# Entanglement-Assisted Capacity of Quantum Multiple Access Channels

Min-Hsiu Hsieh, Igor Devetak and Andreas Winter

Abstract—We find a regularized formula for the entanglement-assisted (EA) capacity region for quantum multiple access channels (QMAC). We illustrate the capacity region calculation with the example of the collective phase-flip channel which admits a single-letter characterization. On the way, we provide a first-principles proof of the EA coding theorem based on a packing argument. We observe that the Holevo-Schumacher-Westmoreland theorem may be obtained from a modification of our EA protocol. We remark on the existence of a family hierarchy of protocols for multiparty scenarios with a single receiver, in analogy to the two-party case. In this way, we relate several previous results regarding QMACs.

Index Terms—Entanglement-assisted capacity, multiple access channels, quantum information, Shannon theory.

#### I. Introduction

HANNON'S classical channel capacity theorem is one of the central results in classical information theory [1]. A single-sender channel is defined by the triple  $(\mathcal{X}, p(y|x), \mathcal{Y})$  where the sets  $\mathcal{X}$  and  $\mathcal{Y}$  represent the input and output alphabets, respectively, and the conditional distribution p(y|x) defines the probability of the output being y given that the input was x. The capacity C of the channel, the maximum rate at which classical information can be transmitted through the channel, is given in terms of the mutual information I(X;Y) = H(X) + H(Y) - H(XY), (here the entropy of a random variable X with probability distribution p(x) is given by  $H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$ ):

$$C = \max_{p(x)} I(X;Y) \tag{1}$$

where the joint distribution of XY is p(x)p(y|x).

The classical multiple-access (MAC) channel  $(\mathcal{X} \times \mathcal{Y}, p(z|x,y), \mathcal{Z})$  is a channel with two senders and one receiver. Now  $\mathcal{X}$  and  $\mathcal{Y}$  are the input alphabets of the first and second sender, respectively. A general overview of MACs can be found in [2], [3]. The capacity problem now involves finding the region of achievable transmission rates  $R_1$  and  $R_2$  for the two senders. The classical capacity region of a MAC

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was found independently by Ahlswede [4] and Liao [5]. It is given by the closure of the convex hull of all  $(R_1,R_2)$  satisfying

$$R_1 \le I(X; Z|Y)$$

$$R_2 \le I(Y; Z|X)$$

$$R_1 + R_2 \le I(XY; Z)$$
(2)

for some product distribution p(x)p(y) on  $\mathcal{X} \times \mathcal{Y}$ . Here the joint distribution of XYZ is p(x)p(y)p(z|x,y), and the conditional mutual information is defined as I(X;Z|Y) = I(X;YZ) - I(X;Y).

The theory of quantum channels is richer, and includes several distinct capacities depending on the type of information one is trying to send and the additional resources one can use. A quantum channel  $\mathcal N$  is modeled as a cptp (completely positive and trace preserving) map. The capacity  $C(\mathcal N)$  of a quantum channel is defined to be the maximum rate at which classical information can be sent through the quantum channel  $\mathcal N$ . This capacity was proved independently by Holevo [6] and Schumacher and Westmoreland [7]. The capacity  $Q(\mathcal N)$  is defined to be the maximum rate at which quantum information can be sent through the quantum channel  $\mathcal N$ , and a formula for it was proven in [8], [9], [10].

Entanglement shared between sender and receiver is a useful resource that generically increases channel capacity. The entanglement-assisted classical capacity  $C_E(\mathcal{N})$  is the maximum rate at which classical information can be transmitted through the quantum channel  $\mathcal{N}$  if the sender and receiver have access to unlimited entanglement. A remarkably simple formula for this capacity was found in [11], [12], to be formally identical to (1), with classical mutual information replaced by the quantum mutual information between quantum systems A and B

$$C_E(\mathcal{N}) = \max_{\rho} I(A; B). \tag{3}$$

The maximization is performed over the sender's input state  $\rho$ , and the quantum mutual information I(A;B) is defined with respect to the purification of  $\rho$  after half of it has passed through the channel  $\mathcal N$ . The system A is the half remaining on the sender's side, and B is the channel output system. Formal definitions of these concepts will be given in Section II.

A quantum multiple access channel (QMAC)  $\mathcal{M}$  is a cptp map with two senders and one receiver. Each sender can transmit either classical or quantum information through the channel  $\mathcal{M}$ . The classical-classical capacity region  $C(\mathcal{M})$  for the case in which both senders transmit classical information through QMAC  $\mathcal{M}$  was found by Winter [13]. Later on, the

classical-quantum capacity region  $CQ(\mathcal{M})$  (where one sender is sending classical, and the other quantum information), and the quantum-quantum channel capacity region  $Q(\mathcal{M})$  were found in [14], [15].

In this work we consider the entanglement-assisted classical-classical capacity region  $C_E(\mathcal{M})$  of a QMAC  $\mathcal{M}$ . In other words, both senders share unlimited entanglement with the receiver and both are sending classical information. We will show it to be the *regularized* closure of the set of all the achievable rate pairs  $(R_1,R_2)$  satisfying

$$R_1 \le I(A; C|B)$$

$$R_2 \le I(B; C|A)$$

$$R_1 + R_2 \le I(AB; C)$$
(4)

for some choice of a product input state  $\rho_1 \otimes \rho_2$  for the two senders. The quantum entropic quantities are defined with respect to the product of purifications of  $\rho_1$  and  $\rho_2$ , after half of it has passed through the channel  $\mathcal{M}$ . The systems A and B are the parts remaining on the senders' sides, and C is the channel output system. A precise statement of the result is given in Theorem 2. The expression (4) thus parallels (2) with the classical mutual information replaced by its quantum counterpart. While our formula does not allow  $C_E(\mathcal{M})$  to be efficiently computed in general, we exhibit a non-trivial example for we can compute  $C_E(\mathcal{M})$  in closed form.

We also provide a new proof of the direct coding theorem for the single-sender entanglement-assisted channel capacity. Our proof is important and necessary in the following sense. First, our proof uses packing lemma that comes from the idea of typical subspaces, which is directly analog to the idea of typical sets Shannon uses to prove the direct coding theorem of single-user channel capacity. The previous proof in [11], [12] is less trivial in the sense that it is based on the Holevo-Schumacher-Westmoreland (HSW) theorem [6], [7], which uses the conditional typical subspaces. Our proof demonstrates our growing understanding of quantum information theory. We believe that our method of proof will not only become a powerful tool but also will find many applications in quantum information theory. Second, our proof provides new properties that can be used to prove the multiparty generalization. These new properties do not exist in the previous proofs. Finally, we show that the HSW theorem is a special case of the two-party entanglement-assisted capacity theorem.

The paper is organized as follows. Section II contains the relevant background material. This includes notational conventions, definitions of the method of types, frequency typical sequences and subspaces, and useful lemmas. Section III contains statements and proofs of our main results. In section IV we compute the capacity region of the collective phase-flip multiple access channel which admits a single-letter expression. In section V we first rewrite our results in the resource inequality framework, from which we recover previously known coding theorems for QMACs. In section VI we conclude by pointing out the open question regarding the single-letter expression for our entanglement-assisted capacity region of quantum multiple access channels. We also give a conjecture on the entanglement-assisted channel capacity with more than two inputs.

