is much easier to compare with the lower bound.

Similar to the derivation of the lower bound (in Appendix B), we consider an $\mathcal{N}(0, \mathbf{K}_z)$ Gaussian random vector \mathbf{z} , independent of \mathbf{x} and all \mathbf{w}_l 's. Defining $\mathbf{y} = \mathbf{x} + \mathbf{z}$, we have the following achievable sum rate:

$$\sum_{l=1}^{L} R_{l} = \sum_{l=1}^{L} h(\mathbf{u}_{l}) - h(\mathbf{u}_{1}, ..., \mathbf{u}_{L} | \mathbf{x})$$

$$= \sum_{l=1}^{L} h(\mathbf{u}_{l}) - h(\mathbf{u}_{1}, ..., \mathbf{u}_{L}) + h(\mathbf{u}_{1}, ..., \mathbf{u}_{L}) - h(\mathbf{u}_{1}, ..., \mathbf{u}_{L} | \mathbf{x})$$

$$= \sum_{l=1}^{L} h(\mathbf{u}_{l}) - h(\mathbf{u}_{1}, ..., \mathbf{u}_{L}) + I(\mathbf{u}_{1}, ..., \mathbf{u}_{L}; \mathbf{x})$$

$$\stackrel{(a)}{=} \sum_{l=1}^{L} h(\mathbf{u}_{l}) - h(\mathbf{u}_{1}, ..., \mathbf{u}_{L}) + I(\mathbf{u}_{1}, ..., \mathbf{u}_{L}; \mathbf{x}) - \left(\sum_{l=1}^{L} h(\mathbf{u}_{l} | \mathbf{y}) - h(\mathbf{u}_{1}, ..., \mathbf{u}_{L} | \mathbf{y})\right)$$

$$= \sum_{l=1}^{L} (h(\mathbf{y}) - h(\mathbf{y} | \mathbf{u}_{l})) - h(\mathbf{y}) + h(\mathbf{y} | \mathbf{u}_{1}, ..., \mathbf{u}_{L}) + h(\mathbf{x}) - h(\mathbf{x} | \mathbf{u}_{1}, ..., \mathbf{u}_{L})$$

$$= h(\mathbf{x}) + (L - 1)h(\mathbf{y}) - \sum_{l=1}^{L} h(\mathbf{y} | \mathbf{u}_{l}) + h(\mathbf{y} | \mathbf{u}_{1}, ..., \mathbf{u}_{L}) - h(\mathbf{x} | \mathbf{u}_{1}, ..., \mathbf{u}_{L})$$

$$= \frac{1}{2} \log \frac{\left|\mathbf{K}_{x}\right| \left|\mathbf{K}_{x} + \mathbf{K}_{z}\right|^{(L-1)} \left|\operatorname{Cov}[\mathbf{x} | \mathbf{u}_{1}, ..., \mathbf{u}_{L}] + \mathbf{K}_{z}\right|}{\left|\operatorname{Cov}[\mathbf{x} | \mathbf{u}_{1}, ..., \mathbf{u}_{L}]\right| \prod_{l=1}^{L} \left|\operatorname{Cov}[\mathbf{x} | \mathbf{u}_{l}] + \mathbf{K}_{z}\right|},$$
(20)

where the last step is from a procedural Gaussian MMSE calculation.

Note that if we have

$$\sum_{l=1}^{L} h(\mathbf{u}_l | \mathbf{y}) - h(\mathbf{u}_1, \dots, \mathbf{u}_L | \mathbf{y}) = 0, \qquad (21)$$

then (a) in (20) is actually an equality. Thus, if our choice of \mathbf{K}_w and \mathbf{K}_z satisfy the following two conditions:

- (21) is true.
- distortion constraints are met with equality, i.e.,

$$Cov[\mathbf{x}|\mathbf{u}_l] = \mathbf{D}_l, \quad l = 1, \dots, L,$$

$$Cov[\mathbf{x}|\mathbf{u}_1, \dots, \mathbf{u}_L] = \mathbf{D}_0,$$
(22)

then the upper bound matches the lower bound and we have characterized the sum rate. In the following we examine under what circumstances the above two conditions are true.

First, we give a necessary and sufficient condition for (21) to be true, delegating the proof to Appendix C.

Proposition 1. There exists some choice of positive definite \mathbf{K}_z such that (21) is true if and only if \mathbf{K}_w , the covariance matrix of $(\mathbf{w}_1, \dots, \mathbf{w}_L)$, takes the following form

$$\mathbf{K}_{w} = \begin{pmatrix} \mathbf{K}_{w_{1}} & -\mathbf{A} & -\mathbf{A} & \dots & -\mathbf{A} \\ -\mathbf{A} & \mathbf{K}_{w_{2}} & -\mathbf{A} & \dots & -\mathbf{A} \\ \dots & \dots & \dots & \dots \\ -\mathbf{A} & \dots & -\mathbf{A} & \mathbf{K}_{w_{L-1}} & -\mathbf{A} \\ -\mathbf{A} & \dots & -\mathbf{A} & -\mathbf{A} & \mathbf{K}_{w_{L}} \end{pmatrix},$$
(23)

where $\mathbf{0} \prec \mathbf{A} \prec \mathbf{K}_x$.

Next, we look at the conditions for (22) to be true. From (18), we have

$$\mathbf{D}_{l}^{-1} = \operatorname{Cov}[\mathbf{x}|\mathbf{u}_{l}]^{-1} = \mathbf{K}_{x}^{-1} + \mathbf{K}_{w_{l}}^{-1}, \quad l = 1, \dots, L$$

$$\mathbf{D}_{0}^{-1} = \operatorname{Cov}[\mathbf{x}|\mathbf{u}_{1}, \dots, \mathbf{u}_{L}]^{-1} = \mathbf{K}_{x}^{-1} + (\mathbf{I}, \dots, \mathbf{I})\mathbf{K}_{w}^{-1}(\mathbf{I}, \dots, \mathbf{I})^{t}.$$
(24)

 $(\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I})\mathbf{K}_{w}^{-1}(\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I})^{t}$, is calculated in the following lemma; the proof is available in Appendix D.

Lemma 2. Let

$$\mathbf{K}_w = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A} & -\mathbf{A} & \dots & -\mathbf{A} \ -\mathbf{A} & \mathbf{K}_{w_2} & -\mathbf{A} & \dots & -\mathbf{A} \ \dots & \dots & \dots & \dots & \dots \ -\mathbf{A} & \dots & -\mathbf{A} & \mathbf{K}_{w_{L-1}} & -\mathbf{A} \ -\mathbf{A} & \dots & -\mathbf{A} & \mathbf{K}_{w_{L-1}} \end{pmatrix}.$$

If $\mathbf{K}_w \succ \mathbf{0}$ and $\mathbf{A} \succeq \mathbf{0}$, then

$$(\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I})\mathbf{K}_{w}^{-1}(\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I})^{t} = \left[\left(\sum_{l=1}^{L} (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1}\right)^{-1} - \mathbf{A}\right]^{-1}.$$

Using this lemma, from (24) we arrive at

$$\left[(\mathbf{D}_0^{-1} - \mathbf{K}_x^{-1})^{-1} + \mathbf{A} \right]^{-1} = \sum_{l=1}^{L} \left[(\mathbf{D}_l^{-1} - \mathbf{K}_x^{-1})^{-1} + \mathbf{A} \right]^{-1}.$$
 (25)

Defining

$$\mathbf{K}_{w_0} = (\mathbf{D}_0^{-1} - \mathbf{K}_x^{-1})^{-1}, \tag{26}$$

(24) is equivalent to

$$\left[\mathbf{K}_{w_0} + \mathbf{A}\right]^{-1} = \sum_{l=1}^{L} \left[\mathbf{K}_{w_l} + \mathbf{A}\right]^{-1}.$$
 (27)

Thus, if there exists a positive definite solution \mathbf{A} to (27), and the corresponding \mathbf{K}_w is positive definite, then the distortion constraints are met with equality, i.e., (22) holds. It turns out that as long as \mathbf{A} is a solution to (27), the resulting \mathbf{K}_w is always positive definite; we state this formally below, delegating the proof to Appendix E.

Lemma 3. If for some $\mathbf{K}_{w_0} \succ \mathbf{0}$ and $\mathbf{A} \succ \mathbf{0}$ (27) is true, then the covariance matrix \mathbf{K}_w defined in (23) is positive definite.

We summarize the state of affairs in the following theorem.

Theorem 3. Given distortion constraints $(\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$, let

$$\mathbf{K}_{w_l} = (\mathbf{D}_l^{-1} - \mathbf{K}_x^{-1})^{-1}, \quad l = 0, \ 1, \ \dots, \ L.$$
(28)

If there exists an solution \mathbf{A}^* to (27) and $\mathbf{0} \prec \mathbf{A}^* \prec \mathbf{K}_x$, then the jointly Gaussian description scheme with \mathbf{K}_w defined in (23) with $\mathbf{A} = \mathbf{A}^*$ achieves the optimal sum rate, and the optimal \mathbf{K}_z for lower bound (11) is $\mathbf{K}_z = \mathbf{K}_x(\mathbf{K}_x - \mathbf{A}^*)^{-1}\mathbf{K}_x - \mathbf{K}_x$.

Thus we show that if the given distortion constraints $(\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$ satisfy the condition for Theorem 3, then the jointly Gaussian description scheme achieves the optimal sum rate and we can calculate the optimal \mathbf{K}_w by solving a matrix equation. However, for arbitrarily given distortion constraints, (27) may not have a solution \mathbf{A}^* such that $\mathbf{0} \prec \mathbf{A}^* \prec \mathbf{K}_x$. In this case, we can show that there exists a jointly Gaussian description scheme that achieves the sum rate lower bound, and resulting in distortions $(\mathbf{D}_1^*, \ldots, \mathbf{D}_L^*, \mathbf{D}_0^*)$ such that $\mathbf{D}_l^* \preccurlyeq \mathbf{D}_l$ for $l = 0, 1, \ldots, L$. In the following we first study the relatively simpler case of scalar Gaussian source, and then move to discuss the vector Gaussian source.

5 Scalar Gaussian Source

Here we suppose that the information source is an i.i.d. sequence of $\mathcal{N}(0, \sigma_x^2)$ scalar Gaussian random variables. Let individual distortion constraints be (d_1, \ldots, d_L) and the central distortion constraints be d_0 , where $0 < d_0 < d_l < \sigma_x^2$ for $l = 1, \ldots, L$. We consider the jointly Gaussian description scheme with the following covariance matrix for w_1, \ldots, w_l .

$$\mathbf{K}_{w} = \begin{pmatrix} \sigma_{1}^{2} & -a & -a & \dots & -a \\ -a & \sigma_{2}^{2} & -a & \dots & -a \\ \dots & \dots & \dots & \dots & \dots \\ -a & \dots & -a & \sigma_{L-1}^{2} & -a \\ -a & \dots & -a & -a & \sigma_{L}^{2} \end{pmatrix}.$$
 (29)

Consider the condition for Theorem 3 to hold: to meet the individual distortion constraint with equality, we need

$$\sigma_l^2 = (d_l^{-1} - \sigma_x^{-2})^{-1} = \frac{d_l \sigma_x^2}{\sigma_x^2 - d_l}, \quad l = 1, \dots, L.$$
(30)

Let

$$\sigma_0^2 \stackrel{\text{def}}{=} (d_0^{-1} - \sigma_x^{-2})^{-1} = \frac{d_0 \sigma_x^2}{\sigma_x^2 - d_0},\tag{31}$$

we need

$$\left[\sigma_0^2 + a\right]^{-1} = \sum_{l=1}^{L} \left[\sigma_l^2 + a\right]^{-1}$$
(32)

to have a solution $a^* \in (0, \sigma_x^2)$, to meet the central distortion constraint with equality. Towards this, define

$$f(a) \stackrel{\text{def}}{=} \frac{1}{\sigma_0^2 + a} - \sum_{l=1}^L \frac{1}{\sigma_l^2 + a},\tag{33}$$

and we have

$$f(0) = \frac{1}{\sigma_0^2} - \sum_{l=1}^{L} \frac{1}{\sigma_l^2} = \frac{1}{d_0} + \frac{L-1}{\sigma_x^2} - \sum_{l=1}^{L} \frac{1}{d_l},$$

$$f(\sigma_x^2) = \frac{1}{\sigma_0^2 + \sigma_x^2} - \sum_{l=1}^{L} \frac{1}{\sigma_l^2 + \sigma_x^2} = \frac{1}{\sigma_x^4} \left(\sum_{l=1}^{L} d_l - d_0 - (L-1)\sigma_x^2 \right).$$
(34)

Using induction, we can show that

$$\left(\sum_{l=1}^{L} \frac{1}{d_l} - \frac{L-1}{\sigma_x^2}\right)^{-1} \ge \sum_{l=1}^{L} d_l - (L-1)\sigma_x^2.$$
(35)

Thus we have

$$f(0) \le 0 \Rightarrow f(\sigma_x^2) \le 0,$$

$$f(\sigma_x^2) \ge 0 \Rightarrow f(0) \ge 0.$$

Then given distortions (d_1, \ldots, d_L, d_0) , f(0) and $f(\sigma_x^2)$ falls into the following three cases.

Case 1: f(0) > 0 and $f(\sigma_x^2) < 0$.

In this case, since f(a) is a continuous function, there exists an $a^* \in (0, \sigma_x^2)$ such that $f(a^*) = 0$. In this case the condition for Theorem 3 holds and from Theorem 3 we know that jointly Gaussian description scheme with covariance matrix for w_1, \ldots, w_l being (29) with $a = a^*$ achieves the optimal sum rate.

Case 2: $f(0) \le 0$. Alternatively, $\frac{1}{d_0} + \frac{L-1}{\sigma_x^2} - \sum_{l=1}^{L} \frac{1}{d_l} \le 0$.

In this case, the condition for Theorem 3 does not hold. But the jointly Gaussian description scheme can still achieve the sum rate. To see this, choosing a = 0 in \mathbf{K}_w we can meet individual distortions with equality and get a central distortion d'_0 . From (24) we have

$$\frac{1}{d'_0} = \frac{1}{\sigma_x^2} + (1 \ 1 \ \dots \ 1) K_w^{-1} (1 \ 1 \ \dots \ 1)^t
= \frac{1}{\sigma_x^2} + \sum_{l=1}^L \frac{1}{\sigma_l^2} = \sum_{l=1}^L \frac{1}{d_l} - \frac{L-1}{\sigma_x^2}
\ge \frac{1}{d_0}.$$
(36)

Hence we have achieved distortion (d_1, \ldots, d_L, d'_0) where $d'_0 \leq d_0$, and from (17) the achievable sum rate is

$$\sum_{l=1}^{L} R_l \ge \frac{1}{2} \log \frac{\sigma_x^{2L}}{d_1 d_2 \cdots d_L},\tag{37}$$

which equals the sum of our bounds on individual rates.

