The Information-Disturbance Tradeoff
and the Continuity of Stinespring's Representation

Dennis Kretschmann,[1,2,*] Dirk Schlingemann,[2,†] and Reinhard F. Werner[2,‡]

[1]*Department of Applied Mathematics and Theoretical Physics,*
University of Cambridge, Wilberforce Road,
Cambridge CB3 0WA, United Kingdom
[2]*Institut für Mathematische Physik, Technische Universität Braunschweig,*
Mendelssohnstraße 3, 38106 Braunschweig, Germany
(Dated: 30 April 2006)

Stinespring's dilation theorem is the basic structure theorem for quantum channels: it states that any quantum channel arises from a unitary evolution on a larger system. Here we prove a continuity theorem for Stinespring's dilation: if two quantum channels are close in cb-norm, then it is always possible to find unitary implementations which are close in operator norm, with dimension-independent bounds. This result generalizes Uhlmann's theorem from states to channels and allows to derive a formulation of the information-disturbance tradeoff in terms of quantum channels, as well as a continuity estimate for the no-broadcasting theorem. We briefly discuss further implications for quantum cryptography, thermalization processes, and the black hole information loss puzzle.

I. INTRODUCTION

According to Stinespring's dilation theorem [1], every completely positive and trace-preserving map, or *quantum channel*, can be built from the basic operations of (*i*) tensoring the input with a second system in a specified state (conventionally called the *ancilla system*), (*ii*) unitary transformation on the combined input – ancilla system, and (*iii*) reduction to a subsystem. Any channel can hence be thought of as arising from a unitary evolution on a larger (*dilated*) system. The theorem comes with a bound on the dimension of the ancilla system. Stinespring's dilation thus not only provides a neat characterization of the set of permissible quantum operations, but is also a most useful tool in quantum information science.

Our contribution is a continuity theorem for Stinespring's dilation: we show that two quantum channels, T_1 and T_2, are close in cb-norm iff we can find dilating unitaries, V_1 and V_2, that are close in operator norm:

$$\inf_{V_1,V_2} \|V_1 - V_2\|_\infty^2 \leq \|T_1 - T_2\|_{cb} \leq 2 \inf_{V_1,V_2} \|V_1 - V_2\|_\infty. \tag{1}$$

The cb-norm $\|\cdot\|_{cb}$ that appears in Eq. (1) is a stabilized version of the standard operator norm $\|\cdot\|_\infty$, as explained in Sec. II B.

Stinespring's representation is unique up to unitary transformations on the ancilla system. So we may just as well fix two Stinespring dilations V_1 and V_2 for T_1 and T_2, respectively, and optimize over all unitaries U on the ancilla system. The continuity estimate Eq. (1) can

then be rewritten as

$$\inf_U \|(\mathbb{1}_B \otimes U)V_1 - V_2\|_\infty^2 \leq \|T_1 - T_2\|_{cb} \leq 2 \inf_U \|(\mathbb{1}_B \otimes U)V_1 - V_2\|_\infty \qquad (2)$$

(cf. Th. 1 in Sec. III C). Hence, the continuity theorem generalizes the uniqueness clause in Stinespring's theorem to cases in which two channels T_1, T_2 differ by a finite amount. For states, i. e., channels with one-dimensional domain, dilations are usually called *purifications*, and in this special case Eq. (2) is an immediate consequence of Uhlmann's theorem. The proof of the continuity theorem relies on a generalization of Uhlmann's theorem from quantum states to quantum channels, and will be presented in Sec. III — preceded by a brief introduction to quantum channels and distance measures in Sec. II. We initially restrict our discussion to finite-dimensional Hilbert spaces. Yet the continuity estimate Eq. (2) has the welcome feature of being completely independent of the dimension of the underlying Hilbert spaces, and is thus perfectly tailored for applications in which this dimension is unknown or large. In Sec. VII we will briefly describe extensions of our results to infinite-dimensional systems.

The ancilla system in Stinespring's representation has a natural interpretation as the environment of the physical system under investigation: the output of the channel T arises from a unitary interaction of the input state with the environment, followed by a partial trace over the degrees of freedom of the environment. Any channel T then has a *complementary channel* T_E, in which the roles of the output system and the environment are interchanged. T_E describes the information flow into the environment. Since complementary channels share a common Stinespring representation, Eq. (2) allows to relate the distance between two quantum channels to the distance between their complementaries. This is particularly fruitful for the noiseless (or *ideal*) channel id, whose complementary channel S is completely depolarizing. The continuity theorem then entails a formulation of the information-disturbance tradeoff, which lies at the heart of quantum physics and explains why quantum information behaves so fundamentally different from its classical counterpart. We prove in Sec. IV that almost all the information can be retrieved from the output of the quantum channel T by means of a decoding operation D iff T releases almost no information to the environment:

$$\frac{1}{4} \inf_D \|TD - \mathrm{id}\|_{cb}^2 \leq \|T_E - S\|_{cb} \leq 2 \inf_D \|TD - \mathrm{id}\|_{cb}^{\frac{1}{2}} \qquad (3)$$

(cf. Th. 3 in Sec. IV). Again, no dimension-dependent factors appear in these bounds. However, we show in Sec. V that this welcome property crucially depends on the choice of the operator topology: if the cb-norm $\|\cdot\|_{cb}$ is replaced by the standard operator norm $\|\cdot\|_\infty$ in Eq. (3), a dimension-independent bound can in general no longer be given.

The tradeoff between information and disturbance guarantees the security of quantum key distribution in a very strong form and implies that quantum information cannot be cloned or distributed. The tradeoff theorem then amounts to a continuity estimate for the no-broadcasting theorem.

Further applications are briefly discussed in Sec. VI, including thermalization processes and the famous black hole information loss puzzle. These have been investigated in more detail by Braunstein and Pati [2]. Finally, we show in a companion paper [3] how the continuity estimate allows to strengthen the impossibility proof for quantum bit commitment.

II.   PRELIMINARIES

We begin with a brief introduction to quantum channels and convenient measures to evaluate their distance. We refer to Davies' textbook [4] and Keyl's survey article [5] for a more extensive discussion.

A. Observables, States, and Channels

The statistical properties of quantum (as well as classical and hybrid) systems are characterized by spaces of operators on a Hilbert space $\mathcal{H}$: The observables of the system are represented by bounded linear operators on $\mathcal{H}$, written $\mathcal{B}(\mathcal{H})$, while the physical states associated with the system are the positive linear functionals $\omega \colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ that satisfy the normalization condition $\omega(\mathbb{1}) = 1$, with the identity operator $\mathbb{1} \in \mathcal{B}(\mathcal{H})$. We restrict this discussion to finite-dimensional Hilbert spaces, for which all linear operators are bounded and every linear functional ω can be expressed in terms of a trace-class operator $\varrho_\omega \in \mathcal{B}_*(\mathcal{H})$ such that $\omega(a) = \mathrm{tr}(\varrho_\omega a)$ for all $a \in \mathcal{B}(\mathcal{H})$. The normalization of the functional ω than translates into the condition $\mathrm{tr}(\varrho_\omega) = 1$. In the finite-dimensional setup, the physical states can thus be identified with the set of normalized density operators $\varrho \in \mathcal{B}_*(\mathcal{H})$. This correspondence no longer holds for infinite-dimensional systems. Some of the necessary amendments will be described in Sec. VII.

A *quantum channel* T which transforms input systems described by a Hilbert space $\mathcal{H}_A$ into output systems described by a (possibly different) Hilbert space $\mathcal{H}_B$ is represented by a completely positive and unital map $T \colon \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$. By unitality we mean that $T(\mathbb{1}_B) = \mathbb{1}_A$, with the identity operator $\mathbb{1}_X \in \mathcal{B}(\mathcal{H}_X)$. Complete positivity means that $\mathrm{id}_\nu \otimes T$ is positive for all $\nu \in \mathbb{N}$, where id_ν denotes the identity operation on the $(\nu \times \nu)$ matrices.

The physical interpretation of the quantum channel T is the following: when the system is initially in the state $\varrho \in \mathcal{B}_*(\mathcal{H}_A)$, the expectation value of the measurement of the observable $b \in \mathcal{B}(\mathcal{H}_B)$ at the output side of the channel is given in terms of T by $\mathrm{tr}(\varrho\, T(b))$.

Alternatively, we can focus on the dynamics of the states and introduce the dual map $T_* \colon \mathcal{B}_*(\mathcal{H}_A) \to \mathcal{B}_*(\mathcal{H}_B)$ by means of the duality relation

$$\mathrm{tr}(T_*(\varrho)\, b) = \mathrm{tr}(\varrho\, T(b)) \quad \forall \; \varrho \in \mathcal{B}_*(\mathcal{H}_A), b \in \mathcal{B}(\mathcal{H}_B). \tag{4}$$

T_* is a completely positive and trace-preserving map and represents the channel in *Schrödinger picture*, while T provides the *Heisenberg picture* representation. For finite-dimensional systems, the Schrödinger and the Heisenberg picture provide a completely equivalent description of physical processes. The interconversion is always immediate from Eq. (4).

