# Low Correlation Sequences over the QAM Constellation 

M. Anand, Student Member, IEEE, and P. Vijay Kumar, Fellow, IEEE


#### Abstract

This paper presents the first concerted look at low correlation sequence families over QAM constellations of size $M^{2}=4^{m}$ and their potential applicability as spreading sequences in a CDMA setting. Five constructions are presented, and it is shown how such sequence families have the ability to transport a larger amount of data as well as enable variable-rate signalling on the reverse link.

Canonical family $\mathcal{C Q}$ has period $N$, normalized maximumcorrelation parameter $\bar{\theta}_{\text {max }}$ bounded above by $\lesssim a \sqrt{N}$, where $a$ ranges from 1.8 in the 16 -QAM case to 3.0 for large $M$. In a CDMA setting, each user is enabled to transfer $2 m$ bits of data per period of the spreading sequence which can be increased to 3 m bits of data by halving the size of the sequence family. The technique used to construct $\mathcal{C Q}$ is easily extended to produce larger sequence families and an example is provided.

Selected family $\mathcal{S Q}$ has a lower value of $\theta_{\max }$ but permits only $(m+1)$-bit data modulation. The interleaved 16-QAM sequence family $\mathcal{I Q}$ has $\bar{\theta}_{\max } \lesssim \sqrt{2} \sqrt{N}$ and supports 3 -bit data modulation.

The remaining two families are over a quadrature-PAM (QPAM) subset of size $2 M$ of the $M^{2}$-QAM constellation. Family $\mathcal{P}$ has a lower value of $\bar{\theta}_{\text {max }}$ in comparison with Family $\mathcal{S} \mathcal{Q}$, while still permitting $(m+1)$-bit data modulation. Interleaved family IP, over the 8 -ary Q-PAM constellation, permits 3 -bit data modulation and interestingly, achieves the Welch lower bound on $\bar{\theta}_{\text {max }}$.


Index Terms-QAM, Q-PAM, low-correlation sequences, CDMA, variable-rate signalling, quaternary sequences, galois rings.

## I. Introduction

In Direct-Sequence Code Division Multiple Access (DSCDMA) systems, low-correlation spreading sequences are employed to separate the signals of different users. In this paper, constructions of families of low-correlation spreading sequences over the $M^{2}$-QAM constellation, $M=2^{m}$, as well as over a quadrature-PAM subconstellation of size $2 M$ are presented.

The periodic correlation between two complex-valued sequences, $\{s(j, t)\}$ and $\{s(k, t)\}$, at time shift $\tau$ is defined

[^0]as
\[

$$
\begin{aligned}
\theta_{s(j), s(k)}(\tau)= & \sum_{t=0}^{N-1} s(j, t+\tau) \overline{s(k, t)} \\
& \text { where } 0 \leq \tau \leq(N-1)
\end{aligned}
$$
\]

with $(t+\tau)$ computed modulo $N$. This form of correlation is also referred to as even-periodic correlation to differentiate it from other forms of correlation between two sequences.

We define the maximum correlation parameter for a family of sequences to be

$$
\theta_{\max }:=\max \left\{\left|\theta_{s(j), s(k)}(\tau)\right| \begin{array}{l}
\text { either } j \neq k  \tag{1}\\
\text { or } \tau \neq 0
\end{array}\right\}
$$

The parameters commonly used to compare sequence families are the size of the symbol alphabet, the period $N$ of each sequence, the number of cyclically-distinct sequences in the family and the value of $\theta_{\text {max }}$.

## A. Motivation

There are several reasons for being interested in lowcorrelation sequences over the QAM and Q-PAM constellations

- the increasing popularity of the QAM alphabet for signalling purposes
- the potential for modulating data at higher data rates
- the potential for variable-rate signalling on the reverse link of a CDMA system
- the potential for larger Euclidean distance between the signals corresponding to different data bits of the same user, thus improving reliability of communication
- the larger symbol alphabet that makes the sequences harder to predict from an intercepted fragment
- the potential for data modulation in ways that are not transparent to a casual observer which makes it harder for the casual observer to recover the data from an observed fragment.
When considering high-order modulation, the first approach that suggests itself is one of "multiplying" the QPSK spreading sequence Family $\mathcal{A}$ sequence by symbols from the $M^{2}$-QAM constellation. This would, however, mean that the transmitted energy per period of the spreading sequence would vary vastly depending upon the particular QAM symbol being transmitted. As a result, each user would experience a varying amount of interference depending upon the particular combination of symbols being transmitted by the other users. The QAM symbols with low magnitude would be far more susceptible to interference than those with larger magnitude. In contrast,
in all of the designs presented here, every spreading sequence has the same energy.

A second alternative one might consider would be to use Family $\mathcal{A}$ and modulate the code sequence with $M^{2}$-ary phase modulation. This would however, lead to smaller Euclidean distance between distinct data symbols of the same user and also make the system more sensitive to phase offsets.

## B. Prior Constructions in the Literature

In this paper, keeping in mind the widespread usage of binary digits to represent data, we restrict our attention to low-correlation sequence families whose symbol-alphabet is a subset of the complex numbers having size that is a power of 2 .

We do not consider sequences over real-valued alphabet such as the BPSK $\{ \pm 1\}$ alphabet or the PAM alphabet since apart from their inherent ability to provide increased spectral efficiency, the corresponding complex counterparts of these alphabets, namely QPSK and QAM, offer better correlation performance in general. For instance, for family sizes that are approximately equal to the sequence period $N, \theta_{\max }$ for the best known BPSK and QPSK sequence families is approximately given by $\sqrt{2 N}$ and $\sqrt{N}$ respectively [4], [5].

Table $\square$ provides a quick overview of some relevant prior constructions:

- The quaternary sequence family, Family $\mathcal{A}$ [2], [4], [14], has the same size as the family of Gold sequences [5], but smaller value of $\theta_{\max }$.
- Quaternary families $\{\mathcal{S}(p)\}, p \geq 1$, [9] are larger families with correspondingly larger values of $\theta_{\max }$ and a member of these families, namely Family $\mathcal{S}(2)$, appears in the WCDMA standard [18] as the short scrambling code.
- In [8], a Galois-ring analogue of the Weil-CarlitzUchiyama (WCU) bound on exponential sums over finite fields is derived and a general technique for constructing low-correlation $2^{m}$-PSK sequences is presented that is based on this bound. In the table, the label WCU is used to refer to sequence families constructed using this technique.
- A 16 -QAM CDMA family $\mathcal{Q}_{B}$ is constructed in [3] by Boztaş. We became aware of this construction only much after the initial writing of this paper, see [1]. As is the case with the sequence families constructed here, Family $\mathcal{Q}_{B}$ is built up of quaternary sequences drawn from QPSK Family $\mathcal{A}$ and is described in greater detail in Section II-E


## C. Notation and Nomenclature

Unless otherwise specified, the word sequence appearing in this paper, will be a reference to a spreading sequence.

We interchangeably use the terms $\mathbb{Z}_{4}$ sequences (i.e., sequences over the integers $(\bmod 4))$, 4-phase sequences or 4-QAM sequences (sequences over $\{ \pm 1 \pm \imath\}$ ) to refer to quaternary sequences in this paper.

The constellation-size parameter $M$ will always be a power of 2 , and more specifically be given by $M=2^{m}$. With the exception of the interleaved sequence families which
have double the period, the period of every sequence family described here is of the form $N=2^{r}-1$. We set $q=2^{r}$ keeping in mind that the finite field of size $2^{r}$ plays a major part in the construction of sequences having this period.

In many of the constructions presented here, each user is assigned a subset of spreading sequences to choose from. Thus such a sequence Family $\mathcal{X}$ is more accurately described as a collection of subsets of sequences. Despite this, to simplify presentation, we will often refer to a selected sequence from one of the subsets assigned to a user as either that user's spreading sequence or else as a sequence belonging to Family $\mathcal{X}$.

Definition 1: We shall say that a sequence over a symbol alphabet $\mathcal{K}$ of size $K$ and of period $N$ is approximately balanced if the number $\mu_{j}$ of times each symbol $\lambda_{j} \in \mathcal{K}$ appears in one period of the sequence, satisfies a bound of the type

$$
\left|\mu_{j}-\frac{N}{K}\right| \leq O(\sqrt{N})
$$

Definition 2: By the data rate of a sequence family, we will mean the maximum number $\nu$ of bits that can be modulated onto each spreading sequence within the family, while leaving the maximum correlation parameter $\theta_{\max }$ undisturbed. Thus the sequence family would be capable of transferring $\nu$ bits per period of the spreading sequence.

## D. Principal Results

The $M^{2}$-QAM constellation is the set

$$
\begin{equation*}
\{a+i b \mid-M+1 \leq a, b \leq M-1, \quad a, b \text { odd }\} \tag{2}
\end{equation*}
$$

When $M=2^{m}$, this constellation can alternately be described as [10], [15]

$$
\begin{equation*}
\left\{\sqrt{2 \imath}\left(\sum_{k=0}^{m-1} 2^{k} \imath^{a_{k}}\right) \mid a_{k} \in \mathbb{Z}_{4}\right\} \tag{3}
\end{equation*}
$$

where by $\sqrt{2 \imath}$ we mean the element $(1+\imath)=\sqrt{2} \exp \left(\frac{\imath 2 \pi}{8}\right)$.
The class of Q-PAM constellations considered in this paper is the subset of the $M^{2}$-QAM constellation of size $2 M=$ $2^{m+1}$ having representation

$$
\left\{\sqrt{2 \imath}\left(\imath^{a_{0}}+\sum_{k=1}^{m-1} 2^{k}(\imath)^{a_{0}+2 a_{k}}\right) \left\lvert\, \begin{array}{c}
a_{0} \in \mathbb{Z}_{4}  \tag{4}\\
a_{k} \in \mathbb{Z}_{2}, k \geq 1
\end{array}\right.\right\}
$$

These representations suggest that quaternary sequences be used in the construction of low correlation sequences over these constellations.

We present five sequence families which adopt this approach, three over the QAM constellation and two over a quadrature-PAM (Q-PAM) subset of the QAM constellation. All sequence families permit data modulation at a rate higher than the 2 bits per sequence period permitted by the use of QPSK spreading sequences. Our initial efforts were directed only at the QAM constellation until we inadvertently discovered that correlation properties could be improved by restricting the alphabet to the Q-PAM constellation, while still retaining the higher data rate property.

TABLE I
Parameters of Some Relevant Prior Constructions in the Literature

| Family | Constellation | Period | Family Size | Data Rate | Asymptotic Upper <br> Bound on $\bar{\theta}_{\max }$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{A}[2],[4],[14]$ | 4-PSK | $N=2^{r}-1$ | $N+2$ | 2 | $\sqrt{N}$ |
| $\mathcal{S}(1)[9]$ | 4-PSK | $N=2^{r}-1$ | $\geq(N+1)^{2}$ | 2 | $2 \sqrt{N}$ |
| $\mathcal{S}(2)[9]$ | 4-PSK | $N=2^{r}-1$ | $\geq(N+1)^{3}$ | 2 | $4 \sqrt{N}$ |
| $\mathcal{S}(p)[9]$ | 4-PSK | $N=2^{r}-1$ | $\geq(N+1)^{p+1}$ | 2 | $2^{p} \sqrt{N}$ |
| WCU Sequences [8] | 4-PSK | $N=2^{r}-1$ | $\geq(N+1)^{2}$ | 2 | $2 \sqrt{N}$ |
|  | 8-PSK | $N=2^{r}-1$ | $\geq(N+1)^{3}$ | 3 | $3 \sqrt{N}$ |
|  | 8-PSK | $N=2^{r}-1$ | $\geq(N+1)^{4}$ | 3 | $4 \sqrt{N}$ |
|  | 8-PSK | $N=2^{r}-1$ | $\geq(N+1)^{5}$ | 3 | $5 \sqrt{N}$ |
|  | 8-PSK | $N=2^{r}-1$ | $\geq(N+1)^{6}$ | 3 | $6 \sqrt{N}$ |
| $\mathcal{Q}_{B}[3]$ | 16-QAM | $N=2^{r}-1$ | $(N+1) / 2$ | (not <br> discussed) | $1.8 \sqrt{N}$ <br> (derived here) |

TABLE II
Parameters of the Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ for various $M$

| Family | $M$ | Constellation | Family Size | Data Rate | Asymptotic Upper <br> Bound on $\bar{\theta}_{\text {max }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{C} \mathcal{Q}_{16}$ | 4 | 16-QAM | $(N+1) / 2$ | 4 | $1.8 \sqrt{N}$ |
| $\mathcal{C} \mathcal{Q}_{64}$ | 8 | 64-QAM | $\lfloor(N+1) / 3\rfloor$ | 6 | $2.33 \sqrt{N}$ |
| $\mathcal{C} \mathcal{Q}_{256}$ | 16 | 256 -QAM | $(N+1) / 4$ | 8 | $2.41 \sqrt{N}$ |
| $\mathcal{C} \mathcal{Q}_{M^{2}}$ | $M$ | $M^{2}$-QAM | $\lfloor(N+1) / m\rfloor$ | $2 m$ | $3 \sqrt{N}$ |

TABLE III
Parameters of the Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ with increased data rate

| Family | $M$ | Constellation | Family Size | Data Rate | Asymptotic Upper <br> Bound on $\bar{\theta}_{\text {max }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{C} \mathcal{Q}_{16}$ | 4 | 16-QAM | $(N+1) / 4$ | 6 | $1.8 \sqrt{N}$ |
| $\mathcal{C} \mathcal{Q}_{64}$ | 8 | $64-$ QAM | $\lfloor(N+1) / 6\rfloor$ | 9 | $2.33 \sqrt{N}$ |
| $\mathcal{C} \mathcal{Q}_{256}$ | 16 | $256-$ QAM | $(N+1) / 8$ | 12 | $2.41 \sqrt{N}$ |
| $\mathcal{C} \mathcal{Q}_{M^{2}}$ | $M$ | $M^{2}$-QAM | $\lfloor(N+1) / 2 m\rfloor$ | $3 m$ | $3 \sqrt{N}$ |

TABLE IV
Parameters of the family $\mathcal{S} \mathcal{Q}_{M^{2}}$ for various $M$

| Family | $M$ | Constellation | Family Size | Data Rate | Asymptotic Upper <br> Bound on $\bar{\theta}_{\text {max }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{S \mathcal { Q } _ { 1 6 }}$ | 4 | 16-QAM | $(N+1) / 2$ | 3 | $1.61 \sqrt{N}$ |
| $\mathcal{S} \mathcal{Q}_{64}$ | 8 | 64-QAM | $(N+1) / 4$ | 4 | $2.10 \sqrt{N}$ |
| $\mathcal{S \mathcal { Q } _ { 2 5 6 }}$ | 16 | $256-$ QAM | $\geq(N+1) / 8$ | 5 | $2.41 \sqrt{N}$ |
| $\mathcal{S} \mathcal{Q}_{1024}$ | 32 | 1024 -QAM | $\geq(N+1) / 8$ | 6 | $2.58 \sqrt{N}$ |
| $\mathcal{S \mathcal { Q } _ { M ^ { 2 } }}$ | $M$ | $M^{2}$-QAM | $\geq(N+1) / 4 m$ | $m+1$ | $2.76 \sqrt{N}$ |

TABLE V
Parameters of the Particular 16-QAM sequence family $\mathcal{I} \mathcal{Q}_{16}$.

| Family | $M$ | Constellation | Period | Family Size | Data Rate | Asymptotic Upper <br> Bound on $\bar{\theta}_{\max }$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{I} \mathcal{Q}_{16}$ | 4 | $16-\mathrm{QAM}$ | $N=2\left(2^{r}-1\right)$ | $(N+2) / 4$ | 3 | $\sqrt{2} \sqrt{N}$ |

All the sequence families constructed here also permit variable-rate signalling. By this we mean that users can adjust
their data rate by switching to a spreading sequence over a constellation of the same type, but of smaller or larger size.

TABLE VI
Parameters of the Family $\mathcal{P}_{2 M}$ For various $M$

| Family | $M$ | Constellation | Family Size | Data Rate | Asymptotic Upper <br> Bound on $\bar{\theta}_{\text {max }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{P}_{8}$ | 4 | 8-ary Q-PAM | $(N+1) / 2$ | 3 | $1.34 \sqrt{N}$ |
| $\mathcal{P}_{16}$ | 8 | 16-ary Q-PAM | $(N+1) / 4$ | 4 | $1.72 \sqrt{N}$ |
| $\mathcal{P}_{32}$ | 16 | 32 -ary Q-PAM | $\geq(N+1) / 7$ and $\leq(N+1) / 5$ | 5 | $1.96 \sqrt{N}$ |
| $\mathcal{P}_{64}$ | 32 | 64 -ary Q-PAM | $\geq(N+1) / 10$ and $\leq(N+1) / 6$ | 6 | $2.09 \sqrt{N}$ |
| $\mathcal{P}_{2 M}$ | $M$ | $2 M$-ary Q-PAM | $\geq(N+1) /\left(m^{2}\right)$ | $m+1$ | $\sqrt{5} \sqrt{N}$ |

TABLE VII
Parameters of the Welch-Bound-Achieving 8-ary Q-PAM Sequence family $\mathcal{I} \mathcal{P} 8$.

| Family | $M$ | Constellation | Period | Family Size | Data Rate | Asymptotic Upper <br> on $\bar{\theta}_{\max }$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{I} \mathcal{P}_{8}$ | 4 | 8 -ary Q-PAM | $N=2\left(2^{r}-1\right)$ | $(N+2) / 4$ | 3 | $\sqrt{N}$ |

Interestingly, as we show, even in the presence of variable-rate signalling, the amount of interference experienced by a user remains essentially unchanged.