Case 3:
$$f(\sigma_x^2) \ge 0$$
, Alternatively, $\sum_{l=1}^{L} d_l - d_0 - (L-1)\sigma_x^2 \ge 0$.

In this case, the conditions for Theorem 3 do not hold as well. But the jointly Gaussian description strategy still achieves the sum rate. To see this, note that we can find a d'_L such that $0 < d'_L \le d_L$ and

$$\sum_{l=1}^{L-1} d_l + d'_L - d_0 - (L-1)\sigma_x^2 = 0,$$
(38)

and we choose $a = \sigma_x^2$, $\sigma_l^2 = (d_l^{-1} - \sigma_x^{-2})^{-1}$ for $l = 1, \dots, L-1$, and $\sigma_L^2 = (d_L'^{-1} - \sigma_x^{-2})^{-1}$ in K_w . Defining $\sigma_0^2 = (d_0^{-1} - \sigma_x^{-2})^{-1}$, (38) is equivalent to the following equation:

$$\left[\sigma_0^2 + \sigma_x^2\right]^{-1} = \sum_{l=1}^{L} \left[\sigma_l^2 + \sigma_x^2\right]^{-1}.$$
(39)

From Lemma 3, our choice of K_w is positive definite. Thus the resulting distortions are $(d_1, \ldots, d_{L-1}, d'_L, d_0)$, where $0 < d'_L \leq d_L$.

Using the determinant equation

$$\begin{vmatrix} \sigma_1^2 & -\sigma_x^2 & -\sigma_x^2 & -\sigma_x^2 & \dots & -\sigma_x^2 \\ -\sigma_x^2 & \sigma_2^2 & -\sigma_x^2 & -\sigma_x^2 & \dots & -\sigma_x^2 \\ -\sigma_x^2 & -\sigma_x^2 & \sigma_x^2 & -\sigma_x^2 & \dots & -\sigma_x^2 \\ \dots & \dots & \dots & \dots & \dots \\ -\sigma_x^2 & \dots & -\sigma_x^2 & -\sigma_x^2 & \sigma_{L-1}^2 & -\sigma_x^2 \\ -\sigma_x^2 & \dots & -\sigma_x^2 & -\sigma_x^2 & -\sigma_x^2 & \sigma_L^2 \end{vmatrix} = \left(1 - \sum_{l=1}^L \frac{\sigma_x^2}{\sigma_l^2 + \sigma_x^2}\right) \prod_{l=1}^L (\sigma_l^2 + \sigma_x^2) \quad (40)$$

and (39), we have an achievable sum rate

$$\sum_{l=1}^{L} R_l = \frac{1}{2} \log \frac{\sigma_x^2}{d_0}.$$
(41)

We conclude that in this case the point-to-point rate-distortion bound for the central receiver is achievable.

In summary, we have shown that the jointly Gaussian description scheme achieves the lower bound on the sum rate. Further, the sum rate can be calculated either trivially (by choosing $a^* = 0$ in case II or $a^* = 1$ in case III) or by solving a polynomial equation in a single variable (case I).

6 Vector Gaussian Source

The essence of our proof of the optimality of jointly Gaussian description scheme for scalar Gaussian sources is the use of the *intermediate value theorem* for scalar continuous functions. However, there is no natural extension of this theorem for vector valued functions. To avoid this problem, we first explicitly solve the two description problem and characterize the optimality of jointly Gaussian description scheme. Next, we show that the jointly Gaussian description scheme is optimal for $L \ge 2$ by showing an equivalence of certain optimization problems. In the last part of this section, we show that the jointly Gaussian description strategy can achieve the optimal rate region for the two description problem.

6.1 Explicit Solutions for Some Cases of Two Description Problem

With only two descriptions, we can explicitly solve (27), thus generalizing the corresponding solution for the scalar Gaussian source, derived in [1].

Suppose the distortion constraints are denoted by $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$ and let

$$\mathbf{K}_w = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A}^* \ -\mathbf{A}^* & \mathbf{K}_{w_2} \end{pmatrix}.$$

We now solve (24), which is equivalent to (27), for \mathbf{K}_{w_1} , \mathbf{K}_{w_2} and \mathbf{A}^* . From (24) we get

$$\mathbf{K}_{w_l} = (\mathbf{D}_l^{-1} - \mathbf{K}_x^{-1})^{-1}, \quad l = 1, \ 2,$$
(42)

and

$$\mathbf{D}_0^{-1} = \mathbf{K}_x^{-1} + (\mathbf{I} \ \mathbf{I}) \mathbf{K}_w^{-1} (\mathbf{I} \ \mathbf{I})^t.$$
(43)

Expanding out \mathbf{K}_w^{-1} using Lemma 6 in Appendix A, we get

$$\mathbf{D}_{0}^{-1} - \mathbf{K}_{x}^{-1} = \mathbf{K}_{w_{1}}^{-1} + (\mathbf{I} + \mathbf{K}_{w_{1}}^{-1}\mathbf{A}^{*})(\mathbf{K}_{w_{2}} - \mathbf{A}^{*}\mathbf{K}_{w_{1}}^{-1}\mathbf{A}^{*})^{-1}(\mathbf{I} + \mathbf{A}^{*}\mathbf{K}_{w_{1}}^{-1}).$$
(44)

Taking inverse on both sides, we have

$$(\mathbf{D}_0^{-1} - \mathbf{K}_x^{-1})^{-1} = \mathbf{K}_{w_1} - (\mathbf{K}_{w_1} + \mathbf{A}^*)(\mathbf{K}_{w_1} + \mathbf{K}_{w_2} + 2\mathbf{A}^*)^{-1}(\mathbf{K}_{w_1} + \mathbf{A}^*).$$
(45)

Defining \mathbf{K}_{w_0} as

$$\mathbf{K}_{w_0} \stackrel{\text{def}}{=} [\mathbf{D}_0^{-1} - \mathbf{K}_x^{-1}]^{-1},\tag{46}$$

(45) is equivalent to

$$\mathbf{K}_{w_1} - \mathbf{K}_{w_0} = (\mathbf{K}_{w_1} + \mathbf{A}^*)(\mathbf{K}_{w_1} + \mathbf{K}_{w_2} + 2\mathbf{A}^*)^{-1}(\mathbf{K}_{w_1} + \mathbf{A}^*).$$
(47)

Defining

$$\mathbf{X} \stackrel{\mathrm{def}}{=} \mathbf{K}_{w_1} + \mathbf{A}^*,$$

(47) is equivalent to

$$\mathbf{K}_{w_1} - \mathbf{K}_{w_0} = \mathbf{X}(2\mathbf{X} + \mathbf{K}_{w_2} - \mathbf{K}_{w_1})^{-1}\mathbf{X},$$
(48)

which is further equivalent to

$$\mathbf{X}(\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{-1}\mathbf{X} = 2\mathbf{X} + \mathbf{K}_{w_2} - \mathbf{K}_{w_1}.$$
(49)

This is a version of the so-called *algebraic Riccati equation*; the corresponding Hamiltonian is readily seen to be positive semidefinite and we can even write down the following explicit solution:

$$\mathbf{X} = \mathbf{K}_{w_1} - \mathbf{K}_{w_0} + (\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{\frac{1}{2}} \left[(\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{-\frac{1}{2}} (\mathbf{K}_{w_2} - \mathbf{K}_{w_0}) (\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{-\frac{1}{2}} \right]^{\frac{1}{2}} (\mathbf{K}_{w_1} - \mathbf{K}_{w_0})^{\frac{1}{2}}.$$
(50)

Thus

$$\mathbf{A}^{*} = (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{\frac{1}{2}} \left[(\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} (\mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}}) (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \right]^{\frac{1}{2}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{\frac{1}{2}} - \mathbf{K}_{w_{0}}.$$
(51)

Now, if $\mathbf{0} \prec \mathbf{A}^* \prec \mathbf{K}_x$ then we can appeal to Theorem 3 and arrive at the explicit jointly Gaussian description scheme parameterized by \mathbf{K}_w that achieves the sum rate. Analogous to the scalar case (cf. [1]), we have the following sufficient condition for when this is true; the proof is available in Appendix F.

Proposition 2. If the distortion constraints $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$ satisfy

$$\mathbf{D}_{0} + \mathbf{K}_{x} - \mathbf{D}_{1} - \mathbf{D}_{2} \succ \mathbf{0}$$

and
$$\mathbf{D}_{0}^{-1} + \mathbf{K}_{x}^{-1} - \mathbf{D}_{1}^{-1} - \mathbf{D}_{2}^{-1} \succ \mathbf{0},$$
 (52)

then $\mathbf{0} \prec \mathbf{A}^* \prec \mathbf{K}_x$.

We now complete the proof by considering the cases that are not covered by the conditions in Proposition 2.

• When

$$\mathbf{D}_0^{-1} + \mathbf{K}_x^{-1} - \mathbf{D}_1^{-1} - \mathbf{D}_2^{-1} \preccurlyeq \mathbf{0},$$

we can choose $\mathbf{A}^* = \mathbf{0}$ to achieve the sum of point-to-point individual rate-distortion functions. Thus in this case, the sum rate is equal to this natural lower bound.

• When

$$\mathbf{D}_0 + \mathbf{K}_x - \mathbf{D}_1 - \mathbf{D}_2 \preccurlyeq \mathbf{0}$$

we can choose $\mathbf{A}^* = \mathbf{K}_x$ to achieve the point-to-point rate distortion-function for central receiver, also a natural lower bound.

• When neither $\mathbf{D}_0 + \mathbf{K}_x - \mathbf{D}_1 - \mathbf{D}_2$ nor $\mathbf{D}_0^{-1} + \mathbf{K}_x^{-1} - \mathbf{D}_1^{-1} - \mathbf{D}_2^{-1}$ is positive or negative semidefinite (this case cannot happen in the scalar case), we cannot use Theorem 3, and the trivial choice of $\mathbf{A}^* = \mathbf{0}$ or $\mathbf{A}^* = \mathbf{K}_x$ does not meet the lower bound. In the next subsection we will address this case and prove that the jointly Gaussian description scheme indeed achieves the lower bound on the sum rate for $L \geq 2$.

If we let the source to be scalar Gaussian, our result reduces to Ozarow's solution of the two description problem for a scalar Gaussian source [1]: this is because the last case described above does not happen in the scalar case.

6.2 Solutions for $L \ge 2$

While we exactly characterized the optimal jointly Gaussian description scheme and used this characterization in arguing that it achieves the fundamental lower bound to the sum rate, such exact calculations do not appear to be as immediate when L > 2. So, we eschew this somewhat brute-force approach and resort to a more subtle proof that involves exploring the structure of the solution to an optimization problem. First, note that by a linear transformation at the encoders and the decoders, we have the following result on rate region for multiple description with individual and central receivers.

Proposition 3.

$$R_*(\mathbf{K}_x, \mathbf{D}_1, \dots, \mathbf{D}_L, \mathbf{D}_0) = R_*(\mathbf{I}, \mathbf{K}_x^{-\frac{1}{2}} \mathbf{D}_1 \mathbf{K}_x^{-\frac{1}{2}}, \dots, \mathbf{K}_x^{-\frac{1}{2}} \mathbf{D}_L \mathbf{K}_x^{-\frac{1}{2}}, \mathbf{K}_x^{-\frac{1}{2}} \mathbf{D}_0 \mathbf{K}_x^{-\frac{1}{2}}).$$
(53)

Thus, throughout this subsection we will suppose, for notation simplicity, that $\mathbf{K}_x = \mathbf{I}$. Given distortion constraints $(\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$, let

$$\mathbf{K}_{w_l} = (\mathbf{D}_l^{-1} - \mathbf{I})^{-1}, \quad l = 0, \ 1, \ \dots, \ L,$$
(54)

and define

$$f(\mathbf{A}) \stackrel{\text{def}}{=} [\mathbf{K}_{w_0} + \mathbf{A}]^{-1} - \sum_{l=1}^{L} [\mathbf{K}_{w_l} + \mathbf{A}]^{-1}, \qquad (55)$$

$$F(\mathbf{A}) \stackrel{\text{def}}{=} \log |\mathbf{K}_{w_0} + \mathbf{A}| - \sum_{l=1}^{L} \log |\mathbf{K}_{w_l} + \mathbf{A}|.$$
(56)

Note that

$$\frac{dF(\mathbf{A})}{d\mathbf{A}} = f(\mathbf{A}). \tag{57}$$

Consider the following optimization problem:

$$\max_{\mathbf{0} \preccurlyeq \mathbf{A} \preccurlyeq \mathbf{I}} \quad F(\mathbf{A}). \tag{58}$$

Now, since $F(\mathbf{A})$ is a continuous map and $\mathbf{0} \preccurlyeq \mathbf{A} \preccurlyeq \mathbf{I}$ is a compact set, there exists an optimal solution \mathbf{A}^* to (58) where \mathbf{A}^* satisfies the Karush-Kuhn-Tucker (KKT) conditions: there exist $\mathbf{\Lambda}_1 \succeq \mathbf{0}$ and $\mathbf{\Lambda}_2 \succeq \mathbf{0}$ such that

$$f(\mathbf{A}^*) + \mathbf{\Lambda}_1 - \mathbf{\Lambda}_2 = \mathbf{0} \tag{59}$$

$$\mathbf{\Lambda}_1 \mathbf{A}^* = \mathbf{0} \tag{60}$$

$$\Lambda_2(\mathbf{A}^* - \mathbf{I}) = \mathbf{0}. \tag{61}$$

Now \mathbf{A}^* falls into the following four cases.

Case 1: $0 \prec \mathbf{A}^* \prec \mathbf{I}$. Alternatively, 0 and 1 are not eigenvalues of \mathbf{A}^* . In this case, $\Lambda_1 = \mathbf{0}$ and $\Lambda_2 = \mathbf{0}$; thus the KKT conditions in (59) reduce to

$$f(\mathbf{A}^*) = \mathbf{0}.$$

Equivalently,

$$[\mathbf{K}_{w_0} + \mathbf{A}^*]^{-1} = \sum_{l=1}^{L} [\mathbf{K}_{w_l} + \mathbf{A}^*]^{-1}.$$
 (62)

From Theorem 3, the jointly Gaussian description scheme with covariance matrix for $\mathbf{w}_1, \ldots, \mathbf{w}_L$ being

$$\mathbf{K}_{w} = \begin{pmatrix} \mathbf{K}_{w_{1}} & -\mathbf{A}^{*} & -\mathbf{A}^{*} & \dots & -\mathbf{A}^{*} \\ -\mathbf{A}^{*} & \mathbf{K}_{w_{2}} & -\mathbf{A}^{*} & \dots & -\mathbf{A}^{*} \\ \dots & \dots & \dots & \dots \\ -\mathbf{A}^{*} & \dots & -\mathbf{A}^{*} & \mathbf{K}_{w_{L-1}} & -\mathbf{A}^{*} \\ -\mathbf{A}^{*} & \dots & -\mathbf{A}^{*} & -\mathbf{A}^{*} & \mathbf{K}_{w_{L}} \end{pmatrix}$$
(63)

achieves the lower bound to the sum rate. Thus in this case, we have characterized the optimality of the jointly Gaussian description scheme parameterized by (63) in terms of achieving the sum rate.