#### II. BACKGROUND

Each quantum system is completely described by the state vector which is a unit vector in Hilbert space  $\mathcal{H}$ . An alternative way to describe a quantum system is by density operator  $\rho:\mathcal{H}\to\mathcal{H}$ , where  $\rho$  has trace equal to one and is a positive operator. If  $\rho$  belongs to a quantum system A we may denote it by  $\rho^A$ . When it is clear from contexts, we will omit the superscript letter that represents the holder of the quantum system. We always use  $\pi$  to denote the maximally mixed state  $\pi=(|\mathcal{H}|)^{-1}I$  where  $|\mathcal{H}|$  represents the dimension of  $\mathcal{H}$ . Given a state  $\rho^A$  whose spectral decomposition is  $\sum_i p_i |i\rangle \langle i|$ , the purification of such state is obtained by introducing a reference system R such that the purified state  $|\psi\rangle^{AR}=\sum_i \sqrt{p_i}|i\rangle^A|i\rangle^R$ . We write the density operator of a pure state  $|\psi\rangle$  as  $\psi\equiv |\psi\rangle \langle \psi|$ .

Saying that  $\mathcal{N}:A\to B$  is a quantum channel, we really mean that  $\mathcal{N}:\mathcal{B}(\mathcal{H}_A)\to\mathcal{B}(\mathcal{H}_B)$  is a cptp (completely positive trace preserving) map, where  $\mathcal{B}(\mathcal{H})$  represents the set of bounded linear operators in  $\mathcal{H}$ . It may be modeled by an isometry  $U_{\mathcal{N}}:A\to BE$  with a larger target space BE, followed by tracing out the "environment" system  $E.U_{\mathcal{N}}$  is known as the Stinespring dilation [16] of  $\mathcal{N}$ . We will often write  $U_{\mathcal{N}}(\rho)$  for  $U_{\mathcal{N}}\rho U_{\mathcal{N}}^{\dagger}$ .

A quantum instrument [17], [18]  $\mathbf{D} = \{\mathcal{D}_m\}_{m \in [\mu]}, [\mu] := \{1, 2, \dots, \mu\}$ , is a set of cp (completely positive) maps  $\mathcal{D}_m$ ,

$$\mathcal{D}_m: \rho \to \sum_k A_{km} \rho A_{km}^{\dagger}.$$

The sum of the cp maps  $\mathcal{D}=\sum_{m\in[\mu]}\mathcal{D}_m$  is trace preserving, and  $\sum_{km}A_{km}^{\dagger}A_{km}=I$ . The instrument has one quantum input and two outputs, classical and quantum. The probability of classical outcome m and corresponding quantum output  $\mathcal{D}_m(\rho)/(\mathrm{Tr}\,\mathcal{D}_m(\rho))$  is  $\mathrm{Tr}\,\mathcal{D}_m(\rho)$ . Ignoring the classical output reduces the instrument to the quantum map  $\mathcal{D}$ . Ignoring the quantum output reduces the instrument to the set of POVMs (positive operator valued measure)  $\{\Lambda_m\}$  with  $\Lambda_m=\sum_k A_{km}^{\dagger}A_{km}$ .

The trace distance is defined as the trace norm of the difference between the two states

$$\|\sigma - \rho\| = \operatorname{Tr} \sqrt{(\sigma - \rho)^2} = \max_{-I \le \Lambda \le I} \operatorname{Tr} [\Lambda(\sigma - \rho)].$$

The method of types is a standard technique of classical information theory. Denote by  $x^n$  a sequence  $x_1x_2\dots x_n$ , where each  $x_i$  belongs to the finite set  $\mathcal{X}$ . Denote by  $|\mathcal{X}|$  the cardinality of  $\mathcal{X}$ . Denote by  $N(a|x^n)$  the number of occurrences of the symbol  $a\in\mathcal{X}$  in the sequence  $x^n$ . The type  $t^{x^n}$  of a sequence  $x^n$  is a probability vector whose elements  $t^{x^n}_a = \frac{N(a|x^n)}{n}$ . Denote the set of sequences of type t by

$$\mathcal{T}_t^n = \{x^n \in \mathcal{X}^n : t^{x^n} = t\}.$$

For the probability distribution p on the set  $\mathcal{X}$  and  $\delta > 0$ , let  $\tau_{\delta} = \{t : \forall a \in \mathcal{X}, |t_a - p_a| \leq \delta\}$ . Define the set of  $\delta$ -typical sequences of length n as

$$\mathcal{T}_{p,\delta}^{n} = \bigcup_{t \in \tau_{\delta}} \mathcal{T}_{t}^{n} 
= \{x^{n} : \forall a \in \mathcal{X}, |t_{a}^{x^{n}} - p_{a}| \leq \delta\}.$$
(5)

Define the probability distribution  $p^n$  on  $\mathcal{X}^n$  to be the tensor power of p. The sequence  $x^n$  is drawn from  $p^n$  if and only if each letter  $x_i$  is drawn independently from p. Typical sequences enjoy many useful properties. Let  $H(p) = -\sum_x p_x \log p_x$  be the Shannon entropy of p. For any  $\epsilon, \delta > 0$ , and all sufficiently large n for which

$$p^n(\mathcal{T}^n_{n\,\delta}) \ge 1 - \epsilon \tag{6}$$

$$2^{-n[H(p)+c\delta]} \le p^n(x^n) \le 2^{-n[H(p)-c\delta]}, \ \forall x^n \in \mathcal{T}_{p,\delta}^n \quad (7)$$

$$|\mathcal{T}_{n,\delta}^n| \le 2^{n[H(p) + c\delta]} \tag{8}$$

for some constant c (see [2] for proofs). For  $t \in \tau_{\delta}$  and for sufficiently large n, the cardinality  $D_t = |\mathcal{T}_t^n|$  is lower bounded as [2]

$$D_t \ge 2^{n[H(p) - \eta(\delta)]} \tag{9}$$

and the function  $\eta(\delta) \to 0$  as  $\delta \to 0$ .

The above concepts generalize to the quantum setting by virtue of the spectral theorem. Let  $\rho = \sum_{x \in \mathcal{X}} p_x |x\rangle \langle x|$  be the spectral decomposition of a given density matrix  $\rho$ . In other words,  $|x\rangle$  is the eigenstate of  $\rho$  corresponding to eigenvalue  $p_x$ . The von Neumann entropy of the density matrix  $\rho$  is

$$H(\rho) = -\operatorname{Tr} \rho \log \rho = H(p).$$

Define the type projector

$$\Pi_t^n = \sum_{x^n \in \mathcal{T}_t^n} |x^n\rangle \langle x^n|.$$

The density operator proportional to the type projector is  $\pi_t = D_t^{-1}\Pi_t^n$ . The typical subspace associated with the density matrix  $\rho$  is defined as

$$\Pi_{\rho,\delta}^{n} = \sum_{x^{n} \in \mathcal{T}_{r,\delta}^{n}} |x^{n}\rangle\langle x^{n}| = \sum_{t \in \tau_{\delta}} \Pi_{t}^{n}.$$

Properties analogous to (6) – (9) hold [19]. For any  $\epsilon, \delta > 0$ , and all sufficiently large n for which

$$\operatorname{Tr} \rho^{\otimes n} \Pi_{\rho,\delta}^n \ge 1 - \epsilon \tag{10}$$

$$2^{-n[H(\rho)+c\delta]}\Pi_{\rho,\delta}^n \leq \Pi_{\rho,\delta}^n \rho^{\otimes n}\Pi_{\rho,\delta}^n \leq 2^{-n[H(\rho)-c\delta]}\Pi_{\rho,\delta}^n, (11)$$

$$\operatorname{Tr} \Pi_{\rho,\delta}^n \le 2^{n[H(\rho) + c\delta]} \tag{12}$$

for some constant c. For  $t \in \tau_{\delta}$  and for sufficiently large n, the dimension of the type projector  $\Pi^n_t$  is lower bounded as

$$\operatorname{Tr} \Pi_t^n \ge 2^{n[H(\rho) - \eta(\delta)]} \tag{13}$$

and the function  $\eta(\delta) \to 0$  as  $\delta \to 0$ .