In all the constructions, the size of the sequence family is of the order of $\frac{N}{\log _{2}(M)}$ where $M$ is the square root of the size of the QAM constellation in the case of a QAM family, and one-half the size of the Q-PAM constellation in the case of a family over the Q-PAM signalling alphabet.

1) Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ : This may be regarded as the canonical QAM sequence family construction. In this family, each sequence $\{s(t)\}$ is of the form

$$
\begin{equation*}
s(t)=\sqrt{2 \imath} \sum_{k=0}^{m-1} 2^{k} \imath^{u_{k}\left(t+\tau_{k}\right)} \tag{5}
\end{equation*}
$$

where the sequences $\left\{u_{k}(t)\right\}$ are drawn from Family $\mathcal{A}$. The phases $\tau_{k}$ are used to ensure that the sequence is approximately balanced over its symbol alphabet.

Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ has period $N=2^{r}-1$ and normalized maximum-correlation parameter $\bar{\theta}_{\max }$ bounded above by $\lesssim$ $a \sqrt{N}$, where $a$ ranges from 1.8 in the 16-QAM case to 3.0 for large $M$. The data rate in a CDMA setting is $2 m$. This number can however, be increased to $3 m$ bits of data by halving the size of the sequence family and assigning double the number of quaternary sequences to each user. Note that in comparison, if one were to attempt to increase data rate with a QPSK sequence family by assigning multiple sequences to each user, then to increase the data rate by $m$, one would have to assign $2^{m}$ sequences to each user, thereby reducing the size of the family by a factor of $2^{m}$.

Parameters of Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ are presented in Table [I] The corresponding parameters for the case when the data rate is increased to 3 m are presented in Table III.

The construction used to construct $\mathcal{C} \mathcal{Q}_{M^{2}}$ is easily extended to produce larger sequence families and an example is provided in Section III-E
2) Family $\mathcal{S Q}_{M^{2}}$ : This family which we call, the "selected" family, has a lower value of $\theta_{\max }$ but permits only $(m+1)$-bit data modulation. The sequences in this family
also can be described by equation (5). The lower value of $\theta_{\text {max }}$ is made possible here by a judicious selection of the component quaternary sequences $\left\{u_{k}(t)\right\}_{k=0}^{m-1}$. Parameters of this construction are presented in Table IV.
3) Family $\mathcal{I} \mathcal{Q}_{16}$ : The 16 -QAM sequence Family $\mathcal{I} \mathcal{Q}_{16}$ has the best correlation properties, having the lowest bound $\theta_{\max } \lesssim \sqrt{2} \sqrt{N}$ amongst all the 16-QAM sequence families constructed in this paper. The family has data rate 3 and is constructed using sequence interleaving. Relevant parameters of the family are listed in Table V
The remaining two families are over the Q-PAM constellation of size $2 M$.
4) Family $\mathcal{P}_{2 M}$ : This family has a lower value of $\theta_{\max }$ in comparison with selected QAM Family $\mathcal{S} \mathcal{Q}_{M^{2}}$, while still maintaining a data rate of $(m+1)$. Lower correlation values are obtained by setting all $\tau_{k}=0$ in (5) followed by adopting the sequence selection used to construct Family $\mathcal{S} \mathcal{Q}_{M^{2}}$. Setting $\tau_{k}=0$ results in a sequence over a Q-PAM constellation (see Table VI for parameters of this sequence family).
5) Family $\mathcal{I P}_{8}$ : This construction combines features of constructions described above, namely, all $\tau_{k}=0$, judicious sequence selection and sequence interleaving. It achieves a data rate of 3 and quite remarkably, achieves the Welch lower bound on $\bar{\theta}_{\text {max }}$ (see Table VII for parameters).

## E. Outline of the Paper

Section 【I provides background material relating to the QAM constellation, to Galois rings, to quaternary Family $\mathcal{A}$ and 16-QAM Family $\mathcal{Q}_{B}$. Sections III, IV discuss the canonical and selected QAM sequence families $\mathcal{C} \mathcal{Q}_{M^{2}}$ and $\mathcal{S} \mathcal{Q}_{M^{2}}$ respectively.

In Section 7 constructions for 16 -QAM sequences are discussed. Family $\mathcal{S} \mathcal{Q}_{16}$, is shown to have correlation properties that improve upon those of Family $\mathcal{Q}_{B}$. A second sequence family, Family $\mathcal{I} \mathcal{Q}_{16}$, introduced in this section and obtained using sequence interleaving, is shown to do even better.

Section VI deals with Q-PAM families, the general construction of Family $\mathcal{P}_{2 M}$ as well as the specific 8 -ary Q-PAM
construction $\mathcal{I} \mathcal{P}_{8}$ that achieves the Welch bound with equality. Most proofs have been moved to the Appendix for the sake of clarity.

## II. BACKGROUND

## A. The $M^{2}-Q A M$ and $2 M$-ary $Q$-PAM Constellations

The equivalence between the two representations of the $M^{2}$ QAM constellations contained in (2) and (3) follows from noting that an odd number, $x$, in the range $[-M+1, M-1]$ can be uniquely expressed as

$$
x=\sum_{k=0}^{m-1} 2^{k}(-1)^{x_{k}}, \quad x_{k} \in \mathbb{F}_{2}
$$

and the relation

$$
(-1)^{x_{i}+x_{j}}+\imath(-1)^{x_{j}}=\sqrt{2 \imath} \imath^{x_{i}+2 x_{j}}, \quad x_{i}, x_{j} \in \mathbb{F}_{2}
$$

As noted in Section I-D the representation in (3) suggests that a sequence over $M^{2}$-QAM can be constructed using a collection of sequences over $\mathbb{Z}_{4}$, of size $m$, and we adopt this approach in the paper. We shall also construct sequences over the Q-PAM constellation described by (4).

The 16-QAM constellation [3], [10], [12]

$$
\left\{\sqrt{2 \imath}\left(\imath^{a_{0}}+2 \imath^{a_{1}}\right) \mid a_{0}, a_{1} \in \mathbb{Z}_{4}\right\}
$$

is shown in Fig. 1 It is easy to check that the average energy of the constellation is 10 .


Fig. 1. 16-QAM Constellation

The 8-ary Q-PAM constellation is a subset of the 16-QAM constellation given by

$$
\left\{\sqrt{2 \imath}\left(\imath^{a_{0}}+2 \imath^{a_{0}+2 a_{1}}\right) \mid a_{0} \in \mathbb{Z}_{4}, \quad a_{1} \in \mathbb{Z}_{2}\right\}
$$

(see Fig. 2b) and has the same average energy.

| $-3+3 \imath$ |  | $\begin{gathered} 3+3 \imath \\ \bullet \end{gathered}$ |
| :---: | :---: | :---: |
| $\begin{gathered} -1+\imath \\ \end{gathered}$ | $1+\imath$ |  |
| $-1-\imath$ | $1-\imath$ |  |
| $-3-3 \imath$ |  | $\begin{gathered} 3-3 \imath \\ \bullet \end{gathered}$ |

Fig. 2. 8-ary Q-PAM Constellation

## B. Galois Rings

Let $\mathbb{Z}_{n}$ denote the ring of integers modulo $n$. In this paper our primary interest is in the ring $\mathbb{Z}_{4}=\{0,1,2,3\}$.

Galois rings [11] are Galois extensions of the prime ring $\mathbb{Z}_{p^{n}} . R \triangleq G R(4, r)$ will denote a Galois extension of $\mathbb{Z}_{4}$ of degree $r . R$ is a commutative ring with identity and contains a unique maximal ideal $M=2 R$ generated by the element 2. Such rings are called local rings. The quotient $R / M$ is isomorphic to $\mathbb{F}_{q}$, the finite field with $q=2^{r}$ elements.

As a multiplicative group, the set $R^{*}$ of units of $R$ has the following structure:

$$
R^{*} \cong \mathbb{Z}_{2^{r}-1} \times \underbrace{\mathbb{F}_{2} \times \mathbb{F}_{2} \ldots \times \mathbb{F}_{2}}_{r \text { times }}
$$

Let $\xi$ be a generator for the multiplicative cyclic subgroup isomorphic to $\mathbb{Z}_{2^{r}-1}$ contained within $R^{*}$. Let $\mathcal{T}$ denote the set $\mathcal{T}=\left\{0,1, \xi, \ldots, \xi^{2^{r}-2}\right\} . \mathcal{T}$ is called the set of Teichmueller representatives (of $\mathbb{F}_{q}$ in $R$ ). It can be shown that every element $z \in R$ can uniquely be expressed as

$$
z=a+2 b, \quad a, b \in \mathcal{T}
$$

This is often referred to as the " 2 -adic expansion" of $z$. Modulo-2 reduction of $z$ is denoted by $\bar{z}$. It can be shown that $\alpha=\bar{\xi}$ is a primitive element in $\mathbb{F}_{q}$.

To every element $a \in \mathbb{F}_{q}$ there exists a unique element $\hat{a}$ in $\mathcal{T}$ such that $\overline{\hat{a}}=a$. The element $\hat{a}$ is called the "lift" of $a$ in R.

Remark 1: To simplify notation, in the sequel, we will often use the same notation to refer to both the finite field element as well as its lift belonging to the associated Teichmuler set $\mathcal{T}$.

The Frobenius automorphism $\sigma: R \rightarrow R$ is given by

$$
\sigma(z)=a^{2}+2 b^{2}
$$

and the trace map from $R$ to $\mathbb{Z}_{4}$ is defined as

$$
T(z)=\sum_{k=0}^{r-1} \sigma^{k}(z)=\sum_{k=0}^{r-1}\left(a^{2^{k}}+2 b^{2^{k}}\right)
$$

Let $\operatorname{tr}: \mathbb{F}_{q} \rightarrow \mathbb{F}_{2}$ denote the binary trace function. More details of Galois rings can be found in [6], [8], [11], [13].

## C. Modifications to the Maximum Correlation Parameter

We make two changes to the maximum correlation parameter. The first change recognizes that when a user is assigned multiple spreading sequences, a bank of correlators is used at the receiver end and the autocorrelation between two such sequences at zero shift does not interfere with the self-synchronization capability of the family. Accordingly, the maximum non-trivial correlation magnitude of a sequence family is given the modified definition:

$$
\theta_{\max }:=\max \left\{\begin{array}{l|l}
\left|\theta_{s(j), s(k)}(\tau)\right| & \begin{array}{l}
\text { either } s(j, t), s(k, t) \\
\text { have been assigned } \\
\text { to distinct users or } \\
\tau \neq 0
\end{array} \tag{6}
\end{array}\right\}
$$

The second change arises from energy considerations. To make a fair comparison between QAM and PSK families, it is required that the correlation magnitude be normalized to take into account the larger energy of the QAM and Q-PAM sequence families. We will use $\bar{\theta}_{\text {max }}$ to denote the maximum correlation magnitude if the sequences have been normalized to have energy $N$, and this will be used as the basis for comparison across signal constellations.

## D. Family $\mathcal{A}$

Family $\mathcal{A}$ is an asymptotically optimal family of quaternary sequences (i.e., over $\mathbb{Z}_{4}$ ) discovered independently by Solé [14] and Boztaş, Hammons and Kumar [2], [4]. A detailed description of their correlation properties appears in [4].

Let $\left\{\gamma_{j}\right\}_{j=1}^{2^{r}}$ denote $2^{r}$ distinct elements in $\mathcal{T}$, i.e., we have the alternate expression $\mathcal{T}=\left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{2^{r}}\right\}$. There are $2^{r}+$ 1 cyclically distinct sequences in Family $\mathcal{A}$, each of period $N=2^{r}-1$. The following representation for sequences in Family $\mathcal{A}$ is used in this paper:

$$
\begin{align*}
s_{j}(t) & =T\left(\left[1+2 \gamma_{j}\right] \xi^{t}\right), \quad 1 \leq j \leq 2^{r} \\
s_{2^{r}+1}(t) & =2 T\left(\xi^{t}\right) \tag{7}
\end{align*}
$$

The maximum non-trivial correlation magnitude for Family $\mathcal{A}$ has the upper bound

$$
\begin{equation*}
\theta_{\max } \leq 1+\sqrt{N+1} \tag{8}
\end{equation*}
$$

More details of the correlation properties of Family $\mathcal{A}$ can be found in Appendix Various desirable properties such as near optimality with respect to correlation, mathematical tractability and ease of generation, make Family $\mathcal{A}$ a prime candidate for use as a building block in constructing sequences over the $M^{2}$-QAM constellation.

Remark 2: In the present paper, we do not make use of the presence of the "binary" sequence $\left\{s_{2^{r}+1}(t)\right\}$ as our sequence constructions require each quaternary sequence employed to take on all possible values over $\mathbb{Z}_{4}$. For this reason, we will treat Family $\mathcal{A}$ as if it were a family composed of $q=2^{r}$ cyclically-distinct sequences. A similar comment applies in Section III-E where we make use of quaternary sequence Family $\mathcal{S}(1)$.

## E. The 16-QAM Sequence Family Constructed by Boztaş

In [3], Boztaş considers the family of sequences

$$
\mathcal{Q}_{B}=\left\{\alpha u_{i}(t)+\beta v_{i}(t) \mid 1 \leq i \leq 2^{r}-2\right\}
$$

where $\alpha, \beta$ are positive real numbers. The sequences $\left\{u_{i}(t)\right\}$ are defined as follows. We adopt the notation introduced in Section II-B relating to a Galois ring $R=G R(4, r)$ of size $4^{r}$. Let the elements $\delta_{i}, \gamma_{i}$ be any selection satisfying

$$
\begin{array}{r}
\left\{\delta_{1}, \delta_{2}, \cdots, \delta_{2^{r-1}-1}\right\} \bigcup\left\{\gamma_{1}, \gamma_{2}, \cdots, \gamma_{2^{r-1}-1}\right\} \\
=\left\{(1-\xi),\left(1-\xi^{2}\right), \cdots,\left(1-\xi^{2^{r}-2}\right)\right\}
\end{array}
$$

where $\xi$ is, as in Section II-B a generator for the multiplicative cyclic subgroup isomorphic to $\mathbb{Z}_{2^{r}-1}$ contained within $R^{*}$. Then the sequences $\left\{u_{i}(t)\right\},\left\{v_{i}(t)\right\}$ are defined by

$$
\begin{aligned}
u_{i}(t) & =\imath^{T\left(\delta_{i} \xi^{t}\right)} \\
v_{i}(t) & =\imath^{T\left(\gamma_{i} \xi^{t}\right)}
\end{aligned}
$$

The resulting sequence family $\mathcal{Q}_{B}$ has in general, a constellation of size 16. Although not explicitly pointed out in [3], by setting $\alpha=1$ and $\beta=2$, one recovers a rotated version of the 16-QAM constellation:

$$
\{a+i b \mid-3 \leq a, b \leq 3, \quad a, b \text { odd }\}
$$

In Table I we have listed parameters of the family $\mathcal{Q}_{B}$ obtained by selecting $\alpha=\sqrt{2 \imath}, \beta=2 \alpha$. While some discussion of the correlation properties of this sequence family is presented in [3], the value of $\theta_{\max }$ for Family $\mathcal{Q}_{B}$ listed in Table $\square$ is derived from the results of the present paper.

## III. CANONICAL FAMILIES OF SEQUENCES OVER $M^{2}$-QAM CONSTELLATION

The equivalent expression for the QAM constellation given in (3) suggests that a family of low correlation sequences can be constructed using, as building blocks, elements of Family A as follows:

$$
s_{p}(t)=\sum_{k=0}^{m-1} 2^{k} \imath^{T\left(a_{p, k} \xi^{t}\right)}
$$

where the coefficients $\left\{a_{p, k}\right\}$ are drawn from $G R(4, r)$. When one considers the crosscorrelation between two sequences $\left\{s_{p}(t)\right\}$ and $\left\{s_{q}(t)\right\}$ having the above form, one quickly realizes that in order to keep correlation values small, no two sequences $\left\{\imath^{T\left(a_{p, k} \xi^{t}\right)}\right\},\left\{\imath^{T\left(a_{q,} \xi^{t}\right)}\right\}$ should be cyclic shifts of one another. This requirement can equivalently be expressed in the form

$$
a_{p, k} \xi^{\tau} \neq a_{q, l}
$$

for any value of the cyclic shift parameter $\tau$, whenever either $p \neq q$ (signifying different users) or whenever $k \neq l, 0 \leq$ $k, l \leq m-1$.

Let us impose the second requirement on the QAM sequence family that every sequence in the family should be approximately balanced as defined in I-C This requires that the number of solutions to the simultaneous equations:

$$
T\left(a_{p, k} x\right)=\nu_{k}, \quad k=0,1, \ldots,(m-1)
$$

be approximately equal for all $m$-tuples $\nu$ $\left(\nu_{0}, \nu_{1}, \ldots, \nu_{m-1}\right)$ in $\mathbb{Z}_{4}^{m}$ as $x$ varies over all of $\mathcal{T}$.