Case 2: $0 \preccurlyeq \mathbf{A}^* \prec \mathbf{I}$. Alternatively, some eigenvalues of \mathbf{A}^* are 0, but no eigenvalues of \mathbf{A}^* are 1. Thus $\Lambda_2 = \mathbf{0}$ and the KKT conditions in (59) reduce to

$$(\mathbf{K}_{w_0} + \mathbf{A}^*)^{-1} - \sum_{l=1}^{L} (\mathbf{K}_{w_l} + \mathbf{A}^*)^{-1} + \mathbf{\Lambda}_1 = \mathbf{0},$$
(64)

for some $\Lambda_1 \succeq 0$ satisfying $\Lambda_1 \mathbf{A}^* = \mathbf{0}$. The key idea now is to see that the distortion constraint on the central receiver is too loose and we can in fact achieve a *lesser* distortion (in the sense of positive semidefinite ordering) for the same sum rate. We first identify this lower distortion: defining

$$\mathbf{K}_{w_0}^* = \left(\mathbf{K}_{w_0}^{-1} + \mathbf{\Lambda}_1
ight)^{-1},$$

consider the smaller distortion matrix on the central receiver

$$\mathbf{D}_{0}^{*} = \left(\mathbf{K}_{w_{0}}^{*}{}^{-1} + \mathbf{I}\right)^{-1} = \left(\mathbf{I} + \mathbf{K}_{w_{0}}^{-1} + \mathbf{\Lambda}_{1}\right)^{-1} = (\mathbf{D}_{0}^{-1} + \mathbf{\Lambda}_{1})^{-1} \prec \mathbf{D}_{0}.$$

This new distortion matrix on the central receiver satisfies two key properties, that we state as a lemma (whose proof is available in Appendix G).

Lemma 4.

$$(\mathbf{K}_{w_0} + \mathbf{A}^*)^{-1} + \Lambda_1 = (\mathbf{K}_{w_0}^* + \mathbf{A}^*)^{-1},$$
(65)

$$\frac{|\mathbf{D}_0 + \mathbf{K}_z|}{|\mathbf{D}_0|} = \frac{|\mathbf{D}_0^* + \mathbf{K}_z|}{|\mathbf{D}_0^*|}.$$
 (66)

Comparing (64) with (65), we have

$$\left[\mathbf{K}_{w_0}^* + \mathbf{A}^*\right]^{-1} = \sum_{l=1}^{L} \left[\mathbf{K}_{w_l} + \mathbf{A}^*\right]^{-1}.$$
 (67)

Now, the corresponding $\mathbf{K}_z = (\mathbf{I} - \mathbf{A}^*)^{-1} - \mathbf{I}$ is singular. If it hadnt been, then by Theorem 3 we could have concluded that jointly Gaussian description scheme achieves the lower bound to the sum rate. We now address this technical difficulty.

Our first observation is that there exists $\delta > 0$ such that for all $\epsilon \in (0, \delta)$ we have $\mathbf{0} \prec \mathbf{A} + \epsilon \mathbf{I} \prec \mathbf{I}$, and $\mathbf{0} \prec \mathbf{K}_{w_0}^* - \epsilon \mathbf{I}$, $\mathbf{0} \prec \mathbf{K}_{w_l} - \epsilon \mathbf{I}$, and we can rewrite (67) as

$$\left[\left(\mathbf{K}_{w_0}^* - \epsilon \mathbf{I} \right) + \left(\mathbf{A}^* + \epsilon \mathbf{I} \right) \right]^{-1} = \sum_{l=1}^{L} \left[\left(\mathbf{K}_{w_l} - \epsilon \mathbf{I} \right) + \left(\mathbf{A}^* + \epsilon \mathbf{I} \right) \right]^{-1}.$$
 (68)

Thus if the distortion constraints were $(\mathbf{D}_1(\epsilon), \ldots, \mathbf{D}_L(\epsilon), \mathbf{D}_0(\epsilon))$ with

$$\mathbf{D}_{l}(\epsilon) = \begin{bmatrix} (\mathbf{K}_{w_{l}} - \epsilon \mathbf{I})^{-1} + \mathbf{I} \end{bmatrix}^{-1}, \quad l = 1, \dots, L, \\ \mathbf{D}_{0}(\epsilon) = \begin{bmatrix} (\mathbf{K}_{w_{0}}^{*} - \epsilon \mathbf{I})^{-1} + \mathbf{I} \end{bmatrix}^{-1}, \end{cases}$$

then $\mathbf{A}^* + \epsilon \mathbf{I}$ is a solution to (68). This situation corresponds to that discussed in Case I; we can conclude that sum rate for this modified distortion multiple description problem is

$$\frac{1}{2}\log\frac{|\mathbf{I} + \mathbf{K}_{z}(\epsilon)|^{(L-1)}|\mathbf{D}_{0}(\epsilon) + \mathbf{K}_{z}(\epsilon)|}{|\mathbf{D}_{0}(\epsilon)|\prod_{l=1}^{L}|\mathbf{D}_{l}(\epsilon) + \mathbf{K}_{z}(\epsilon)|},$$
(69)

where $\mathbf{K}_{z}(\epsilon) = [\mathbf{I} - (\mathbf{A}^{*} + \epsilon \mathbf{I})]^{-1} - \mathbf{I}$. We would like to let ϵ approach zero and consider the limiting multiple description problem. In particular, we show that

$$\mathbf{D}_l(\epsilon) \rightarrow \mathbf{D}_l, \quad l = 1, \dots, L,$$
 (70)

$$\mathbf{D}_0(\epsilon) \quad \to \quad \mathbf{D}_0^*, \tag{71}$$

as $\epsilon \to 0$ in Appendix H. Further, we show that

$$\mathbf{K}_{z}(\epsilon) \to (\mathbf{I} - \mathbf{A}^{*})^{-1} - \mathbf{I},$$
(72)

as $\epsilon \to 0$ in Appendix I. Thus we can conclude that the sum rate approaches, using (66),

$$\frac{1}{2}\log\frac{|\mathbf{I} + \mathbf{K}_z|^{(L-1)}|\mathbf{D}_0 + \mathbf{K}_z|}{|\mathbf{D}_0|\prod_{l=1}^L |\mathbf{D}_l + \mathbf{K}_z|},\tag{73}$$

as $\epsilon \to 0$; here $\mathbf{K}_z = (\mathbf{I} - \mathbf{A}^*)^{-1} - \mathbf{I}$. We observe that this sum rate is achievable using the jointly Gaussian multiple scheme. Further, this sum rate is identical to the lower

bound to sum rate for the original distortions $(\mathbf{D}_1, \ldots, \mathbf{D}_L, \mathbf{D}_0)$. Thus we conclude the optimality of the jointly Gaussian description scheme in this case as well.

Case 3: $0 \prec A^* \preccurlyeq I$. Alternatively, some eigenvalues of A^* are 1, but no eigenvalues of A^* are 0. In this case, the $\Lambda_1 = 0$ and the KKT conditions in (59) reduce to

$$(\mathbf{K}_{w_0} + \mathbf{A}^*)^{-1} - \sum_{l=1}^{L} (\mathbf{K}_{\mathbf{w}_l} + \mathbf{A}^*)^{-1} - \mathbf{\Lambda}_2 = 0,$$
(74)

for some $\Lambda_2 \succeq 0$ satisfying $\Lambda_2(\mathbf{A}^* - \mathbf{I}) = \mathbf{0}$. Defining

$$\mathbf{K}^*_{w_l} = \left[(\mathbf{K}_{\mathbf{w}_l} + \mathbf{I})^{-1} + \mathbf{\Lambda}_2 \right]^{-1} - \mathbf{I},$$

we have, as in (65), that

$$(\mathbf{K}_{\mathbf{w}_{l}} + \mathbf{A}^{*})^{-1} + \mathbf{\Lambda}_{2} = (\mathbf{K}_{\mathbf{w}_{l}}^{*} + \mathbf{A}^{*})^{-1}.$$
(75)

The observation

$$(\mathbf{K}_{\mathbf{w}_l} + \mathbf{A}^*)^{-1} + \mathbf{\Lambda}_2 = [(\mathbf{K}_{\mathbf{w}_l} + \mathbf{I}) + (\mathbf{A}^* - \mathbf{I})]^{-1} + \mathbf{\Lambda}_2$$

combined with the proof of (65) suffices to justify (75). Now, from (75),

$$(\mathbf{K}_{w_0} + \mathbf{A}^*)^{-1} - \sum_{l=1}^{L-1} (\mathbf{K}_{w_l} + \mathbf{A}^*)^{-1} - (\mathbf{K}_{w_L}^* + \mathbf{A}^*)^{-1} = \mathbf{0}.$$
 (76)

As in the previous case, the key step is to identify smaller distortion matrices at each of the individual receivers (ordered in the positive semidefinite sense) that is achievable at the same sum rate:

$$\mathbf{D}_{l}^{*} = \begin{bmatrix} \mathbf{K}_{w_{l}}^{*-1} + \mathbf{I} \end{bmatrix}^{-1}, \quad l = 1, \dots, L.$$

To see that this is indeed a smaller distortion matrix, observe that since \mathbf{K}_w is positive definite, it follows that $\mathbf{K}_{w_l}^* \succ 0$ and

$$\mathbf{D}_{l}^{*} = \left[\mathbf{K}_{w_{l}}^{*}^{-1} + \mathbf{I}\right]^{-1}$$

$$= \left[\left(\left((\mathbf{K}_{w_{l}} + \mathbf{I})^{-1} + \mathbf{\Lambda}_{2}\right)^{-1} - \mathbf{I}\right)^{-1} + \mathbf{I}\right]^{-1}$$

$$= \left[\mathbf{I} - (\mathbf{K}_{w_{l}} + \mathbf{I})^{-1} - \mathbf{\Lambda}_{2}\right]$$

$$= \left[\mathbf{I} + \mathbf{K}_{w_{l}}\right]^{-1} - \mathbf{\Lambda}_{2}$$

$$= \mathbf{D}_{l} - \mathbf{\Lambda}_{2}, \quad l = 1, \dots, L.$$
(77)

Since $\Lambda_2 \geq 0$, it follows that $0 \prec \mathbf{D}_l^* \preccurlyeq \mathbf{D}_l, \ l = 1, \ldots, L$. Define

$$\mathbf{D}_{l}(\epsilon) = \begin{bmatrix} (\mathbf{K}_{w_{l}} + \epsilon \mathbf{I})^{-1} + \mathbf{I} \end{bmatrix}^{-1}, \quad l = 0, 1, \dots, L - 1, \\ \mathbf{D}_{L}(\epsilon) = \begin{bmatrix} (\mathbf{K}_{w_{L}}^{*} + \epsilon \mathbf{I})^{-1} + \mathbf{I} \end{bmatrix}^{-1},$$
(78)

then there exists $\delta > 0$ such that for all $\epsilon \in (0, \delta)$ we have $\mathbf{0} \prec \mathbf{A}^* - \epsilon \mathbf{I} \prec \mathbf{I}$, and $\mathbf{0} \prec \mathbf{D}_l(\epsilon) \prec \mathbf{I}$. We can rewrite (76) as

$$\left[\left(\mathbf{K}_{w_{0}}+\epsilon\mathbf{I}\right)+\left(\mathbf{A}^{*}-\epsilon\mathbf{I}\right)\right]^{-1}=\sum_{l=1}^{L-1}\left[\left(\mathbf{K}_{w_{l}}+\epsilon\mathbf{I}\right)+\left(\mathbf{A}^{*}-\epsilon\mathbf{I}\right)\right]^{-1}+\left[\left(\mathbf{K}_{w_{L}}^{*}+\epsilon\mathbf{I}\right)+\left(\mathbf{A}^{*}-\epsilon\mathbf{I}\right)\right]^{-1}$$
(79)

Thus if the distortion constraints were $(\mathbf{D}_1(\epsilon), \ldots, \mathbf{D}_L(\epsilon), \mathbf{D}_0(\epsilon))$, then $\mathbf{A}^* - \epsilon \mathbf{I}$ is a solution to (79). This situation corresponds to that discussed in Case I; we conclude that the sum rate for this modified distortion multiple description problem is

$$\frac{1}{2}\log\frac{|\mathbf{I} + \mathbf{K}_{z}(\epsilon)|^{(L-1)}|\mathbf{D}_{0}(\epsilon) + \mathbf{K}_{z}(\epsilon)|}{|\mathbf{D}_{0}(\epsilon)|\prod_{l=1}^{L}|\mathbf{D}_{l}(\epsilon) + \mathbf{K}_{z}(\epsilon)|},$$
(80)

where $\mathbf{K}_{z}(\epsilon) = [\mathbf{I} - (\mathbf{A}^{*} - \epsilon \mathbf{I})]^{-1} - \mathbf{I}$. We would like to let ϵ approach zero and consider the limiting multiple description problem. Similar to equations (70) and (71), we have

$$\mathbf{D}_{l}(\epsilon) \to \mathbf{D}_{l}, \quad l = 1, \dots L, \\
\mathbf{D}_{0}(\epsilon) \to \mathbf{D}_{0}^{*}.$$
(81)

Further, we show that

$$\lim_{\epsilon \to 0} \frac{|\mathbf{I} + \mathbf{K}_z(\epsilon)|^{(L-1)} |\mathbf{D}_0(\epsilon) + \mathbf{K}_z(\epsilon)|}{|\prod_{l=1}^L |\mathbf{D}_l(\epsilon) + \mathbf{K}_z(\epsilon)|} = 1$$
(82)

in Appendix J. We can now conclude that the sum rate approaches

$$\frac{1}{2}\log\frac{1}{|\mathbf{D}_0|}\tag{83}$$

as ϵ approaches 0. In other words, the point-to-point rate-distortion function for central receiver with distortion \mathbf{D}_0 can be achieved by using the jointly Gaussian description scheme, and the resulting distortion is $(\mathbf{D}_1, \ldots, \mathbf{D}_L^*, \mathbf{D}_0)$ where $\mathbf{0} \prec \mathbf{D}_L^* \preccurlyeq \mathbf{D}_L$. In conclusion, the jointly Gaussian description scheme is also optimal in this case.