B. Distance Measures

For both the continuity theorem and the tradeoff theorem we will need to evaluate the distance between different quantum channels on the one hand, and different Stinespring isometries on the other. There are several candidates for such distance measures, which are adapted to different scenarios.

Assume two quantum channels, T_1 and T_2, with common input and output spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Since these T_i are (in Heisenberg picture) operators between normed spaces $\mathcal{B}(\mathcal{H}_B)$ and $\mathcal{B}(\mathcal{H}_A)$, the natural choice to quantify their distance is the *operator norm*,

$$\|T_1 - T_2\|_\infty := \sup_{b \neq 0} \frac{\|T_1(b) - T_2(b)\|_\infty}{\|b\|_\infty}. \tag{5}$$

The norm distance Eq. (5) has a neat operational characterization: it is just twice the largest difference between the overall probabilities in two statistical quantum experiments differing only in replacing one use of T_1 with one use of T_2.

However, in many applications it is more appropriate to allow for more general experiments, in which the two channels are only applied to a sub-system of a larger system. This requires *stabilized* distance measures [6], and naturally leads to the so-called *norm of complete boundedness* (or *cb-norm*, for short) [7]:

$$\|T_1 - T_2\|_{cb} := \sup_{\nu \in \mathbb{N}} \|\mathrm{id}_\nu \otimes (T_1 - T_2)\|_\infty, \tag{6}$$

where id_ν again denotes the *ideal* (or *noiseless*) channel on the ν-dimensional Hilbert space $\mathbb{C}^\nu$. Useful properties of the cb-norm include *multiplicativity*, i.e., $\|T_1 \otimes T_2\|_{cb} = \|T_1\|_{cb} \|T_2\|_{cb}$, and *unitality*, $\|T\|_{cb} = 1$ for any channel T.

Obviously, $\|T\|_{cb} \geq \|T\|_\infty$ for every linear map T. If either the input or output space is a classical system, we even have equality: $\|T\|_{cb} = \|T\|_\infty$ (cf. Ch. 3 in [7]). Fully quantum systems generically show a separation between these two norms. However, in the vicinity of the noiseless channel id the operator norm and the cb-norm may always be estimated in terms of each other with dimension-independent bounds [8], and can thus be considered equivalent, even when the dimensions of the underlying Hilbert spaces are not known and possibly large:

$$\|T - \mathrm{id}\|_\infty \leq \|T - \mathrm{id}\|_{cb} \leq 8 \|T - \mathrm{id}\|_\infty^{\frac{1}{4}}. \tag{7}$$

Examples which show that this equivalence does *not* hold generally will be provided in Sec. V. Thus, in a quantum world correlations may help to distinguish locally akin quantum channels. This plays an important role for the interpretation of the tradeoff theorem in Sec. IV.

States are channels with one-dimensional input space, $\mathcal{H}_A = \mathbb{C}$. Since this is a classical system, there is no need to distinguish between stabilized and non-stabilized distance measures. The so-called *trace norm* $\|\varrho\|_1 = \mathrm{tr}\sqrt{\varrho^* \varrho}$ is a convenient measure for the distance between two density operators. The trace norm difference $\|\varrho - \sigma\|_1$ is equivalent to the *fidelity* $f(\varrho, \sigma) := \mathrm{tr}\sqrt{\sqrt{\varrho}\,\sigma\sqrt{\varrho}}$ by means of the relation [9]

$$1 - f(\varrho, \sigma) \leq \frac{1}{2} \|\varrho - \sigma\|_1 \leq \sqrt{1 - f^2(\varrho, \sigma)}. \tag{8}$$

Finally, we note that for any linear operator T the operator norm $\|T\|_\infty$ equals the norm of the Schrödinger adjoint T_* on the space of trace class operators, i.e.,

$$\|T\|_\infty = \sup_{\|\varrho\|_1 \leq 1} \|T_*(\varrho)\|_1 \tag{9}$$

(cf. Ch. VI of [10] and Sec. 2.4 of [11] for details), which is the usual way to convert norm estimates from the Heisenberg picture into the Schrödinger picture and vice versa. For states $T_* = \varrho$, the operator norm then indeed just coincides with the trace norm: $\|T\|_\infty = \|T_*\|_1 = \|\varrho\|_1$.

III. CONTINUITY OF STINESPRING'S REPRESENTATION

A. Stinespring's Representation

Stinespring's famous representation theorem [1, 7], as adapted to maps between finite-dimensional quantum systems, states that any completely positive (not necessarily unital)

map $T\colon\mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ can be written as

$$T(b) = V^* \left(b \otimes \mathbb{1}_E\right) V \quad \forall \; b \in \mathcal{B}(\mathcal{H}_B) \tag{10}$$

with a linear operator $V\colon\mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$. The finite-dimensional Hilbert space $\mathcal{H}_E$ is usually called the *dilation space*, and the pair $(\mathcal{H}_E, V)$ a *Stinespring representation* for T. If T is unital (and thus a quantum channel), then V is an *isometry*, i.e., $V^*V = \mathbb{1}_A$.

By means of the duality relation Eq. (4), in the Schrödinger picture Stinespring's theorem gives rise to the so-called *ancilla representation* of the quantum channel T_*:

$$T_*(\varrho) = \operatorname{tr}_E V \varrho V^* \quad \forall \; \varrho \in \mathcal{B}_*(\mathcal{H}_A), \tag{11}$$

where tr_E denotes the *partial trace* over the system $\mathcal{H}_E$ [9]. In the physical interpretation of Stinespring's theorem the dilation space $\mathcal{H}_E$ represents the *environment*. Stinespring's isometry V transforms the input state ϱ into the state $V \varrho V^*$ on $\mathcal{H}_B \otimes \mathcal{H}_E$, which is correlated between the output and the environment. The output state $T_*(\varrho) \in \mathcal{B}_*(\mathcal{H}_B)$ is then obtained by tracing out the degrees of freedom of the environment. Physically, one would expect a unitary operation U instead of an isometric V. However, the initial state of the environment can be considered fixed, effectively reducing U to an isometry, $V\psi := U(\psi \otimes \psi_0)$ for some fixed initial pure state $|\psi_0\rangle$ of the environment system.

The Stinespring representation $(\mathcal{H}_E, V)$ is called *minimal* iff the set of vectors $(b \otimes \mathbb{1}_E) V\varphi$ with $b \in \mathcal{B}(\mathcal{H}_B)$ and $\varphi \in \mathcal{H}_A$ spans $\mathcal{H}_B \otimes \mathcal{H}_E$. In this case the dilation space $\mathcal{H}_E$ can be chosen such that $\dim \mathcal{H}_E \le \dim \mathcal{H}_A \dim \mathcal{H}_B$.

B. Uniqueness and Unitary Equivalence

Stinespring's representation is not at all unique: if $(\mathcal{H}_E, V)$ is a representation for the completely positive map $T\colon\mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$, then it is easily seen that a further representation for T is given by $(\mathcal{H}_E, (\mathbb{1}_B \otimes U)V)$ with any unitary $U \in \mathcal{B}(\mathcal{H}_E)$.

However, it is straightforward to show that the minimal Stinespring representation is unique up to such unitary equivalence: Assume that the completely positive map $T\colon\mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ has a minimal Stinespring dilation $(\mathcal{H}_E, V)$ as in Eq. (10) as well as a further, not necessarily minimal one $(\mathcal{H}_{\widetilde{E}}, \widetilde{V})$, i.e.,

$$T(b) = \widetilde{V}^* \left(b \otimes \mathbb{1}_{\widetilde{E}}\right) \widetilde{V} \quad \forall \; b \in \mathcal{B}(\mathcal{H}_B) \tag{12}$$

with another linear map $\widetilde{V}\colon\mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_{\widetilde{E}}$ and a possibly different dilation space $\mathcal{H}_{\widetilde{E}}$. Since the representation $(\mathcal{H}_E, V)$ is chosen to be minimal, we conclude that $\dim \mathcal{H}_E \le \dim \mathcal{H}_{\widetilde{E}}$. Setting

$$\widetilde{U}(b \otimes \mathbb{1}_E)V\varphi := (b \otimes \mathbb{1}_{\widetilde{E}})\widetilde{V}\varphi \tag{13}$$

for $b \in \mathcal{B}(\mathcal{H}_B)$ and $\varphi \in \mathcal{H}_A$ then yields a well-defined isometry $\widetilde{U}\colon\mathcal{H}_B \otimes \mathcal{H}_E \to \mathcal{H}_B \otimes \mathcal{H}_{\widetilde{E}}$. In particular, by choosing $b = \mathbb{1}_B$ in Eq. (13) we see that $\widetilde{U}V = \widetilde{V}$. From the definition of $\widetilde{U}$ we immediately find the intertwining relation

$$\widetilde{U}(b \otimes \mathbb{1}_E) = (b \otimes \mathbb{1}_{\widetilde{E}})\widetilde{U} \quad \forall \; b \in \mathcal{B}(\mathcal{H}_B). \tag{14}$$

Hence $\widetilde{U}$ must be decomposable as $\widetilde{U} = \mathbb{1}_B \otimes U$ for some isometry $U\colon\mathcal{H}_E \to \mathcal{H}_{\widetilde{E}}$. If both representations $(\mathcal{H}_E, V)$ and $(\mathcal{H}_{\widetilde{E}}, \widetilde{V})$ are minimal, the dimensions of the dilation spaces $\mathcal{H}_E$ and $\mathcal{H}_{\widetilde{E}}$ coincide, and U is unitary, as suggested.