Lemma 3.1: The sequence

$$
s_{p}(t)=\sum_{k=0}^{m-1} 2^{k} \imath^{T\left(a_{p, k} \xi^{t}\right)}
$$

is approximately balanced over the $4^{m}$-QAM alphabet if the coefficients $a_{p, k}$ are linearly independent over $\mathbb{Z}_{4}$, i.e.,

$$
\sum_{k=0}^{m-1} \omega_{k} a_{p, k} \neq 0
$$

for any choice $\left\{\omega_{k} \in \mathbb{Z}_{4}\right\}_{k=0}^{m-1}$ of coefficients, where at least one of the $\omega_{k}$ 's is non-zero.

## Proof: Please see Appendix 【

From the above discussion we arrive at the twin conditions

$$
\begin{equation*}
a_{p, k} \xi^{\tau} \neq a_{q, l} \tag{9}
\end{equation*}
$$

whenever either $p \neq q$ or $k \neq l$ and

$$
\begin{equation*}
\sum_{k=0}^{m-1} \omega_{k} a_{p, k} \neq 0 \tag{10}
\end{equation*}
$$

for any non-zero coefficient set $\left\{\omega_{k}\right\}_{k=0}^{m-1}$; which we will respectively term as the cyclic distinctness and linear independence conditions to be satisfied by the coefficients $\left\{a_{p, k}\right\}$.

One means of constructing coefficient sets $\left\{a_{p, k}\right\}$ satisfying the twin conditions in (9) and (10) is described below.

Let $\left\{\tau_{0}=0, \tau_{1}, \tau_{2}, \ldots, \tau_{m-1}\right\}$ be integers $0 \leq \tau_{i} \leq 2^{r}-$ 2 , such that $\left\{\alpha^{\tau_{0}}=1, \alpha^{\tau_{1}}, \alpha^{\tau_{2}}, \ldots, \alpha^{\tau_{m-1}}\right\}$ form a linearly independent set. Let the elements of the Teichmuller $\mathcal{T}$ set be divided into disjoint (ordered) subsets, each of size $m$, of the form $g=\left(g_{0}, g_{1}, \ldots, g_{m-1}\right)$. Let $G$ refer to the collection of all such $g$ 's. Note that

$$
|G|=\left\lfloor\frac{q}{m}\right\rfloor=\left\lfloor\frac{N+1}{m}\right\rfloor .
$$

Set

$$
a_{g, k}=\left(1+2 g_{k}\right) \xi^{\tau_{k}}
$$

It is straightforward to verify that the coefficients $a_{g, k}$ satisfy the cyclic distinctness requirement. To see that the linear independence requirement is also met, note that

$$
\sum_{k} \omega_{k} a_{g, k}=0
$$

implies

$$
\sum_{k} \omega_{k} \xi^{\tau_{k}}=0 \quad(\bmod 2)
$$

which is not possible by the choice of $\left\{\tau_{k}\right\}$ unless all the $\omega_{k} \in\{0,2\}$. But this possibility can also be dismissed using a similar argument.

This leads to the construction of a family of $M^{2}$-QAM sequences which we shall term the canonical construction and denote by Family $\mathcal{C} \mathcal{Q}_{M^{2}}$.
$=\quad$ Let $\kappa=\left(\kappa_{0}, \kappa_{1}, \ldots, \kappa_{m-1}\right) \in \mathbb{Z}_{4} \times \mathbb{F}_{2}^{m-1}$. A mathematical expression for Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ is provided below.

$$
\begin{equation*}
\mathcal{C} \mathcal{Q}_{M^{2}}=\left\{\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4}^{m}\right\} \mid g \in G\right\} . \tag{11}
\end{equation*}
$$

Each user is thus assigned the set

$$
\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4}^{m}\right\}
$$

of sequences with the $\kappa$-th sequence given by

$$
s(g, \kappa, t)=\sqrt{2 \imath}\left(\sum_{k=0}^{m-1} 2^{k} \imath^{u_{k}(t)} \imath^{\kappa_{k}}\right)
$$

where

$$
\begin{align*}
u_{k}(t)=T\left(\left[1+2 g_{k}\right] \xi^{t+\tau_{k}}\right) & \\
& k=0,2, \ldots, m-1 \tag{12}
\end{align*}
$$

The main properties of Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ are summarized in the following theorem:

Theorem 3.2: Let $m \geq 2$ be a positive integer and let $\mathcal{C} \mathcal{Q}_{M^{2}}$ be the family of sequences over the $M^{2}$-QAM constellation defined in 11). Then,

1) All sequences in $\mathcal{C} \mathcal{Q}_{M^{2}}$ have period $N=2^{r}-1$.
2) For large values of $N$, the energy of the sequences in the family is given by

$$
\mathcal{E} \approx \frac{2}{3}\left(M^{2}-1\right) N
$$

(which is what one would expect if the average energy of a symbol across the constellation were equal to the average symbol energy across one period of the sequence).
3) The maximum correlation parameter of the family can be bounded as

$$
\theta_{\max } \lesssim 2\left(2^{m}-1\right)^{2} \sqrt{N+1}
$$

For large values of $M$ and $N$, the normalized maximum correlation parameter of the family can be bounded as

$$
\bar{\theta}_{\max } \lesssim 3 \sqrt{N}
$$

4) Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ can support $\lfloor(N+1) / m\rfloor$ distinct users.
5) Each user can transmit $2 m$ bits of information per sequence period.
6) The normalized minimum squared Euclidean distance between all sequences assigned to a user is given by

$$
\bar{d}_{\min }^{2} \approx \frac{6}{M^{2}-1} N
$$

7) The sequences in Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ are approximately balanced.

Proof: Property (1) follows from the periodicity properties of Family $\mathcal{A}$ sequences. Properties (2) and (3) follow from the correlation properties of Family $\mathcal{A}$ sequences and the derivation may be found in Appendices III and IV respectively.

In order to prove property (4), we note that each user is assigned $m$ cyclically distinct sequences from Family $\mathcal{A}$, namely the sequence set

$$
\left\{T\left(\left[1+2 g_{k}\right] \xi^{t+\tau_{k}}\right\}_{k=0}^{m-1}\right.
$$

Since there are $q$ possible choices for $g_{k}$ in $\mathcal{T}$, it follows that the maximum number of users that can be supported is given by $\left\lfloor 2^{r} / m\right\rfloor=\lfloor(N+1) / m\rfloor$. Property (5) follows from the definition of the sequences. The symbols $\left\{\kappa_{k}\right\}_{k=0}^{m-1}$ are the $m$ information-bearing symbols. Property (6) is concerned with the Euclidean-distance of the sequences assigned to a user and is proved in Appendix $\overline{\text { Property }}$ (7) follows from our earlier arguments.

Remark 3: An examination of the proof of Property (2) will reveal that the result is valid for any $M^{2}$-QAM, $\left(M^{2}=4^{m}\right)$, sequence $\{s(t)\}$, given by an expression of the form:

$$
s(t)=\sqrt{2 \imath} \sum_{k=0}^{m-1} 2^{k} \imath^{u_{k}(t)}
$$

where the component quaternary sequences $\left\{u_{k}(t)\right\}$, are distinct elements of Family $\mathcal{A}$.

## A. Variable-Rate Signalling

The asynchronous nature of the reverse link (mobile to base station) in a CDMA system makes it difficult to accommodate users having differing data-rate requirements i.e., users who wish to communicate a different number of bits of data per sequence period. It precludes, for example, the use of orthogonal-variable-spreading-factor (OVSF) channelization (Walsh) codes that are part of the WCDMA standard.

One of the advantages of the structure of the sequences in Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ (and others presented here) is that it is possible to place an upper bound on the crosscorrelation of sequences over QAM constellations of different size, thereby enabling variable-rate signalling on the reverse link.

Enabling variable-rate signalling in the case of Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ is fairly straightforward as we shall see. One first partitions the entire finite field into subsets, and the subsets will typically be of different sizes. The elements in each subset are then ordered in some arbitrary fashion, and if $\left\{g_{k}\right\}_{k=0}^{m-1} \subseteq \mathbb{F}_{q}$ is the ordered subset, then this subset is associated with the $4^{m}$-QAM sequence

$$
\sqrt{2 \imath} \sum_{k=0}^{m-1} 2^{k} \imath^{T\left(\left[1+2 g_{k}\right] \xi^{t+\tau_{k}}\right)}
$$

Thus every partition of the elements of the Teichmuller set corresponds to an assignment of variable rates to the users, with the number of users equal to the number of subsets in the partition. It follows that we can support $n_{i}, i=1,2, \cdots, p$ users with constellations of size $4^{m_{i}}$ iff

$$
\sum_{k=1}^{p} n_{i} m_{i} \leq|\mathcal{T}|=q
$$

If two users have been assigned tuples enabling them to transmit sequences from Families $\mathcal{C} \mathcal{Q}_{M_{1}^{2}}$ and $\mathcal{C} \mathcal{Q}_{M_{2}^{2}}$, with
$M_{1}>M_{2}$, then the user assigned sequences from Family $\mathcal{C} \mathcal{Q}_{M_{2}^{2}}$ will experience a marginally increased amount of interference from the user assigned sequences from Family $\mathcal{C} \mathcal{Q}_{M_{1}^{2}}$. The reverse is true in the case of the interference experienced by the user having the larger constellation. This is based on the bounds on normalized crosscorrelation ${ }^{2}$ derived in Appendix VI The marginal change is by a factor of

$$
\sqrt{\frac{\left(M_{1}-1\right)\left(M_{2}+1\right)}{\left(M_{1}+1\right)\left(M_{2}-1\right)}}
$$

It is this essentially-unchanged level of interference that enables variable-rate signalling.

## B. Euclidean Distance Comparison with $M^{2}-P S K$ Constellation

Each sequence belonging to Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ can be modulated by $\log M^{2}=2 m$ data bits. An alternative means of transporting $2 m$ data bits per period of spreading sequence, is to use a QPSK code sequence family and then use $M^{2}$-ary phase data modulation which corresponds to multiplication of the code sequence by a complex symbol drawn from the set

$$
\left\{\exp \left(\imath \frac{2 \pi a}{M^{2}}\right), \quad a \in \mathbb{Z}_{M^{2}}\right\}
$$

We compare the two schemes in terms of the minimum Euclidean distance between the same code sequence when modulated by two different $2 m$-tuples of data. In the case of $M^{2}$-ary PSK modulation, the minimum squared Euclidean distance between two distinct modulations of a sequence over $M^{2}$-PSK can be shown to be given by

$$
2\left(1-\cos \left(\frac{2 \pi}{M^{2}}\right)\right) N
$$

where $N$ is the period of the code sequence. For large $M$ the right hand side can be approximated by

$$
\begin{aligned}
2\left(1-\cos \left(\frac{2 \pi}{M^{2}}\right)\right) N & \left.\approx 2\left(1-\left(1-\frac{\left(\frac{2 \pi}{M^{2}}\right)^{2}}{2!}\right)\right)\right) N \\
& =\frac{4 \pi^{2}}{M^{4}} N
\end{aligned}
$$

In comparison, the normalized minimum squared Euclidean distance between two sequences assigned to a user in Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ is given by

$$
\frac{6}{\left(M^{2}-1\right)} N
$$

and it is clear from this that Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ has significantly larger separation between different data sets which makes for increased reliability.

[^1]
## C. Further Increasing the Data Rate

In the present construction, Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ is a family of $\left\lfloor\frac{N+1}{m}\right\rfloor$ sequences in which each user can transmit $2 m$ bits of data. The sequence $\{s(g, \kappa, t)\}$ of each user is built up of $m$ quaternary sequences $\left\{u_{k}(t)\right\}_{k=0}^{m-1}$ drawn from Family $\mathcal{A}$ and is of the form

$$
s(g, \kappa, t)=\sum_{k=0}^{m-1} 2^{k} \imath^{u_{k}(t)+\kappa_{k}}
$$

where

$$
u_{k}(t)=T\left(\left[1+2 g_{k}\right] \xi^{t+\tau_{k}}\right)
$$

Suppose, we were to assign additional $m$ sequences $\left\{v_{k}(t)\right\}_{k=0}^{m-1}$ from Family $\mathcal{A}$ to each user where

$$
v_{k}(t)=T\left(\left[1+2 h_{k}\right] \xi^{t+\tau_{k}}\right)
$$

This would, on the one hand, reduce the family size by a factor of 2 to $\left\lfloor\frac{N+1}{2 m}\right\rfloor$. On the other hand, this would enable each user to transmit an additional $m$ bits of data per period of the code sequence. The user could simply select between the pair $\left\{u_{k}(t)\right\}$ and $\left\{v_{k}(t)\right\}$ for the $k$-th component sequence. There is no penalty to be paid in terms of increased correlations since, as can easily be verified, the maximum normalized correlation magnitude bound remains unchanged. Decorrelation at the receiver end can be accomplished with the aid of $2 m$ decorrelators in place of the $m$ previously needed.

Note that this feature is peculiar to the structure of the signalling set used here. If one were to attempt something similar in conjunction with a QPSK sequence family, then in order to send an additional $m$ data bits, one would have to assign each user an additional $2^{m}-1$ code sequences and employ $2^{m}-1$ additional de-correlators at the receiver!

## D. Compatibility with Quaternary Sequence Families

Being built up of quaternary sequences gives Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ the added advantage of being compatible with QPSK Families $\mathcal{S}(p)$ in the sense that the value of maximum correlation magnitude is increased only slightly if one enlarges Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ to include quaternary sequences drawn from $\mathcal{S}(p) \backslash \mathcal{A}$. We omit the details.

## E. Larger Canonical Families over the QAM Alphabet

The canonical sequence family $\mathcal{C} \mathcal{Q}_{M^{2}}$ described in Theorem 3.2 was based on the use of Family $\mathcal{A}$ as the source for the component quaternary sequences $\left\{u_{k}(t)\right\}$ (see (12)). The construction extends easily to the case when the component sequences are drawn from any low-correlation quaternary family. In particular, one could construct larger, low-correlation $M^{2}$-QAM families from the large collection of low-correlation WCU quaternary sequence families (see Table $\square$ ). We illustrate by considering the case when Family $\mathcal{A}$ is replaced by quaternary sequence family $\mathcal{S}(1)$ and leave the details in the other cases to the reader. We will use the notation $\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{S}(1))$ to describe this sequence family. Under this notation, $\mathcal{C} \mathcal{Q}_{M^{2}}$ is shorthand for Family $\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{A})$.

Family $\mathcal{S}(1)$ contains $q^{2}$ cyclically distinct sequence families. Let $P=\left\lfloor\frac{q^{2}}{m}\right\rfloor$ and let a subset of Family $\mathcal{S}(1)$ of size $P m$
be selected. Only this subset will be used in the construction. Then it can be shown that this collection of Pm cyclicallydistinct sequences can be placed into an array of size $P \times m$ in which the $(p, k)$-th element is of the form

$$
T\left(\left[1+2 g_{p, k}\right] \xi^{t+\tau_{k}}+2 h_{p, k} \xi^{3 t}\right)
$$

Let $\kappa=\left(\kappa_{0}, \kappa_{1}, \ldots, \kappa_{m-1}\right) \in \mathbb{Z}_{4}^{m}$. The signal of the $p$-th user, $1 \leq p \leq P$, is then given by

$$
s(p, \kappa, t)=\sqrt{2 \imath}\left(\sum_{k=0}^{m-1} 2^{k} \imath^{u_{k}(t)} \imath^{\kappa_{k}}\right)
$$

where

$$
u_{k}(t)=T\left(\left[1+2 g_{p, k}\right] \xi^{t+\tau_{k}}+2 h_{p, k} \xi^{3 t}\right)
$$

Then, Family $\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{S}(1))$ is the collection of sequences
$\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{S}(1))=\left\{\left\{s(p, \kappa, t) \mid \kappa \in \mathbb{Z}_{4}^{m}\right\} \mid 1 \leq p \leq P\right\}$.
The main properties of Family $\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{S}(1))$ are summarized in the following theorem:

Theorem 3.3: Let $m \geq 2$ be a positive integer and let $\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{S}(1))$ be the family of sequences over the $M^{2}$-QAM constellation defined in (13). Then,

1) All sequences in $\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{S}(1))$ have period $N=2^{r}-1$.
2) For large values of $N$, the energy of the sequences in the family is given by

$$
\mathcal{E} \approx \frac{2}{3}\left(M^{2}-1\right) N
$$

3) The maximum correlation parameter of the family can be bounded as

$$
\theta_{\max } \lesssim 4\left(2^{m}-1\right)^{2} \sqrt{N+1}
$$

For large values of $M$ and $N$, the normalized maximum correlation parameter of the family can be bounded as

$$
\bar{\theta}_{\max } \lesssim 6 \sqrt{N} .
$$

4) Family $\mathcal{C} \mathcal{Q}_{M^{2}}(\mathcal{S}(1))$ can support $\left\lfloor\left(q^{2}\right) / m\right\rfloor$ distinct users.
5) Each user can transmit $2 m$ bits of information per sequence period.
6) The normalized minimum squared Euclidean distance between all sequences assigned to a user is given by

$$
\bar{d}_{\min }^{2} \approx \frac{6}{M^{2}-1} N
$$

Proof: The proof is along the same lines as used to prove the properties of Family $\mathcal{C} \mathcal{Q}_{M^{2}}$. The principal difference is that $\theta_{\max }(\mathcal{S}(1)) \leq 2 \sqrt{N+1}$ in place of $\theta_{\max }(\mathcal{A}) \leq \sqrt{N+1}$, see [8], [9].