Case 4: $\mathbf{0} \preccurlyeq \mathbf{A}^* \preccurlyeq \mathbf{I}$. i.e., both 0 and 1 are eigenvalues of \mathbf{A}^* . In this case, the KKT conditions are: there exist $\mathbf{\Lambda}_1 \succeq 0$ and $\mathbf{\Lambda}_2 \succeq 0$ such that equations (59), (60) and (61) hold. We can combine equations (65) and (75) to get

$$(\mathbf{K}_{w_0}^* + \mathbf{A}^*)^{-1} = \sum_{l=1}^{L-1} (\mathbf{K}_{w_l} + \mathbf{A}^*)^{-1} + (\mathbf{K}_{w_L}^* + \mathbf{A}^*)^{-1},$$
(84)

where

$$egin{aligned} \mathbf{K}^*_{w_0} &= \left(\mathbf{K}^{-1}_{w_0} + \mathbf{\Lambda}_1
ight)^{-1}, \ \mathbf{K}^*_{w_L} &= \left[(\mathbf{K}_{w_L} + \mathbf{I})^{-1} + \mathbf{\Lambda}_2
ight]^{-1} - \mathbf{I}. \end{aligned}$$

As in cases 2 and 3, we want to show the optimality of the jointly Gaussian multiple description scheme through a limiting procedure. We do this by first perturbing \mathbf{A}^* so that it has no eigenvalue equal to 0 or 1 as follows.

Without loss of generality, suppose that \mathbf{A}^* has p eigenvalues equal to 0 and q eigenvalues equal 1, where p > 0 and q > 0, and there exists $N \times N$ orthogonal matrix \mathbf{Q} such that

$$\mathbf{Q}\mathbf{A}^*\mathbf{Q}^t = \operatorname{diag}\{\underbrace{0, \ldots, 0}_{p}, \underbrace{1, \ldots, 1}_{q}, a_{p+q+1}, \ldots, a_N\},\$$

with $0 < a_{p+q+1} < 1, \ldots, 0 < a_N < 1$. We need to perturb the eigenvalues of \mathbf{A}^* away from both 0 and 1. Towards this, we define two $N \times N$ diagonal matrices:

$$\mathbf{E}_{1} = \operatorname{diag}(\underbrace{1, \ \dots, \ 1}_{p}, \underbrace{0, \ \dots, \ 0, \ 0, \ \dots, \ 0}_{N-p}), \\ \mathbf{E}_{2} = \operatorname{diag}(\underbrace{0, \ \dots, \ 0}_{p}, \underbrace{1, \ \dots, \ 1}_{q}, \ 0, \ \dots, \ 0),$$

Also define

$$\mathbf{A}^{*}(\epsilon_{1}, \epsilon_{2}) = \mathbf{A}^{*} + \mathbf{Q}^{t}(\epsilon_{1}\mathbf{E}_{1} - \epsilon_{2}\mathbf{E}_{2})\mathbf{Q},$$

$$\mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2}) = (\mathbf{I} - \mathbf{A}^{*}(\epsilon_{1}, \epsilon_{2}))^{-1} - \mathbf{I},$$

$$\mathbf{K}_{w_{l}}(\epsilon_{1}, \epsilon_{2}) = \mathbf{K}_{w_{l}} - \mathbf{Q}^{t}(\epsilon_{1}\mathbf{E}_{1} - \epsilon_{2}\mathbf{E}_{2})\mathbf{Q}, \quad l = 1, \dots, L - 1,$$

$$\mathbf{K}_{w_{L}}(\epsilon_{1}, \epsilon_{2}) = \mathbf{K}_{w_{L}}^{*} - \mathbf{Q}^{t}(\epsilon_{1}\mathbf{E}_{1} - \epsilon_{2}\mathbf{E}_{2})\mathbf{Q},$$

$$\mathbf{K}_{w_{0}}(\epsilon_{1}, \epsilon_{2}) = \mathbf{K}_{w_{0}}^{*} - \mathbf{Q}^{t}(\epsilon_{1}\mathbf{E}_{1} - \epsilon_{2}\mathbf{E}_{2})\mathbf{Q}.$$

Further, defining

$$\mathbf{D}_{l}(\epsilon_{1},\epsilon_{2}) = (\mathbf{I} + \mathbf{K}_{w_{l}}(\epsilon_{1},\epsilon_{2}))^{-1}, \quad l = 1, \dots, L,$$
(85)

there exists $\delta > 0$ such that for all $\epsilon_1 \in (0, \delta)$ and $\epsilon_2 \in (0, \delta)$ we have $\mathbf{0} \prec \mathbf{A}^*(\epsilon_1, \epsilon_2) \prec \mathbf{I}$, and $\mathbf{0} \prec \mathbf{D}_l(\epsilon_1, \epsilon_2) \prec \mathbf{I}$. Now, we can rewrite (84) as

$$\left[\mathbf{K}_{w_0}(\epsilon_1, \epsilon_2) + \mathbf{A}^*(\epsilon_1, \epsilon_2)\right]^{-1} = \sum_{l=1}^{L} \left[\mathbf{K}_{w_l}(\epsilon_1, \epsilon_2) + \mathbf{A}^*(\epsilon_1, \epsilon_2)\right]^{-1}.$$
(86)

Thus if the distortion constraints were $(\mathbf{D}_1(\epsilon_1, \epsilon_2), \ldots, \mathbf{D}_L(\epsilon_1, \epsilon_2), \mathbf{D}_0(\epsilon_1, \epsilon_2))$, then $\mathbf{A}^*(\epsilon_1, \epsilon_2)$ is a solution to (86). This situation corresponds to that discussed in Case I; we conclude that the sum rate for this modified distortion multiple description problem is

$$\frac{1}{2}\log\frac{|\mathbf{I} + \mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2})|^{(L-1)}|\mathbf{D}_{0}(\epsilon_{1}, \epsilon_{2}) + \mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2})|}{|\mathbf{D}_{0}(\epsilon_{1}, \epsilon_{2})|\prod_{l=1}^{L}|\mathbf{D}_{l}(\epsilon_{1}, \epsilon_{2}) + \mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2})|},$$
(87)

where $\mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2}) = [\mathbf{I} - \mathbf{A}^{*}(\epsilon_{1}, \epsilon_{2})]^{-1} - \mathbf{I}$. We would like to let ϵ_{1} and ϵ_{2} approach zero and consider the limiting multiple description problem. Similar to equations (70) and (71), when ϵ_{1} and ϵ_{2} approach 0, we get

$$\mathbf{D}_{l}(\epsilon_{1},\epsilon_{2}) \to \mathbf{D}_{l}, \quad l = 1, \dots, L-1, \\
\mathbf{D}_{L}(\epsilon_{1},\epsilon_{2}) \to \mathbf{D}_{L}^{*}, \\
\mathbf{D}_{0}(\epsilon_{1},\epsilon_{2}) \to \mathbf{D}_{0}^{*},$$
(88)

where $\mathbf{D}_L^* = \mathbf{D}_L - \mathbf{\Lambda}_2$ as in case 3 and $\mathbf{D}_0^* = [\mathbf{D}_0^{-1} + \mathbf{\Lambda}_1^{-1}]^{-1}$ as in case 2. Further, we show that

$$\lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \frac{1}{2} \log \frac{|\mathbf{I} + \mathbf{K}_z(\epsilon_1, \epsilon_2)|^{(L-1)} |\mathbf{D}_0(\epsilon_1, \epsilon_2) + \mathbf{K}_z(\epsilon_1, \epsilon_2)|}{|\mathbf{D}_0(\epsilon_1, \epsilon_2)| \prod_{l=1}^L |\mathbf{D}_l(\epsilon_1, \epsilon_2) + \mathbf{K}_z(\epsilon_1, \epsilon_2)|} = \frac{1}{2} \log \frac{1}{|\mathbf{D}_0|}$$
(89)

in Appendix K. We conclude that the sum rate approaches

$$\frac{1}{2}\log\frac{1}{|\mathbf{D}_0|}\tag{90}$$

as ϵ_1 and ϵ_2 approach 0. Thus the point-to-point rate-distortion function for central receiver with distortion \mathbf{D}_0 can be achieved by using the jointly Gaussian description scheme, and the resulting distortions are $(\mathbf{D}_1, \ldots, \mathbf{D}_L^*, \mathbf{D}_0^*)$ where $\mathbf{0} \prec \mathbf{D}_L^* \preccurlyeq \mathbf{D}_L$ and $\mathbf{0} \prec \mathbf{D}_0^* \preccurlyeq \mathbf{D}_0$. In other words, the jointly Gaussian multiple description scheme is also optimal in this case.

To summarize, we see that the jointly Gaussian description scheme achieves the limiting sum rate. The limiting sum rate is the solution to an optimization problem. For some specific distortion constraints, the sum rate can be characterized as the solution to a matrix polynomial equation (Case I).

6.3 Rate Region for Two Descriptions

Applying the result in Section 6.2 to the case of L = 2, i.e., the two description problem, we can see that jointly Gaussian description scheme achieves the optimal sum rate. This resolves the case left out in Section 6.1. It also turns out that in the two description problem, we can show that jointly Gaussian description strategy achieves the entire rate region. This is the main result of this subsection.

From Section 3 we have a outer bound to the rate region for the two description problem

$$\mathcal{R}_{out}(\mathbf{K}_{x}, \mathbf{D}_{1}, \mathbf{D}_{2}, \mathbf{D}_{0}) = \begin{cases} (R_{1}, R_{2}) : \\ R_{l} \geq \frac{1}{2} \log \frac{|\mathbf{K}_{x}|}{|\mathbf{D}_{l}|}, & l = 1, 2 \\ R_{1} + R_{2} \geq \sup_{\mathbf{K}_{z} \succ \mathbf{0}} \frac{1}{2} \log \frac{|\mathbf{K}_{x}||\mathbf{K}_{x} + \mathbf{K}_{z}||\mathbf{D}_{0} + \mathbf{K}_{z}|}{|\mathbf{D}_{0}||\mathbf{D}_{1} + \mathbf{K}_{z}||\mathbf{D}_{2} + \mathbf{K}_{z}|} \end{cases} \right\}.$$
(91)

Following the discussion in Section 6.2, we show in the following that the jointly Gaussian description strategy (jointly Gaussian multiple description schemes and the time sharing between them) achieves the outer bound to the rate region.

Let

$$\mathbf{K}_{w_l} = (\mathbf{D}_l^{-1} - \mathbf{K}_x^{-1})^{-1}, \quad l = 0, \ 1, \ 2$$

and

$$F(\mathbf{A}) = \log |\mathbf{K}_{w_0} + \mathbf{A}| - \log |\mathbf{K}_{w_1} + \mathbf{A}| - \log |\mathbf{K}_{w_2} + \mathbf{A}|.$$

Now consider the optimization problem:

$$\max_{\mathbf{0} \preccurlyeq \mathbf{A} \preccurlyeq \mathbf{K}_x} \quad F(\mathbf{A}). \tag{92}$$

As in Section 6.2, the optimal solution \mathbf{A}^* falls into four cases.

Case 1: $0 \prec A^* \prec K_x$. In this case, we know from Section 4 that the rate pair (R_1, R_2) satisfying

$$\left\{\begin{array}{l}
\left(R_{1}, R_{2}\right):\\
R_{l} \geq \frac{1}{2}\log\frac{|\mathbf{K}_{x} + \mathbf{K}_{w_{l}}|}{|\mathbf{K}_{w_{l}}|}, \quad l = 1, 2\\
R_{1} + R_{2} \geq \frac{1}{2}\log\frac{|\mathbf{K}_{x} + \mathbf{K}_{w_{1}}||\mathbf{K}_{x} + \mathbf{K}_{w_{2}}|}{|\mathbf{K}_{w}|}\end{array}\right\}$$
(93)

is achievable using the jointly Gaussian multiple description scheme with the covariance matrix of \mathbf{w}_1 , \mathbf{w}_2 being

$$\mathbf{K}_w = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A}^* \ -\mathbf{A}^* & \mathbf{K}_{w_2} \end{pmatrix}.$$

Denoting the resulting distortions as $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$, we readily calculate

$$\frac{1}{2}\log\frac{|\mathbf{K}_x + \mathbf{K}_{w_l}|}{|\mathbf{K}_{w_l}|} = \frac{1}{2}\log|\mathbf{K}_x||\mathbf{K}_{w_l}^{-1} + \mathbf{K}_x^{-1}| = \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_l|}$$

for l = 1, 2. From the discussion in Section 6.2, we know that the lower bound to sum rate is achieved using this jointly Gaussian description scheme. Thus, in this case, the jointly Gaussian description scheme achieves the rate region. As an aside, we note in this case that, \mathbf{A}^* satisfies

$$\left[\mathbf{K}_{w_0}^* + \mathbf{A}^*\right]^{-1} = \left[\mathbf{K}_{w_1} + \mathbf{A}^*\right]^{-1} + \left[\mathbf{K}_{w_2} + \mathbf{A}^*\right],$$

and, from the discussion in Section 6.1, that a sufficient condition for this case to happen is (52).

Case 2: $0 \preccurlyeq A^* \prec K_x$. This case is similar to case 1: the jointly Gaussian description scheme with covariance matrix for \mathbf{w}_1 , \mathbf{w}_2 being

$$\mathbf{K}_w = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A}^* \ -\mathbf{A}^* & \mathbf{K}_{w_2} \end{pmatrix}.$$

achieves the lower bound on the rate region. We note that in this case the resulting distortions are $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0^*)$, with $\mathbf{D}_0^* \preccurlyeq \mathbf{D}_0$. Further, we know from the discussion in 6.1, that a sufficient condition for this case to happen is

$$\mathbf{D}_0^{-1} + \mathbf{K}_x^{-1} - \mathbf{D}_1^{-1} - \mathbf{D}_2^{-1} \preccurlyeq \mathbf{0}$$

Case 3: $\mathbf{0} \prec \mathbf{A}^* \preccurlyeq \mathbf{K}_x$. In this case, we know from the discussion in Section 6.2 that for another two description problem with distortions $(\mathbf{D}_1, \mathbf{D}_2^*, \mathbf{D}_0)$ such that $\mathbf{D}_2^* \preccurlyeq \mathbf{D}_2$, the jointly Gaussian description scheme with covariance matrix for $\mathbf{w}_1, \mathbf{w}_2$ being

$$\mathbf{K}_w = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A}^* \ -\mathbf{A}^* & \mathbf{K}_{w_2}^* \end{pmatrix}$$

achieves the lower bound to sum rate $\left(\frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_0|}\right)$ to the original distortions $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$. We can see, from the contra-polymatroid structure of the achievable region of jointly Gaussian description scheme, that the corner point

$$B_1 = \left(\frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_1|}, \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_0|} - \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_1|}\right)$$

in Figure 3 is achievable by this jointly Gaussian description scheme.