For a multipartite state  $\rho^{ABC}$ , we write  $H(A)_{\rho} = H(\rho^{A})$ , etc. We omit the subscript if the state is clear from the context. Define the quantum mutual information by

$$I(A; B) = H(A) + H(B) - H(AB)$$

and the quantum conditional mutual information by

$$I(A; C|B) = H(AB) + H(BC) - H(ABC) - H(B).$$

These are non-negative by strong subadditivity [20]. If I(A;B)=0 then

$$I(A; C|B) = I(A; CB)$$

is easy to verify.

The set of generalized Pauli matrices  $\{U_m\}_{m\in[d^2]}$  is defined by  $U_{l\cdot d+k}=\hat{Z}_d(l)\hat{X}_d(k)$  for  $k,l=0,1,\cdots,d-1$  and

$$\hat{X}_d(k) = \sum_{s} |s\rangle\langle s + k| = \hat{X}_d(1)^k,$$

$$\hat{Z}_d(l) = \sum_{s} e^{i2\pi sl/d} |s\rangle\langle s| = \hat{Z}_d(1)^l.$$
(14)

The + sign denotes addition modulo d.

We will always use  $|\Phi\rangle$  to represent the maximally entangled state. Then the maximally entangled state  $|\Phi\rangle^{AB}$  on a pair of d-dimensional quantum systems A and B is given as:

$$|\Phi\rangle^{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle^{A} |i\rangle^{B}.$$
 (15)

We have the following result (see [11] for a proof):

$$\frac{1}{d^2} \sum_{m=1}^{d^2} (U_m \otimes I) \Phi^{AB} (U_m^{\dagger} \otimes I) = \pi^A \otimes \pi^B, \quad (16)$$

where  $\pi^A = \pi^B = \frac{I}{d}$ . We will also need the following equality:

$$(I \otimes U)|\Phi\rangle = (U^{tr} \otimes I)|\Phi\rangle \tag{17}$$

for any operator U, and  $U^{tr}$  denotes transposition of U.

Next is a coherent version of the gentle operator lemma ([21], Lemma 9). It states that a measurement which is likely to be successful in identifying a state tends not to significantly disturb the state.

Lemma 1 (Gentle coherent measurement): Let  $\{\rho_k^A\}_{k\in[K]}$  be a collection of density operators and  $\{\Lambda_k\}_{k\in[K]}$  be a set of POVMs on quantum system A such that

$$\operatorname{Tr} \rho_k \Lambda_k \geq 1 - \epsilon$$

for all k. Let  $|\phi_k\rangle^{RA}$  be a purification of  $\rho_k^A$ . Then there exists an isometric quantum operation  $\mathcal{D}:A\to AJ$  such that

$$\|(I^R \otimes \mathcal{D})(\phi_k^{RA}) - \phi_k^{RA} \otimes |k\rangle\langle k|^J\| \le \sqrt{8\epsilon}.$$

*Proof:* Every POVM can be written as an isometry followed by projective measurement on a subsystem. In particular, there exists an isometry  $\mathcal{D}: A \to AJ$  such that

$$(I^R \otimes \mathcal{D}) |\phi\rangle^{RA} = \sum_{j} [(I^R \otimes \sqrt{\Lambda_j}) |\phi\rangle^{RA}] |j\rangle^J.$$

Thus

$$\langle k | \langle \phi_k | (I \otimes \mathcal{D}) | \phi_k \rangle = \langle \phi_k | (I \otimes \sqrt{\Lambda_k}) | \phi_k \rangle$$

$$\geq \langle \phi_k | (I \otimes \Lambda_k) | \phi_k \rangle$$

$$= \operatorname{Tr} \rho_k \Lambda_k$$

$$\geq 1 - \epsilon.$$
(18)

The first inequality uses that  $\Lambda_k \leq \sqrt{\Lambda_k}$  when  $0 \leq \Lambda_k \leq I$ . The statement of the lemma follows from the fact that for pure states  $|\zeta\rangle$  and  $|\psi\rangle$ ,

$$\|\zeta - \psi\| = 2\sqrt{1 - |\langle \zeta | \psi \rangle|^2}.$$

The packing lemma below will prove to be a powerful tool in quantum information theory. The technique used here is simple, directly analogous to the classical coding theorem.

Lemma 2 (Packing): We are given an ensemble  $\{\lambda_m, \sigma_m\}_{m \in \mathcal{S}}$  with average density operator

$$\sigma = \sum_{m \in \mathcal{S}} \lambda_m \sigma_m.$$

Assume the existence of projectors  $\Pi$  and  $\{\Pi_m\}_{m\in\mathcal{S}}$  with the following properties:

$$\operatorname{Tr} \sigma_m \Pi_m \geq 1 - \epsilon,$$
 (19)

$$\operatorname{Tr} \sigma_m \Pi \geq 1 - \epsilon,$$
 (20)

$$\operatorname{Tr} \Pi_m \leq d,$$
 (21)

$$\Pi \sigma \Pi \quad < \quad D^{-1} \Pi \tag{22}$$

for all  $m \in \mathcal{S}$  and some positive integers D and d. Let  $N = \lfloor \gamma D/d \rfloor$  for some  $0 < \gamma < 1$  where  $\lfloor r \rfloor$  represents the largest integer less than r. Then there exists a map  $f: [N] \to \mathcal{S}$ , and a corresponding set of POVMs  $\{\Lambda_k\}_{k \in [N]}$  which reliably distinguishes between the states  $\{\sigma_{f(k)}\}_{k \in [N]}$  in the sense that

$$\operatorname{Tr} \sigma_{f(k)} \Lambda_k \ge 1 - 4(\epsilon + \sqrt{8\epsilon}) - 8\gamma$$

for all  $k \in [N]$ .

Proof: See Appendix A.

Lemma 3: If  $|\psi\rangle^{ABE}$  is a pure state then

$$H(B|E)_{\psi} = -H(B|A)_{\psi}.$$

*Proof:* Since  $|\psi\rangle^{ABE}$  is pure, we have  $H(A)_{\psi}=H(BE)_{\psi}$  and  $H(E)_{\psi}=H(AB)_{\psi}.$  Then

$$H(B|E)_{\psi} = H(BE)_{\psi} - H(E)_{\psi}$$

$$= H(A)_{\psi} - H(AB)_{\psi}$$

$$= -H(B|A)_{\psi}.$$
(23)

Lemma 4: For any state  $\sigma^{ABE}$ ,

$$I(A;B)_{\sigma} \leq H(B)_{\sigma} + H(B|E)_{\sigma}$$
.