## IV. A "Selected" Construction of Sequences Over $M^{2}$-QAM

We now introduce a second family, Family $\mathcal{S}_{M^{2}}$, of sequences over the $M^{2}$-QAM constellation having a lower value of normalized correlation parameter $\bar{\theta}_{\text {max }}$, and twice the squared-Euclidean distance between different data modulations of the same spreading sequence. As against this, Family
$\mathcal{S} \mathcal{Q}_{M^{2}}$ permits users to transmit only $(m+1)$ bits of data per sequence period in place of the $2 m$ bits allowed by Family $\mathcal{C} \mathcal{Q}_{M^{2}}$.

Lower correlation values are achieved by judicious selection of the component quaternary sequences constituting a QAMsequence.

## A. Definition of Family $\mathcal{S} \mathcal{Q}_{M^{2}}$

Let $\left\{\delta_{0}=0, \delta_{1}, \delta_{2}, \ldots, \delta_{m-1}\right\}$ be elements from $\mathbb{F}_{q}$ such that $\operatorname{tr}\left(\delta_{k}\right)=1, \forall k \geq 1$. Set

$$
H=\left\{\delta_{0}, \delta_{1} \cdots, \delta_{m-1}\right\}
$$

Let $G=\left\{g_{k}\right\}$ be the largest subset of $\mathbb{F}_{q}$ having the property that

$$
\begin{equation*}
g_{k}+\delta_{p} \neq g_{l}+\delta_{q}, \quad g_{k}, g_{l} \in G, \quad \delta_{p}, \delta_{q} \in H \tag{14}
\end{equation*}
$$

unless $g_{k}=g_{l}$ and $\delta_{p}=\delta_{q}$. Then the corresponding GilbertVarshamov and Hamming bounds on the size of $G$ are given by

$$
\begin{equation*}
\frac{2^{r}}{1+\binom{m-1}{1}+\binom{m-1}{2}} \leq|G| \leq \frac{2^{r}}{1+\binom{m-1}{1}} \tag{15}
\end{equation*}
$$

1) A Subspace-Based Construction for the $H$ and $G$ : Given constellation parameter $m$, let $2^{l}$ denote the smallest power of 2 greater than $(m-1)$, i.e., $l$ is defined by

$$
\begin{equation*}
2^{l-1}<(m-1) \leq 2^{l} \tag{16}
\end{equation*}
$$

For reasons that will shortly become clear, we will refer to the integer $l$ as the subspace-size exponent (sse) associated with the constellation parameter (c-p) $m$. Thus $l$ will lie in the range $0 \leq l \leq(r-1)$. Let $\mu$ denote the function that, given c-p $m$ in the range $1 \leq m \leq 2^{r-1}+1$, maps $m$ to the corresponding sse $l$ given above, i.e.,

$$
\mu(m)=l
$$

Treating $\mathbb{F}_{q}$ as a vector space over $\mathbb{F}_{2}$ of dimension $r$, let $W_{r-1}$ denote the subspace of $\mathbb{F}_{q}$ of dimension $(r-1)$ corresponding to the elements of trace $=0$. Let $W_{l}$ denote a subspace of $W_{r-1}$ having dimension $l$. Let $\zeta$ be an element in $\mathbb{F}_{q}$ having trace 1 and let $V_{l}$ denote the subspace

$$
V_{l}=W_{l} \cup\left\{W_{l}+\zeta\right\}
$$

of size $2^{l+1}$. Noting that every element in the coset $W_{l}+\zeta$ of $W_{l}$ has trace 1 , we select as the elements $\left\{\delta_{k}\right\}_{k=1}^{m-1}$ to be used in the construction of Family $\mathcal{S Q}_{M^{2}}$, an arbitrary collection of $(m-1) \leq 2^{l}$ elements selected from the set $W_{l}+\zeta$.

Next, we partition $W_{r-1}$ into the $2^{r-l-1}$ cosets $W_{l}+g$ of $W_{l}$. With each coset, we associate a distinct user. To this user, we assign the coefficient set

$$
\left\{g, g+\delta_{1}, g+\delta_{2}, \ldots, g+\delta_{l}\right\}
$$

The coefficients $\left\{g+\delta_{k}\right\}_{k=1}^{m-1}$ belong to the coset $W_{l}+(g+\zeta)$ of $W_{l}$. Thus in general, each user is assigned $m$ coefficients, with one coefficient $g$, belonging to the coset $W_{l}+g$ of $W_{l}$ lying in $W_{r-1}$ and the remaining drawn from the coset $W_{l}+$
$g+\zeta$ of $W_{l}$. Since $V_{l}=W_{l} \cup\left(W_{l}+\zeta\right)$, all $m$ coefficients taken together belong to the coset $V_{l}+g$ of $V_{l}$. Note that

$$
V_{l}+g=V_{l}+g^{\prime}
$$

implies

$$
\left\{W_{l}+g\right\} \cup\left\{W_{l}+\zeta+g\right\}=\left\{W_{l}+g^{\prime}\right\} \cup\left\{W_{l}+\zeta+g^{\prime}\right\}
$$

But this is impossible since $g, g^{\prime}$ belong to different cosets of $W_{l}$ and $g, g^{\prime}$ have trace zero, whereas, $\operatorname{tr}(\zeta)=1$. It follows that the coefficient sets of distinct users belong to different cosets of $V_{l}$ and are hence distinct.

Thus, the basic sequence $\{s(g, 0, t)\}$ assigned to user $g$ will take on the form

$$
s(g, 0, t)=\sqrt{2 \imath} \sum_{k=0}^{m-1} 2^{m-1-k} \imath^{T\left(\left[1+2\left(g+\delta_{k}\right)\right] \xi^{t+\tau_{k}}\right)}
$$

with both $\delta_{0}$ and $\tau_{0}$ equal to 0 .
Let $G$ be the set of all such coset representatives of $W_{l}$ in $W_{r-1}$. Since each user is associated to a unique coset representative, the number of users is given by

$$
|G|=2^{r-l-1}
$$

When combined with (16), we obtain

$$
\frac{2^{r}}{4(m-1)}<|G| \leq \frac{2^{r}}{2(m-1)}
$$

Thus the size of $G$ is at most a factor of 4 smaller than the best possible suggested by the Hamming bound, see (15).

Let $\left\{\tau_{1}, \tau_{2}, \ldots, \tau_{m-1}\right\}$ be a set of non-zero, distinct timeshifts with $\left\{1, \alpha^{\tau_{1}}, \alpha^{\tau_{2}}, \ldots, \alpha^{\tau_{m-1}}\right\}$ being a linearly independent set. Let $\kappa=\left(\kappa_{0}, \kappa_{1}, \ldots, \kappa_{m-1}\right) \in \mathbb{Z}_{4} \times \mathbb{F}_{2}^{m-1}$. Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ is then defined as follows:

$$
\begin{equation*}
\mathcal{S} \mathcal{Q}_{M^{2}}=\left\{\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4} \times \mathbb{F}_{2}^{m-1}\right\} \mid g \in G\right\} \tag{17}
\end{equation*}
$$

so that each user is identified by an element of $G$. Each user is assigned the collection

$$
\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4} \times \mathbb{F}_{2}^{m-1}\right\}
$$

of sequences with the $\kappa$-th sequence given by

$$
\begin{array}{r}
s(g, \kappa, t)=\sqrt{2 \imath}\left(\sum_{k=1}^{m-1} 2^{m-k-1} \imath^{u_{k}(t)}(-1)^{\kappa_{k}}+\right. \\
\left.2^{m-1} \imath^{u_{0}(t)}\right) \imath^{\kappa_{0}} \tag{18}
\end{array}
$$

where

$$
\begin{aligned}
& u_{0}(t)=T\left([1+2 g] \xi^{t}\right) \\
& u_{k}(t)= T\left(\left[1+2\left(g+\delta_{k}\right)\right] \xi^{t+\tau_{k}}\right) \\
& \quad k=1,2, \ldots, m-1
\end{aligned}
$$

We will refer to the element $g$ as the ground coefficient. Note that given the ground coefficient and the set $\left\{\delta_{1}, \cdots, \delta_{m-1}\right\}$, the set of coefficients used by a user are uniquely determined. The elements $\left\{\delta_{k}\right\}$ will turn out to provide a selection of the component sequences that leads to lower correlation values.

Within the subset of sequences assigned to a particular user, the sequences corresponding to $\kappa=0$ will be termed basic sequences. Basic sequences have a simpler representation and correlations involving basic sequences turn out to be representative of the general case.

Theorem 4.1: Sequences in Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ satisfy the following properties:

1) All sequences in the family have period $N=2^{r}-1$.
2) For large $N$, the energy of any sequence in the family is given by

$$
\mathcal{E} \approx \frac{2}{3}\left(M^{2}-1\right) N
$$

3) The correlation parameter $\theta_{\max }$ has the upper bound:

$$
\theta_{\max } \lesssim \sqrt{\frac{61}{18}} M^{2} \sqrt{N+1}
$$

For large $M$ and $N$, the normalized maximum correlation parameter of the family satisfies the bound

$$
\bar{\theta}_{\max } \lesssim 2.76 \sqrt{N}
$$

4) The family can support

$$
\frac{2^{r}}{4(m-1)}<|G| \leq \frac{2^{r}}{2(m-1)}
$$

distinct users. (Note from (15) that this can potentially be improved by a different construction of the set $G$ ).
5) Each user can transmit $(m+1)$ bits of data per sequence period.
6) The normalized minimum squared Euclidean distance between all sequences assigned to a user is given by

$$
\bar{d}_{\min }^{2} \approx \frac{12}{M^{2}-1} N
$$

7) The number $\mathbf{N}$ of times an element from the $M^{2}$-QAM constellation occurs in sequences of large period can be bounded as:

$$
\left|\mathbf{N}-\frac{N+1}{M^{2}}\right| \leq \frac{M^{2}-1}{M^{2}} \sqrt{N+1}
$$

i.e., the sequences in Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ are approximately balanced.

Proof: Property (1) follows from the periodicity of the sequences in Family $\mathcal{A}$. The proof of Property (2) is identical to the proof concerning the energy of sequences in Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ (see Remark 3).

Property (3) is proved in detail in Appendix VII
Properties (4) and (5) follow directly from the definition of the sequence family.

Property (6) can be proved using techniques similar to those in Appendix $\bar{V}$, as it turns out, the minimum Euclidean distance is associated with data sets $\left(\kappa, \kappa^{\prime}\right)$ where

$$
\begin{aligned}
\kappa_{m-1} & =\kappa_{m-1}^{\prime}+2 \\
\kappa_{k} & =\kappa_{k}^{\prime}, \quad k=0,1, \ldots,(m-2)
\end{aligned}
$$

The proof of Property (7) concerning symbol balance is identical to the proof in the case of Family $\mathcal{C} \mathcal{Q}_{M^{2}}$.

## B. Variable-Rate Signalling on the Reverse Link Using Family

 $\mathcal{S} \mathcal{Q}_{M^{2}}$In this section, we show how Families $\left\{\mathcal{S} \mathcal{Q}_{M^{2}}\right\}$ can also be used to provide variable-rate signalling on a CDMA reverse link. We retain the notation of Section IV-A.

We begin by constructing a chain of subspaces

$$
W_{0} \subseteq W_{1} \subseteq \cdots \subseteq W_{r-1}
$$

in which each subspace $W_{k}$ contains only elements of trace 0 . Let the elements $\rho_{k}$ be such that

$$
W_{k+1}=W_{k} \cup\left\{W_{k}+\rho_{k}\right\}
$$

i.e., $\rho_{k}$ is a coset representative of the coset of $W_{k}$ in $W_{k+1}$ other than $W_{k}$ itself.

For each $k, 0 \leq k \leq(r-1)$, the set

$$
V_{k}=W_{k} \cup\left(W_{k}+\zeta\right)
$$

is also a subspace of $\mathbb{F}_{q}$. Each element in the coset $W_{k}+\zeta$ has trace equal to 1 . Let $\left\{\delta_{1}, \delta_{2}, \ldots, \delta_{2^{r-1}}\right\}$ be an ordering of the elements in $W_{r-1}+\zeta$, obtained by imposing the condition that the elements of the coset $W_{k}+\zeta$ precede the elements of $W_{l}+\zeta$ if $k<l$.

A user is permitted to pick a c-p $m$ in the range, $1 \leq m \leq$ $2^{r-1}+1$, and this choice will permit him to communicate $(m+1)$ bits per period of the spreading sequence.

Let there be $N_{l}$ users wishing to communicate using c-p $m$ satisfying

$$
\mu(m)=l
$$

i.e., associated to sse $l$. Our construction below will require that the inequality

$$
\begin{equation*}
\sum_{l=0}^{r-1} N_{l} 2^{l+1} \leq 2^{r} \tag{19}
\end{equation*}
$$

hold and we will assume that this is the case. Let

$$
l_{1}>l_{2}>\cdots>l_{K}
$$

be an ordering of subspace-size exponents. The goal here is to provide each user with a ground coefficient $g$ which will enable him to construct his particular QAM sequence.

We begin with the c-ps associated to largest sse $l_{1}$. We begin by partitioning $W_{r-1}$ into disjoint cosets of the subspace $W_{l_{1}}$ and assign a coset of $W_{l_{1}}$ to each of these users. Each such coset is of the form $g+W_{l_{1}}$ and the user then constructs the user's QAM sequence using ground coefficient $g$. The set of all coefficients assigned to the user, namely the set

$$
\left\{g, g+\delta_{1}, \cdots, g+\delta_{m-1}\right\}
$$

then belongs to the coset of $V_{l_{1}+1}$ given by

$$
V_{l_{1}+1}+g=\left\{g+W_{l_{1}}\right\} \cup\left\{(g+\zeta)+W_{l_{1}}\right\}
$$

Having in this way made an assignment of coefficients to the users with largest c-p, we next move on to the users with next largest c-p. Suppose that $l_{2}=l_{1}-1$. In this case, we can partition one of the unused cosets of $W_{l_{1}}$, say $h+W_{l_{1}}$, according to

$$
W_{l_{1}}+h=\left\{W_{l_{2}}+h\right\} \cup\left\{W_{l_{2}}+\rho_{l_{2}}+h\right\}
$$

We can then assign either $h$ or $h+\rho_{l_{2}}$ as the ground coefficient for the user with sse $l_{2}$. If $l_{2}=l_{1}-2$, then we continue the process by further partitioning each coset $W_{l_{1}-1}+h, W_{l_{1}-1}+$ $h+\zeta$ of $W_{l_{1}-1}$ into two cosets of $W_{l_{1}-2}$ and assigning a coset of $W_{l_{1}-2}$ to that user etc. This process can clearly be continued to satisfy all users provided that the inequality in (19) is satisfied.

We illustrate with the help of an example for the case $r=4$.
Example 1: Let $q=16$ so that $r=4$. Let the primitive element $\alpha \in \mathbb{F}_{16}$ satisfy $\alpha^{4}+\alpha+1=0$. It is known that the element $\alpha^{3}$ has trace 1 , so we make the selection $\zeta=\alpha^{3}$. Let $W_{3}, W_{3}+\zeta$ denote the subsets of $\mathbb{F}_{16}$ having trace 0 and 1 respectively. Then it can be verified that

$$
\begin{aligned}
W_{3} & =\left\{0,1, \alpha, \alpha^{2}, \alpha^{4}, \alpha^{5}, \alpha^{8}, \alpha^{10}\right\} \\
W_{3}+\zeta & =\left\{\alpha^{3}, \alpha^{6}, \alpha^{7}, \alpha^{9}, \alpha^{11}, \alpha^{12}, \alpha^{13}, \alpha^{14}\right\} .
\end{aligned}
$$

Let the values

$$
N_{0}=N_{1}=N_{2}=1
$$

satisfying (19) be given. The largest value of $l$ such that $N_{l} \neq$ 0 is $l_{1}=2$. We also have $l_{2}=1, l_{3}=0$. We begin by considering the sequence of subspaces

$$
W_{0} \subseteq W_{1} \subseteq W_{2}
$$

We choose

$$
\begin{aligned}
& W_{0}=\{0\} \\
& W_{1}=W_{0} \cup\left\{W_{0}+\alpha\right\}=\{0, \alpha\} \\
& W_{2}=W_{1} \cup\left\{W_{1}+\alpha^{2}\right\}=\left\{0, \alpha, \alpha^{2}, \alpha^{5}\right\}
\end{aligned}
$$

Thus $\rho_{0}=\alpha$ and $\rho_{1}=\alpha^{2}$. This leads to

$$
\left(\delta_{1}, \delta_{2}, \delta_{3}, \delta_{4}\right)=\left(\alpha^{3}, \alpha^{9}, \alpha^{6}, \alpha^{11}\right)
$$

Since $l_{1}=2$, we begin by considering cosets of $W_{2}$ in $W_{r-1}=W_{3}$. It can be verified that

$$
W_{2} \cup\left\{W_{2}+1\right\}=W_{3} .
$$

We first select the coset $\left\{W_{2}+1\right\}$ (either coset could have been chosen at this step). The corresponding ground coefficient equals 1 and this is assigned to the user with sse $=l_{1}=2$. Since there is only one user with sse equal to 2 , we move on to consider the user with sse $=l_{2}=1$. Our next step is to partition the remaining coset of $W_{2}$, namely, in this case, $W_{2}$ itself. Since $\rho_{1}=\alpha^{2}$, we can partition $W_{2}$ into

$$
W_{2}=W_{1} \cup\left\{W_{1}+\alpha^{2}\right\}
$$

Again faced with a choice, we choose to assign coset $W_{1}+$ $\alpha^{2}$ to the user with sse $l_{2}=1$, corresponding to choice of $\alpha^{2}$ as the ground coefficient. This leaves us with the coset $W_{1}$. There is one remaining user with sse $=l_{0}=1$. Since $\rho_{0}=\alpha$, we have the partitioning

$$
W_{1}=W_{0} \cup\left\{W_{0}+\alpha\right\}
$$

Again we choose to assign coset $W_{0}+\alpha$ to the last remaining user, whose ground coefficient thus is set equal to $\alpha$.