Now observe that the discussion in case 3 of Section 6.2 is symmetric with respect to the individual receivers. Thus, by exchanging the role of receiver 1 and receiver 2, we can achieve the other corner point

$$B_2 = \left(\frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_0|} - \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_2|}, \frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_2|}\right)$$

in Figure 3 by another appropriate jointly Gaussian description scheme. Finally, time sharing between these two jointly Gaussian multiple description schemes allows us to achieve the lower bound on the rate region. As an aside, we note, as a consequence of the discussion in Section 6.1, that a sufficient condition for this case to happen is

$$\mathbf{D}_0 + \mathbf{K}_x - \mathbf{D}_1 - \mathbf{D}_2 \preccurlyeq \mathbf{0}.$$

Case 4: $\mathbf{0} \preccurlyeq \mathbf{A}^* \preccurlyeq \mathbf{K}_x$. In this case, we know, from the discussion in Section 6.2, that for another two description problem with distortions $(\mathbf{D}_1, \mathbf{D}_2^*, \mathbf{D}_0^*)$ such that $\mathbf{D}_2^* \preccurlyeq \mathbf{D}_2$ and $\mathbf{D}_0^* \preccurlyeq \mathbf{D}_0$, the jointly Gaussian description scheme with covariance matrix for $\mathbf{w}_1, \mathbf{w}_2$ being

$$\mathbf{K}_w = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A}^* \ -\mathbf{A}^* & \mathbf{K}_{w_2}^* \end{pmatrix}$$

achieves the lower bound to sum rate $\left(\frac{1}{2}\log\frac{|\mathbf{K}_x|}{|\mathbf{D}_0|}\right)$ to the original distortions $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$. Using an argument entirely analogous to that applied that the jointly Gaussian description strategy achieves the rate region. To summarize: the jointly Gaussian description strategy achieves the rate region for the two description problem. For a class of distortion constraints, the corner points of the rate region can be characterized by solving a matrix polynomial equation, as already seen in Section 6.1.

7 Discussions

Although multiple description for individual and central receivers is a special case of the most general multiple description problem, the solution to this problem sheds substantial insight to the issue-at-large. In this section, we discuss two instances of other multiple description problems that can be resolved using the insights developed so far. In particular, we discuss the problem of two descriptions with separate distortion constraints and the general multiple description problem for some special sets of distortion constraints.

7.1 Two Description with Separate Distortion Constraints

The problem of two descriptions with separate distortion constraints is illustrated in Figure 2. Suppose the vector Gaussian source $\mathbf{x}[m] = (\mathbf{x}_1[m], \mathbf{x}_2[m])$, the dimension of $\mathbf{x}_1[m]$ is N_1 and the dimension of $\mathbf{x}_2[m]$ is N_2 . This implies that the dimension of $\mathbf{x}[m]$ is $N = N_1 + N_2$. Let $\mathbf{K}_x = \mathbb{E}[\mathbf{x}[m]^t \mathbf{x}[m]]$, $\mathbf{K}_{x_1} = \mathbb{E}[\mathbf{x}_1[m]^t \mathbf{x}_1[m]]$, and $\mathbf{K}_{x_2} = \mathbb{E}[\mathbf{x}_2[m]^t \mathbf{x}_2[m]]$. There are two encoders at the source providing two descriptions of $\mathbf{x}[m]$. There are three receivers: the individual receivers 1 and 2 are only interested in generating reproduction of $\mathbf{x}_1[m]$ with mean square distortion constraint \mathbf{D}_1 (an $N_1 \times N_1$ positive definite matrix) from description 1 and $\mathbf{x}_2[m]$ with mean square distortion constraint \mathbf{D}_2 (an $N_2 \times N_2$ positive definite matrix) from description 2, respectively. The central receiver uses both descriptions to generate a reproduction of $\mathbf{x}[m]$ with the error covariance meeting a distortion constraint \mathbf{D}_0 (an $N \times N$ positive definite matrix) from both descriptions.

This situation is closely related to the two description problem and we can harness our results thus far to completely characterize the rate region of the problem at hand.

Theorem 4. The rate region of two description with separate distortion constraints is

$$\mathcal{R}(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0) = \bigcup_{\Upsilon(\mathbf{D}_1', \mathbf{D}_2')} \mathcal{R}_*(\mathbf{D}_1', \mathbf{D}_2', \mathbf{D}_0),$$
(94)

where $\Upsilon(\mathbf{D}'_1, \mathbf{D}'_2)$ is defined as

$$\Upsilon(\mathbf{D}'_{1}, \mathbf{D}'_{2}) \stackrel{def}{=} \left\{ (\mathbf{D}'_{1}, \mathbf{D}'_{2}) : (\mathbf{D}'_{1})_{\{1, \dots, N_{1}\}} \preccurlyeq \mathbf{D}_{1}, (\mathbf{D}'_{2})_{\{N_{1}+1, \dots, N\}} \preccurlyeq \mathbf{D}_{2} \right\}.$$
(95)

Proof. It is clear that any rate pair $(R_1, R_2) \in \mathcal{R}_*(\mathbf{D}'_1, \mathbf{D}'_2, \mathbf{D}_0)$ for some $(\mathbf{D}'_1, \mathbf{D}'_2) \in \Upsilon(\mathbf{D}'_1, \mathbf{D}'_2)$ is in the rate region for the two description with separate distortion constraints, and so

$$\mathcal{R}_*(\mathbf{D}_1', \mathbf{D}_2', \mathbf{D}_0) \subseteq \mathcal{R}(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0).$$

On the other hand, although receiver 1 (2) is only interested in reconstructing \mathbf{x}_1 (\mathbf{x}_2), they can actually reconstruct the entire source \mathbf{x} based on their received descriptions. Hence, any coding scheme for the two description with separate distortion constraints will result in some achievable distortions (\mathbf{D}'_1 , \mathbf{D}'_2 , \mathbf{D}'_0) with (\mathbf{D}'_1 , \mathbf{D}'_2) $\in \Upsilon(\mathbf{D}'_1$, \mathbf{D}'_2) and $\mathbf{D}'_0 \preccurlyeq \mathbf{D}_0$. Thus any rate pair (R_1, R_2) $\in \mathcal{R}(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_0)$ achieved by this coding scheme is in the rate region $\mathcal{R}_*(\mathbf{D}'_1, \mathbf{D}'_2, \mathbf{D}_0)$ for the two description problem. Thus

$$\mathcal{R}(\mathbf{D}_1, \ \mathbf{D}_2, \ \mathbf{D}_0) \subseteq \bigcup_{\Upsilon(\mathbf{D}_1', \ \mathbf{D}_2')} \mathcal{R}_*(\mathbf{D}_1', \ \mathbf{D}_2', \ \mathbf{D}_0).$$

From equivalence of the two regions in (94), the proof is now complete.

7.2 General Gaussian Multiple Description Problem for Special Choices of Distortion Constraints

Consider the general Gaussian multiple description problem with source covariance \mathbf{K}_x and $2^L - 1$ distortion constraints D_S for each $S \subseteq \{1, \ldots, L\}$.

Following arguments similar to that used in arriving at the lower bound (11) for sum rate, we have an outer bound on the rate region:

$$\mathcal{R}_{out}(\mathbf{K}_x, \mathbf{D}_1, \dots, \mathbf{D}_L, \mathbf{D}_0) = \left\{ \begin{array}{l} (R_1, \dots, R_L) :\\ \sum_{l \in S} R_l \ge \frac{1}{2} \log \frac{|\mathbf{K}_x||\mathbf{K}_x + \mathbf{K}_z|^{(|S|-1)}|\mathbf{D}_S + \mathbf{K}_z|}{|\mathbf{D}_S| \prod_{l \in S} |\mathbf{D}_l + \mathbf{K}_z|}, \quad \forall S \subseteq \{1, \dots, L\} \end{array} \right\}.$$

$$(96)$$

Following arguments similar to those used in arriving at the upper bound (20) for the sum rate, we can use a jointly Gaussian description scheme with covariance matrix of \mathbf{w}_l 's (\mathbf{K}_w) taking the form (23), any tuple (R_1, \ldots, R_L) satisfying

$$\left\{\begin{array}{l}
\left(R_{1}, \ldots, R_{L}\right): \\
\sum_{l \in S} R_{l} \geq \frac{1}{2} \log \frac{\left|\mathbf{K}_{x}\right| \left|\mathbf{K}_{x} + \mathbf{K}_{z}\right|^{\left(|S|-1\right)} \left|\operatorname{Cov}[\mathbf{x}|\mathbf{u}_{l}, l \in S] + \mathbf{K}_{z}\right|}{\left|\operatorname{Cov}[\mathbf{x}|\mathbf{u}_{l}, l \in S]\right| \prod_{l \in S} \left|\operatorname{Cov}[\mathbf{x}|\mathbf{u}_{l}] + \mathbf{K}_{z}\right|}, \quad \forall S \subseteq \{1, \ldots, L\}\end{array}\right\}\right\}$$
(97)

is achievable. Thus if we can find a \mathbf{K}_w of the form in (23) such that all of the $2^L - 1$ distortion constraints are met with equality, i.e.,

$$\mathbf{D}_{S} = \operatorname{Cov}[\mathbf{x}|\mathbf{u}_{l}, \ l \in S] = [\mathbf{K}_{x}^{-1} + (\mathbf{I}, \ \dots, \ \mathbf{I})\mathbf{K}_{w_{S}}^{-1}(\mathbf{I}, \ \dots, \ \mathbf{I})^{t}]^{-1}, \quad \forall S \subseteq \{1, \ \dots, \ L\},$$
(98)

where \mathbf{K}_{w_S} is the covariance matrix for all \mathbf{K}_{w_l} , $l \in S$, then the achievable region matches the outer bound and we would have characterized the rate region of the multiple description problem.

From the above discussion, we see that for some choice of distortion constraints of the multiple description problem, we can indeed do this: First choose L + 1 distortions $(\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_L, \mathbf{D}_0)$ such that they satisfy the condition for Theorem 3 for the multiple description problem with individual and central receivers. Next we can solve for the \mathbf{K}_w which is the covariance matrix of $(\mathbf{w}_1, \ldots, \mathbf{w}_L)$ for the sum-rate-achieving jointly Gaussian description scheme. For any other $S \subseteq \{1, \ldots, L\}$, this scheme results in distortion $\mathbf{D}_S = [\mathbf{K}_x^{-1} + (\mathbf{I}, \ldots, \mathbf{I})\mathbf{K}_{w_S}^{-1}(\mathbf{I}, \ldots, \mathbf{I})^t]^{-1}$. Finally we choose these \mathbf{D}_S 's as the other distortion constraints. Now we have a general multiple description problem with $2^L - 1$ distortion constraints D_S for each $S \subseteq \{1, \ldots, L\}$, and hence we can find a \mathbf{K}_w of form (23) such that all of the $2^L - 1$ distortion constraints are met with equality. Thus (96) is actually the rate region and it can be achieved by a jointly Gaussian description scheme.

Appendix

A Useful Matrix Lemmas

In this appendix we provide some useful results in matrix analysis that are extensively used in this paper.

Lemma 5 (Matrix Inversion Lemma). [25, Theorem 2.5] Let \mathbf{A} be an $m \times m$ nonsingular matrix and \mathbf{B} be an $n \times n$ nonsingular matrix and let \mathbf{C} and \mathbf{D} be $m \times n$ and $n \times m$ matrices, respectively. If the matrix $\mathbf{A} + \mathbf{CBD}$ is nonsingular, then

$$(\mathbf{A} + \mathbf{C}\mathbf{B}\mathbf{D})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{B}^{-1} + \mathbf{D}\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{D}\mathbf{A}^{-1}$$

Lemma 6. [25, Theorem 2.3] Suppose that the partitioned matrix

$$\mathbf{M} = egin{pmatrix} \mathbf{A} & \mathbf{B} \ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

is invertible and that the inverse is conformally partitioned as

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{U} & \mathbf{V} \end{pmatrix}.$$

If \mathbf{A} is a nonsingular principal sub-matrix of \mathbf{M} , then

$$\begin{aligned} \mathbf{X} &= \mathbf{A}^{-1} + \mathbf{A}^{-1} \mathbf{B} (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \mathbf{C} \mathbf{A}^{-1}, \\ \mathbf{Y} &= -\mathbf{A}^{-1} \mathbf{B} (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1}, \\ \mathbf{U} &= -(\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \mathbf{C} \mathbf{A}^{-1}, \\ \mathbf{V} &= (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1}. \end{aligned}$$
(99)

Lemma 7. [25, Theorem 6.13] Let $\mathbf{E} \in \mathbb{M}_n$ be a positive definite matrix and let \mathbf{F} be an $n \times m$ matrix. Then for any $m \times m$ positive definite matrix \mathbf{G} ,

$$\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{F}^t & \mathbf{G} \end{pmatrix} \succ 0 \iff \mathbf{G} \succ \mathbf{F}^t \mathbf{E}^{-1} \mathbf{F}.$$
 (100)

Lemma 8. [25, Theorem 6.8 and 6.9] Let \mathbf{A} and \mathbf{B} be positive definite matrices such that $\mathbf{A} \succ \mathbf{B} (\mathbf{A} \succeq \mathbf{B})$. Then,

$$|\mathbf{A}| \succ |\mathbf{B}| \quad (|\mathbf{A}| \succcurlyeq |\mathbf{B}|),$$

$$\mathbf{A}^{-1} \prec \mathbf{B}^{-1} \quad (\mathbf{A}^{-1} \preccurlyeq \mathbf{B}^{-1}),$$

$$\mathbf{A}^{1/2} \succ \mathbf{B}^{1/2} \quad (\mathbf{A}^{1/2} \succcurlyeq \mathbf{B}^{1/2}).$$
(101)

B Proof of Lemma 1

Define an i.i.d. random process $\{\mathbf{z}[m]\}, m = 1, \ldots, n \text{ of } \mathcal{N}(0, \mathbf{K}_z)$ Gaussian random vectors, where $\mathbf{z}[m], m = 1, \ldots, n$ are independent of \mathbf{x}^n and $C_l, l = 1, \ldots, L$. Form a random process $\mathbf{y}^n = (\mathbf{y}[1], \ldots, \mathbf{y}[n])^t$ by

$$\mathbf{y}[m] = \mathbf{x}[m] + \mathbf{z}[m], \quad m = 1, \dots, n.$$

It follows that $\{\mathbf{y}[m]\}\$ is an i.i.d. random process of $\mathcal{N}(0, \mathbf{K}_y)$ Gaussian random vectors, where $\mathbf{K}_y = \mathbf{K}_x + \mathbf{K}_z$. Then

$$I(C_{1}; C_{2}; ...; C_{L}) + I(C_{1}, ..., C_{L}; \mathbf{x}^{n})$$

$$= \sum_{l=1}^{L} H(C_{l}) - H(C_{1}, ..., C_{L}) + I(C_{1}, ..., C_{L}; \mathbf{x}^{n})$$

$$\geq \sum_{l=1}^{L} H(C_{l}) - H(C_{1}, ..., C_{L}) + I(C_{1}, ..., C_{L}; \mathbf{x}^{n})$$

$$- \left(\sum_{l=1}^{L} H(C_{l}|\mathbf{y}^{n}) - H(C_{1}, ..., C_{L}|\mathbf{y}^{n})\right)$$

$$= \sum_{l=1}^{L} \left(h(\mathbf{y}^{n}) - h(\mathbf{y}^{n}|C_{l})\right) - h(\mathbf{y}^{n}) + h(\mathbf{y}^{n}|C_{1}, ..., C_{L}) + h(\mathbf{x}^{n}) - h(\mathbf{x}^{n}|C_{1}, ..., C_{L})$$

$$= h(\mathbf{x}^{n}) + (L-1)h(\mathbf{y}^{n}) - \sum_{l=1}^{L} h(\mathbf{y}^{n}|C_{l}) + h(\mathbf{y}^{n}|C_{1}, ..., C_{L}) - h(\mathbf{x}^{n}|C_{1}, ..., C_{L}).$$
(102)