Proof: Introduce a reference system R that purifies the state  $\sigma^{ABE},$  then

$$I(A;B)_{\sigma} = H(B)_{\sigma} - H(B|A)_{\sigma}$$

$$= H(B)_{\sigma} + H(B|ER)_{\sigma}$$

$$\leq H(B)_{\sigma} + H(B|E)_{\sigma}.$$
(24)

The first equality follows from the definition of quantum mutual information. The second equality follows from Lemma 3. The first inequality uses the fact that conditioning reduces entropy [20].

# III. MAIN RESULT

#### A. Two party entanglement-assisted coding

Before attacking the multiuser problem we give a new proof of the two-party entanglement-assisted direct coding theorem. This theorem was first proved in [11] and subsequently in [12]. Both proofs invoke the HSW theorem. The HSW theorem uses the method of conditionally typical subspaces. We give a

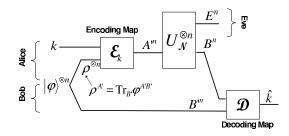


Fig. 1. Two-party entanglement-assisted communication

direct proof based on the packing lemma which only uses typical subspaces. The proof perhaps sheds more light on why achievable rates take on the form of mutual information. Furthermore, our proof provides new properties (ii) and iii) below) that serve as a bridge to the proof of multiparty coding theorem.

As shown in Fig. 1, Alice and Bob are connected by a large number n uses of the quantum channel  $\mathcal{N}:A'\to B$ . Alice controls the channel input system A' and Bob has access to the channel output B. They also have entanglement in the form of n copies of some pure bipartite state  $\varphi^{A'B'}$ . Any such state is determined upto a local unitary transformation by the local density operator  $\rho^{A'}=\operatorname{Tr}_{B'}\varphi^{A'B'}$ . Alice and Bob use these resources to communicate, in analogy to superdense coding [22]. Based on her message Alice performs a quantum operation on her share of the entanglement. She then sends it through the quantum channel. Bob performs a decoding measurement on the channel output plus his share of the entanglement. They endeavor to maximize the communication rate

We formalize the above information processing task. Define an  $[n,R,\rho,\epsilon]$  entanglement-assisted code by

- a set of unitary encoding maps  $\{\mathcal{E}_k\}_{k\in[2^{nR}]}$  acting on  $A'^n:=A'_1\ldots A'_n$  for Alice;
- Bob's decoding instrument  $\mathbf{D} = \{\mathcal{D}_k\}_{k \in [2^{nR}]}$  acting on  $B^n B'^n$ .

such that for all  $k \in [2^{nR}]$ 

- i) Tr  $\{ [\mathcal{D}_k \circ ((\mathcal{N}^{\otimes n} \circ \mathcal{E}_k) \otimes I)](\varphi^{\otimes n}) \} \ge 1 \epsilon;$
- ii) the encoded density operator satisfies  $\mathcal{E}_k(\rho^{\otimes n}) = \rho^{\otimes n}$ ;

iii)

$$\left\| [(\mathcal{D} \otimes I^{E^n}) \circ ((U_{\mathcal{N}}^{\otimes n} \circ \mathcal{E}_k) \otimes I) - (U_{\mathcal{N}}^{\otimes n} \otimes I)](\varphi^{\otimes n}) \right\|$$

$$\leq \epsilon$$

where o represents composition of two maps.

Condition i) means that Bob correctly decodes Alice's message with high probability. This condition suffices for two-party entanglement-assisted communication. The remaining two properties, which were not shown in [11], [12], are important for the multiparty generalization. Condition ii) means that Alice always inputs a tensor power state into the channel. Condition iii) says that the encoding and decoding operations in effect cancel each other out. So it is as if Alice just sent the state  $\rho^{\otimes n}$  down the channel without any coding. In reality, she has also managed to convey the message to Bob.

Theorem 5: Define  $\theta^{AB}=(I\otimes \mathcal{N})\varphi^{AA'}$  and  $R=I(A;B)_{\theta}$ . For every  $\epsilon,\delta>0$  and n sufficiently large, there exists an  $[n,R-\delta,\rho,\epsilon]$  entanglement-assisted code.

*Proof:* Let  $t(1),\ldots,t(a)$  be an ordering of the distinct types  $t^{x^n}$ . Define the maximally mixed state  $\pi^n_\alpha=1/d_\alpha\Pi^n_{t(\alpha)}$ , where  $d_\alpha=\operatorname{Tr}\Pi^n_{t(\alpha)}$ . Define  $|\Phi_\alpha\rangle$  to be the maximally entangled state on a pair of  $d_\alpha$ -dimensional quantum systems  $A'^n$  and  $B'^n$ 

$$|\Phi_{\alpha}\rangle^{A^{\prime n}B^{\prime n}} = \frac{1}{\sqrt{d_{\alpha}}} \sum_{x^n \in \mathcal{T}_{t(\alpha)}^n} |x^n\rangle^{A^{\prime n}} |x^n\rangle^{B^{\prime n}}.$$
 (25)

In the beginning Alice and Bob share the entangled state

$$|\Psi\rangle^{A'^n B'^n} = |\varphi\rangle^{\otimes n}$$

$$= \sum_{\alpha} \sqrt{p_{\alpha}} |\Phi_{\alpha}\rangle, \tag{26}$$

where  $p_{\alpha} = \sum_{x^n \in \mathcal{T}^n_{t(\alpha)}} p^n(x^n)$ . The type projectors  $\Pi^n_{t(\alpha)}$  induce a decomposition of the Hilbert space  $\mathcal{H}^{\otimes n}$  of  $A'^n$  (correspondingly of  $B'^n$ ) into a direct sum

$$\mathcal{H}^{\otimes n} = \bigoplus_{\alpha=1}^{a} \mathcal{H}_{t(\alpha)}.$$

Let  $\mathcal{G} = \{(g_1,g_2,\cdots,g_a): g_\alpha \in [d_\alpha^2], \alpha \in [a]\}, \mathcal{B} = \{(b_1,b_2,\cdots,b_a): b_\alpha \in \{0,1\}\}, \text{ and } \mathcal{S} = \mathcal{G} \times \mathcal{B}.$  Every element  $s^a \in \mathcal{S}$  is uniquely determined by  $g^a \in \mathcal{G}$  and  $b^a \in \mathcal{B}$ . Given an element  $s^a \in \mathcal{S}$ , define a unitary operation  $U_{s^a}$  to be

$$U_{s^{a}} \equiv U_{g^{a},b^{a}} = \bigoplus_{\alpha=1}^{a} (-1)^{b_{\alpha}} U_{g_{\alpha}}$$
 (27)

where  $\{U_{g_{\alpha}}\}$  are the  $d_{\alpha}^2$  generalized Pauli operators (14) defined on  $\mathcal{H}_{t(\alpha)}$ . Define

$$\sigma_{s^a}^{B^n B'^n} := (\mathcal{N}^{\otimes n} \otimes I) \left[ (U_{s^a} \otimes I) \Psi^{A'^n B'^n} (U_{s^a}^{\dagger} \otimes I) \right]$$

$$= (I \otimes U_{s^a}^{tr}) \theta^{\otimes n} (I \otimes U_{s^a}^*).$$
(28)