C.   A Continuity Theorem

We have seen from Stinespring's theorem that two minimal dilations of a given quantum channel are unitarily equivalent. The uniqueness clause is a powerful tool and has proved helpful in the investigation of localizable quantum channels [12] and the structure theorem for quantum memory channels [13]. We will now generalize the uniqueness clause to cases in which two quantum channels differ by a finite amount, and prove continuity: if two quantum channels are close in cb-norm, we may find Stinespring isometries that are close in operator norm.

The converse also holds, and is in fact much simpler to show. So we start with this part: Assume two quantum channels $T_1, T_2 \colon \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ with Stinespring isometries $V_1, V_2 \colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$. We can always assume that T_1 and T_2 share a common dilation space $\mathcal{H}_E$, possibly after adding some extra dimensions to one of the dilation spaces and some unitary transformations. We do not assume that either dilation $(\mathcal{H}_E, V_1)$ or $(\mathcal{H}_E, V_2)$ is minimal.

A straightforward application of the triangle inequality shows that for all $X \in \mathcal{B}(\mathbb{C}^\nu) \otimes \mathcal{B}(\mathcal{H}_B)$ we have

$$
\begin{aligned}
\left\| \left(\mathrm{id}_\nu \otimes (T_1 - T_2) \right) X \right\|_\infty \\
= \left\| (\mathbb{1}_\nu \otimes V_1^*) X (\mathbb{1}_\nu \otimes V_1) - (\mathbb{1}_\nu \otimes V_2^*) X (\mathbb{1}_\nu \otimes V_2) \right\|_\infty \\
\leq \left\| \left[\mathbb{1}_\nu \otimes (V_1^* - V_2^*) \right] X (\mathbb{1}_\nu \otimes V_1) \right\|_\infty + \left\| (\mathbb{1}_\nu \otimes V_2^*) X \left[\mathbb{1}_\nu \otimes (V_1 - V_2) \right] \right\|_\infty \\
\leq 2 \left\| V_1 - V_2 \right\|_\infty \left\| X \right\|_\infty,
\end{aligned}
\tag{15}
$$

independently of $\nu \in \mathbb{N}$, which immediately implies that

$$
\| T_1 - T_2 \|_{cb} \leq 2 \| V_1 - V_2 \|_\infty .
\tag{16}
$$

Thus, if we can find Stinespring isometries V_1 and V_2 for the channels T_1 and T_2 which are close in operator norm, the channels will be close in cb-norm (and hence also in operator norm, cf. Sec. II B).

As advertised, we will now show the converse implication. As Stinespring isometries are unique only up to unitary equivalence, we cannot expect that *any* two given Stinespring isometries V_1, V_2 are close. The best we can hope for is that these isometries can be *chosen* close together, with dimension-independent bounds. This is the essence of the continuity theorem:

Theorem 1 (Continuity)
Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces, and suppose that

$$
T_1, T_2 \colon \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)
\tag{17}
$$

are quantum channels with Stinespring isometries $V_1, V_2 \colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ and a common dilation space $\mathcal{H}_E$. We then have:

$$
\inf_U \left\| (\mathbb{1}_B \otimes U) V_1 - V_2 \right\|_\infty^2 \leq \| T_1 - T_2 \|_{cb} \leq 2 \inf_U \left\| (\mathbb{1}_B \otimes U) V_1 - V_2 \right\|_\infty ,
\tag{18}
$$

where the minimization is with respect to all unitary $U \in \mathcal{B}(\mathcal{H}_E)$.

As a first step towards the proof of Th. 1 we will lift the equivalence Eq. (8) of fidelity and trace norm from quantum states to quantum channels. The stabilized version of the

fidelity for two quantum channels T_1, T_2 has been called *operational fidelity* [14]:

$$F(T_1, T_2) := \inf \left\{ f(\text{id} \otimes T_{1*}\, \varrho,\ \text{id} \otimes T_{2*}\, \varrho) \ \middle|\ \varrho \in \mathcal{B}_*(\mathcal{H}_A)^{\otimes 2}, \|\varrho\|_1 \leq 1 \right\}$$
$$= \inf \left\{ f(\text{id} \otimes T_{1*}\, |\psi\rangle\langle\psi|,\ \text{id} \otimes T_{2*}\, |\psi\rangle\langle\psi|) \ \middle|\ \psi \in \mathcal{H}_A^{\otimes 2}, \|\psi\| \leq 1 \right\}, \tag{19}$$

where minimization over pure states is sufficient by the joint concavity of the fidelity f (cf. [9], Th. 9.7).

Since quantum states are quantum channels with one-dimensional domain (and stabilization is not needed in this case), we have $F(\varrho, \sigma) = f(\varrho, \sigma)$ for any two quantum states $\varrho, \sigma \in \mathcal{B}_*(\mathcal{H}_A)$. The following Lemma, which we again cite from [14], is then a straightforward generalization of the equivalence relation Eq. (8):

Lemma 2 (Equivalence)
For any two quantum channels $T_1, T_2 \colon \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ we have:

$$1 - F(T_1, T_2) \leq \frac{1}{2}\, \|T_1 - T_2\|_{cb} \leq \sqrt{1 - F^2(T_1, T_2)}, \tag{20}$$

where $F(T_1, T_2)$ denotes the operational fidelity *introduced in Eq. (19).*

Proof of Lemma 2: The channel difference $\Phi := T_1 - T_2$ is a linear map into the $\dim \mathcal{H}_A$-dimensional system $\mathcal{B}(\mathcal{H}_A)$. Note that for such linear maps $\Phi \colon \mathcal{B} \to \mathcal{B}(\mathbb{C}^\nu)$, stabilization with a ν-dimensional bystander system is sufficient, $\|\Phi\|_{cb} = \|\text{id}_\nu \otimes \Phi\|_\infty$ (cf. [7], Prop. 8.11). Conversion into the Schrödinger picture via the duality relation Eq. (9) then yields

$$\|T_1 - T_2\|_{cb} = \sup \left\{ \|\text{id} \otimes (T_{1*} - T_{2*})\, \varrho\|_1 \ \middle|\ \varrho \in \mathcal{B}_*(\mathcal{H}_A)^{\otimes 2}, \|\varrho\|_1 \leq 1 \right\}. \tag{21}$$

The statement of the lemma now immediately follows by combining Eqs. (19) and (21) with the equivalence relation Eq. (8). ∎

Lemma 2 allows us to concentrate entirely on fidelity estimates in the

Proof of Th. 1: According to Uhlmann's theorem [9, 15], the fidelity $f(\varrho, \sigma)$ of two quantum states $\varrho, \sigma \in \mathcal{B}_*(\mathcal{H}_A)$ has a natural interpretation as the maximal overlap of all purifying vectors ψ_ϱ, $\psi_\sigma \in \mathcal{H}_A \otimes \mathcal{H}_R$, with the purification or reference system $\mathcal{H}_R \cong \mathcal{H}_A$:

$$f(\varrho, \sigma) = \max_{\psi_\varrho, \psi_\sigma} |\langle \psi_\varrho | \psi_\sigma \rangle|. \tag{22}$$

In particular, it is possible to fix one of the purifications in Eq. (22), ψ_ϱ say, and maximize over the purifications of σ. Since any two purifications of the given state σ are identical up to a unitary rotation $U \in \mathcal{B}(\mathcal{H}_R)$ on the purifying system, Uhlmann's theorem can be given the alternative formulation

$$f(\varrho, \sigma) = \max_{U \in \mathcal{B}(\mathcal{H}_R)} |\langle \psi_\varrho | (\mathbb{1}_A \otimes U) \psi_\sigma \rangle|, \tag{23}$$

where now ψ_ϱ and ψ_σ are any two *fixed* purifications of $\varrho \in \mathcal{B}_*(\mathcal{H}_A)$ and $\sigma \in \mathcal{B}_*(\mathcal{H}_A)$, respectively.