Since \mathbf{x}^n and \mathbf{y}^n are Gaussian vectors, for the first two terms in (102), we have

$$h(\mathbf{x}^{n}) = \frac{1}{2} \log(2\pi e)^{Nn} |\mathbf{K}_{x}|^{n},$$

$$h(\mathbf{y}^{n}) = \frac{1}{2} \log(2\pi e)^{Nn} |\mathbf{K}_{y}|^{n} = \frac{1}{2} \log(2\pi e)^{Nn} |\mathbf{K}_{x} + \mathbf{K}_{z}|^{n}.$$
(103)

We also have the following bound on $h(\mathbf{y}^n|C_l)$ for $l = 1, \ldots, L$:

$$h(\mathbf{y}^{n}|C_{l}) \leq \sum_{m=1}^{n} h(\mathbf{y}[m]|C_{l})$$

$$\leq \sum_{m=1}^{n} \frac{1}{2} \log(2\pi e)^{N} |\operatorname{Cov}[\mathbf{y}[m]|C_{l}]|$$

$$\leq \frac{1}{2} \log(2\pi e)^{Nn} + \frac{n}{2} \log \left|\frac{1}{n} \sum_{m=1}^{n} \operatorname{Cov}[\mathbf{y}[m]|C_{l}]\right|$$

$$= \frac{1}{2} \log(2\pi e)^{Nn} + \frac{n}{2} \log \left|\frac{1}{n} \sum_{m=1}^{n} \operatorname{Cov}[(\mathbf{x}[m] + \mathbf{z}[m])|C_{l}]\right|$$

$$= \frac{1}{2} \log(2\pi e)^{Nn} + \frac{n}{2} \log \left|\frac{1}{n} \sum_{m=1}^{n} \operatorname{Cov}[\mathbf{x}[m]|C_{l}] + \mathbf{K}_{z}\right|$$

$$\leq \frac{1}{2} \log(2\pi e)^{Nn} + \frac{n}{2} \log |\mathbf{D}_{l} + \mathbf{K}_{z}|$$

$$= \frac{1}{2} \log(2\pi e)^{Nn} |\mathbf{D}_{l} + \mathbf{K}_{z}|^{n}.$$
(104)

Next we bound the last two terms of (102) as follows.

$$h(\mathbf{y}^{n}|C_{1}, \ldots, C_{L}) - h(\mathbf{x}^{n}|C_{1}, \ldots, C_{L})$$

= $h(\mathbf{y}^{n}|C_{1}, \ldots, C_{L}) - h(\mathbf{x}^{n}|\mathbf{z}^{n}, C_{1}, \ldots, C_{L})$
= $h(\mathbf{y}^{n}|C_{1}, \ldots, C_{L}) - h(\mathbf{y}^{n}|\mathbf{z}^{n}, C_{1}, \ldots, C_{L})$
= $I(\mathbf{y}^{n}; \mathbf{z}^{n}|C_{1}, \ldots, C_{L}).$ (105)

Letting

$$\mathbf{K}_{c}[m] \stackrel{\text{def}}{=} \operatorname{Cov}[\mathbf{x}[m] - \hat{\mathbf{x}}_{0}[m]], \qquad (106)$$

we have

$$I(\mathbf{y}^{n}; \mathbf{z}^{n} | C_{1}, ..., C_{L}) = h(\mathbf{z}^{n} | C_{1}, ..., C_{L}) - h(\mathbf{z}^{n} | \mathbf{y}^{n}, C_{1}, ..., C_{L})$$

$$= h(\mathbf{z}^{n}) - h(\mathbf{z}^{n} | \mathbf{y}^{n} - \hat{\mathbf{x}}_{0}^{n}, C_{1}, ..., C_{L})$$

$$\geq h(\mathbf{z}^{n}) - h(\mathbf{z}^{n} | \mathbf{y}^{n} - \hat{\mathbf{x}}_{0}^{n})$$

$$= \sum_{m=1}^{n} \left(h(\mathbf{z}[m]) - h(\mathbf{z}[m] | \mathbf{z}[1], ..., \mathbf{z}[m-1], \mathbf{y}^{n} - \hat{\mathbf{x}}_{0}^{n}) \right)$$

$$\geq \sum_{m=1}^{n} \left(h(\mathbf{z}[m]) - h(\mathbf{z}[m] | \mathbf{y}[m] - \hat{\mathbf{x}}_{0}[m]) \right)$$

$$= \sum_{m=1}^{n} I(\mathbf{z}[m]; \mathbf{x}[m] - \hat{\mathbf{x}}_{0}[m] + \mathbf{z}[m])$$

$$\stackrel{(a)}{\geq} \sum_{m=1}^{n} \frac{1}{2} \log \frac{|\mathbf{K}_{c}[m] + \mathbf{K}_{z}[m]|}{|\mathbf{K}_{c}[m]|}$$

$$\stackrel{(b)}{\geq} \frac{n}{2} \log \frac{|\mathbf{D}_{0} + \mathbf{K}_{z}|}{|\mathbf{D}_{0}|},$$

where (a) is from (106) and [26, Lemma II.2]. The justification for (b) is from the convexity of $\log \frac{|\mathbf{A}+\mathbf{B}|}{|\mathbf{B}|}$ in **A** and (9). From (105) and (107) we have

$$h(\mathbf{y}^{n}|C_{1}, \ldots, C_{L}) - h(\mathbf{x}^{n}|C_{1}, \ldots, C_{L}) \ge \frac{n}{2}\log\frac{|\mathbf{D}_{0} + \mathbf{K}_{z}|}{|\mathbf{D}_{0}|}.$$
 (108)

Combining (102), (103) and (108), we have

$$I(C_1; C_2; \ldots; C_L) + I(C_1, \ldots, C_L; \mathbf{x}^n) \ge \frac{n}{2} \log \frac{|\mathbf{K}_x| |\mathbf{K}_x + \mathbf{K}_z|^{(L-1)} |\mathbf{D}_0 + \mathbf{K}_z|}{|\mathbf{D}_0| \prod_{l=1}^L |\mathbf{D}_l + \mathbf{K}_z|}.$$
 (109)

Taking the supremum over all positive definite \mathbf{K}_z , we can sharpen the lower bound in (109):

$$\sum_{l=1}^{L} I(C_1; C_2; \dots; C_L) + I(C_1, \dots, C_L; \mathbf{x}^n) \ge \sup_{\mathbf{K}_z \succ \mathbf{0}} \frac{n}{2} \log \frac{|\mathbf{K}_x| |\mathbf{K}_x + \mathbf{K}_z|^{(L-1)} |\mathbf{D}_0 + \mathbf{K}_z|}{|\mathbf{D}_0| \prod_{l=1}^{L} |\mathbf{D}_l + \mathbf{K}_z|}.$$
(110)

Proof of Proposition 1 \mathbf{C}

Conditioned on y, the collection of random variables $(\mathbf{u}_1, \ldots, \mathbf{u}_L)$ are jointly Gaussian and thus we have

$$\sum_{l=1}^{L} h(\mathbf{u}_{l}|\mathbf{y}) - h(\mathbf{u}_{1}, \ldots, \mathbf{u}_{L}|\mathbf{y}) = \frac{1}{2} \log \frac{\prod_{l=1}^{L} |\operatorname{Cov}[\mathbf{u}_{l}|\mathbf{y}]|}{\left|\operatorname{Cov}[\mathbf{u}_{1}, \ldots, \mathbf{u}_{L}|\mathbf{y}]\right|}.$$
 (111)

From MMSE of \mathbf{u}_l from \mathbf{y} we have

$$\operatorname{Cov}[\mathbf{u}_{l}|\mathbf{y}] = \mathbf{K}_{x} + \mathbf{K}_{w_{l}} - \mathbf{K}_{x}(\mathbf{K}_{x} + \mathbf{K}_{z})^{-1}\mathbf{K}_{x}, \quad l = 1, \dots, L$$
(112)

and

$$\operatorname{Cov}(\mathbf{u}_1, \ldots, \mathbf{u}_L | \mathbf{y}) = \mathbf{J} \otimes \mathbf{K}_x + \mathbf{K}_w - \mathbf{J} \otimes \left(\mathbf{K}_x (\mathbf{K}_x + \mathbf{K}_z)^{-1} \mathbf{K}_x \right),$$
(113)
where **J** is an $L \times L$ matrix of all ones and \otimes is the Kronecker Product.

By Fischer inequality (the block matrix version of Hadamard inequality, see [25, Theorem 6.10]) we know that $\prod_{l=1}^{L} |\operatorname{Cov}[\mathbf{u}_l|\mathbf{y}]| = |\operatorname{Cov}[\mathbf{u}_1, \ldots, \mathbf{u}_L|\mathbf{y}]|$ if and only if the off-diagonal block matrices of $\operatorname{Cov}[\mathbf{u}_1, \ldots, \mathbf{u}_L|\mathbf{y}]$ are all zero matrices. Thus we have

$$\sum_{l=1}^{L} h(\mathbf{u}_l | \mathbf{y}) - h(\mathbf{u}_1, \ldots, \mathbf{u}_L | \mathbf{y}) = 0$$

if and only if

$$\mathbf{K}_x - \mathbf{A} = \mathbf{K}_x (\mathbf{K}_x + \mathbf{K}_z)^{-1} \mathbf{K}_x, \qquad (114)$$

or equivalently, if and only if

$$\mathbf{K}_z = \mathbf{K}_x (\mathbf{K}_x - \mathbf{A})^{-1} \mathbf{K}_x - \mathbf{K}_x.$$
(115)

To get a valid $\mathbf{K}_z \succ \mathbf{0}$, we need the additional condition $\mathbf{0} \prec \mathbf{A} \prec \mathbf{K}_x$.

Proof of Lemma 2 D

First we assume $\mathbf{A} \succ 0$, and hence

$$\begin{bmatrix} \mathbf{A}^{-1} + (\mathbf{I} \mathbf{I} \dots \mathbf{I}) \mathbf{K}_{w}^{-1} (\mathbf{I} \mathbf{I} \dots \mathbf{I})^{t} \end{bmatrix}^{-1}$$

= $\mathbf{A} - \mathbf{A} (\mathbf{I} \mathbf{I} \dots \mathbf{I}) \begin{bmatrix} \mathbf{K}_{w} + (\mathbf{I} \mathbf{I} \dots \mathbf{I})^{t} \mathbf{A} (\mathbf{I} \mathbf{I} \dots \mathbf{I}) \end{bmatrix}^{-1} (\mathbf{I} \mathbf{I} \dots \mathbf{I})^{t} \mathbf{A}$
= $\mathbf{A} - \mathbf{A} (\mathbf{I} \mathbf{I} \dots \mathbf{I}) \begin{bmatrix} \text{diag} \{ \mathbf{K}_{w_{1}} + \mathbf{A}, \mathbf{K}_{w_{2}} + \mathbf{A}, \dots \mathbf{K}_{w_{L}} + \mathbf{A} \} \end{bmatrix}^{-1} (\mathbf{I} \mathbf{I} \dots \mathbf{I})^{t} \mathbf{A}$ (116)
= $\mathbf{A} - \mathbf{A} \sum_{l=1}^{L} [\mathbf{K}_{w_{l}} + \mathbf{A}]^{-1} \mathbf{A}.$

Thus,

$$(\mathbf{I} \mathbf{I} \dots \mathbf{I}) \mathbf{K}_{w}^{-1} (\mathbf{I} \mathbf{I} \dots \mathbf{I})^{t}$$

$$= \left[\mathbf{A} - \mathbf{A} \sum_{l=1}^{L} (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1} \mathbf{A} \right]^{-1} - \mathbf{A}^{-1}$$

$$= \mathbf{A}^{-1} - \mathbf{A}^{-1} \mathbf{A} \left[- \left(\sum_{l=1}^{L} (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1} \right)^{-1} + \mathbf{A} \mathbf{A}^{-1} \mathbf{A} \right]^{-1} \mathbf{A} \mathbf{A}^{-1} - \mathbf{A}^{-1} \qquad (117)$$

$$= \left[\left(\sum_{l=1}^{L} (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1} \right)^{-1} - \mathbf{A} \right]^{-1}.$$

When **A** is singular, we can choose $\delta > 0$ such that $\mathbf{A} + \epsilon \mathbf{I} \succ 0$ for $\epsilon \in (0, \delta)$, and thus we can apply the previous argument and let $\epsilon \to 0^+$ in the end.

E Proof of Lemma 3

We use induction. First consider the matrix

$$oldsymbol{\Delta}_2 = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A} \ -\mathbf{A} & \mathbf{K}_{w_2} \end{pmatrix}.$$

.

We have

$$\begin{split} \mathbf{\Delta}_{2} \succ \mathbf{0} & \Longleftrightarrow \mathbf{K}_{w_{2}} \succ \mathbf{A}\mathbf{K}_{w_{1}}^{-1}\mathbf{A} \\ & \Leftrightarrow \mathbf{K}_{w_{2}} + \mathbf{A} \succ \mathbf{A}\mathbf{K}_{w_{1}}^{-1}\mathbf{A} + \mathbf{A} \\ & \Leftrightarrow (\mathbf{K}_{w_{2}} + \mathbf{A})^{-1} \prec (\mathbf{A}\mathbf{K}_{w_{1}}^{-1}\mathbf{A} + \mathbf{A})^{-1} \\ & \Leftrightarrow (\mathbf{K}_{w_{2}} + \mathbf{A})^{-1} \prec (\mathbf{A}\mathbf{K}_{w_{1}}^{-1}\mathbf{A} + \mathbf{A})^{-1} \\ & \Leftrightarrow (\mathbf{K}_{w_{1}} + \mathbf{A})^{-1} \prec \mathbf{A}^{-1} - (\mathbf{K}_{w_{1}} + \mathbf{A})^{-1} \prec \mathbf{A}^{-1} \\ & \leftarrow \sum_{l=1}^{L} (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1} \prec \mathbf{A}^{-1} \\ & \leftarrow \mathbf{K}_{w_{0}} + \mathbf{A} \succ \mathbf{A} \\ & \leftrightarrow \mathbf{K}_{w_{0}} \succ \mathbf{0}, \end{split}$$
(118)

where (a) is from (27).