The last equality follows from (17). Let  $\sigma$  to be the average of  $\sigma_{s^a}$  over S, then we get (29). The last equality comes from (30) and (31) below. When  $\alpha = \alpha'$ ,

$$\frac{1}{|\mathcal{B}||\mathcal{G}|} \sum_{g^{a} \in \mathcal{G}} \sum_{b^{a} \in \mathcal{B}} p_{\alpha}(\mathcal{N}^{\otimes n} \otimes I) \left[ (U_{g^{a}, b^{a}} \otimes I) \Phi_{\alpha}(U_{g^{a}, b^{a}}^{\dagger} \otimes I) \right]$$

$$= (\mathcal{N}^{\otimes n} \otimes I) \frac{1}{|\mathcal{G}|} \sum_{g_{1}} \cdots \sum_{g_{a}} p_{\alpha}(U_{g_{\alpha}} \otimes I) \Phi_{\alpha}(U_{g_{\alpha}}^{\dagger} \otimes I)$$

$$= (\mathcal{N}^{\otimes n} \otimes I) p_{\alpha}(\pi_{\alpha}^{n} \otimes \pi_{\alpha}^{n}). \tag{30}$$

The last equality follows from (16). When  $\alpha \neq \alpha'$ , we get (31). Define the projectors on  $B'^nB^n$ 

$$\Pi_{s^a} = (I \otimes U_{s^a}^{tr}) \Pi_{\theta,\delta}^n (I \otimes U_{s^a}^*), 
\Pi = \Pi_{\mathcal{N}(a)}^n \delta \otimes \Pi_{a,\delta}^n.$$
(32)

The following properties are proved in Appendix B. For all  $\epsilon > 0, \delta > 0$  and all sufficiently large n,

$$\operatorname{Tr} \sigma_{s^a} \Pi \geq 1 - \epsilon$$
 (33)

$$\operatorname{Tr} \sigma_{s^a} \Pi_{s^a} \geq 1 - \epsilon \tag{34}$$

$$\operatorname{Tr} \Pi_{s^a} \leq 2^{n[H(AB)_{\theta} + c\delta]} \tag{35}$$

$$\Pi \sigma \Pi \leq 2^{n[H(A)_{\theta} + H(B)_{\theta} + c\delta]} \Pi.$$
 (36)

Let  $\lambda_{s^a}=\frac{1}{|\mathcal{S}|}$  and  $R=I(A;B)_{\theta}-(2c+1)\delta$ . We now apply the packing lemma to the ensemble  $\{\lambda_{s^a},\sigma_{s^a}\}_{s^a\in\mathcal{S}}$  and projectors  $\Pi$  and  $\Pi_{s^a}$ . Thus there exist a map  $f:[2^{nR}]\to\mathcal{S}$  and a POVM  $\{\Lambda_k\}_{k\in[2^{nR}]}$  such that

$$\operatorname{Tr} \sigma_{f(k)} \Lambda_k \ge 1 - \epsilon', \tag{37}$$

with

$$\epsilon' = 4(\epsilon + \sqrt{8\epsilon}) + 16 \times 2^{-n\delta}$$

Define the encoding operation by  $\mathcal{E}_k = U_{f(k)}$ . Including the environment system, the state of  $B^n B'^n E^n$  after the application of the channel  $U_N$  is

$$|\Upsilon_{k}\rangle^{B^{n}B^{\prime n}E^{n}} = (U_{\mathcal{N}}^{\otimes n} \otimes I)(U_{f(k)} \otimes I)|\Psi\rangle^{A^{\prime n}B^{\prime n}}$$
$$= (U_{\mathcal{N}}^{\otimes n} \otimes U_{f(k)}^{tr})|\Psi\rangle^{A^{\prime n}B^{\prime n}}.$$
 (38)

 $|\Upsilon_k\rangle$  is a purification of  $\sigma_{f(k)}$ . By Lemma 1, there exists an isometry  $\mathcal{D}':B^nB'^n\to B^nB'^nJ$  such that

$$\|(I\otimes \mathcal{D}')(\Upsilon_k) - \Upsilon_k \otimes |k\rangle\langle k|^J\| \leq \sqrt{8\epsilon'}.$$

Bob performs the controlled unitary

$$W^{JB^{\prime n}} = \sum_{k} |k\rangle\langle k|^{J} \otimes (U_{f(k)}^{*})^{B^{\prime n}}.$$

Defining  $\mathcal{D}'' = (W \otimes I^{B^n}) \circ \mathcal{D}'$ , this implies

$$\|(I \otimes \mathcal{D}'')(\Upsilon_k) - [(U_{\mathcal{N}}^{\otimes n} \otimes I)(\varphi^{\otimes n})] \otimes |k\rangle\langle k|\| \leq \sqrt{8\epsilon'}.$$
(39)

The instrument  $\mathbf{D} = \{\mathcal{D}_k\}$  is defined by  $\mathcal{D}''$  followed by a von Neumann measurement of the system J. Equation (39) expresses the fact that the classical communication being performed is almost decoupled from all the quantum systems involved in the protocol, including ancillas and the inaccessible environment. We remark that this guarantees the ability to "coherify" the protocol in the sense of [23].

Condition i) in the form

Tr 
$$\{ [\mathcal{D}_k \circ ((\mathcal{N}^{\otimes n} \circ \mathcal{E}_k) \otimes I)] (\varphi^{\otimes n}) \} \geq 1 - \epsilon'$$

is immediate from (37). Condition ii) follows from the construction (27). Condition iii) in the form

$$\left\| [(\mathcal{D} \otimes I^{E^n}) \circ ((U_{\mathcal{N}}^{\otimes n} \circ \mathcal{E}_k) \otimes I) - (U_{\mathcal{N}}^{\otimes n} \otimes I)](\varphi^{\otimes n}) \right\| \leq \sqrt{8\epsilon'}$$
 follows from (39).

#### B. Remark on the HSW theorem

Suppose that Alice and Bob are connected by a special cq channel of the form

$$\mathcal{N} = \mathcal{N}' \circ \Delta$$
,

where  $\Delta$  is the dephasing channel

$$\Delta: \rho \to \sum_{x} |x\rangle\langle x|\rho|x\rangle\langle x|.$$

A  $\{c \to q\}$  channel is equivalent to one with classical inputs and quantum outputs. The HSW coding theorem states that rates  $R = I(A;B)_{\theta}, \; \theta^{AB} = (I \otimes \mathcal{N})\varphi^{AA'}$  are achievable even without entanglement assistance. We show that this fact follows from our construction in two steps.