Since $(\mathbb{1}_{A'} \otimes V_i)\psi$ is a purification of the output state $\text{id}_{A'} \otimes T_{i*}(|\psi\rangle\langle\psi|)$, the operational fidelity $F(T_1, T_2)$ can then be expressed in terms of the Stinespring isometries V_1 and V_2 as follows:

$$
\begin{aligned}
F(T_1, T_2) &= \inf_{\psi} \; f(\text{id}_{A'} \otimes T_{1*} |\psi\rangle\langle\psi|, \; \text{id}_{A'} \otimes T_{2*} |\psi\rangle\langle\psi|) \\
&= \inf_{\psi} \sup_{U} |\langle (\mathbb{1}_{A'} \otimes V_1)\psi | (\mathbb{1}_{A'} \otimes \mathbb{1}_B \otimes U)(\mathbb{1}_{A'} \otimes V_2)\psi\rangle| \\
&= \inf_{\varrho \in \mathcal{B}_*(\mathcal{H}_A)} \sup_{U} |\text{tr}\, \varrho\, V_1^*(\mathbb{1}_B \otimes U)V_2| \\
&= \inf_{\varrho \in \mathcal{B}_*(\mathcal{H}_A)} \sup_{U} \text{Re}(\text{tr}\, \varrho\, V_1^*(\mathbb{1}_B \otimes U)V_2),
\end{aligned}
\tag{24}
$$

where maximization is over all unitary $U \in \mathcal{B}(\mathcal{H}_E)$.

This representation is almost what we need for the desired norm estimate, since only the (fixed) Stinespring isometries V_1, V_2 and the unitary operations U on the ancilla system appear. However, from the order in which the optimization in Eq. (24) is performed it is clear that the optimal unitary U for the inner maximization will in general depend on the quantum state ϱ, $U = U(\varrho)$. In order to obtain a universal unitary, observe that for fixed $\varrho \in \mathcal{B}_*(\mathcal{H}_A)$ the inner variation can be written as $\sup_U |\text{tr}\, XU|$ with $X := \text{tr}_B V_2 \varrho V_1^* \in \mathcal{B}(\mathcal{H}_E)$. It is easily seen that this supremum is attained when U is the unitary from the polar decomposition [9] of X, and equals $\|X\|_1$. However, since $|\text{tr}\, XY| \leq \|X\|_1 \|Y\|_\infty$ for all $Y \in \mathcal{B}(\mathcal{H}_E)$, we can replace the supremum over all unitaries in Eq. (24) by a supremum over all $U \in \mathcal{B}(\mathcal{H}_E)$ such that $\|U\|_\infty \leq 1$.

With this modification both variations in Eq. (24) range over convex sets, and the target function is linear is both inputs. Von Neumann's minimax theorem [16, 17] then allows us to interchange the infimum and supremum to obtain:

$$
F(T_1, T_2) = \sup_{\|U\|_\infty \leq 1} \inf_{\varrho \in \mathcal{B}_*(\mathcal{H}_A)} \text{Re}(\text{tr}\, \varrho\, V_1^*(\mathbb{1}_B \otimes U)V_2).
\tag{25}
$$

The optimization now yields a universal $U \in \mathcal{B}(\mathcal{H}_E)$. In addition we know from our discussion above that U can always be chosen to be unitary in Eq. (25). Since $\|Y\|_\infty = \sup_{\|\varrho\|_1 \leq 1} |\text{tr}\, \varrho Y|$ for any $Y \in \mathcal{B}(\mathcal{H}_A)$, we may now conclude that

$$
\begin{aligned}
\inf_{U} \|(\mathbb{1}_B \otimes U)V_1 - V_2\|_\infty^2 &= \inf_{U} \|\left(V_1^*(\mathbb{1}_B \otimes U^* - V_2^*) \right)\left((\mathbb{1}_B \otimes U)V_1 - V_2 \right)\|_\infty \\
&= \inf_{U} \sup_{\varrho} \text{tr}\, \varrho \left(V_1^*(\mathbb{1}_B \otimes U^*) - V_2^* \right)\left((\mathbb{1}_B \otimes U)V_1 - V_2 \right) \\
&= 2 - 2 \sup_{U} \inf_{\varrho} \text{Re}(\text{tr}\, \varrho V_1^*(\mathbb{1}_B \otimes U)V_2) \\
&= 2\left(1 - F(T_1, T_2) \right) \\
&\leq \|T_1 - T_2\|_{cb},
\end{aligned}
\tag{26}
$$

where in the last step we have applied Lemma 2. This proves the left half of Eq. (18). ▲

The right half, which we have seen is the easier part, follows immediately from our discussion leading to Eq. (16) above. Alternatively, one could apply the right half of the equivalence lemma Eq. (20) to obtain that

$$
\|T_1 - T_2\|_{cb} \leq 2\sqrt{1 - F^2(T_1, T_2)} \leq 2\sqrt{2}\sqrt{1 - F(T_1, T_2)}.
\tag{27}
$$

Note that without any need to invoke the minimax theorem, we can now directly conclude from Eq. (24) that

$$
1 - F(T_1, T_2) \leq 1 - \sup_{U} \inf_{\varrho} \text{Re}\,\text{tr}\, \varrho\, V_1^*(\mathbb{1}_B \otimes U)V_2 = \frac{1}{2}\inf_{U} \|(\mathbb{1}_B \otimes U)V_1 - V_2\|_\infty^2.
\tag{28}
$$

Substituting Eq. (28) into Eq. (27), we then find

$$\|T_1 - T_2\|_{cb} \leq 2 \inf_U \; \|(\mathbb{1}_B \otimes U)V_1 - V_2\|_\infty, \tag{29}$$

and so we have in fact rediscovered the familiar upper bound on the cb-norm distance. ∎

IV.   INFORMATION-DISTURBANCE TRADEOFF

Due to the essential uniqueness of the Stinespring dilation $(\mathcal{H}_E, V)$, to every quantum channel $T: \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$ we may associate a *complementary channel* $T_E: \mathcal{B}(\mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_A)$, in which the roles of the output system $\mathcal{H}_B$ and the environment system $\mathcal{H}_E$ are interchanged:

$$T_E(e) := V^* \left(\mathbb{1}_B \otimes e\right) V \quad \forall \; e \in \mathcal{B}(\mathcal{H}_E). \tag{30}$$

The channel T_E describes the information flow from the input system $\mathcal{H}_A$ to the environment $\mathcal{H}_E$. In the Schrödinger picture representation, it is obtained by tracing out the output system $\mathcal{H}_B$ instead of $\mathcal{H}_E$:

$$T_{E*}(\varrho) = \mathrm{tr}_B \, V \varrho V^* \qquad \Longleftrightarrow \qquad T_*(\varrho) = \mathrm{tr}_E \, V \varrho V^* \tag{31}$$

for all $\varrho \in \mathcal{B}_*(\mathcal{H}_A)$. Henceforth, we will usually write T_B for the channel T to better distinguish it from its complementary channel T_E.

The name *complementary channel* has been suggested by I. Devetak and P. Shor in the course of their investigation of quantum degradable channels [18]. Recently A. Holevo [19] has shown that the classical channel capacity of a quantum channel T_B is additive iff the capacity of its complementary channel T_E is additive. Analogous results have been obtained independently by C. King *et al.* [20] (who chose the term *conjugate channels*).

Since two complementary channels share a common Stinespring isometry, the continuity theorem relates the cb-norm distance between two quantum channels to the cb-norm distance between the complementary channels. The complementary channel of the noiseless channel is completely depolarizing. The continuity theorem then allows us to give a dimension-independent estimate for the information-disturbance tradeoff in terms of quantum channels:

Theorem 3 (Information-Disturbance Tradeoff)
Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces, and suppose that $T_B: \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$ is a quantum channel with Stinespring dilation $(\mathcal{H}_E, V)$. Let $T_E: \mathcal{B}(\mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_A)$ be the complementary channel, as defined in Eq. (30) above. We then have the following tradeoff estimate:

$$\frac{1}{4} \inf_D \|T_B D - \mathrm{id}_A\|_{cb}^2 \leq \|T_E - S\|_{cb} \leq 2 \inf_D \|T_B D - \mathrm{id}_A\|_{cb}^{\frac{1}{2}}, \tag{32}$$

where the infimum is over all decoding channels $D: \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$. In Eq. (32), $S: \mathcal{B}(\mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_A)$ denotes a completely depolarizing channel, *i.e.,*

$$S(e) := \mathrm{tr}(\sigma e) \, \mathbb{1}_A \quad \forall \, e \in \mathcal{B}(\mathcal{H}_E) \tag{33}$$

for some fixed quantum state $\sigma \in \mathcal{B}_*(\mathcal{H}_E)$.