Thus the signals of the 3 users are given by

$$
\begin{aligned}
s(1,0, t)= & \sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{12}\right] x \xi^{\tau_{4}}\right)}+2 \imath^{T\left(\left[1+2 \alpha^{13}\right] x \xi^{\tau_{3}}\right)}+\right. \\
& +4 \imath^{T\left(\left[1+2 \alpha^{7}\right] x \xi^{\tau_{2}}\right)}+8 \imath^{T\left(\left[1+2 \alpha^{14}\right] x \xi^{\tau_{1}}\right)} \\
& \left.+16 \imath^{T([1+2] x)}\right) \\
s\left(\alpha^{2}, 0, t\right)= & \sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{11}\right] x \xi^{\tau_{2}}\right)}+2 \imath^{T\left(\left[1+2 \alpha^{6}\right] x \xi^{\tau_{1}}\right)}+\right. \\
& \left.+4 \imath^{T\left(\left[1+2 \alpha^{2}\right] x\right)}\right) \\
s(\alpha, 0, t)= & \left.\sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{9}\right] x \xi^{\tau_{1}}\right.}\right)+2 \imath^{T([1+2 \alpha] x)}\right),
\end{aligned}
$$

where $x=\xi^{t}$ and $\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)=(1,2,3,4)$.
Figure 3 graphically depicts the assignment of ground coefficients. In the tree, the root node corresponds to the subspace $W_{r-1}=W_{3}$ in the example. Each node in the tree corresponds to a coset $W_{l}+g$ of some subspace $W_{l}$ of $W_{3}$. The nodes one level down from the root node corresponds to the two cosets of $W_{2}$ (one of them of course is $W_{2}$ itself). The nodes two levels down from the root node correspond to cosets of $W_{1}$. The leaf nodes correspond to cosets $W_{0}+g$ of $W_{0}=\{0\}$ in $W_{3}$. Each user is assigned a distinct node in the tree. The ground coefficient assigned to the particular user can be chosen to be any coset representative of the coset associated to that node. Given that a node is assigned to a user, no descendant of that node can be assigned to any other user. The coefficients assigned to the user are of the form

$$
\left\{g, g+\delta_{1}, \cdots, g+\delta_{m-1}\right\}
$$

The tree only depicts how $g$ is to be selected. Given $g$, the remaining coefficients are obtained by adding elements $\delta_{k}$ to $g$. The elements $\left\{\delta_{k}\right\}$ are themselves drawn from the coset $W_{3}+\zeta$ of $W_{3}$. This coset is not depicted in the tree.

In this example, the reader will have noticed that there are $16-10=6$ unused sequences remaining in Family $\mathcal{A}$. These may be added to the existing list of sequences as users wishing to use a 4-QAM constellation:

$$
\begin{aligned}
s(0,0, t) & =\sqrt{2 \imath}\left(\imath^{T(x)}\right) \\
s\left(\alpha^{3}, 0, t\right) & =\sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{3}\right] x\right)}\right) \\
s\left(\alpha^{4}, 0, t\right) & =\sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{4}\right] x\right)}\right) \\
s\left(\alpha^{5}, 0, t\right) & =\sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{5}\right] x\right)}\right) \\
s\left(\alpha^{8}, 0, t\right) & =\sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{8}\right] x\right)}\right) \\
s\left(\alpha^{10}, 0, t\right) & =\sqrt{2 \imath}\left(\imath^{T\left(\left[1+2 \alpha^{10}\right] x\right)}\right) .
\end{aligned}
$$

## V. An Interleaved Construction for 16-QAM

Setting $M=4$ in the $\mathcal{S} \mathcal{Q}_{M^{2}}$ construction yields Family $\mathcal{S} \mathcal{Q}_{16}$. The associated normalized maximum correlation parameter $\bar{\theta}_{\text {max }}$ for this family is upper bounded by $1.61 \sqrt{N}$ which is lower than the upper bound on $\bar{\theta}_{\max }$ of $1.8 \sqrt{N}$ for sequence family $\mathcal{Q}_{B}$. In the next subsection, we interleave sequences to construct a 16 -QAM sequence family whose upper bound on $\bar{\theta}_{\text {max }}$ is further lowered to $\sqrt{2 N}$.


Fig. 3. Variable-Rate Signalling with Three Users

## A. Family $\mathcal{I} \mathcal{Q}_{16}$

Let $\delta_{1}$ be an element of $\mathbb{F}_{q}$ with $\operatorname{tr}\left(\delta_{1}\right)=1$ and $H$ the additive subgroup $\left\{0, \delta_{1}\right\}$. Let $G$ be the set obtained by picking one coset representative from each of the coset representatives of $H$ in $\mathbb{F}_{q}$. Thus $G$ is of size

$$
|G|=\frac{q}{2}
$$

Let $\tau_{1}$ be such that $\left\{1, \alpha^{\tau_{1}}\right\}$ is a linearly independent set over $\mathbb{F}_{2}$.

Family $\mathcal{I} \mathcal{Q}_{16}$ is then defined as the collection of sequences:

$$
\begin{equation*}
\mathcal{I} \mathcal{Q}_{16}=\left\{\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4} \times \mathbb{F}_{2}\right\} \mid g \in G\right\} \tag{20}
\end{equation*}
$$

with the $\kappa$-th sequence assigned to the $g$-th user sequence given by 21. Sequences $\left\{u_{0}(t)\right\},\left\{u_{1}(t)\right\}$ are given by

$$
\begin{aligned}
& u_{0}(t)=T\left([1+2 g] \xi^{t}\right) \\
& u_{1}(t)=T\left(\left[1+2\left(g+\delta_{1}\right)\right] \xi^{t+\tau_{1}}\right)
\end{aligned}
$$

The theorem below identifies the principal properties of Family $\mathcal{I} \mathcal{Q}_{16}$.

Theorem 5.1: Let $\mathcal{I} \mathcal{Q}_{16}$ be the family of sequences over 16-QAM constellation defined in (20). Then,

1) All sequences in $\mathcal{I} \mathcal{Q}_{16}$ have period $N=2\left(2^{r}-1\right)$.
2) For large values of $N$, the energy of the sequences in the family is given by

$$
\mathcal{E} \approx 10 N
$$

3) For large values of $N$, the normalized maximum correlation parameter of the family can be bounded as

$$
\bar{\theta}_{\max } \lesssim \sqrt{2} \sqrt{N}
$$

4) Family $\mathcal{I} \mathcal{Q}_{16}$ can support $(N+2) / 4$ distinct users.
5) Each user can transmit 3 bits of data per sequence period.
6) The normalized minimum squared Euclidean distance between all sequences assigned to a user is given by

$$
\bar{d}_{\min }^{2} \approx 2 N
$$

7) The sequences in Family $\mathcal{I} \mathcal{Q}_{16}$ are approximately balanced.

Proof: A proof of the Property (3) can be found in Appendix VIII The remaining properties can be established in essentially the same manner as was done in the case of Family $\mathcal{S} \mathcal{Q}_{M^{2}}$.

## VI. FAMILIES OF SEQUENCES OVER Q-PAM CONSTELLATION

So far all our constructions have been for sequences over the $M^{2}$-QAM constellation. In this section, we shall show that by restricting the symbol alphabet to a size $2 M$ subset of the QAM constellation, the maximum correlation magnitude can be further lowered. This subset of the $M^{2}$-QAM constellation is given by (4) and for obvious reasons, will be referred to as the quadrature-PAM or Q-PAM constellation.

We first present a general technique for constructing families $\mathcal{P}_{2 M}$ of sequences over the Q-PAM constellation. Subsequently, we shall use a different interleaving technique to construct a family $\mathcal{I P}_{8}$ of sequences over the specific 8 -ary Q-PAM constellation. Remarkably, this latter sequence family achieves the Welch bound [16] on maximum magnitude of correlation with equality. To our knowledge, this family is the only-known non-trivial optimal family of sequences over a non-PSK symbol alphabet, i.e., over an alphabet not comprised of roots of unity.

## A. Family $\mathcal{P}_{2 M}$

Let $\delta_{1}, \delta_{2}, \ldots, \delta_{m-1}$ be trace 1 elements of $\mathbb{F}_{q}$ having the property that $\left\{1, \delta_{1}, \delta_{2}, \ldots, \delta_{m-1}\right\}$ is a linearly independent

$$
s(g, \kappa, t)= \begin{cases}\sqrt{2 \imath}\left(\imath^{u_{1}(t)}(-1)^{\kappa_{1}}+2 \imath^{u_{0}(t)}\right) \imath^{\kappa_{0}}, & t \text { even }  \tag{21}\\ \sqrt{2 \imath \imath}\left(\imath^{u_{0}(t)}-2 \imath^{u_{1}(t)}(-1)^{\kappa_{1}}\right) \imath^{\kappa_{0}}, & t \text { odd. }\end{cases}
$$

set over $\mathbb{F}_{2}$. Set $\delta_{0}=0$. Let $G=\left\{g_{a}\right\}$ be the largest subset of $\mathbb{F}_{q}$ having the property that

$$
g_{a}+\delta_{k} \neq g_{b}+\delta_{l}, \quad g_{a}, g_{b} \in G, \quad 0 \leq k, l \leq(m-1),
$$

unless $g_{a}=g_{b}$ and $\delta_{k}=\delta_{l}$. As before, the corresponding Gilbert-Varshamov and Hamming bounds on the size of $G$ are given by

$$
\begin{equation*}
\frac{2^{r}}{1+\binom{m-1}{1}+\binom{m-1}{2}} \leq|G| \leq \frac{2^{r}}{1+\binom{m-1}{1}} \tag{22}
\end{equation*}
$$

Family $\mathcal{P}_{2 M}$ is defined as follows:

$$
\begin{equation*}
\mathcal{P}_{2 M}=\left\{\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4} \times \mathbb{F}_{2}^{m-1}\right\} \mid g \in G\right\} . \tag{23}
\end{equation*}
$$

Each user is thus assigned the set

$$
\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4} \times \mathbb{F}_{2}^{m-1}\right\}
$$

of sequences with the $(\kappa)$-th sequence given by

$$
\begin{array}{r}
s(g, \kappa, t)=\sqrt{2 \imath}\left(\sum_{k=1}^{m-1} 2^{m-k-1} \imath^{u_{k}(t)}(-1)^{\kappa_{k}}+\right. \\
\left.2^{m-1} \imath^{u_{0}(t)}\right) \imath^{\kappa_{0}}
\end{array}
$$

where

$$
\begin{aligned}
& u_{0}(t)=T\left([1+2 g] \xi^{t}\right) \\
& u_{k}(t)= T\left(\left[1+2\left(g+\delta_{k}\right)\right] \xi^{t}\right) \\
& \quad k=1,2, \ldots, m-1
\end{aligned}
$$

Note that in relation to the definition of the sequence Family $\mathcal{S} \mathcal{Q}_{M^{2}}$, the time-shift parameters $\tau_{k}$ are absent in the present construction. It is the absence of the terms $\xi^{\tau_{k}}$ as we inadvertently discovered, that causes the sequence symbol alphabet to lie in the Q-PAM subconstellation. Nevertheless, as we see below, Family $\mathcal{P}_{2 m}$ has essentially the same properties as does Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ while enjoying the added advantage of a lower value of maximum correlation magnitude. The principal properties of Family $\mathcal{P}_{2 M}$ are summarized in the following theorem:

Theorem 6.1: Let $m \geq 2$ be a positive integer and let $\mathcal{P}_{2 M}$ be the family of sequences over $2 M$-ary Q-PAM constellation defined in (23). Then,

1) All sequences in $\mathcal{P}_{2 M}$ have period $N=2^{r}-1$.
2) For large values of $N$, the energy of the sequences in the family is given by

$$
\mathcal{E} \approx \frac{2}{3}\left(M^{2}-1\right) N
$$

3) The maximum correlation parameter of the family can be bounded as

$$
\theta_{\max } \lesssim \sqrt{\frac{20}{9}} M^{2} \sqrt{N+1}
$$

For large values of $M$ and $N$, the normalized maximum correlation parameter of the family can be bounded as

$$
\bar{\theta}_{\max } \lesssim \sqrt{5} \sqrt{N}
$$

4) Family $\mathcal{P}_{2 M}$ can support $|G|$ distinct users where $|G|$ lies in the range given in (22).
5) Each user can transmit $m+1$ bits of data per sequence period.
6) The normalized minimum squared Euclidean distance between all sequences assigned to a user is given by

$$
\bar{d}_{\min }^{2} \approx \frac{12}{M^{2}-1} N
$$

7) The sequences in Family $\mathcal{P}_{2 M}$ are approximately balanced.

Proof: The above properties of Family $\mathcal{P}_{2 M}$ can be established using the same techniques used to prove properties of Families $\mathcal{C} \mathcal{Q}_{M^{2}}$ and $\mathcal{S} \mathcal{Q}_{M^{2}}$, and are hence omitted.

The only difference in the correlation computations for Families $\mathcal{P}_{2 M}$ and $\mathcal{S} \mathcal{Q}_{M^{2}}$ is that, in this case, $\theta_{u_{k}, u_{0}^{\prime}}(\tau)$ is also at right angles with $\theta_{u_{0}, u_{k}^{\prime}}(\tau)$. This is in addition to $\theta_{u_{0}, u_{0}^{\prime}}(\tau)$ being in right angles with $\theta_{u_{k}, u_{k}^{\prime}}(\tau)$ (see Appendix VII).

1) Variable-Rate Signalling with Family $\mathcal{P}_{2 M}$ : In Section IV-B, we described in detail a technique to allow different users to transmit at variable rates by choosing sequences from various members of Families $\left\{\mathcal{S} \mathcal{Q}_{M^{2}}\right\}$ corresponding to constellations of different sizes. Similar techniques can be used to permit variable-rate signalling on the reverse link of a CDMA system in which users are permitted to choose spreading sequences from Families $\mathcal{P}_{2 M}$ for different values of parameter $M$. One obvious difference from the previous case is the linear independence of the set $\left\{1, \delta_{1}, \ldots, \delta_{m-1}\right\}$. A key ingredient of the sequence assignment in the variable-rate signalling scheme involving Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ was the identification, for every collection $H$ of elements $H=\left\{\delta_{1}, \delta_{2}, \cdots \delta_{m-1}\right\}$, of the smallest subspace $V_{l+1}$ containing $H$. In the case of Family $\mathcal{S} \mathcal{Q}_{M^{2}}$, this subspace was of dimension $l+1$ where $l=\mu(m)$. In the present case, the linear independence of the $\left\{\delta_{j}\right\}$ forces $l+1=(m-1)$ and the choice

$$
V_{l+1}=\left\langle\delta_{1}, \delta_{2}, \cdots, \delta_{m-1}\right\rangle .
$$

Given the subspaces $V_{l+1}$ the assignment proceeds as earlier. We omit the details.

## B. Family $\mathcal{I P}_{8}$

There are not many families of sequences that (asymptotically) meet the Welch lower bound [16] on sequence correlation, see [7]. To the authors' knowledge, those that do achieve the Welch bound with equality, are over a signal constellation associated to $M$-ary phase-shift keying for some $M \geq 2$. The asymptotically optimal family of sequences constructed in this section, Family $\mathcal{I P}_{8}$, is, however, over the

$$
s(g, \kappa, t)= \begin{cases}\sqrt{2 \imath}\left(\imath^{u_{1}(t)}(-1)^{\kappa_{1}}+2 \imath^{u_{0}(t)}\right) \imath^{\kappa_{0}} & , \quad t \text { even }  \tag{24}\\ \sqrt{2 \imath \imath}\left(\imath^{u_{0}(t)}-2 \imath^{u_{1}(t)}(-1)^{\kappa_{1}}\right) \imath^{\kappa_{0}} & , \quad t \text { odd }\end{cases}
$$

TABLE VIII
Simulation results for various sequence families.

| Family | Constellation | Period <br> $(N)$ | Family Size <br> Rate | Data | $\theta_{\max }$ | $\frac{\bar{\theta}_{\max }}{\sqrt{N}}$ | $d_{\min }^{2}$ | $\frac{\bar{d}_{\min }^{2}}{N}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{I} \mathcal{Q}_{16}$ | 16-QAM | 30 | 8 | 3 | 100 | 1.82 | 600 | 2 |
| $\mathcal{S} \mathcal{Q}_{16}$ | 16-QAM | 15 | 8 | 3 | 82.38 | 2.04 | 120 | 0.75 |
| $\mathcal{I} \mathcal{P}_{8}$ | 8-ary Q-PAM | 30 | 8 | 3 | 72.11 | 1.31 | 600 | 2 |
| $\mathcal{P}_{8}$ | 8-ary Q-PAM | 15 | 8 | 3 | 66.48 | 1.81 | 120 | 0.79 |
| $\mathcal{C} \mathcal{Q}_{16}$ | 16-QAM | 15 | 8 | 4 | 84.21 | 2.1 | 60 | 0.34 |

8-ary Q-PAM alphabet (see Fig. 2) and is constructed using sequence interleaving.