$$\sigma = \frac{1}{|\mathcal{S}|} \sum_{s^{a} \in \mathcal{S}} \sigma_{s^{a}}$$

$$= \frac{1}{|\mathcal{B}||\mathcal{G}|} \sum_{g^{a} \in \mathcal{G}} \sum_{b^{a} \in \mathcal{B}} \sum_{\alpha, \alpha'} \sqrt{p_{\alpha} p_{\alpha'}} (\mathcal{N}^{\otimes n} \otimes I) \left[ (U_{g^{a}, b^{a}} \otimes I) |\Phi_{\alpha}\rangle \langle \Phi_{\alpha'} | (U_{g^{a}, b^{a}}^{\dagger} \otimes I) \right].$$

$$= \sum_{\alpha} p_{\alpha} \left( \mathcal{N}^{\otimes n} (\pi_{\alpha}^{n}) \otimes \pi_{\alpha}^{n} \right).$$
(29)

$$\frac{1}{|\mathcal{B}||\mathcal{G}|} \sum_{g^{a} \in \mathcal{G}} \sum_{b^{a} \in \mathcal{B}} \sqrt{p_{\alpha} p_{\alpha'}} (\mathcal{N}^{\otimes n} \otimes I) \left[ (U_{g^{a}, b^{a}} \otimes I) |\Phi_{\alpha}\rangle \langle \Phi_{\alpha'}| (U_{g^{a}, b^{a}}^{\dagger} \otimes I) \right]$$

$$= \frac{1}{d_{\alpha}^{2} d_{\alpha'}^{2}} \sqrt{p_{\alpha} p_{\alpha'}} \sum_{b_{\alpha} b_{\alpha'}} \frac{(-1)^{b_{\alpha} + b_{\alpha'}}}{4} \left\{ \sum_{g_{\alpha} g_{\alpha'}} (\mathcal{N}^{\otimes n} \otimes I) \left[ (U_{g_{\alpha}} \otimes I) |\Phi_{\alpha}\rangle \langle \Phi_{\alpha'}| (U_{g_{\alpha'}}^{\dagger} \otimes I) \right] \right\}$$

$$= 0 \tag{31}$$

The first step is to replace the entanglement used by classical common randomness. Observe that the encoding operations  $U_{s^a}$  all satisfy

$$\Delta^{\otimes n} \circ U_{s^a} = \Delta^{\otimes n} \circ U_{s^a} \circ \Delta^{\otimes n}.$$

This follows from the corresponding property of the generalized Pauli operators (14). Hence for cq channels  $\mathcal{N}$ 

$$\sigma_{f(k)} = [(\mathcal{N}^{\otimes n} \circ \mathcal{E}_k) \otimes I](\varphi^{\otimes n})$$

$$= [(\mathcal{N}^{\otimes n} \circ \mathcal{E}_k \circ \Delta^{\otimes n}) \otimes I](\varphi^{\otimes n})$$

$$= [(\mathcal{N}^{\otimes n} \circ \mathcal{E}_k) \otimes I](\overline{\varphi}^{\otimes n}),$$
(40)

where

$$\overline{\varphi} = (\Delta \otimes I)\varphi = \sum_{x} p_{x} |x\rangle\langle x| \otimes |x\rangle\langle x|$$

is the dephased version of  $\varphi$ . The state  $\overline{\varphi}^{\otimes n}$  can be constructed from classical common randomness like that used in Shannon's original coding theorem.

The second step is showing that common randomness is not needed. The argument parallels the derandomization step from the proof of the packing lemma (Appendix A). We have thus recovered the HSW coding theorem.

The benefit of the above proof is its close analogy to Shannon's joint typicality decoding. We only made use of typical subspaces and not conditionally typical subspaces.

#### C. Multiple-Access Channel

We turn to the communication scenario with two senders, Alice and Bob, and one receiver, Charlie. They are connected by a large number n of uses of the *multiple-access* quantum channel  $\mathcal{M}: A'B' \to C$ . Alice and Bob control the channel input systems A' and B', respectively. Charlie has access to the channel output C. Each sender also shares unlimited entanglement with the receiver, in the form of arbitrary pure states  $|\Gamma_1\rangle^{AC_A}$  and  $|\Gamma_2\rangle^{BC_B}$ . The system A is held by Alice, B by Bob, and  $C_AC_B$  by Charlie. Based on her message Alice performs a quantum operation on her share of the entanglement, and likewise for Bob. These are then sent through the quantum channel. Charlie performs a decoding

measurement on the channel output plus his share of the entanglement. Now both Alice's and Bob's communication rates need to be optimized.

We formalize the above information processing task. Define an  $(n, R_1, R_2, \epsilon)$  entanglement-assisted code by

- two sets of encoding cptp maps:  $\{\mathcal{E}^1_k\}_{k\in[2^{nR_1}]}$  taking A to  $A'^n$  for Alice, and  $\{\mathcal{E}^2_l\}_{l\in[2^{nR_2}]}$  taking B to  $B'^n$  for Bob:
- Charlie's decoding POVM  $\{\Lambda_{k,l}\}_{k\in[2^{nR_1}],l\in[2^{nR_2}]}$  on  $C_AC_BC$ ,

such that

$$\operatorname{Tr} \left\{ \Lambda_{k,l} [((\mathcal{M}^{\otimes n} \circ (\mathcal{E}_k^1 \otimes \mathcal{E}_l^2)) \otimes I^{C_A C_B}) (\Gamma_1^{A C_A} \otimes \Gamma_2^{B C_B})] \right\}$$

$$\geq 1 - \epsilon. \quad (41)$$

We say that  $(R_1,R_2)$  is an achievable rate pair if for all  $\epsilon>0,\delta>0$  and sufficiently large n there exists an  $(n,R_1-\delta,R_2-\delta,\epsilon)$  entanglement-assisted code. The entanglement-assisted capacity region  $C_E(\mathcal{M})$  is defined to be the closure of the set of all achievable rate pairs.

Theorem 6: Consider a quantum multiple access channel  $\mathcal{M}:A'B'\to C.$  For some states  $\rho_1^{A'}$  and  $\rho_2^{B'}$  define

$$\theta^{ABC} = (I^{AB} \otimes \mathcal{M})(\varphi_1^{AA'} \otimes \varphi_2^{BB'}) \tag{42}$$

where  $|\varphi_1\rangle^{AA'}$  and  $|\varphi_2\rangle^{BB'}$  are purifications of  $\rho_1^{A'}$  and  $\rho_2^{B'}$  respectively. Define the two-dimensional region  $C_E(\mathcal{M},\rho_1,\rho_2)$ , shown in Fig. 2, by the set of pairs of nonnegative rates  $(R_1,R_2)$  satisfying

$$R_1 \le I(A; C|B)_{\theta}$$

$$R_2 \le I(B; C|A)_{\theta}$$

$$R_1 + R_2 \le I(AB; C)_{\theta}.$$
(43)

Define  $\widetilde{C}_E(\mathcal{M})$  as the union of the  $C_E(\mathcal{M}, \rho_1, \rho_2)$  regions taken over all states  $\rho_1, \rho_2$ . Then the entanglement-assisted capacity region  $C_E(\mathcal{M})$  is given by the regularized expression

$$C_E(\mathcal{M}) = \overline{\bigcup_{n=1}^{\infty} \frac{1}{n} \widetilde{C}_E(\mathcal{M}^{\otimes n})}$$
(44)

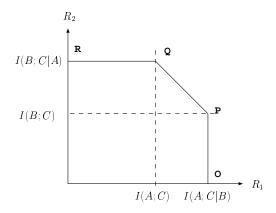


Fig. 2. Capacity region of multiple access channel for fixed input states  $\rho_1$ and  $\rho_2$ 

where the bar indicates taking closure. There is an additional single-letter upper bound on the sum rate

$$R_1 + R_2 \le \max_{\rho_1, \rho_2} I(AB; C)_{\theta}.$$
 (45)

*Proof:* (direct coding theorem) Let the entanglement be given in a tensor power form, as in Theorem 1. Define a  $[n, R_1, R_2, \rho_1, \rho_2, \epsilon]$  entanglement-assisted code as a special case of an  $(n, R_1, R_2, \epsilon)$  code: specify  $\Gamma_1 = \phi_1^{\otimes n}$  and  $\Gamma_2 = \phi_2^{\otimes n}$ , and identify  $A := A'^n$  and  $B := B'^n$ .