The interpretation of the tradeoff theorem is straightforward: Whenever we may find a decoding channel D such that almost all the information can be retrieved from the output of the quantum channel T_B, the norm difference $\|T_B D - \mathrm{id}_A\|_{cb}$ will be small. By the right half of Eq. (32), we may then conclude that the complementary channel T_E is very well approximated by a completely depolarizing channel S, and thus releases almost no information to the environment. Consequently, if a non-negligible amount of information escapes to the environment, for instance by means of a measurement performed by an eavesdropper, this will inevitably disturb the system. Hence, in quantum physics there is "no measurement without perturbation". We know from Eq. (7) that cb-norm and operator norm are completely equivalent in the vicinity of the noiseless channel. So any disturbance in the transmission can always be detected locally.

On the other hand, if we are assured that the channel T_E is close to some depolarizing channel S in cb-norm, the left half of Eq. (32) guarantees that we may find a decoding channel D which retrieves almost all the information from the B-branch of the system. Consequently, there is "no perturbation without measurement". However, in this case it is usually not enough to verify that T_E erases information locally; the channel also needs to destroy correlations. We will come back to this distinction and its implications for the interpretation of the tradeoff theorem in Sec. V.

Proof of Th. 3: It is easily verified that a Stinespring isometry for the completely depolarizing channel $S: \mathcal{B}(\mathcal{H}_E) \to \mathcal{B}(\mathcal{H}_A)$, as given in Eq. (33), is the isometric embedding

$$V_S: \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_{E'} \otimes \mathcal{H}_E \qquad |\varphi\rangle \mapsto |\varphi\rangle \otimes |\psi_\sigma\rangle, \tag{34}$$

where $\mathcal{H}_{E'} \cong \mathcal{H}_E$, and $|\psi_\sigma\rangle \in \mathcal{H}_{E'} \otimes \mathcal{H}_E$ is a purification of $\sigma \in \mathcal{B}_*(\mathcal{H}_E)$. Thus, the completely depolarizing channel $S \equiv S_{E'E}$ and the ideal channel id_A are indeed complementary.

The tradeoff theorem is then a straightforward consequence of the continuity theorem. Let us focus on the left half of Eq. (32) first, and assume that $V_T: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ is a Stinespring dilation for the quantum channel T_E (and its complementary channel T_B, respectively). Let $V_S: \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_{E'} \otimes \mathcal{H}_E$ be the Stinespring isometry of $S_{E'E}$ given by Eq. (34). Note that the dilation spaces $\mathcal{H}_B$ and $\mathcal{H}_A \otimes \mathcal{H}_{E'}$ are not necessarily of the same size. However, we can easily correct for this by suitably enlarging the smaller system, $\mathcal{H}_B$ say. The left half of the continuity estimate Eq. (18) then guarantees the existence of an isometry $V_D: \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_{E'}$ such that

$$\|(V_D \otimes \mathbb{1}_E)V_T - V_S\|_\infty \leq \|T_E - S_{E'E}\|_{cb}^{\frac{1}{2}}. \tag{35}$$

As illustrated in Fig. 1, the isometry V_D defines a decoding channel

$$D_A: \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B) \qquad D_A(a) := V_D^*(a \otimes \mathbb{1}_{E'})V_D, \tag{36}$$

and by the right half of the continuity estimate Eq. (18) we may now conclude that

$$\|T_B D_A - \mathrm{id}_A\|_{cb} \leq 2\,\|T_E - S_{E'E}\|_{cb}^{\frac{1}{2}}, \tag{37}$$

which proves the left half of Eq. (32). ▲

The proof of the right half of Eq. (32) proceeds very much along the same lines: Assume that $V_T: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ and $V_D: \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_{E'}$ are Stinespring isometries for the quantum channels T_B and D_A, respectively. As in Eq. (34), let $V_S: \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_{E'} \otimes \mathcal{H}_E$ denote the Stinespring isometry of the ideal channel id_A and its complementary channel, the completely depolarizing channel $S_{E'E}$. Just as before, the left half of the continuity estimate Eq. (18) assures us that we may find a unitary operator $U \in \mathcal{B}(\mathcal{H}_{E'E})$ such that (cf. Fig. 2)

$$\|(\mathbb{1}_A \otimes U)(V_D \otimes \mathbb{1}_E)V_T - V_S\|_\infty \leq \|T_B D_A - \mathrm{id}_A\|_{cb}^{\frac{1}{2}}, \tag{38}$$
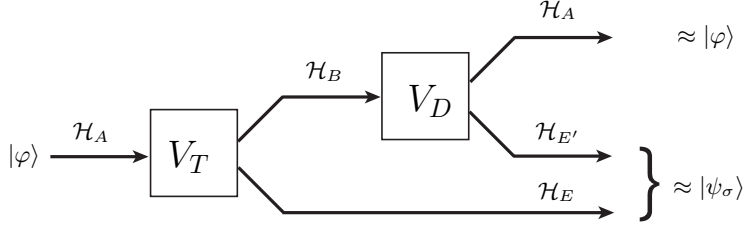
FIG. 1: *"No perturbation without measurement."* Whenever the cb-norm difference $\|T_E - S_{E'E}\|_{cb}$ is small, we may find a decoding channel D_A with Stinespring isometry V_D such that the concatenated isometry $(V_D \otimes \mathbb{1}_E)V_T$ hardly differs from the Stinespring isometry of a noiseless channel, with some fixed $|\psi_\sigma\rangle \in \mathcal{H}_{E'} \otimes \mathcal{H}_E$.
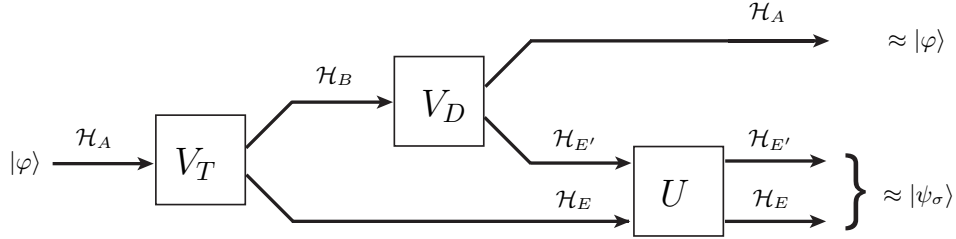

FIG. 2: *"No measurement without perturbation."* If there exists a decoding channel D_A with Stinespring isometry V_D such that the cb-norm difference $\|T_B D_A - \mathrm{id}_A\|_{cb}$ is small, we may find a unitary $U \in \mathcal{B}(\mathcal{H}_{E'}) \otimes \mathcal{B}(\mathcal{H}_E)$ such that the concatenated isometry $(\mathbb{1}_A \otimes U)(V_D \otimes \mathbb{1}_E)V_T$ is close to the Stinespring isometry of a completely depolarizing channel, with some fixed $|\psi_\sigma\rangle \in \mathcal{H}_{E'} \otimes \mathcal{H}_E$.

where again we have suitably enlarged the dilation space $\mathcal{H}_{E'}$, if necessary.