Next we define

$$oldsymbol{\Delta}_k = egin{pmatrix} \mathbf{K}_{w_1} & -\mathbf{A} & -\mathbf{A} & \ldots & -\mathbf{A} \ -\mathbf{A} & \mathbf{K}_{w_2} & -\mathbf{A} & \ldots & -\mathbf{A} \ \ldots & \ldots & \ldots & \ldots \ -\mathbf{A} & \ldots & -\mathbf{A} & \mathbf{K}_{w_{k-1}} & -\mathbf{A} \ -\mathbf{A} & \ldots & -\mathbf{A} & \mathbf{K}_{w_k} \end{pmatrix}$$

and suppose $\Delta_k \succ 0$ for $k = 3, \ldots, l - 1$. Then

$$\begin{split} \mathbf{\Delta}_{l} \succ 0 & \iff \mathbf{K}_{w_{l}} \succ \mathbf{A}(\mathbf{I}, \mathbf{I}, \dots, \mathbf{I}) \Delta_{l-1}^{-1}(\mathbf{I}, \mathbf{I}, \dots, \mathbf{I})^{t} \mathbf{A} \\ & \iff \mathbf{K}_{w_{l}} \succ \mathbf{A} \left[\left(\sum_{k=1}^{l-1} (\mathbf{K}_{W_{k}} + \mathbf{A})^{-1} \right)^{-1} - \mathbf{A} \right]^{-1} \mathbf{A} \\ & \iff \mathbf{K}_{w_{l}} + \mathbf{A} \succ \mathbf{A} \left[\left(\sum_{k=1}^{l-1} (\mathbf{K}_{w_{k}} + \mathbf{A})^{-1} \right)^{-1} - \mathbf{A} \right]^{-1} \mathbf{A} + \mathbf{A} \\ & \iff (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1} \prec \left[\mathbf{A} \left[\left(\sum_{k=1}^{l-1} (\mathbf{K}_{w_{k}} + \mathbf{A})^{-1} \right)^{-1} - \mathbf{A} \right]^{-1} \mathbf{A} + \mathbf{A} \right]^{-1} \\ & \iff (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1} \prec \mathbf{A}^{-1} - \left[\left(\sum_{k=1}^{l-1} (\mathbf{K}_{w_{k}} + \mathbf{A})^{-1} \right)^{-1} - \mathbf{A} + \mathbf{A} \right]^{-1} \\ & \iff (\mathbf{K}_{w_{l}} + \mathbf{A})^{-1} \prec \mathbf{A}^{-1} - \sum_{k=1}^{l-1} (\mathbf{K}_{w_{k}} + \mathbf{A})^{-1} \\ & \iff (\mathbf{K}_{w_{k}} + \mathbf{A})^{-1} \prec \mathbf{A}^{-1} \\ & \iff \sum_{k=1}^{L} (\mathbf{K}_{w_{k}} + \mathbf{A})^{-1} \prec \mathbf{A}^{-1} \\ & \iff \mathbf{K}_{w_{0}} + \mathbf{A} \succ \mathbf{A} \\ & \iff \mathbf{K}_{w_{0}} \succ \mathbf{0}, \end{split}$$

where (b) is from (27).

F Proof of Proposition 2

First we prove that

$$\mathbf{D}_0^{-1} + \mathbf{K}_x^{-1} - \mathbf{D}_1^{-1} - \mathbf{D}_2^{-1} \succ \mathbf{0} \Rightarrow \mathbf{A}^* \succ \mathbf{0}.$$

Proof. We have

$$\mathbf{A}^{*} = (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{\frac{1}{2}} \left[(\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} (\mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}}) (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \right]^{\frac{1}{2}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{\frac{1}{2}} - \mathbf{K}_{w_{0}}.$$
(120)

Thus

$$\begin{aligned} \mathbf{A}^{*} \succ \mathbf{0} \\ \Leftrightarrow (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{\frac{1}{2}} \left[(\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} (\mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}}) (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \right]^{\frac{1}{2}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{\frac{1}{2}} \succ \mathbf{K}_{w_{0}} \\ \Leftrightarrow \left[(\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} (\mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}}) (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \right]^{\frac{1}{2}} \succ (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \mathbf{K}_{w_{0}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \\ \Leftrightarrow (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} (\mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}}) (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \\ \Leftrightarrow (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \mathbf{K}_{w_{0}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-1} \mathbf{K}_{w_{0}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-\frac{1}{2}} \\ \Leftrightarrow (\mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}} \succ \mathbf{K}_{w_{0}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-1} \mathbf{K}_{w_{0}} \\ \Leftrightarrow \mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}} \succ \mathbf{K}_{w_{0}} (-\mathbf{I} + (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-1} \mathbf{K}_{w_{1}} \\ \Leftrightarrow \mathbf{K}_{w_{2}} - \mathbf{K}_{w_{0}} \succ - \mathbf{K}_{w_{0}} + \mathbf{K}_{w_{0}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-1} \mathbf{K}_{w_{1}} \\ \Leftrightarrow \mathbf{K}_{w_{2}} \succ \mathbf{K}_{w_{0}} (\mathbf{K}_{w_{1}} - \mathbf{K}_{w_{0}})^{-1} \mathbf{K}_{w_{1}} \\ \Leftrightarrow \mathbf{K}_{w_{2}} \succ \mathbf{K}_{w_{0}} \mathbf{K}_{w_{0}}^{-1} (\mathbf{K}_{w_{0}}^{-1} - \mathbf{K}_{w_{1}}^{-1})^{-1} \mathbf{K}_{w_{1}}^{-1} \mathbf{K}_{w_{1}} \\ \Leftrightarrow \mathbf{K}_{w_{2}} \succ (\mathbf{K}_{w_{0}}^{-1} - \mathbf{K}_{w_{0}}^{-1})^{-1} \\ \Leftrightarrow \mathbf{K}_{w_{1}}^{-1} + \mathbf{K}_{w_{1}}^{-1} \prec \mathbf{K}_{w_{0}}^{-1} \\ \Leftrightarrow \mathbf{D}_{0}^{-1} + \mathbf{K}_{w}^{-1} - \mathbf{D}_{1}^{-1} - \mathbf{D}_{2}^{-1} \succ \mathbf{0}. \end{aligned}$$
(121)

The proof of

$$\mathbf{D}_0 + \mathbf{K}_x - \mathbf{D}_1 - \mathbf{D}_2 \succ \mathbf{0} \Rightarrow \mathbf{A}^* \prec \mathbf{K}_x$$

is similar and hence is omitted.

G Proof of Lemma 4

$$[(\mathbf{K}_{w_{0}} + \mathbf{A}^{*})^{-1} + \mathbf{\Lambda}_{1}]^{-1} = [(\mathbf{K}_{w_{0}} + \mathbf{A}^{*})^{-1}(\mathbf{I} + (\mathbf{K}_{w_{0}} + \mathbf{A}^{*})\mathbf{\Lambda}_{1})]^{-1} \stackrel{(a)}{=} (\mathbf{I} + \mathbf{K}_{w_{0}}\mathbf{\Lambda}_{1})^{-1}(\mathbf{K}_{w_{0}} + \mathbf{A}^{*}) = (\mathbf{I} + \mathbf{K}_{w_{0}}\mathbf{\Lambda}_{1})^{-1}(\mathbf{K}_{w_{0}} + \mathbf{A}^{*} - (\mathbf{I} + \mathbf{K}_{w_{0}}\mathbf{\Lambda}_{1})\mathbf{A}^{*}) + \mathbf{A}^{*} \stackrel{(b)}{=} (\mathbf{I} + \mathbf{K}_{w_{0}}\mathbf{\Lambda}_{1})^{-1}\mathbf{K}_{w_{0}} + \mathbf{A}^{*} = (\mathbf{K}_{w_{0}}^{-1}(\mathbf{I} + \mathbf{K}_{w_{0}}\mathbf{\Lambda}_{1}))^{-1} + \mathbf{A}^{*} = (\mathbf{K}_{w_{0}}^{-1} + \mathbf{\Lambda}_{1})^{-1} + \mathbf{A}^{*},$$

$$(122)$$

where (a) and (b) are from $\Lambda_1 \mathbf{A}^* = \mathbf{0}$.

$$\frac{|\mathbf{D}_{0}^{*} + \mathbf{K}_{z}|}{|\mathbf{D}_{0}^{*}|} = |\mathbf{I} + \mathbf{D}_{0}^{*-1}\mathbf{K}_{z}|
= |\mathbf{I} + (\mathbf{D}_{0}^{-1} + \mathbf{\Lambda}_{1})\mathbf{K}_{z}|
= |\mathbf{I} + \mathbf{D}_{0}^{-1}\mathbf{K}_{z} + \mathbf{\Lambda}_{1}\mathbf{K}_{z}|
= |\mathbf{I} + \mathbf{D}_{0}^{-1}\mathbf{K}_{z} + \mathbf{\Lambda}_{1} ((\mathbf{I} - \mathbf{A}^{*})^{-1} - \mathbf{I})|$$
(123)
$$\frac{^{(c)}}{=} |\mathbf{I} + \mathbf{D}_{0}^{-1}\mathbf{K}_{z} + \mathbf{\Lambda}_{1} (\mathbf{I} - \mathbf{A}^{*}) ((\mathbf{I} - \mathbf{A}^{*})^{-1} - \mathbf{I})|
= |\mathbf{I} + \mathbf{D}_{0}^{-1}\mathbf{K}_{z}|
= \frac{|\mathbf{D}_{0} + \mathbf{K}_{z}|}{|\mathbf{D}_{0}|},$$

where (c) is from $\Lambda_1 \mathbf{A}^* = \mathbf{0}$.

H Proof of Equations (70) **and** (71)

We first prove the following lemma.

Lemma 9. Let **D** be an $N \times N$ matrix such that $\mathbf{0} \prec \mathbf{D} \prec \mathbf{I}$. Let $\mathbf{K} = (\mathbf{D}^{-1} - \mathbf{I})^{-1}$. Choose $\epsilon > 0$ such that $\mathbf{K} - \epsilon \mathbf{I} \succ \mathbf{0}$. Define

$$\mathbf{D}(\epsilon) \stackrel{def}{=} \left[(\mathbf{K} - \epsilon \mathbf{I})^{-1} + \mathbf{I} \right]^{-1}.$$

Then, there exist constants $b_1 \ge b_2 > 0$, such that

$$\mathbf{D} - b_1 \epsilon \mathbf{I} + o(\epsilon) \prec \mathbf{D}(\epsilon) \prec \mathbf{D} - b_2 \epsilon \mathbf{I} + o(\epsilon)$$

Proof. There exists an $N \times N$ orthogonal matrix **Q** such that

$$\mathbf{Q}\mathbf{K}\mathbf{Q}^t = \operatorname{diag}\{k_1, \ldots, k_N\},\$$

where $k_i > 0$ are eigenvalues of **K**. We have

$$\mathbf{QDQ}^{t} = \mathbf{Q}(\mathbf{K}^{-1} + \mathbf{I})^{-1}\mathbf{Q}^{t}$$
$$= \operatorname{diag}\left\{\frac{k_{1}}{1+k_{1}}, \dots, \frac{k_{N}}{1+k_{N}}\right\},$$

and

$$\mathbf{QD}(\epsilon)\mathbf{Q}^{t} = \mathbf{Q}\left[(\mathbf{K} - \epsilon \mathbf{I})^{-1} + \mathbf{I}\right]^{-1}\mathbf{Q}^{t}$$

= $\left[(\operatorname{diag}\{k_{1}, \ldots, k_{N}\} - \epsilon \mathbf{I})^{-1} + \mathbf{I}\right]^{-1}$
= $\operatorname{diag}\left\{\frac{k_{1} - \epsilon}{1 + k_{1} - \epsilon}, \ldots, \frac{k_{N} - \epsilon}{1 + k_{N} - \epsilon}\right\}$
= $\operatorname{diag}\left\{\frac{k_{1}}{1 + k_{1}} - \frac{\epsilon}{(1 + k_{1})^{2}} + o(\epsilon), \ldots, \frac{k_{N}}{1 + k_{N}} - \frac{\epsilon}{(1 + k_{N})^{2}} + o(\epsilon)\right\}.$

We now have

$$\mathbf{QDQ}^t - b_1 \epsilon \mathbf{I} + o(\epsilon) \prec \mathbf{QD}(\epsilon) \mathbf{Q}^t \prec \mathbf{QDQ}^t - b_2 \epsilon \mathbf{I} + o(\epsilon),$$

where $b_1 \ge b_2 > 0$ are some constants. Hence

$$\mathbf{D} - b_1 \epsilon \mathbf{I} + o(\epsilon) \prec \mathbf{D} - b_2 \epsilon \mathbf{I} + o(\epsilon).$$

Equations (70) and (71) are a direct consequence of this lemma.

I Proof of Equation (72)

We first prove the following lemma.

Lemma 10. Let \mathbf{A} be an $N \times N$ matrix such that $\mathbf{0} \preccurlyeq \mathbf{A} \prec \mathbf{I}$. Let $\mathbf{K}_z = (\mathbf{I} - \mathbf{A})^{-1} - \mathbf{I}$. Choose $\epsilon > 0$ such that $\mathbf{A} + \epsilon \mathbf{I} \prec \mathbf{I}$. Define

$$\mathbf{K}_{z}(\epsilon) \stackrel{def}{=} \left[\mathbf{I} - (\mathbf{A} + \epsilon \mathbf{I}) \right]^{-1} - \mathbf{I}.$$

Then, there exist constants $c_1 \ge c_2 > 0$ such that

$$\mathbf{K}_z - c_1 \epsilon \mathbf{I} + o(\epsilon) \prec \mathbf{K}_z(\epsilon) \prec \mathbf{K}_z - c_2 \epsilon \mathbf{I} + o(\epsilon).$$

Proof. There exists an $N \times N$ orthogonal matrix \mathbf{Q} such that

$$\mathbf{QAQ}^t = \operatorname{diag}\{a_1, \ldots, a_N\}$$

where $a_i > 0$ are the eigenvalues of **A**. We have

$$\mathbf{Q}\mathbf{K}_{z}\mathbf{Q}^{t} = \mathbf{Q}((\mathbf{I} - \mathbf{A})^{-1} - \mathbf{I})\mathbf{Q}^{t}$$
$$= \operatorname{diag}\left\{\frac{a_{1}}{1 - a_{1}}, \dots, \frac{a_{N}}{1 - a_{N}}\right\},$$

and

$$\begin{aligned} \mathbf{Q}\mathbf{K}_{z}(\epsilon)\mathbf{Q}^{t} &= \mathbf{Q}((\mathbf{I} - (\mathbf{A} + \epsilon \mathbf{I}))^{-1} - \mathbf{I})\mathbf{Q}^{t} \\ &= \operatorname{diag}\left\{\frac{a_{1} + \epsilon}{1 - a_{1} - \epsilon}, \dots, \frac{a_{N} + \epsilon}{1 - a_{N} - \epsilon}\right\} \\ &= \operatorname{diag}\left\{\frac{a_{1}}{1 - a_{1}} - \frac{(2a_{1} - 1)\epsilon}{(1 - a_{1})^{2}} + o(\epsilon), \dots, \frac{a_{N}}{1 - a_{N}} - \frac{(2a_{N} - 1)\epsilon}{(1 - a_{N})^{2}} + o(\epsilon)\right\}.\end{aligned}$$

We now have

$$\mathbf{Q}\mathbf{K}_{z}\mathbf{Q}^{t} - c_{1}\epsilon\mathbf{I} + o(\epsilon) \prec \mathbf{Q}\mathbf{K}_{z}(\epsilon)\mathbf{Q}^{t} \prec \mathbf{Q}\mathbf{K}_{z}\mathbf{Q}^{t} - c_{2}\epsilon\mathbf{I} + o(\epsilon),$$

where $c_1 \ge c_2 > 0$ are some constants. Hence

$$\mathbf{K}_z - c_1 \epsilon \mathbf{I} + o(\epsilon) \prec \mathbf{K}_z(\epsilon) \prec \mathbf{K}_z - c_2 \epsilon \mathbf{I} + o(\epsilon).$$

Equation (72) is a direct result of this lemma.