To show the achievability of every rate pair  $(R_1, R_2)$  in the convex hull of the  $C_E(\mathcal{M}, \rho_1, \rho_2)$ , it suffices to show that the corner points are achievable. Once we show that, the non-corner points can be achieved by time-sharing (see, e.g., [24]). Consider the corner point O. For all  $\epsilon > 0, \delta > 0$ and n sufficiently large, we show below that there exists a  $[n, I(A; C)_{\theta} - \delta, I(B; C|A)_{\theta} - \delta, \rho_1, \rho_2, \epsilon]$  entanglementassisted code  $(\mathcal{E}^1, \mathcal{E}^2, \mathcal{D})$ .

The point Q corresponds to the maximum rate that at which Alice can send as long as Bob sends at his maximum rate. This is the rate that is achieved when Bob's input is considered as noise for the channel from Alice to Charlie. From the two party direct coding theorem, Alice can send at a rate I(A; C) and Charlie can decode the message with arbitrarily low probability. Charlie then knows which encoding operation Alice used and can subtract its effect from the channel. Therefore, Bob can achieve the rate I(B; C|A). This outlines the proof of the achievability of point Q.

Define the channel  $\mathcal{N}_1: A' \to C$  by

$$\mathcal{N}_1: \omega \mapsto \mathcal{M}(\omega \otimes \rho_2).$$

 $\mathcal{N}_1^{\otimes n}$  is the effective channel from Alice to Charlie when Bob's input to  $\mathcal{M}^{\otimes n}$  is  $\rho_2^{\otimes n}$ . Define  $\hat{\mathcal{N}}_1:A'\to C_BC$  by

$$\hat{\mathcal{N}}_1: \omega \mapsto (I \otimes \mathcal{M})(\omega \otimes \varphi_2).$$

Observe that  $\hat{\mathcal{N}}_1$  is an extension of  $\mathcal{N}_1$ . Hence it is a restriction

Define the channel  $\mathcal{N}_2: B' \to C_A C$  by

$$\mathcal{N}_2: \omega \mapsto (I \otimes \mathcal{M})(\varphi_1 \otimes \omega).$$

 $\mathcal{N}_2^{\otimes n}$  is effective the channel from Bob to Charlie if Alice simply inputs the A' part of the entangled state/purification  $(\varphi_1^{A'C_A})^{\otimes n}$  without encoding.

Fix  $\epsilon > 0, \delta > 0$ . Define  $R_1 = I(A; C)_{\theta} - \delta$ and  $R_2 = I(B; C|A)_{\theta} - \delta$ , with  $\theta$  defined in (42). By Theorem 1, for sufficiently large n there exists an  $[n, R_1, \rho_1, \epsilon]$  entanglement-assisted code  $(\mathcal{E}^1, \mathcal{D}^1)$  for  $\mathcal{N}_1$  and an  $[n, R_2, \rho_2, \epsilon]$  entanglement-assisted code  $(\mathcal{E}^2, \mathcal{D}^2)$  for  $\mathcal{N}_2$ such that for all  $k \in [2^{nR_1}], l \in [2^{nR_2}],$ 

i) Tr 
$$\{[\mathcal{D}_k^1 \circ ((\mathcal{N}_1^{\otimes n} \circ \mathcal{E}_k^1) \otimes I^{C_A})](\varphi_1^{\otimes n})\} \ge 1 - \epsilon;$$

$$\left\| \left[ (\mathcal{D}^1 \otimes I) \circ ((\hat{\mathcal{N}}_1^{\otimes n} \circ \mathcal{E}_k^1) \otimes I^{C_A}) - (\hat{\mathcal{N}}_1^{\otimes n} \otimes I^{C_A}) \right] (\varphi_1^{\otimes n}) \right\|$$

iii) Tr 
$$\{ [\mathcal{D}_{I}^{2} \circ ((\mathcal{N}_{2}^{\otimes n} \circ \mathcal{E}_{I}^{2}) \otimes I^{C_{B}})](\varphi_{2}^{\otimes n}) \} > 1 - \epsilon$$

iii)  $\operatorname{Tr}\left\{\left[\mathcal{D}_l^2\circ((\mathcal{N}_2^{\otimes n}\circ\mathcal{E}_l^2)\otimes I^{C_B})\right](\varphi_2^{\otimes n})\right\}\geq 1-\epsilon;$  iv) the encoded density operator satisfies  $\mathcal{E}_l^2(\rho_2^{\otimes n})=\rho_2^{\otimes n}.$ 

We now define our code for the multiple access channel  $\mathcal{M}$ . Alice and Bob encode according to  $\{\mathcal{E}_k^1\}$  and  $\{\mathcal{E}_l^2\}$ , respectively. Define the instrument  $\mathbf{D} = \{\mathcal{D}_{k,l}\}$  on  $CC_AC_B$ 

$$\mathcal{D}_{k,l}=\mathcal{D}_l^2\circ(\mathcal{D}_k^1\otimes I^{C_B}).$$

Then Charlie's decoding POVM  $\{\Lambda_{k,l}\}$  is the restriction of  $\{\mathcal{D}_{k,l}\}$ . Examining the success probability of decoding Alice's message k:

$$\operatorname{Tr}\left\{ (\mathcal{D}_{k}^{1} \otimes I^{C_{B}}) \circ ((\mathcal{M}^{\otimes n} \circ (\mathcal{E}_{k}^{1} \otimes \mathcal{E}_{l}^{2})) \otimes I^{C_{A}C_{B}}) (\varphi_{1}^{\otimes n} \otimes \varphi_{2}^{\otimes n}) \right\}$$

$$= \operatorname{Tr}\left\{ \mathcal{D}_{k}^{1} \circ ((\mathcal{M}^{\otimes n} \circ (\mathcal{E}_{k}^{1} \otimes \mathcal{E}_{l}^{2})) \otimes I^{C_{A}}) (\varphi_{1}^{\otimes n} \otimes \rho_{2}^{\otimes n}) \right\}$$

$$= \operatorname{Tr}\left\{ \mathcal{D}_{k}^{1} \circ ((\mathcal{M}^{\otimes n} \circ (\mathcal{E}_{k}^{1} \otimes I^{B'^{n}})) \otimes I^{C_{A}}) (\varphi_{1}^{\otimes n} \otimes \rho_{2}^{\otimes n}) \right\}$$

$$= \operatorname{Tr}\left\{ [\mathcal{D}_{k}^{1} \circ ((\mathcal{N}_{1}^{\otimes n} \circ \mathcal{E}_{k}^{1}) \otimes I)] (\varphi_{1}^{\otimes n}) \right\}$$

$$\geq 1 - \epsilon. \tag{46}$$

The second equality follows from iv) and the third from i). Next examining the success probability of decoding Bob's message l: Rewrite ii) in terms of  $\mathcal{M}$ :

$$\|[(\mathcal{D}^{1} \otimes I^{C_{B}}) \circ ((\mathcal{M}^{\otimes n} \circ (\mathcal{E}_{k}^{1} \otimes I^{B'^{n}})) \otimes I^{C_{A}C_{B}}) - (\mathcal{M}^{\otimes n} \otimes I^{C_{A}C_{B}})](\varphi_{1}^{\otimes n} \otimes \varphi_{2}^{\otimes n})\| \leq \epsilon;$$