Let $\delta_{1}$ be a trace 1 element of $\mathbb{F}_{q}$ such that $\left\{1, \delta_{1}\right\}$ is a linearly independent set over $\mathbb{F}_{2}$. Let $H=\left\{0, \delta_{1}\right\}$ denote the additive subgroup generated by $\delta$ and let $G$ be the set of coset representatives of $H$ in $\mathbb{F}_{q}$.

Family $\mathcal{I P}_{8}$ is defined as follows:

$$
\begin{equation*}
\mathcal{I P}_{8}=\left\{\left\{s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_{4} \times \mathbb{F}_{2}\right\} \mid g \in G\right\} \tag{25}
\end{equation*}
$$

with the $\kappa$-th sequence given by (24) and with

$$
\begin{aligned}
u_{0}(t) & \left.=T([1+2 g)] \xi^{t}\right) \\
u_{1}(t) & =T\left(\left[1+2\left(g+\delta_{1}\right)\right] \xi^{t}\right)
\end{aligned}
$$

As would be evident from the definition of the sequence family, we have interleaved two sequences over the 8 -ary QPAM alphabet to generate a single sequence over the same alphabet.

Most of the properties of Family $\mathcal{I P}_{8}$ like period of the sequences, family size, data rate, Euclidean distance, and balance are exactly the same as that of Family $\mathcal{I} \mathcal{Q}_{16}$ (see Section V-A). The main difference between the two families is that in case of Family $\mathcal{I} \mathcal{P}_{8}$, for large values of $N$, the normalized maximum correlation parameter achieves the Welch bound, i.e., can be bounded as

$$
\bar{\theta}_{\max } \lesssim \sqrt{N}
$$

and this is established in Appendix IX For sake of brevity, we omit the proofs of the other properties.

## VII. Simulation Results

We have simulated one member of each of the various sequence families that we have constructed in this paper. The results of the simulation are available in Table VIII The underlying finite field that is used to construct all the families is the same: $\mathbb{F}_{16}$. We have chosen the minimal polynomial $X^{4}+X+1$ to construct $\mathbb{F}_{16}$. The construction of the families mirrors their definition in the paper. Since we have chosen sequences of short period, the results are not completely indicative of the asymptotic behavior of the families.

## Appendix I <br> Closed-Form Expression for Family $\mathcal{A}$ Correlation

A useful closed-form expression for the pairwise-correlation between a pair of sequences drawn from Family $\mathcal{A}$ is given below.

Lemma 1.1: [17] Let $a+2 b \in R$ with $a, b \in \mathcal{T}, a \neq 0$. Define

$$
\Gamma(a+2 b):=\sum_{x \in \mathcal{T}} \imath^{T([a+2 b] x)} .
$$

Then

$$
\Gamma(1)=\left\{\begin{array}{cc}
\sqrt{2^{r}} \epsilon^{r}, & r \text { is odd }  \tag{26}\\
-\sqrt{2^{r}} \epsilon^{r}, & r \text { is even }
\end{array}\right.
$$

where

$$
\epsilon=\frac{1+\imath}{\sqrt{2}}
$$

Further,

$$
\begin{equation*}
\Gamma(a+2 b)=\Gamma(1) \imath^{-T\left(\frac{b}{a}\right)} \tag{27}
\end{equation*}
$$

Proof: A proof of 27) can be found in [17] and is presented below for the sake of completeness.

$$
\begin{align*}
\Gamma(a+2 b) & =\sum_{x \in \mathcal{T}} \imath^{T([a+2 b] x)} \\
& =\sum_{x \in \mathcal{T}} \imath^{T\left(\left[1+2 \frac{b}{a}\right] a x\right)} \\
& =\sum_{x \in \mathcal{T}} \imath^{T\left(\left[1+2 \frac{b}{a}\right] x\right)} \\
& =\Gamma(1+2 \gamma), \quad \text { where } \gamma=\frac{b}{a} \in \mathcal{T} . \tag{28}
\end{align*}
$$

The third equation is obtained by replacing $x$ by $a x$. Now, if $x$ and $\gamma$ are two elements in $\mathcal{T}$, then $(x+\gamma+2 \sqrt{x \gamma})$ also belongs to $\mathcal{T}$. If $x$ runs over all elements of $\mathcal{T}$, then $(x+\gamma+2 \sqrt{x \gamma})$
also runs over all elements of $\mathcal{T}$. Therefore

$$
\begin{align*}
\Gamma(1) & =\sum_{x \in \mathcal{T}} \imath^{T(x)} \\
& =\sum_{x \in \mathcal{T}} \imath^{T(x+\gamma+2 \sqrt{x \gamma})} \\
& =\sum_{x \in \mathcal{T}} \imath^{T(x+2 \sqrt{x \gamma})} \imath^{T(\gamma)} \\
& =\sum_{x \in \mathcal{T}} \imath^{T(x+2 x \gamma)} \imath^{T(\gamma)} \\
& =\sum_{x \in \mathcal{T}} \imath^{T([1+2 \gamma] x)} \imath^{T(\gamma)} \\
& =\Gamma(1+2 \gamma) \imath^{T(\gamma)} \tag{29}
\end{align*}
$$

Proof of the lemma is completed by comparing the two expressions in (28) and (29).

We refer the reader to [17] for the algebraic-geometric proof of (26).

The following lemma summarizes the correlation properties of sequences from Family $\mathcal{A}$.

Lemma 1.2: Consider two sequences from Family $\mathcal{A}$ defined as

$$
\begin{aligned}
s(a, t) & =\imath^{T\left([1+2 a] \xi^{t}\right)} \quad \text { and } \\
s(b, t) & =\imath^{T\left([1+2 b] \xi^{t}\right)}, \quad a, b \in \mathcal{T} .
\end{aligned}
$$

Then

$$
\theta_{s(a), s(b)}(\tau)= \begin{cases}2^{r}-1, & a=b, \tau=0 \\ -1, & a \neq b, \tau=0 \\ -1+\Gamma(1) \imath^{-T(z)}, & \tau \neq 0\end{cases}
$$

where

$$
z=a+\frac{a+b}{y}+\frac{1}{\sqrt{y}}+2 \mu(a, b, y)
$$

with

$$
y=\xi^{\tau}+1+2 \sqrt{\xi^{\tau}}
$$

and $\mu$ some function of $a, b, y$.
Proof: Set $x=\xi^{\tau}$. The proof follows from an application of Lemma 1.1, cf. [4], [17], and noting that

$$
\begin{aligned}
{[1} & +2 a] x-[1+2 b] \\
& =(x-1)+2(a x+b) \\
& =x+1+2 \sqrt{x}+2[a x+b+1+\sqrt{x}] \\
& =y+2[a(y+1)+b+1+\sqrt{y}+1] \\
& =y+2[a y+a+b+\sqrt{y}],
\end{aligned}
$$

where we have set

$$
y \triangleq x+1+2 \sqrt{x}
$$

Thus the " $\gamma$ " in Lemma 1.1 is given by

$$
a+\frac{a+b}{y}+\frac{1}{\sqrt{y}}+2 \mu(a, b, y)
$$

## Appendix II

Balance of Sequences over $M^{2}$-QAM
(Proof of Lemma 3.1).
Let $\mathbf{N}(\nu)$ be the number of times the element $\sum_{k=0}^{m-1} 2^{k} \imath^{\nu_{k}}$ from the $M^{2}$-QAM constellation occurs in one period of the sequence $\left\{s_{p}(t)\right\}$, where $\nu=\left(\nu_{0}, \nu_{1}, \ldots, \nu_{m-1}\right)$. We have
$\mathbf{N}(\nu)=\left|\left\{x \in \mathcal{T} \mid T\left(a_{p, k} x\right)=\nu_{k}, k=0,1, \ldots, m-1\right\}\right|$
We rewrite the expression for $\mathbf{N}(\nu)$ with the aid of exponential sums to get

$$
\begin{align*}
& \mathbf{N}(\nu) \\
&= \frac{1}{4^{m}} \sum_{x \in \mathcal{T}} \sum_{\omega \in \mathbb{Z}_{4}^{m}} \imath^{\sum_{k=0}^{m-1}\left(T\left(a_{p, k} x\right)-\nu_{k}\right) \omega_{k}}, \\
&= \frac{1}{M^{2}} \sum_{\omega \in \mathbb{Z}_{4}^{m}} \imath^{-\sum_{k=0}^{m-1} \nu_{k} \omega_{k}} . \\
& \sum_{x \in \mathcal{T}} \imath^{T\left(\left[\sum_{k=0}^{m-1} \omega_{k} a_{p, k}\right] x\right)} \\
&= \frac{q}{M^{2}}+\frac{1}{M^{2}} \sum_{\omega \in \mathbb{Z}_{4}^{m}, \omega \neq 0} \imath^{-\sum_{k=0}^{m-1} \nu_{k} \omega_{k}} . \\
&\left(\sum_{x \in \mathcal{T}} \imath^{T\left(\left[\sum_{k=0}^{m-1} \omega_{k} a_{p, k}\right] x\right)}\right) . \tag{30}
\end{align*}
$$

By the linear independence of the coefficients $a_{p, k}$, $\sum_{k=0}^{m-1} \omega_{k} a_{p, k}$ does not equal 0 for any choice of $\omega$ and, hence, the magnitude of

$$
\left(\sum_{x \in \mathcal{T}} \imath^{T\left(\left[\sum_{k=0}^{m-1} \omega_{k} a_{p, k}\right] x\right)}\right)
$$

is bounded from above by $\sqrt{q}=\sqrt{N+1}$ (see Appendix $\square$ ). With this, we get the bound

$$
\left|\mathbf{N}(\nu)-\frac{q}{M^{2}}\right| \leq \frac{M^{2}-1}{M^{2}} \sqrt{N+1}
$$

thus proving approximate balance.

## Appendix III

Energy of Sequences in Canonical Family $\mathcal{C} \mathcal{Q}_{M^{2}}$
(Proof of Property (2) in Theorem 3.2)
Consider two sequences in Family $\mathcal{C} \mathcal{Q}_{M^{2}},\{s(g, \kappa, t)\}$ and $\left\{s\left(g^{\prime}, \kappa^{\prime}, t\right)\right\}$, given by

$$
\begin{align*}
s(g, \kappa, t) & =\sqrt{2 \imath} \sum_{k=0}^{m-1} 2^{k} \imath^{u_{k}(t)} \imath^{\kappa_{k}}  \tag{31}\\
s\left(g^{\prime}, \kappa^{\prime}, t\right) & =\sqrt{2 \imath} \sum_{l=0}^{m-1} 2^{l} \imath^{\prime} u_{l}^{\prime}(t) \tag{32}
\end{align*} \imath^{\kappa_{l}^{\prime}} .
$$

The correlation between the two sequences is given by

$$
\begin{aligned}
& \theta_{s(g, \kappa), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau)= \\
& 2 \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} 2^{k+l} u^{u_{k}(t+\tau)-u_{l}^{\prime}(t)} \imath^{\kappa_{k}-\kappa_{l}^{\prime}}
\end{aligned}
$$

Thus, energy of a sequence $\{s(g, \kappa, t)\}$ is given by

$$
\begin{aligned}
\mathcal{E} & (s(g, \kappa)) \\
& =\theta_{s(g, \kappa), s(g, \kappa)}(0) \\
& =2 \sum_{t} \sum_{k} \sum_{l} 2^{k+l} \imath^{u_{k}(t)-u_{l}(t)} \imath^{\kappa_{k}-\kappa_{l}} \\
& =2 \sum_{k=0}^{m-1} 4^{k} N+2 \sum_{k, l, k \neq l} 2^{k+l} \theta_{u_{k}, u_{l}}(0) \imath^{\kappa_{k}-\kappa_{l}}
\end{aligned}
$$

It follows from this that

$$
\begin{aligned}
\mid \mathcal{E} & \left.(s(g, \kappa))-2 \frac{\left(M^{2}-1\right)}{3} N \right\rvert\, \\
& \leq 2\left(\sum_{k, l} 2^{k+l}-\sum_{k=l} 2^{k}\right) \sqrt{N+1} \\
& =2\left(\left(2^{m}-1\right)^{2}-\frac{4^{m}-1}{3}\right) \sqrt{N+1} \\
& =2\left(\frac{2}{3} 4^{m}-2^{m+1}+\frac{4}{3}\right\} \sqrt{N+1}
\end{aligned}
$$

Thus, for large $N$, the energy is approximately given by

$$
\mathcal{E}(s(g, \kappa)) \approx 2 \frac{\left(M^{2}-1\right)}{3} N
$$

## Appendix IV

Correlation of Sequences in Canonical Family

$$
\mathcal{C} \mathcal{Q}_{M^{2}}
$$

(Proof of Property (3) in Theorem 3.2)
The correlation between two sequences, $\{s(g, \kappa, t)\}$ and $\left\{s\left(g^{\prime}, \kappa^{\prime}, t\right)\right\}$ (see (31) and (32), from Family $\mathcal{C} \mathcal{Q}_{M^{2}}$ is given by

$$
\begin{aligned}
& \theta_{s(g, \kappa), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau)= \\
& \quad 2 \sum_{t} \sum_{k} \sum_{l} 2^{k+l} \imath^{u_{k}(t+\tau)-u_{l}^{\prime}(t)} \imath^{\kappa_{k}-\kappa_{l}^{\prime}} .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \left|\theta_{s(g, \kappa), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau)\right| \\
& \quad \leq 2 \sum_{k} \sum_{l} 2^{k+l}\left|\theta_{u_{k}, u_{l}^{\prime}}(\tau)\right| \\
& \quad \leq 2 \sum_{k, l} 2^{k+l}(1+\sqrt{N+1}) \\
& \quad \lesssim 2\left(2^{m}-1\right)^{2} \sqrt{N+1} \\
& \quad=2(M-1)^{2} \sqrt{N+1} .
\end{aligned}
$$

Normalizing with the energy of the concerned sequences, we obtain

$$
\begin{aligned}
\bar{\theta}_{\max } & \lesssim \frac{2(M-1)^{2}}{2\left(M^{2}-1\right) / 3} \sqrt{N+1} \\
& \lesssim \frac{3(M-1)^{2}}{\left(M^{2}-1\right)} \sqrt{N+1} \\
& \lesssim 3 \sqrt{N}
\end{aligned}
$$

for large $N$ and $M$.

## Appendix V

Minimum-SQuared Euclidean Distance for CANONICAL FAMILY $\mathcal{C} \mathcal{Q}_{M^{2}}$
(Proof of Property (6) in Theorem 3.2)
We are interested in computing the minimum squared Euclidean distance between all the sequences assigned to the same user. Consider the two sequences assigned to a user: $\{s(g, \kappa, t)\}$ and $\left\{s\left(g, \kappa^{\prime}, t\right)\right\}$.