Setting $\mathrm{Ad}_{V_T} := V_T^*(\cdot)V_T$ and $\mathrm{Ad}_{U^*} := U(\cdot)U^*$, we may now conclude from the right half of Eq. (18) that

$$\|\mathrm{Ad}_{V_T} \circ (D_{E'} \otimes \mathrm{id}_E) - S_{E'E} \circ \mathrm{Ad}_{U^*}\|_{cb} \leq 2\,\|T_B D_A - \mathrm{id}_A\|_{cb}^{\frac{1}{2}}, \tag{39}$$

which is almost the desired result. It only remains to restrict the depolarizing channel $S_{E'E}$ to the E-branch of the output system. Obviously, since $S_{E'E}$ is completely depolarizing on the combined output system $\mathcal{H}_{E'} \otimes \mathcal{H}_E$, the same holds true after a unitary rotation by U^* and the restriction to one of the branches. In particular, by setting

$$\widetilde{S}_E : \mathcal{B}(\mathcal{H}_E) \to \mathcal{B}(\mathcal{H}_A) \qquad \widetilde{S}_E(e) := S_{E'E} \circ \mathrm{Ad}_{U^*}(\mathbb{1}_{E'} \otimes e) \tag{40}$$

we obtain a completely depolarizing channel on the restricted system $\mathcal{H}_E$ such that

$$\widetilde{S}_E(e) = \mathrm{tr}(\widetilde{\sigma}\,e)\,\mathbb{1}_A \quad \forall\; e \in \mathcal{B}(\mathcal{H}_E) \tag{41}$$

for $\widetilde{\sigma} := \mathrm{tr}_{E'}\,U^*|\psi_\sigma\rangle\langle\psi_\sigma|U$. It then immediately follows from Eq. (39) that

$$\|T_E - \widetilde{S}_E\|_{cb} \leq 2\,\|T_B D_A - \mathrm{id}_A\|_{cb}^{\frac{1}{2}}, \tag{42}$$

as advertised. ∎

The tradeoff theorem amounts to a simple continuity estimate for the no-broadcasting and no-cloning theorems: A quantum channel $T\colon \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H})$ with a triple of isomorphic Hilbert spaces $\mathcal{H}_1 \cong \mathcal{H}_2 \cong \mathcal{H}$ is said to *broadcast* the quantum state $\varrho \in \mathcal{B}_*(\mathcal{H})$ iff the restrictions of the output state $T_*(\varrho)$ to both subsystems coincide with the input ϱ, $\mathrm{tr}_2 T_*(\varrho) = \varrho = \mathrm{tr}_1 T_*(\varrho)$. The only way to broadcast a pure state $\varrho = |\psi\rangle\langle\psi|$ is to generate the product state $|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$. Thus, broadcasting of pure states is equivalent to cloning. H. Barnum *et al.* [21] have shown that a quantum channel T can broadcast two quantum states ϱ_1 and ϱ_2 iff they commute — an extension of the famous no-cloning theorem [22, 23] to mixed states.

The tradeoff theorem immediately shows that approximate broadcasting is also impossible, and provides dimension-independent bounds:

Corollary 4 (No Broadcasting)
Let $V\colon \mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_E$ be a Stinespring isometry for the quantum channel $T\colon \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H})$ with local restrictions $T_1(a) := V^*(a \otimes \mathbb{1}_2 \otimes \mathbb{1}_E)V$ and $T_2(a) := V^*(\mathbb{1}_1 \otimes a \otimes \mathbb{1}_E)V$. Then there exists a completely depolarizing channel $S\colon \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H})$ defined as in Eq. (33) such that

$$\|T_2 - S\|_{cb} \leq 2\, \|T_1 - \mathrm{id}\,\|_{cb}^{\frac{1}{2}}. \tag{43}$$

Hence, any broadcast channel that has reasonably high fidelity in one of the output branches releases little information to the other branch (and the environment). While Corollary 4 shows that neither perfect nor approximate broadcasting is possible, the bound is certainly not tight. The merit of the tradeoff theorem is a dimension-independent estimate, while optimal cloning bounds are known to depend strongly on the dimension of the underlying Hilbert space [24, 25].

V. WEAKER NOTIONS OF DISTURBANCE AND ERASURE

The tradeoff estimate established in Sec. IV has the somewhat surprising and very welcome feature of being completely independent of the dimensions of the underlying Hilbert spaces, which makes it ideally suited for applications in which these dimensions are unknown and possibly very large, as in black hole evaporation. However, this property depends crucially on the choice of the distance measure: in this Section we will give an example of a quantum channel $T\colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ such that $\|T - S\|_\infty \approx 0$ for $\nu \to \infty$, but $\|T - S\|_{cb} \geq 1$. The example shows that operator norm and cb-norm are in general inequivalent in the vicinity of the completely depolarizing channel S. In contrast, we know from Eq. (7) that equivalence does hold in the neighborhood of the noiseless channel.

An example for a channel which nicely demonstrates this separation is

$$T\colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu) \qquad T := \frac{\nu}{\nu+1} S + \frac{1}{\nu+1} \Theta, \tag{44}$$

where $S\colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ is the completely depolarizing channel given again by

$$S(e) = \frac{1}{\nu}\mathrm{tr}(e)\,\mathbb{1} \quad \forall\, e \in \mathcal{B}(\mathbb{C}^\nu) \qquad \Longleftrightarrow \qquad S_*(\varrho) = \frac{1}{\nu}\mathbb{1} \quad \forall\, \varrho \in \mathcal{B}_*(\mathbb{C}^\nu), \tag{45}$$

and $\Theta\colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ is the so-called *transpose map:* $\Theta(e) = e^t$, the matrix transpose of $e \in \mathcal{B}(\mathbb{C}^\nu)$. While Θ is linear, unital and positive, it is not completely positive, and thus cannot be implemented as a quantum channel [7]. However, we will show in Prop. 5 below that T nonetheless remains a valid quantum channel.

Noting that $\|\Theta\|_\infty = 1$ and $\|\Theta\|_{cb} = \nu$ [7], it then immediately follows that

$$\|T - S\|_\infty = \frac{1}{\nu + 1} \|\Theta - S\|_\infty \leq \frac{1}{\nu + 1} \left(\|\Theta\|_\infty + \|S\|_\infty \right) = \frac{2}{\nu + 1}, \tag{46}$$

and thus $\lim_{\nu \to \infty} \|T - S\|_\infty = 0$. On the other hand, making again use of the triangle inequality we have the lower bound

$$\|T - S\|_{cb} = \frac{1}{\nu + 1} \|\Theta - S\|_{cb} \geq \frac{1}{\nu + 1} \left(\|\Theta\|_{cb} - \|S\|_{cb} \right) = \frac{\nu - 1}{\nu + 1}. \tag{47}$$

This demonstrates the suggested separation between cb-norm and operator norm as $\nu \to \infty$. It only remains to show that T is a quantum channel, which will become clear from the proof of

Proposition 5 *Let* $\Theta \colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ *be the transpose map, and let* $S \colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ *be the completely depolarizing channel as given in Eq. (45). Then*

$$T_p \colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu) \qquad T_p := (1 - p) \, S + p \, \Theta \tag{48}$$

for $p \in [0, 1]$ *defines a quantum channel iff* $p \leq \frac{1}{\nu + 1}$.

Proof: While T_p is clearly linear, unital and positive for all $p \in [0, 1]$, it is not necessarily completely positive, since the transpose map Θ does not have this property. In order to test for complete positivity, it is sufficient to apply the Schrödinger dual T_{p*} to half of a maximally entangled state $|\Omega\rangle := \frac{1}{\sqrt{\nu}} \sum_{i=1}^{\nu} |i, i\rangle$ on $\mathbb{C}^\nu \otimes \mathbb{C}^\nu$. In fact, it follows from Jamiolkowski's duality theorem (cf. [26] and Th. 2.3.4 in [5]) that a linear map $R \colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ is completely positive iff $\varrho := R_* \otimes \mathrm{id} \, |\Omega\rangle\langle\Omega|$ is a quantum state.

We will now apply this statement to the family T_p. It is easily seen from Eq. (4) that the Schrödinger dual Θ_* coincides with Θ, $\Theta_* = \Theta$. Straightforward calculation shows that

$$\varrho_p := T_{p*} \otimes \mathrm{id} \, |\Omega\rangle\langle\Omega| = \frac{1 - p}{\nu^2} \, \mathbb{1} \otimes \mathbb{1} + \frac{p}{\nu} \, \mathbb{F}, \tag{49}$$

where $\mathbb{F} := \sum_{i,j} |i, j\rangle\langle j, i|$ is the so-called *flip operator*. Note that $\mathbb{F}^* = \mathbb{F}$ and $\mathbb{F}^2 = \mathbb{1}$. Quantum states of the form

$$\varrho = \alpha \, \mathbb{1} + \beta \, \mathbb{F} \tag{50}$$

are usually called *Werner states* [27]. In order to see for which values of the parameters α and β the operator ϱ describes a quantum state, it is useful to rewrite Eq. (50) in terms of the eigenprojections $P_\pm$ of the flip operator $\mathbb{F}$, i. e., $\mathbb{F} \, P_\pm \, |\psi\rangle = \pm \, P_\pm |\psi\rangle$ with

$$P_+ := \frac{\mathbb{1} + \mathbb{F}}{2} \qquad \text{and} \qquad P_- := \frac{\mathbb{1} - \mathbb{F}}{2}. \tag{51}$$

P_+ is the projection onto the symmetric (Bose) subspace, while P_- describes the projection onto the antisymmetric (Fermi) subspace. Observing that $P_+ + P_- = \mathbb{1}$ and $P_+ - P_- = \mathbb{F}$ and substituting these expressions into Eq. (50), we see that

$$\varrho = (\alpha + \beta) \, P_+ + (\alpha - \beta) \, P_-, \tag{52}$$

which is positive iff $\alpha \geq \beta$. This implies that the output state ϱ_p, as given in Eq. (49), is a quantum state (and thus T_p is completely positive, by the Jamiolkowski duality) iff $p \leq \frac{1}{\nu + 1}$, as suggested. ∎