J Proof of equation (82)

We first prove the following lemma.

Lemma 11. Let **A** be an $N \times N$ matrix such that $\mathbf{0} \prec \mathbf{A} \preccurlyeq \mathbf{I}$. Choose $\epsilon > 0$ such that $\mathbf{A} - \epsilon \mathbf{I} \succ \mathbf{0}$. Define

$$\mathbf{K}_{z}(\epsilon) \stackrel{def}{=} \left[\mathbf{I} - (\mathbf{A} - \epsilon \mathbf{I})\right]^{-1} - \mathbf{I}.$$

Then, for any E and F such that $0\prec E\preccurlyeq I$ and $0\prec F\preccurlyeq I,$ we have

$$\lim_{\epsilon \to 0} \frac{|\mathbf{E} + \mathbf{K}_z(\epsilon)|}{|\mathbf{F} + \mathbf{K}_z(\epsilon)|} = 1$$

Proof. There exists an $N \times N$ orthogonal matrix \mathbf{Q} such that

$$\mathbf{QAQ}^t = \operatorname{diag}\{a_1, \ldots, a_N\},\$$

where $0 < a_i \leq 1$ are eigenvalues of **A**. Without loss of generality, we suppose $a_1 = 1, \ldots, a_p = 1, a_{p+1} < 1, \ldots, a_N < 1$.

We have

$$\mathbf{Q}\mathbf{K}_{z}(\epsilon)\mathbf{Q}^{t} = \mathbf{Q}((\mathbf{I} - (\mathbf{A} - \epsilon\mathbf{I}))^{-1} - \mathbf{I})\mathbf{Q}^{t}$$
$$= \operatorname{diag}\left\{\frac{1-\epsilon}{\epsilon}, \dots, \frac{1-\epsilon}{\epsilon}, \frac{a_{p+1}-\epsilon}{1-a_{p+1}+\epsilon}, \frac{a_{N}-\epsilon}{1-a_{N}+\epsilon}\right\},\$$

and since

$$\frac{|\mathbf{I} + \mathbf{K}_z(\epsilon)|}{|\mathbf{K}_z(\epsilon)|} \ge \frac{|\mathbf{E} + \mathbf{K}_z(\epsilon)|}{|\mathbf{F} + \mathbf{K}_z(\epsilon)|} \ge \frac{|\mathbf{K}_z(\epsilon)|}{|\mathbf{I} + \mathbf{K}_z(\epsilon)|},$$

we have

$$\lim_{\epsilon \to 0} \frac{|\mathbf{E} + \mathbf{K}_z(\epsilon)|}{|\mathbf{F} + \mathbf{K}_z(\epsilon)|} = 1.$$

Equation (82) is a direct consequence of this lemma.

K Proof of Equation (89)

We would like to have a property similar to (66), as ϵ_1 approaches zero, and a property similar to (82), as ϵ_2 approaches zero. To see this is the case, we need the following lemma.

Lemma 12.

$$\mathbf{\Lambda}_1 \mathbf{K}_z(\epsilon_1 = 0, \epsilon_2) = \mathbf{0}$$

Proof. Since

$$\mathbf{Q} \mathbf{\Lambda}_1 \mathbf{Q}^t \mathbf{Q} \mathbf{A}^* \mathbf{Q}^t = \mathbf{0}$$

and

$$\mathbf{QA}^*\mathbf{Q}^t = \operatorname{diag}(\underbrace{0, \ldots, 0}_{p}, \underbrace{1, \ldots, 1}_{q}, a_{p+q+1}, \ldots, a_s)$$
$$\mathbf{QA}^*\mathbf{Q}^t - \epsilon_2 \mathbf{E}_2 = \operatorname{diag}(\underbrace{0, \ldots, 0}_{p}, \underbrace{1-\epsilon_2, \ldots, 1-\epsilon_2}_{q}, a_{p+q+1}, \ldots, a_s),$$

we have that

$$\mathbf{Q}\mathbf{A}^{*}\mathbf{Q}^{t}(\mathbf{Q}\mathbf{A}^{*}\mathbf{Q}^{t}-\epsilon_{2}\mathbf{E}_{2})=\mathbf{0}.$$

Thus

$$\begin{aligned} \mathbf{Q} \mathbf{\Lambda}_1 \mathbf{K}_z(\epsilon_1 = 0, \epsilon_2) \mathbf{Q}^t &= \mathbf{Q} \mathbf{\Lambda}_1 \mathbf{Q}^t \mathbf{Q} \left((\mathbf{I} - \mathbf{A}^* + \mathbf{Q}^t \epsilon_2 \mathbf{E}_2 \mathbf{Q})^{-1} - \mathbf{I} \right) \mathbf{Q}^t \\ &= \mathbf{Q} \mathbf{\Lambda}_1 \mathbf{Q}^t \left((\mathbf{I} - \mathbf{Q} \mathbf{A}^* \mathbf{Q}^t + \epsilon_2 \mathbf{E}_2)^{-1} - \mathbf{I} \right) \\ &= \mathbf{Q} \mathbf{\Lambda}_1 \mathbf{Q}^t (\mathbf{I} - \mathbf{Q} \mathbf{A}^* \mathbf{Q}^t + \epsilon_2 \mathbf{E}_2) \left((\mathbf{I} - \mathbf{Q} \mathbf{A}^* \mathbf{Q}^t + \epsilon_2 \mathbf{E}_2)^{-1} - \mathbf{I} \right) \\ &= \mathbf{0}. \end{aligned}$$

г		
н		
L		

Using this lemma, we can show a property similar to (66) as ϵ_1 approaches zero. First note that similar to case 2, we have

$$\mathbf{D}_0^{-1} + \mathbf{\Lambda}_1 - e_2 \epsilon_2 \mathbf{I} + o(\epsilon_2) \prec \mathbf{D}_0^{-1}(\epsilon_1, \epsilon_2) \prec \mathbf{D}_0^{-1} + \mathbf{\Lambda}_1 + e_1 \epsilon_1 \mathbf{I} + o(\epsilon_1)$$

where $e_1 > 0$ and $e_2 > 0$ are constants. Hence we have

$$\frac{|\mathbf{D}_{0}(\epsilon_{1}=0,\epsilon_{2})+\mathbf{K}_{z}(\epsilon_{1}=0,\epsilon_{2})|}{|\mathbf{D}_{0}(\epsilon_{1}=0,\epsilon_{2})|} = |\mathbf{I}+\mathbf{D}_{0}^{-1}(\epsilon_{1}=0,\epsilon_{2})\mathbf{K}_{z}(\epsilon_{1}=0,\epsilon_{2})|$$
$$\geq |\mathbf{I}+(\mathbf{D}_{0}^{-1}+\mathbf{\Lambda}_{1}-e_{2}\epsilon_{2}\mathbf{I})\mathbf{K}_{z}(\epsilon_{1}=0,\epsilon_{2})|$$
$$= |\mathbf{I}+\mathbf{D}_{0}^{-1}\mathbf{K}_{z}(\epsilon_{1}=0,\epsilon_{2})-e_{2}\epsilon_{2}\mathbf{K}_{z}(\epsilon_{1}=0,\epsilon_{2})|$$
$$= \frac{|\mathbf{D}_{0}+\mathbf{K}_{z}(\epsilon_{1}=0,\epsilon_{2})-e_{2}\epsilon_{2}\mathbf{D}_{0}\mathbf{K}_{z}(\epsilon_{1}=0,\epsilon_{2})|}{|\mathbf{D}_{0}|}.$$

Similarly, we have

$$\frac{|\mathbf{D}_0(\epsilon_1=0,\epsilon_2)+\mathbf{K}_z(\epsilon_1=0,\epsilon_2)|}{|\mathbf{D}_0(\epsilon_1=0,\epsilon_2)|} \le \frac{|\mathbf{D}_0+\mathbf{K}_z(\epsilon_1=0,\epsilon_2)|}{|\mathbf{D}_0|}.$$

Thus

$$\lim_{\epsilon_{2}\to 0} \lim_{\epsilon_{1}\to 0} \frac{1}{2} \log \frac{|\mathbf{I} + \mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2})|^{(L-1)} |\mathbf{D}_{0}(\epsilon_{1}, \epsilon_{2}) + \mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2})|}{|\mathbf{D}_{0}(\epsilon_{1}, \epsilon_{2})| \prod_{l=1}^{L} |\mathbf{D}_{l}(\epsilon_{1}, \epsilon_{2}) + \mathbf{K}_{z}(\epsilon_{1}, \epsilon_{2})|}$$

$$= \lim_{\epsilon_{2}\to 0} \frac{1}{2} \log \frac{|\mathbf{I} + \mathbf{K}_{z}(\epsilon_{1} = 0, \epsilon_{2})|^{(L-1)} |\mathbf{D}_{0} + \mathbf{K}_{z}(\epsilon_{1} = 0, \epsilon_{2})|}{|\mathbf{D}_{0}| \prod_{l=1}^{L} |\mathbf{D}_{l}(\epsilon_{1} = 0, \epsilon_{2}) + \mathbf{K}_{z}(\epsilon_{1} = 0, \epsilon_{2})|} \qquad (124)$$

$$= \frac{1}{2} \log \frac{1}{|\mathbf{D}_{0}|},$$

where the last step is similar to (82).

References

- [1] L. Ozarow, "On a source-coding problem with two channels and three receivers," *Bell Syst. Tech. J.*, vol. 59, no. 10, pp. 1909–1921, Dec. 1980.
- [2] A. E. Gamal and T. M. Cover, "Achievable rates for multiple descriptions," *IEEE Trans. Inform. Theory*, vol. 28, no. 6, pp. 851–857, Nov. 1982.
- [3] R. Ahlswede, "The rate-distortion region for multiple descriptions without excess rate," *IEEE Trans. Inform. Theory*, vol. 31, no. 6, pp. 721–726, Nov. 1985.

- [4] Z. Zhang and T. Berger, "New results in binary multiple descriptions," IEEE Trans. Inform. Theory, vol. 33, no. 4, pp. 502–521, July 1987.
- [5] R. Zamir, "Gaussian codes and shannon bounds for multiple descriptions," *IEEE Trans. Inform. Theory*, vol. 45, no. 7, pp. 2629–2635, Nov. 1999.
- [6] F. W. Fu and R. W. Yeung, "On the rate-distortion region for multiple descriptions," *IEEE Trans. Inform. Theory*, vol. 48, no. 7, pp. 2012–2021, July 2002.
- [7] R. Venkataramani, G. Kramer, and V. K. Goyal, "Multiple description coding with many channels," *IEEE Trans. Inform. Theory*, vol. 49, no. 9, pp. 2106–2114, Sept. 2003.
- [8] S. S. Pradhan, R. Puri, and K. Ramchandran, "n-channel symmetric multiple descroption-part I: (n,k) source-channel erasure codes," *IEEE Trans. Inform. The*ory, vol. 50, no. 1, pp. 47–61, Jan. 2004.
- [9] H. Feng and M. Effros, "On the rate loss of multiple description source codes," *IEEE Trans. Inform. Theory*, vol. 51, no. 2, pp. 671–683, Feb. 2005.
- [10] R. Puri, S. S. Pradhan, and K. Ramchandran, "n-channel symmetric multiple descroption-part II: an achievable rate-distortion region," *IEEE Trans. Inform. The*ory, vol. 51, no. 4, pp. 1377–1392, Apr. 2005.
- [11] V. A. Vaishampayan, "Design of multiple description scalar quantizers," IEEE Trans. Inform. Theory, vol. 39, no. 3, pp. 821–834, May 1993.
- [12] V. A. Vaishampayan and J. Domaszewicz, "Design of entropy-constrained multipledescription scalar quantizers," *IEEE Trans. Inform. Theory*, vol. 40, no. 1, pp. 245–250, Jan. 1994.
- [13] V. A. Vaishampayan and J. C. Batllo, "Asymptotic analysis of multiple-description quantizers," *IEEE Trans. Inform. Theory*, vol. 44, no. 1, pp. 278–284, Jan. 1998.
- [14] V. A. Vaishampayan, N. Sloane, and S. Servetto, "Multiple description vector quantizers with lattice codebooks: design and analysis," *IEEE Trans. Inform. Theory*, vol. 47, no. 4, pp. 1718–1734, July 2001.
- [15] V. K. Goyal, "Multiple description coding: compression meets the network," *IEEE Signal Processing Mag.*, vol. 18, pp. 74–93, Sept. 2001.
- [16] S. N. Diggavi, N. J. A. Sloane, and V. A. Vaishampayan, "Asymmetric multiple description lattice vector quantizers," *IEEE Trans. Inform. Theory*, vol. 48, no. 1, pp. 174–191, Jan. 2002.

- [17] V. K. Goyal, J. A. Kelner, and J. Kovacevic, "Multiple description vector quantization with a coarse lattice," *IEEE Trans. Inform. Theory*, vol. 48, no. 3, pp. 781–788, Mar. 2002.
- [18] C. Tian and S. S. Hemami, "Universal multiple description scalar quantizer: analysis and design," *IEEE Trans. Inform. Theory*, vol. 50, no. 9, pp. 2737–2751, Sept. 2004.
- [19] P. Ishwar, R. Puri, S. S. Pradhan, and K. Ramchandran, "On compression for robust estimation in sensor networks," in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, Yokohama, Japan, June-July 2003.
- [20] J. Chen and T. Berger, "Robust distributed source coding," submitted to IEEE Trans. Inform. Theory, 2005.
- [21] N. Alon and J. H. Spencer, *The probabilistic Method*, 2nd edition. New York: Wiley, 2000.
- [22] D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.
- [23] D. Tse and S. Hanly, "Multi-access fading channels: part I: polymatroid structure,optimal resource allocation and throughput capacities," *IEEE Trans. Inform. Theory*, vol. 44, no. 7, pp. 2796–2815, Nov. 1998.
- [24] P. Viswanath, "Sum rate of a class of gaussian multiterminal source coding problems," in Advances in Network Information Theory, P. Gupta, G. Kramer and A. Wijngaarden editors, Rutgers, NJ, 2004, pp. 43–60.
- [25] F. Zhang, Matrix Theory: Basic Results and Techniques. Springer, 1999.
- [26] S. Diggavi and T. M. Cover, "Worst additive noise under covariance constraints," *IEEE Trans. Inform. Theory*, vol. 47, no. 7, pp. 3072–3081, Nov. 2001.