Let $L$ be the set of all indices $k$ such that

$$
\kappa_{k} \neq \kappa_{k}^{\prime} .
$$

Then

$$
\begin{aligned}
& d_{E}^{2}\left(s(g, \kappa, t), s\left(g, \kappa^{\prime}, t\right)\right) \\
& =2 \sum_{t}\left|\sum_{k \in L} 2^{k} \imath^{u_{k}(t)}\left(\imath^{\kappa_{k}}-\imath^{\kappa_{k}^{\prime}}\right)\right|^{2} \\
& =2 \sum_{t} \sum_{k, l \in L} 2^{k+l} \imath^{u_{k}(t)-u_{l}(t)} \text {. } \\
& \left(\imath^{\kappa_{k}}-\imath^{\kappa_{k}^{\prime}}\right)\left(\imath^{-\kappa_{l}}-\imath^{-\kappa_{l}^{\prime}}\right) \\
& =2 \sum_{t} \sum_{k \in L} 4^{k}\left|\imath^{\kappa_{k}}-\imath^{\kappa_{k}^{\prime}}\right|^{2}+ \\
& 2 \sum_{t} \sum_{k, l \in L, k \neq l} 2^{k+l} \imath^{u_{k}(t)-u_{l}(t)} . \\
& \left(\imath^{\kappa_{k}}-\imath^{\kappa_{k}^{\prime}}\right)\left(\imath^{-\kappa_{l}}-\imath^{-\kappa_{l}^{\prime}}\right) \\
& \geq 2 N \sum_{k \in L} 4^{k}\left|\imath^{\kappa_{k}}-\imath^{\kappa_{k}^{\prime}}\right|^{2}- \\
& 2 \sqrt{N+1} \sum_{k, l \in L, k \neq l} 2^{k+l}\left(\imath^{\kappa_{k}}-\imath^{\kappa_{k}^{\prime}}\right)\left(\imath^{\kappa_{l}}-\imath^{\kappa_{l}^{\prime}}\right)
\end{aligned}
$$

For large $N$, this bound is minimized when

$$
\begin{aligned}
& \kappa_{k}=\kappa_{k}^{\prime} \quad 1 \leq k \leq(m-1) \\
& \kappa_{0}=\kappa_{0}^{\prime}+1
\end{aligned}
$$

leading to

$$
d_{\min }^{2}=4 N
$$

Upon normalization with the energy of the sequences, we obtain

$$
\bar{d}_{\min }^{2} \approx \frac{4 N}{\frac{2\left(M^{2}-1\right)}{3}}=\frac{6 N}{M^{2}-1}
$$

## Appendix VI

Bounding the Correlation under Variable Rate Signalling for Families $\left\{\mathcal{C} \mathcal{Q}_{M^{2}}\right\}$
Let the sequences assigned to two users from Families $\mathcal{C} \mathcal{Q}_{M_{1}^{2}}$ and $\mathcal{C} \mathcal{Q}_{M_{2}^{2}}$ be given by

$$
\begin{aligned}
s(g, \kappa, t) & =\sqrt{2 \imath} \sum_{k=0}^{m_{1}-1} 2^{k} \imath^{u_{k}(t)} \imath^{\kappa_{k}} \\
s\left(g^{\prime}, \kappa^{\prime}, t\right) & =\sqrt{2 \imath} \sum_{l=0}^{m_{2}-1} 2^{l} \imath^{u_{l}^{\prime}(t)} \imath^{\kappa_{l}^{\prime}}
\end{aligned}
$$

We assume, without loss of generality, that $m_{1}>m_{2}$. Then

$$
\theta_{s(g, \kappa), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau)=2 \sum_{k=0}^{m_{1}-1} \sum_{l=0}^{m_{2}-1} 2^{k+l} \theta_{u_{k}, u_{l}^{\prime}}(\tau) \imath^{\kappa_{k}-\kappa_{l}^{\prime}}
$$

Each $\theta_{u_{k}, u_{l}^{\prime}}(\tau)$ corresponds to correlation of a pair of sequences from Family $\mathcal{A}$ and has magnitude bounded by $(1+\sqrt{N+1})$. Therefore, the magnitude of $\theta_{s(g, \kappa), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau)$ can be bounded as

$$
\begin{aligned}
\left|\theta_{s(g, \kappa), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau)\right| & \lesssim 2\left(2^{m_{1}}-1\right)\left(2^{m_{2}}-1\right) \sqrt{N+1} \\
& =2\left(M_{1}-1\right)\left(M_{2}-1\right) \sqrt{N+1}
\end{aligned}
$$

The sequences $\{s(g, \kappa, t)\},\left\{s\left(g^{\prime}, \kappa^{\prime}, t\right)\right\}$ have energy $\frac{2\left(M_{1}^{2}-1\right)}{3}$ and $\frac{2\left(M_{2}^{2}-1\right)}{3}$ respectively. Upon normalization, we obtain the bound on normalized correlation

$$
\begin{aligned}
\bar{\theta}_{s(g, \kappa), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau) & \lesssim \frac{2\left(M_{1}-1\right)\left(M_{2}-1\right)}{\frac{2}{3} \sqrt{\left(M_{1}^{2}-1\right)\left(M_{2}^{2}-1\right)}} \sqrt{N+1} \\
& =\frac{3 \sqrt{\left(M_{1}-1\right)\left(M_{2}-1\right)}}{\sqrt{\left(M_{1}+1\right)\left(M_{2}+1\right)}} \sqrt{N+1}
\end{aligned}
$$

The normalized non-trivial autocorrelation functions are bounded respectively by

$$
\begin{aligned}
\bar{\theta}_{s(g, \kappa), s(g, \kappa)}(\tau) & \lesssim \frac{3\left(M_{1}-1\right)^{2}}{\left(M_{1}^{2}-1\right)} \sqrt{N+1} \\
& =\frac{3\left(M_{1}-1\right)}{\left(M_{1}+1\right)} \sqrt{N+1}
\end{aligned}
$$

for sequences over QAM constellations of size $M_{1}^{2}$, and

$$
\bar{\theta}_{s\left(g^{\prime}, \kappa^{\prime}\right), s\left(g^{\prime}, \kappa^{\prime}\right)}(\tau) \lesssim \frac{3\left(M_{2}-1\right)}{\left(M_{2}+1\right)} \sqrt{N+1}
$$

for sequences over QAM constellations of size $M_{2}^{2}$. It follows that the normalized maximum correlation magnitude $\bar{\theta}_{\max }$ experienced by the user with a smaller constellation is given by

$$
\bar{\theta}_{\max } \lesssim \frac{3 \sqrt{\left(M_{1}-1\right)\left(M_{2}-1\right)}}{\sqrt{\left(M_{1}+1\right)\left(M_{2}+1\right)}} \sqrt{N+1}
$$

whereas the value of $\bar{\theta}_{\text {max }}$ experienced by the user with larger constellation size remains unchanged at

$$
\bar{\theta}_{\max } \lesssim \frac{3\left(M_{1}-1\right)}{\left(M_{1}+1\right)} \sqrt{N+1}
$$

## Appendix VII

## Correlation of SEQUENCES IN FAmily $\mathcal{S} \mathcal{Q}_{M^{2}}$

(Proof of Property (3) in Theorem 4.1)
We establish properties for the basic sequences and leave the straightforward extension of the results to the general case to the reader.

For a fixed $M$, let $\{s(g, 0, t)\}$ and $\left\{s\left(g^{\prime}, 0, t\right)\right\}$ be two basic sequences from Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ where

$$
\begin{align*}
& s(g, 0, t)=\sqrt{2 \imath} \sum_{k=0}^{m-1} 2^{m-k-1} \imath^{u_{k}(t)}  \tag{33}\\
& s\left(g^{\prime}, 0, t\right)=\sqrt{2 \imath} \sum_{l=0}^{m-1} 2^{m-l-1} \imath^{u_{l}^{\prime}(t)} \tag{34}
\end{align*}
$$

with

$$
\begin{align*}
& u_{0}(t)=T\left([1+2 g] \xi^{t}\right) \\
& u_{k}(t)= T\left(\left[1+2\left(g+\delta_{k}\right)\right] \xi^{t+\tau_{k}}\right)  \tag{35}\\
& \quad k=1,2, \ldots, m-1 \\
& u_{0}^{\prime}(t)=T\left(\left[1+2 g^{\prime}\right] \xi^{t}\right) \\
& u_{k}^{\prime}(t)= T\left(\left[1+2\left(g^{\prime}+\delta_{k}\right)\right] \xi^{t+\tau_{k}}\right)  \tag{36}\\
& k=1,2, \ldots, m-1
\end{align*}
$$

The correlation between the two sequences, $\{s(g, 0, t)\}$ and $\left\{s\left(g^{\prime}, 0, t\right)\right\}$, is given by

$$
\begin{align*}
& \theta_{s(g, 0), s\left(g^{\prime}, 0\right)}(\tau) \\
& \quad=\sum_{t=0}^{N-1}\left(\sqrt{2 \imath} \sum_{k=0}^{m-1} 2^{m-k-1} \imath^{u_{k}(t+\tau)}\right) \\
& \left.\quad=2 \sum_{l=0}^{m-1} 2^{m-l-1} \imath^{u_{l}^{\prime}(t)}\right) \\
& \quad=2 \sum_{k, l=0}^{m-1} 2^{2 m-k-l-2}\left(\sum_{t=0}^{N-1} \imath^{u_{k}(t+\tau)-u_{l}^{\prime}(t)}\right)  \tag{37}\\
& \quad 2 m-k-l-2
\end{align*} \theta_{u_{k}, u_{l}^{\prime}}(\tau) .
$$

The correlation properties of sequences in Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ are handled in two separate cases.

1) Zero time shift $(\tau=0)$ : Only the case $g \neq g^{\prime}$ is of interest here. Then,

$$
\theta_{s(g, 0), s\left(g^{\prime}, 0\right)}(0)=2 \sum_{k, l=0}^{m-1} 2^{2 m-k-l-2} \theta_{u_{k}, u_{l}^{\prime}}(0)
$$

For the case $k=l$ we have:

$$
\begin{aligned}
& \theta_{u_{k}} u_{k}^{\prime}(0) \\
&=\sum_{t} \imath^{T\left(\left[1+2\left(g_{k}+\delta_{k}\right)\right] \xi^{t+\tau_{k}}\right)-T\left(\left[1+2\left(g_{k}^{\prime}+\delta_{k}\right)\right] \xi^{t+\tau_{k}}\right)} \\
& \quad=\sum_{t} \imath^{2 T\left(\left[g_{k}-g_{k}^{\prime}\right] \xi^{t+\tau_{k}}\right)} \\
& \quad=-1 \quad \text { (see Appendix 【I). }
\end{aligned}
$$

For the case $k \neq l$, we have the bound

$$
\left|\theta_{u_{k}, u_{l}^{\prime}}(0)\right| \leq(1+\sqrt{N+1})
$$

This leads to

$$
\begin{align*}
& \left|\theta_{s(g), s\left(g^{\prime}\right)}(0)\right| \\
& \quad \leq 2\left(2^{m}-1\right)^{2}+2\left(\frac{2}{3} 4^{m}-2^{m+1}+\frac{4}{3}\right) \sqrt{N+1} \\
& \quad \lesssim 2\left(\frac{2}{3} 4^{m}-2^{m+1}+\frac{4}{3}\right) \sqrt{N} \tag{38}
\end{align*}
$$

2) Non-zero time shift $(\tau \neq 0)$ : Here we need to consider, in addition, the case when $g=g^{\prime}$.

We rewrite the expression for $\theta_{s(g, 0), s\left(g^{\prime}, 0\right)}(\tau)$ from (37) to get

$$
\begin{gather*}
\theta_{s(g, 0), s\left(g^{\prime}, 0\right)}(\tau) \\
=2\left[\sum_{k, l=0, k \neq l}^{m-1}\left(2^{2(m-1)-k-l} \theta_{u_{k}, u_{l}^{\prime}}(\tau)\right)+\right. \\
\left.\sum_{k=0}^{m-1} 2^{2(m-k-1)} \theta_{u_{k}, u_{k}^{\prime}}(\tau)\right] . \tag{39}
\end{gather*}
$$

Using Lemma 1.2 in Appendix $\mathbb{\square}$ we can rewrite the expressions for $\theta_{u_{k}, u_{k}^{\prime}}(\tau)$ as:

$$
\theta_{u_{k}, u_{k}^{\prime}}(\tau)=-1+\Gamma(1) \imath^{-T\left(z_{k}\right)}, \quad k=0,1, \ldots, m-1
$$

where

$$
\begin{aligned}
z_{0} & =g+\frac{g+g^{\prime}}{y}+\frac{1}{\sqrt{y}}+2 \mu\left(g, g^{\prime}, y\right) \\
z_{k} & =\left(g+\delta_{k}\right)+\frac{\left(g+g^{\prime}\right)}{y}+\frac{1}{\sqrt{y}}+2 \mu\left(g+\delta_{k}, g^{\prime}+\delta_{k}, y\right) \\
& =z_{0}+\delta_{k}+2 \mu^{\prime}\left(g+\delta_{k}, g^{\prime}+\delta_{k}, y\right)
\end{aligned}
$$

The element $y$ is a function of the time shift $\tau$.
Substituting the expressions for $\theta_{u_{k}, u_{k}^{\prime}}(\tau)$ in (39), we get

$$
\begin{aligned}
& \theta_{s(g, 0), s\left(g^{\prime}, 0\right)}(\tau) \\
& =2\left(\sum_{k, l=0, k \neq l}^{m-1}\left(2^{2(m-1)-k-l} \theta_{u_{k}, u_{l}^{\prime}}(\tau)\right)+\right. \\
& \sum_{k=1}^{m-1} 2^{2(m-k-1)}(-1+ \\
& \left.\Gamma(1) \imath^{-T\left(z_{0}+\delta_{k}+2 \mu^{\prime}\left(g+\delta_{k}, g^{\prime}+\delta_{k}, y\right)\right)}\right)+ \\
& =2\left(\sum_{k, l=0, k \neq l}^{m-1}\left(2^{2(m-1)-k-l} \theta_{u_{k}, u_{l}^{\prime}}(\tau)\right)+\right. \\
& \sum_{k=1}^{m-1} 2^{2(m-k-1)}(-1+ \\
& \\
& \left.\Gamma(1) \imath^{-T\left(z_{0}\right)} \imath^{-T\left(\delta_{k}\right)}(-1)^{t r\left(\mu^{\prime}\left(g+\delta_{k}, g^{\prime}+\delta_{k}, y\right)\right)}\right)+ \\
& \left.2^{2 m-2}\left(-1+\Gamma(1) \imath^{-T\left(z_{0}\right)}\right)\right) .
\end{aligned}
$$

We know that $\left|\theta_{u_{k}, u_{l}^{\prime}}(\tau)\right|$ can be bounded as $(1+|\Gamma(1)|)=$
$(1+\sqrt{N+1})$. Also, $\operatorname{tr}\left(\delta_{k}\right)=1$. It follows that

$$
\begin{align*}
& \left|\theta_{s(g, 0), s\left(g^{\prime}, 0\right)}(\tau)\right| \\
& \leq \mid 2\left(2^{m}-1\right)^{2}+2 \Gamma(1)\left(\sum_{k, l=0, k \neq l}^{m-1} 2^{2(m-1)-k-l}+\right. \\
& \left.\sum_{k=0}^{m-2} 2^{2 k}(\imath)+2^{2 m-2}(1)\right) \\
& =\left\lvert\, 2\left(2^{m}-1\right)^{2}+2 \Gamma(1)\left(4^{m-1}+\frac{2}{3} 4^{m}-2^{m+1}+\right.\right. \\
& \frac{4}{3}+{\left.\frac{4^{m-1}-1}{3} \imath\right) \mid}_{2} \\
& \lesssim\left(\frac{61}{18} 16^{m}-\frac{44}{3} 8^{m}+\frac{230}{9} 4^{m}-\frac{64}{3} 2^{m}+\frac{68}{9}\right)^{\frac{1}{2}} \sqrt{N} \text {. } \tag{40}
\end{align*}
$$

To determine $\theta_{\max }$ (and hence $\bar{\theta}_{\max }$ as well), for Family $\mathcal{S} \mathcal{Q}_{M^{2}}$, it turns out to be enough to restrict attention to correlations amongst basic sequences. With the aid of Property (2) dealing with the energy of sequences of Family $\mathcal{S} \mathcal{Q}_{M^{2}}$ and the correlation bounds in (38) and 40), we arrive at the following upper bounds on $\theta_{\max }$ and $\bar{\theta}_{\max }$ for large $M, N$ :

$$
\theta_{\max } \lesssim \frac{\sqrt{122}}{6} M^{2} \sqrt{N}
$$

and

$$
\bar{\theta}_{\max } \lesssim \frac{\sqrt{122}}{4} \sqrt{N}
$$

## Appendix VIII

## Correlation of SEQUENCES in Family $\mathcal{I} \mathcal{Q}_{16}$

(Proof of Property (3) of Theorem 5.1)
As in the case of Family $\mathcal{S} \mathcal{Q}_{16}$, we define basic sequences in Family $\mathcal{I} \mathcal{Q}_{16}$ as sequences corresponding to assigning $\kappa_{0}=\kappa_{1}=0$ in 21). We analyze the correlation between two basic sequences from Family $\mathcal{I} \mathcal{Q}_{16}$ and it is straightforward to extend the results to the case of modulated sequences.

Let $\left\{s\left(g_{1}, 0, t\right)\right\}$ and $\left\{s\left(g_{2}, 0, t\right)\right\}$ be two basic sequences belonging to Family $\mathcal{I} \mathcal{Q}_{16}$, i.e.,

$$
\begin{align*}
& s\left(g_{1}, 0, t\right)=\left\{\begin{array}{l}
\sqrt{2 \imath}\left(\imath^{u_{1}(t)}+2 \imath^{u_{0}(t)}\right), t \text { even } \\
\sqrt{2 \imath} \imath\left(\imath^{u_{0}(t)}-2 \imath^{u_{1}(t)}\right), t \text { odd }
\end{array}\right.  \tag{41}\\
& s\left(g_{2}, 0, t\right)=\left\{\begin{array}{l}
\sqrt{2 \imath}\left(\imath^{v_{1}(t)}+2 \imath^{v_{0}(t)}\right), t \text { even } \\
\sqrt{2 \imath} \imath\left(\imath^{v_{0}(t)}-2 \imath^{v_{1}(t)}\right), t \text { odd }
\end{array}\right. \tag{42}
\end{align*}
$$

where

$$
\begin{aligned}
u_{0}(t) & =T\left(\left[1+2 g_{1}\right] \xi^{t}\right) \\
u_{1}(t) & =T\left(\left[1+2\left(g_{1}+\delta_{1}\right)\right] \xi^{t+\tau_{1}}\right) \\
v_{0}(t) & =T\left(\left[1+2 g_{2}\right] \xi^{t}\right) \text { and } \\
v_{1}(t) & =T\left(\left[1+2\left(g_{2}+\delta_{1}\right)\right] \xi^{t+\tau_{1}}\right)
\end{aligned}
$$

Lemma 8.1: Let $\left\{s\left(g_{1}, 0, t\right)\right\}$ and $\left\{s\left(g_{2}, 0, t\right)\right\}$ be two sequences defined in (41) and (42). Then

$$
\theta_{s\left(g_{1}\right), s\left(g_{2}\right)}(\tau) \lesssim \sqrt{2} \sqrt{N}, \quad 0 \leq \tau<N
$$

Proof: The expression for the correlation between the two Q-PAM sequences will take on one of two forms depending on if $\tau=0(\bmod 2)$ or if $\tau=1(\bmod 2)$.