Hayden *et al.* [28] have recently proven that random selections of unitary matrices generically show an even stronger separation: in their terminology, a quantum channel $R \colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ is called *ε-randomizing* iff

$$\left\| R_*(\varrho) - \frac{1}{\nu}\,\mathbb{1} \right\|_\infty \leq \frac{\varepsilon}{\nu} \quad \forall\ \varrho \in \mathcal{B}_*(\mathbb{C}^\nu). \tag{53}$$

For sufficiently large ν, Hayden *et al.* show that a random selection of $\mu \sim \frac{1}{\varepsilon^2}\nu\log\nu$ unitary operators $\{U_i\}_{i=1}^\mu \subset \mathcal{B}(\mathbb{C}^\nu)$ with high probability yields an ϵ-randomizing quantum channel,

$$R(e) := \frac{1}{\mu}\sum_{i=1}^\mu U_i^*\,e\,U_i \quad \forall\ e \in \mathcal{B}(\mathbb{C}^\nu). \tag{54}$$

In striking contrast, exact randomization of quantum states (such that $\varepsilon = 0$ in Eq. (53)) is known to require an ancilla system of dimension at least $\nu^2 \gg \mu$ [29].

The definition of approximate randomization given in Eq. (53) implies the strictly weaker estimate $\|R - S\|_\infty \leq \varepsilon$, with the completely depolarizing channel S as in Eq. (45) above. However, Eq. (53) does not amount to the stabilized version $\|R - S\|_{cb} \leq \varepsilon$. Hayden *et al.* explicitly demonstrate this by showing that for ε-randomizing channels R randomly generated as in Eq. (54) above, one always has the upper bound

$$\|(R_* - S_*) \otimes \mathrm{id}\,(|\Omega\rangle\!\langle\Omega|)\|_1 \geq 2\left(1 - \frac{\mu}{\nu^2}\right) \xrightarrow{\nu\to\infty} 2, \tag{55}$$

where $|\Omega\rangle := \frac{1}{\sqrt{\nu}}\sum_{i=1}^\nu |i,i\rangle$ again denotes the maximally entangled state on $\mathbb{C}^\nu \otimes \mathbb{C}^\nu$.

The bound in Eq. (55) implies that $\lim_{\nu\to\infty}\|R - S\|_{cb} = 2$, and the same holds true for any other channel R with an ancilla system of dimension $o(\nu^2)$. From the right half of the tradeoff theorem Eq. (32) we may then conclude that for none of these channels will it be possible to find a decoding channel D such that the randomized information can be recovered from the ancilla system alone. Information may remain hidden in quantum correlations and cannot be retrieved locally.

Note that while these examples demonstrate that it is in general not possible to upper bound the cb-norm $\|\cdot\|_{cb}$ in terms of the operator norm $\|\cdot\|_\infty$ with a dimension-independent estimate, a dimension-*dependent* bound can of course be given. In fact, for any linear map $R \colon \mathcal{A} \to \mathcal{B}(\mathbb{C}^\nu)$ with an arbitrary (possibly infinite) C^*-algebra $\mathcal{A}$ we have $\|R\|_{cb} \leq \nu\,\|R\|_\infty$ [7]. The transpose map Θ shows that this bound can be tight.

VI. FURTHER APPLICATIONS

We have shown in Sec. IV how the continuity theorem entails dimension-independent bounds for the information-disturbance tradeoff in terms of stabilized operator norms. These results go beyond the standard measurement-based approach and can be seen as complementary to the entropic bounds obtained recently by Christandl and Winter [30]: Assume that a uniform quantum ensemble $E_1 := \{\frac{1}{\nu}, |i\rangle\}$ of basis states of the Hilbert space $\mathcal{H} \cong \mathbb{C}^\nu$ and the Fourier-rotated ensemble $E_2 := \{\frac{1}{\nu}, U|i\rangle\}$ have both nearly maximal Holevo information when sent through the quantum channel $T_B \colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$:

$$\chi\big(T_B(E_k)\big) \geq \mathrm{ld}\,\nu - \varepsilon \tag{56}$$

for $k = 1, 2$ and some (small) $\varepsilon > 0$, where $\operatorname{ld} \nu$ is the dual logarithm of ν and

$$\chi\big(T_B(E_1)\big) := S\bigg(\frac{1}{\nu}\sum_{i=1}^{\nu} T_{B*}\big(|i\rangle\langle i|\big)\bigg) - \frac{1}{\nu}\sum_{i=1}^{\nu} S\big(T_{B*}\big(|i\rangle\langle i|\big)\big) \tag{57}$$

denotes the Holevo information of the output ensemble $T_B(E_1) := \{\frac{1}{\nu}, T_{B*}(|i\rangle\langle i|)\}$. Analogously, $\chi\big(T_B(E_2)\big)$ is the Holevo information of the rotated ensemble $T_B(E_2) := \{\frac{1}{\nu}, T_{B*}(U|i\rangle\langle i|U^*)\}$. Christandl and Winter then conclude from their entropic uncertainty relation that the *coherent information*

$$I_c\Big(T_B, \frac{1}{\nu}\mathbb{1}\Big) := S\bigg(\frac{1}{\nu}\sum_{i=1}^{\nu} T_B\big(|i\rangle\langle i|\big)\bigg) - S\big(T_{B*}\otimes\operatorname{id}\big(|\Omega\rangle\langle\Omega|\big)\big) \geq \operatorname{ld}\nu - 2\varepsilon \tag{58}$$

is likewise large, where again $|\Omega\rangle := \frac{1}{\sqrt{\nu}}\sum_{i=1}^{\nu}|i,i\rangle$ denotes a maximally entangled state on $\mathbb{C}^\nu \otimes \mathbb{C}^\nu$. As a consequence, there exists a decoding operation $D\colon \mathcal{B}(\mathbb{C}^\nu) \to \mathcal{B}(\mathbb{C}^\nu)$ such that $F_c(T_B D) \geq 1 - 2\sqrt{2\varepsilon}$ [31], where $F_c(R)$ denotes the *channel fidelity* of the quantum channel R, $F_c(R) := \langle\Omega|R_* \otimes \operatorname{id}(|\Omega\rangle\langle\Omega|)|\Omega\rangle$. However, the faithful transmission of the maximally entangled state $|\Omega\rangle$ alone is not sufficient to conclude that $T_B D \approx \operatorname{id}$ in operator norm with dimension-independent bounds [8]. But it is always possible to find a subspace $\mathcal{H}' \subset \mathcal{H}$ with $\dim \mathcal{H}' \geq \frac{1}{2}\dim\mathcal{H}$ such that $\|T_B' D' - \operatorname{id}'\|_{cb} \leq 13\,\varepsilon^{\frac{1}{8}}$ [8, 32], where T_B' and D' are channels whose range and domain, respectively, are restricted to $\mathcal{H}'$. The tradeoff theorem then guarantees that $\|T_E' - S\|_{cb} \leq 8\,\varepsilon^{\frac{1}{16}}$ for some completely depolarizing channel S. Thus, in combination with Th. 3 the existence of highly reliable detectors for a basis and its conjugate alone imply a stabilized version of privacy, which is in general much stronger than the entropic version that appears in [30]. The improvement comes at the expense of a smaller code space. However, in many applications this is an exponentially large space, hence its reduction by a factor $1/2$ does not affect the rate of the protocol.

The information-disturbance tradeoff also plays the central role in the infamous black hole information loss puzzle: black holes emit thermal Hawking radiation [33], which contains (almost) no information about the previously absorbed quantum states. Hawking's derivation is perturbative, but we can nonetheless try to model this evaporation process as an (almost) completely depolarizing quantum channel, $T_E \approx S$. The tradeoff theorem then suggests that all the data about the formation of the black hole reside inside the event horizon, and could at least in principle be retrieved from there. However, the black hole may eventually evaporate completely, seemingly erasing all this information in the process and hence violating the unitarity of quantum mechanics.

The tradeoff theorem provides the explicit bounds for this estimate. Our results also show that for large systems with many internal degrees of freedom — such as all the information swallowed by a black hole —, the estimate crucially depends on the choice of the operator topology. If only an unstabilized estimate $\|T_E - S\|_\infty \leq \varepsilon$ can be guaranteed, information may remain hidden in quantum correlations between the thermal radiation and the black hole final state.

Similar conclusions apply to thermalization processes, in which a quantum system approaches an equilibrium state via repeated interaction with an environment. In so-called *collision models* [34, 35] the evolution of the thermalizing quantum system is described in terms of a quantum channel T_E. If T_E is almost completely depolarizing in cb-norm, all the information about the initial state of the system will have dissipated into the environment, and can at least in principle be retrieved from there.

Braunstein and Pati [2] have explored the consequences of the information-disturbance tradeoff for the physics of black holes and thermalization in greater detail.

VII. CONCLUSIONS

In conclusion, we have shown and explored a continuity theorem for Stinespring's dilation theorem: two quantum channels, T_1, T_2 are close in cb-norm iff there exist corresponding Stinespring isometries, V_1, V_2, which are close in operator norm.

When applied to the noiseless channel $T_1 = \mathrm{id}$, the continuity theorem yields a formulation of the information-disturbance tradeoff in which both information gain and disturbance are measured in terms of operator norms, complementing recently obtained entropic bounds.

In the form we have presented it, the continuity theorem applies to quantum channels on finite-dimensional quantum systems and yields dimension-independent bounds. This makes the result ideally suited for applications to situations in which these dimensions are large or possibly unknown.

The absence of dimension-dependent factors in the continuity bounds Eq. (18) seems to indicate that the result is not restricted to the finite-dimensional setting. Extensions of the continuity theorem to completely positive maps between arbitrary C^*-algebras are currently under investigation.

Infinite dimensions lead to a number of complications. The neat and useful one-to-one correspondence between states and density operators fails to hold in infinite-dimensional Hilbert spaces: there are always positive linear functionals on $\mathcal{B}(\mathcal{H})$ which cannot be represented as trace class operators [36]. The dual space of $\mathcal{B}_*(\mathcal{H})$ is the space of compact operators on $\mathcal{B}(\mathcal{H})$, not the full operator space $\mathcal{B}(\mathcal{H})$. The states that do allow a tracial representation are called *normal*. A positive functional $\omega \colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is normal iff $\lim_{n \to \infty} \omega(b_n) = \omega(b)$ for every sequence $(b_n)_n$ of norm-bounded increasing operators with least upper bound $b \in \mathcal{B}(\mathcal{H})$ (cf. [4], Ch. 1.6). More generally, a quantum channel $T \colon \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ is *normal* iff $\lim_{n \to \infty} T(b_n) = T(b)$. The normal channels T are then precisely those for which the duality relation Eq. (4) continues to hold (cf. [4], Ch. 9). Non-normal (or *singular*) channels do not have a Schrödinger dual.

However, as long as the Hilbert spaces are separable and all systems obey generic energy constraints, the state space will be compact [37], the channels respecting these energy constraints will be normal, and our proof of the continuity theorem and the tradeoff bounds then goes through unchanged. Thus, all the results presented in this work continue to hold in the practically relevant settings.

[1] W. F. Stinespring: *Positive Functions on C*-Algebras*, Proc. Amer. Math. Soc. **6** (1955) 211
[2] S. L. Braunstein, A. K. Pati: *Quantum Information Cannot be Completely Hidden in Correlations: Implications for the Black-Hole Information Paradox*, gr-qc/0603046 (March 2006)
[3] G. M. D'Ariano, D. Kretschmann, D. Schlingemann, R. F. Werner: *Quantum Bit Commitment Revisited: the Possible and the Impossible*, quant-ph/0605xyz
[4] E. B. Davies: *Quantum Theory of Open Systems* (Academic, London, 1976)
[5] M. Keyl: *Fundamentals of Quantum Information Theory*, Phys. Rep. **369** (2002) 431 (quant-ph/0202122)

[6] A. Gilchrist, N. K. Langford, M. A. Nielsen: *Distance Measures to Compare Real and Ideal Quantum Processes*, Phys. Rev. A **71** (2005) 062310 (quant-ph/0408063)

[7] V. I. Paulsen: *Completely Bounded Maps and Operator Algebras* (Cambridge University Press, Cambridge, 2002)

[8] D. Kretschmann, R. F. Werner: *Tema Con Variazioni: Quantum Channel Capacity*, New Jour. Phys. **6** (2004) 26 (quant-ph/0311037)

[9] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000)

[10] M. Reed, B. Simon: *Methods of Modern Mathematical Physics I: Functional Analysis* (Academic, New York, 1980)

[11] O. Bratteli, D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics 1*, 2nd edition (Springer, Berlin, 1987)

[12] T. Eggeling, D. Schlingemann, R. F. Werner: *Semicausal Operations are Semilocalizable*, Europhys. Lett. **57** (2001) 782 (quant-ph/0104027)

[13] D. Kretschmann, R. F. Werner: *Quantum Channels with Memory*, Phys. Rev. A **72** (2005) 062323 (quant-ph/0502106)

[14] V. P. Belavkin, G. M. D'Ariano, M. Raginsky: *Operational Distance and Fidelity for Quantum Channels*, J. Math. Phys. **46** (2005) 062106 (quant-ph/0408159)

[15] A. Uhlmann: *The 'Transition Probability' in the State Space of a *-Algebra*, Rep. Math. Phys. **9** (1976) 273

[16] J. von Neumann: *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928) 295

[17] S. Simons: *Minimax and Monotonicity* (Springer, Berlin, 1998)

[18] I. Devetak, P. Shor: *The Capacity of a Quantum Channel for Simultaneous Transmission of Classical and Quantum Information*, Comm. Math. Phys. **256** (2005) 287 (quant-ph/0311131)

[19] A. S. Holevo: *On Complementary Channels and the Additivity Problem*, Probability Theory and Applications **51** (2006) 133 (quant-ph/0509101)

[20] C. King, K. Matsumoto, M. Nathanson, M. B. Ruskai: *Properties of Conjugate Channels with Applications to Additivity and Multiplicativity*, quant-ph/0509126 v2 (Nov. 2005)

[21] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, B. Schumacher: *Noncommuting Mixed States Cannot Be Broadcast*, Phys. Rev. Lett. **76** (1996) 2818 (quant-ph/9511010)

[22] W. K. Wootters, W. H. Zurek: *A Single Quantum Cannot Be Cloned*, Nature **299** (1982) 802

[23] D. Dieks: *Communication by EPR Devices*, Phys. Lett. **92** (1982) 271

[24] V. Scarani, S. Iblisdir, N. Gisin: *Quantum Cloning*, Rev. Mod. Phys. **77** (2005) 1225 (quant-ph/0511088)

[25] N. J. Cerf, J. Fiurášek: *Optical Quantum Cloning — a Review*, quant-ph/0512172 (Dec. 2005)

[26] A. Jamiolkowski: *Linear Transformations which Preserve Trace and Positive Semidefiniteness of Operators*, Rep. Math. Phys. **3** (1972) 275

[27] R. F. Werner: *Quantum States with Einstein-Podolsky-Rosen Correlations Admitting a Hidden-Variable Model*, Phys. Rev. A **40** (1989) 4277

[28] P. Hayden, D. Leung, P. W. Shor, A. Winter: *Randomizing Quantum States: Constructions and Applications*, Comm. Math. Phys. **250** (2004) 371 (quant-ph/0307104)

[29] A. Ambainis, M. Mosca, A. Tapp, R. de Wolf: *Private Quantum Channels*; in *IEEE Symposium on Foundations of Computer Science (FOCS)*, pp. 547-553, 2000 (quant-ph/0003101)

[30] M. Christandl, A. Winter: *Uncertainty, Monogamy and Locking of Quantum Correlations*, IEEE Trans. Inf. Th. **51** (2005) 3159 (quant-ph/0501090)

[31] B. Schumacher, M. D. Westmoreland: *Approximate Quantum Error Correction*, Quantum Inf. Processing **1** (2002) 5 (quant-ph/0112106)

[32] H. Barnum, E. Knill, M. A. Nielsen: *On Quantum Fidelities and Channel Capacities*, IEEE Trans. Inf. Th. **46** (2000) 1317 (quant-ph/9809010)

[33] S. W. Hawking: *Black Hole Explosions?*, Nature **248** (1974) 30

[34] V. Scarani, M. Ziman, P. Štelmachovic, N. Gisin, V. Bužek: *Thermalizing Quantum Machines: Dissipation and Entanglement*, Phys. Rev. Lett. **88** (2002) 097905 (quant-ph/0110088)

[35] M. Ziman, P. Štelmachovic, V. Bužek, M. Hillery, V. Scarani, N. Gisin: *Diluting Quantum Information: An Analysis of Information Transfer in System-Reservoir Interactions*, Phys. Rev. A **65** (2002) 042105 (quant-ph/0110164)

[36] I. E. Segal: *Irreducible Representations of Operator Algebras*, Bull. Am. Math. Soc. **53** (1947)

73

[37] A. S. Holevo: *Classical Capacities of Quantum Channels with Constrained Inputs*, Probability Theory and Applications **48** (2003) 359 (quant-ph/0211170)