Let us suppose that $\tau=0(\bmod 2)$. In that case, the correlation between the two sequences can be written as:

$$
\begin{aligned}
& \theta_{s\left(g_{1}\right), s\left(g_{2}\right)}(\tau) \\
& =\sum_{t=0}^{N-1} s\left(g_{1}, 0, t+\tau\right) \overline{s\left(g_{2}, 0, t\right)} \\
& =2 \sum_{t \text { even }}\left(\imath^{u_{1}(t+\tau)}+2 \imath^{u_{0}(t+\tau)}\right)\left(\imath^{-v_{1}(t)}+2 \imath^{-v_{0}(t)}\right)+ \\
& \quad 2 \sum_{t \text { odd }}\left(\imath^{u_{0}(t+\tau)}-2 \imath^{u_{1}(t+\tau)}\right)\left(\imath^{-v_{0}(t)}-2 \imath^{-v_{1}(t)}\right) \\
& = \\
& \quad 2\left(\theta_{u_{1}, v_{1}}(\tau)+2 \theta_{u_{0}, v_{1}}(\tau)+2 \theta_{u_{1}, v_{0}}(\tau)+\right. \\
& \left.\quad 4 \theta_{u_{0}, v_{0}}(\tau)\right)+2\left(\theta_{u_{0}, v_{0}}(\tau)-2 \theta_{u_{0}, v_{1}}(\tau)-\right. \\
& = \\
& \left.=10 \theta_{u_{1}, v_{0}}(\tau)+4 \theta_{u_{1}, v_{1}}(\tau)\right) \\
& \left.=1 \theta_{u_{0}, v_{0}}(\tau)+\theta_{u_{1}, v_{1}}(\tau)\right)
\end{aligned}
$$

Using the results from Appendix VII we can see that $\theta_{u_{0}, v_{0}}(\tau)$ and $\theta_{u_{1}, v_{1}}(\tau)$ are at right angles to each other. We can bound the magnitude of $\theta_{s\left(g_{1}\right), s\left(g_{2}\right)}(\tau)$ in the above expression as

$$
\begin{align*}
\left|\theta_{s\left(g_{1}\right), s\left(g_{2}\right)}(\tau)\right| & \leq|10+10 \Gamma(1)(1+\imath)| \\
& \lesssim 10 \sqrt{2} \sqrt{N / 2} \\
& \lesssim 10 \sqrt{N} . \tag{43}
\end{align*}
$$

Now, if $\tau=1(\bmod 2)$, we get

$$
\begin{align*}
& \theta_{s\left(g_{1}\right), s\left(g_{2}\right)}(\tau) \\
& =\sum_{t=0}^{N-1} s\left(g_{1}, 0, t+\tau\right) \overline{s\left(g_{2}, 0, t\right)} \\
& =2 \imath \sum_{t \text { even }}\left(\imath^{u_{0}(t+\tau)}-2 \imath^{u_{1}(t+\tau)}\right)\left(\imath^{-v_{1}(t)}+2 \imath^{-v_{0}(t)}\right)- \\
& \quad 2 \imath \sum_{t \text { odd }}\left(\imath^{u_{1}(t+\tau)}+2 \imath^{u_{0}(t+\tau)}\right)\left(\imath^{-v_{0}(t)}-2 \imath^{-v_{1}(t)}\right) \\
& = \\
& \quad 2 \imath\left(\theta_{u_{0}, v_{1}}(\tau)-2 \theta_{u_{1}, v_{1}}(\tau)+2 \theta_{u_{0}, v_{0}}(\tau)-\right. \\
& \\
& \left.\quad 4 \theta_{u_{1}, v_{0}}(\tau)\right)-2 \imath\left(\theta_{u_{1}, v_{0}}(\tau)-2 \theta_{u_{1}, v_{1}}(\tau)+\right.  \tag{44}\\
& = \\
& =10 \imath\left(\theta_{u_{0}, v_{0}}(\tau)-4 \theta_{u_{0}, v_{1}}(\tau)-\theta_{u_{1}, v_{0}}(\tau)\right)
\end{align*}
$$

The two correlations appearing in the above expression, viz. $\theta_{u_{0}, v_{1}}(\tau)$ and $\theta_{u_{1}, v_{0}}(\tau)$ are not aligned with respect to each other. In the worst case, both of them will contribute $(1+\Gamma(1))$ to the final correlation expression. With that, we can bound the magnitude of $\theta_{s\left(g_{1}\right), s\left(g_{2}\right)}(\tau)$ in the above expression as

$$
\begin{align*}
\left|\theta_{s\left(g_{1}\right), s\left(g_{2}\right)}(\tau)\right| & \leq|10 \Gamma(1)(1+1)| \\
& \lesssim 20 \sqrt{N / 2} \\
& \lesssim 10 \sqrt{2} \sqrt{N} \tag{45}
\end{align*}
$$

With the bounds in (43) and (45), and by normalizing with the energy of the sequences, we get the statement of the Lemma.

## Appendix IX

Correlation of SEQUENCES IN FAmily $\mathcal{I P}_{8}$
As with other sequence families and for the same reasons, we analyze the correlation between two basic sequences in Family $\mathcal{I} \mathcal{P}_{8}$ corresponding to the assignment $\kappa_{0}=\kappa_{1}=0$ in (24).

Let $\{s(g, 0, t)\}$ and $\left\{s\left(g^{\prime}, 0, t\right)\right\}$ be two basic sequences belonging to Family $\mathcal{I} \mathcal{P}_{8}$, i.e.,

$$
\left.\begin{array}{rl}
s(g, 0, t) & =\left\{\begin{array}{l}
\sqrt{2 \imath}\left(\imath^{u_{1}(t)}+2 \imath^{u_{0}(t)}\right), t \text { even } \\
\sqrt{2 \imath} \imath\left(\imath^{u_{0}(t)}-2 \imath^{u_{1}(t)}\right),
\end{array}, t\right. \text { odd }
\end{array}\right\} \begin{aligned}
& \sqrt{2 \imath}\left(\imath^{u_{1}^{\prime}(t)}+2 \imath^{u_{0}^{\prime}(t)}\right), t \text { even } \\
& \sqrt{2 \imath} \imath\left(\imath^{u_{0}^{\prime}(t)}-2 \imath^{u_{1}^{\prime}(t)}\right), t \text { odd } \tag{47}
\end{aligned}
$$

where

$$
\begin{aligned}
u_{0}(t) & =T\left([1+2 g] \xi^{t}\right) \\
u_{1}(t) & =T\left(\left[1+2\left(g+\delta_{1}\right)\right] \xi^{t}\right) \\
u_{0}^{\prime}(t) & =T\left(\left[1+2 g^{\prime}\right] \xi^{t}\right) \text { and } \\
u_{1}^{\prime}(t) & =T\left(\left[1+2\left(g^{\prime}+\delta_{1}\right)\right] \xi^{t}\right)
\end{aligned}
$$

The expression for the correlation between the two Q-PAM sequences will take on one of two forms depending on if $\tau=$ $0(\bmod 2)$ or if $\tau=1(\bmod 2)$.

Let us suppose that $\tau=0(\bmod 2)$. In that case, the correlation between the two sequences can be written as (see Appendix VIII for details):

$$
\theta_{s(g), s\left(g^{\prime}\right)}(\tau)=10\left(\theta_{u_{0}, u_{0}^{\prime}}(\tau)+\theta_{u_{1}, u_{1}^{\prime}}(\tau)\right)
$$

Using the results from Appendix VII we can see that $\theta_{u_{0}, u_{0}^{\prime}}(\tau)$ and $\theta_{u_{1}, u_{1}^{\prime}}(\tau)$ are at right angles to each other. We can bound the magnitude of $\theta_{s(g), s\left(g^{\prime}\right)}(\tau)$ in the above expression as

$$
\begin{align*}
\left|\theta_{s(g), s\left(g^{\prime}\right)}(\tau)\right| & \leq|10+10 \Gamma(1)(1+\imath)| \\
& \lesssim 10 \sqrt{2} \sqrt{N / 2} \\
& \lesssim 10 \sqrt{N} . \tag{48}
\end{align*}
$$

Now, if $\tau=1(\bmod 2)$, we get (see Appendix VIII for details)

$$
\begin{equation*}
\theta_{s(g), s\left(g^{\prime}\right)}(\tau)=10 \imath\left(\theta_{u_{0}, u_{1}^{\prime}}(\tau)-\theta_{u_{1}, u_{0}^{\prime}}(\tau)\right) \tag{49}
\end{equation*}
$$

From the results in Appendix the two correlations appearing in the above expression, viz. $\theta_{u_{0}, u_{1}^{\prime}}(\tau)$ and $\theta_{u_{1}, u_{0}^{\prime}}(\tau)$ can be rewritten as

$$
\begin{aligned}
& \theta_{u_{0}, u_{1}^{\prime}}(\tau)=-1+\Gamma(1) \imath^{-T\left(z_{0}\right)} \\
& \theta_{u_{1}, u_{0}^{\prime}}(\tau)=-1+\Gamma(1) \imath^{-T\left(z_{1}\right)}
\end{aligned}
$$

where

$$
\begin{aligned}
z_{0} & =g+\frac{g+g^{\prime}+\delta_{1}}{y}+\frac{1}{\sqrt{y}}+2 \mu\left(g, g^{\prime}+\delta_{1}, y\right) \\
z_{1} & =g+\delta_{1}+\frac{g+\delta_{1}+g^{\prime}}{y}+\frac{1}{\sqrt{y}}+2 \mu\left(g+\delta_{1}, g^{\prime}, y\right) \\
& =z_{0}+\delta_{1}+2 \mu^{\prime}\left(g+\delta_{1}, g^{\prime}, y\right)
\end{aligned}
$$

Since $\operatorname{tr}\left(\delta_{1}\right)=1$, we can see that the two correlation terms appearing in 49) are at right angles and we can bound the magnitude of $\theta_{s(g), s\left(g^{\prime}\right)}(\tau)$ as

$$
\begin{align*}
\left|\theta_{s(g), s\left(g^{\prime}\right)}(\tau)\right| & \leq|10+10 \Gamma(1)(1+\imath)| \\
& \lesssim 10 \sqrt{2} \sqrt{N / 2} \\
& \lesssim 10 \sqrt{N} . \tag{50}
\end{align*}
$$

With the bounds in (48) and (50), and by normalizing with the energy of the sequences, we can bound $\theta_{\max }$ and $\bar{\theta}_{\text {max }}$ for Family $\mathcal{I P}_{8}$ as:

$$
\begin{aligned}
\theta_{\max } & \lesssim 10 \sqrt{N} \\
\bar{\theta}_{\max } & \lesssim \sqrt{N} .
\end{aligned}
$$

## REFERENCES

[1] M. Anand and P. Vijay Kumar, "16-QAM Sequences with Low Periodic Correlation," Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore, Technical Report No. TR-PME-2006-05, May 2006.
[2] S. Boztaş, "Near-Optimal $4 \phi$ (four-phase) sequences and optimal binary sequences for CDMA," Ph. D. dissert. Univ of Southern California, Los Angeles, CA, 1990.
[3] S. Boztaş, "CDMA over QAM and other Arbitrary Energy Constellations," Proc. IEEE International Conf. on Comm. Systems, vol. 2, pp. 21.7.1-21.7.5, Singapore, 1996.
[4] S. Boztaş, R. Hammons, and P. V. Kumar, "4-phase sequences with near-optimum correlation properties," IEEE Trans. Inform. Theory, vol. 38, no. 3, pp 1101-1113, May 1992.
[5] R. Gold, "Maximal recursive sequences with 3-valued recursive crosscorrelation functions," IEEE Trans. Inform. Theory, vol. 14, pp. 154156, Jan. 1968.
[6] A.R. Hammons Jr., P.V. Kumar, A.R. Calderbank, N.J.A. Sloane, and P. Solé, "The $\mathbb{Z}_{4}$-linearity of Kerdock, Preparata, Goethals, and related codes," IEEE Trans. Inform. Theory, vol. 40, no. 2, pp. 301-319, Mar. 1994.
[7] T. Helleseth and P.V. Kumar, "Sequences with low correlation," in Handbook of Coding Theory, V.S. Pless and W.C. Huffman, Eds. Amsterdam, The Netherlands: Elsevier, 1998.
[8] P.V. Kumar, T. Helleseth, and A.R. Calderbank, "An upper bound for Weil exponential sums over Galois rings and applications," IEEE Trans. Inform. Theory, vol. 41, no. 2, pp. 456-468, Mar. 1995.
[9] P.V. Kumar, T. Helleseth, A.R. Calderbank, and A.R. Hammons Jr., "Large families of quaternary sequences with low correlation," IEEE Trans. Inform. Theory, vol. 42, no. 2, pp. 579-592, Mar. 1996.
[10] H. F. Lu and P. V. Kumar, "A Unified Construction of Space-Time Codes With Optimal Rate-Diversity Tradeoff," IEEE Trans. Inform. Theory, vol. 51, no. 5, pp. 1709-1730, May 2005.
[11] B.R. MacDonald, Finite Rings with Identity. New York: Marcel Dekker, 1974.
[12] C. Rößing and V. Tarokh, "A Construction of OFDM 16-QAM Sequences Having Low Peak Powers," IEEE Trans. Inform. Theory, vol. 47, no. 5, pp 2091-2094, Jul. 2001.
[13] P. Shankar, "On BCH codes over arbitrary integer rings," IEEE Trans. Inform. Theory, vol. 25, no. 4, pp. 480-483, 1979.
[14] P. Solé, "A quaternary cyclic code and a family of quadriphase sequences with low correlation properties," Coding Theory and Applications, Lecture Notes in Computer Science, vol. 388. Berlin: Springer-Verlag, 1989.
[15] B. Tarokh and H. R. Sadjadpour, "Construction of OFDM $M$-QAM Sequences With Low Peak-to-Average Power Ratio," IEEE Trans. Comm., vol. 51, no. 1, pp. 25-28, Jan. 2003.
[16] L. R. Welch, "Lower bounds on the maximum cross correlation of signals," IEEE Trans. Inform. Theory, vol. 20, no. 3, pp. 397-399, May 1974.
[17] K. Yang, T. Helleseth, P. V. Kumar and A. Shanbhag, "The weight hierarchy of Kerdock codes over $\mathbb{Z}_{4}$," IEEE Trans. Inform. Theory, vol. 42, no. 5, pp. 1587-1593, Sep. 1996.
[18] Release 99 document 3G TS 25.213 of the 3rd Generation Partnership Project, http://www.3gpp.org)
M. Anand received the B.E. degree in Electronics and Communication Engineering from the Visvesvaraya Technological University, Belgaum, in 2003 and the M.Sc.(Engg.) degree from the Indian Institute of Science, Bangalore, in 2007.

He is currently a Ph.D. student at the Coordinated Science Laboratory, University of Illinois, Urbana, IL. His research interests include multi-user information theory and code design for wireless communications.
P. Vijay Kumar (S'80-M'82-SM'01-F'02) received the B.Tech. and M.Tech. degrees from the Indian Institutes of Technology (Kharagpur and Kanpur), and the Ph.D. Degree from the University of Southern California (USC), Los Angeles, in 1983, all in electrical engineering. Since 1983 he has been on the faculty of the EE-Systems Department of USC. He is presently on leave of absence from USC at the Indian Institute of Science, Bangalore. His research interests include space-time codes for cooperative communication networks, low-correlation sequences for wireless and optical CDMA and sensor networks. A low-correlation sequence family co-designed by him is now part of the 3G-WCDMA standard. He was an Associate Editor for Coding Theory for the IEEE Transactions on Information Theory, 1993-1996. In 1994, he received the USC School-of-Engineering Senior Research Award for contributions to coding theory. He is a co-recipient of the IEEE Information Theory Society 1995 Prize Paper Award.


[^0]:    M. Anand is with the Department of Electrical and Computer Engineering, and the Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801 USA (email:amurali2@uiuc.edu). This work was carried out while M. Anand was with the Department of Electrical Communication Engineering, Indian Institute of Science Bangalore, 560012 India.
    P. Vijay Kumar is with the Department of EE-Systems, University of Southern California, Los Angeles, CA 90089 USA (email: vijayk@usc.edu). This work was carried out while P. Vijay Kumar was on leave of absence at the Indian Institute of Science, Bangalore.

    This research is supported in part by NSF-ITR CCR-0326628 and in part by the DRDO-IISc Program on Advanced Research in Mathematical Engineering.
    ${ }^{1}$ For the sake of brevity, we abbreviate and use $s(j), s(k)$ throughout in place of $s(j, t), s(k, t)$ whenever these terms appear in the subscript.

[^1]:    ${ }^{2}$ The reader can readily verify that normalized crosscorrelation is the right measure to employ here.