Since  $\mathcal{E}_l^2$  is unitary and satisfies iv),

$$\begin{aligned} & \| [ (\mathcal{D}^1 \otimes I^{C_B}) \circ ((\mathcal{M}^{\otimes n} \circ (\mathcal{E}_k^1 \otimes \mathcal{E}_l^2)) \otimes I^{C_A C_B}) \\ & - ((\mathcal{M}^{\otimes n} \circ (I^{A'^n} \otimes \mathcal{E}_l^2)) \otimes I^{C_A C_B}) ] (\varphi_1^{\otimes n} \otimes \varphi_2^{\otimes n}) \| \leq \epsilon; \end{aligned}$$

Rewrite iii) in terms of  $\mathcal{M}$ 

$$\operatorname{Tr} \{ [\mathcal{D}_{l}^{2} \circ ((\mathcal{M}^{\otimes n} \circ (I^{A'^{n}} \otimes \mathcal{E}_{l}^{2})) \otimes I^{C_{A}C_{B}})] (\varphi_{1}^{\otimes n} \otimes \varphi_{2}^{\otimes n}) \}$$

$$> 1 - \epsilon.$$

Define

$$\Omega^{CC_AC_B} = (\mathcal{D}^1 \otimes I^{C_B}) \circ ((\mathcal{M}^{\otimes n} \circ (\mathcal{E}_k^1 \otimes \mathcal{E}_l^2)) \otimes I^{C_AC_B}) (\varphi_1^{\otimes n} \otimes \varphi_2^{\otimes n})$$

Hence

$$\operatorname{Tr}\left[\mathcal{D}_{l}^{2} \Omega^{CC_{A}C_{B}}\right] \geq 1 - 2\epsilon.$$

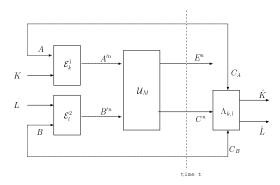


Fig. 3. A general protocol for multiple-access entanglement-assisted classical communication

Now (41) follows. This concludes the achievability of point O.

Corner point P can be shown in the same manner. Corner point R corresponds to the maximum rate achievable from Bob to Charlie when Alice is not sending any information. The proof is obvious since we can assume that Alice is throwing the same state into the channel all the time. The corner point O follows from the same reasoning. This concludes the proof of direct coding theorem.

**Remark**. The entanglement assistance may be phrased in terms of tensor powers of ebit states  $|\Phi_+\rangle=\frac{1}{\sqrt{2}}(|0\rangle|0\rangle+|1\rangle|1\rangle)$  instead of the arbitrary  $|\Gamma_1\rangle$  and  $|\Gamma_2\rangle$ . The protocol achieving the corner points of the region  $C_E(\mathcal{M},\rho_1,\rho_2)$  uses  $|\Gamma_1\rangle=|\phi_1\rangle^{\otimes n}$  and  $|\Gamma_2\rangle=|\phi_2\rangle^{\otimes n}$ . By entanglement dilution [25],  $|\Gamma_1\rangle$  may be asymptotically obtained from an ebit rate of  $E_1=H(A)_\theta$  shared between Alice and Charlie. Likewise  $|\Gamma_2\rangle$  may be asymptotically obtained from an ebit rate of  $E_2=H(B)_\theta$  shared between Bob and Charlie. Entanglement dilution additionally requires an arbitrarily small rate of classical communication. This resource is obtained by applying the HSW theorem to an arbitrarily small fraction of the n channels  $\mathcal{M}$ . Doing so has no effect on the capacity region.

Proof: (converse) Start with some  $(n,R_1,R_2,\epsilon)$  entanglement-assisted code (see Fig. 3). Assume Alice's message k and Bob's message l are picked according to the uniform distributions on  $[2^{nR_1}]$  and  $[2^{nR_2}]$ , respectively. These correspond to random variables K and L. Alice performs the encoding operation  $\mathcal{E}^1_k$  on the A part of  $|\Gamma_1\rangle^{AC_A}$  conditioned on K=k. Bob performs the encoding operation  $\mathcal{E}^1_l$  on the B part of  $|\Gamma_2\rangle^{BC_B}$  conditioned on L=l. The output of  $\mathcal{E}^1_k\otimes\mathcal{E}^2_l$  is sent through the multiple access channel  $\mathcal{M}^{\otimes n}$  just after time  $t_0$ . The channel output  $C^n$  is acquired by Charlie at time t. Charlie performs a POVM on the channel output and his part of the entanglement  $C_AC_B$ . The measurement outcome is a random variable  $W=(\hat{K},\hat{L})$ . By the condition (41),

$$\Pr\{K \neq \hat{K} \text{ and } L \neq \hat{L}\} < \epsilon. \tag{47}$$

The protocol ends at time  $t_f$ . We first obtain an upper bound on the sum rate  $R_1 + R_2$ . At this time

$$n(R_1 + R_2) = H(KL) \le I(KL; \hat{K}\hat{L}) + n\eta(n, \epsilon),$$
 (48)

where the function  $\eta(n,\epsilon)$  tends to 0 as  $\epsilon$  tends to 0 and n tends to infinity. The inequality is standard in classical information theory [2]. It is obtained by applying Fano's inequality [2] to (47). Denote the state of the system at time t by

$$\omega^{KLC_AC_BC^nE^n} = (I^{KLC_AC_B} \otimes U_{\mathcal{M}}^{\otimes n})(\xi_1 \otimes \xi_2),$$
  
$$\xi_1^{A'^nKC_A} = 2^{-nR_1} \sum_k |k\rangle\langle k|^K \otimes (\mathcal{E}_k^1 \otimes I^{C_A})(\Gamma_1^{AC_A}),$$
  
$$\xi_2^{B'^nLC_B} = 2^{-nR_2} \sum_l |l\rangle\langle l|^L \otimes (\mathcal{E}_l^2 \otimes I^{C_B})(\Gamma_2^{BC_B}).$$

Denote by  $A^n$  the system which purifies the restriction of the  $A'^n$  parts of the state  $\xi_1$  at time  $t_0$ . Then  $A^n$  contains K and  $C_A$  as subsystems. Define  $B^n$  in a similar fashion.

The Holevo bound reads

$$I(KL; \hat{K}\hat{L}) \le I(KL; C_A C_B C^n)_{\omega}. \tag{49}$$

The entropic quantities below refer to the state  $\omega$ 

$$I(KL; C_{A}C_{B}C^{n})$$

$$= I(C^{n}; C_{A}C_{B}KL) - I(C_{A}C_{B}; C^{n}) + I(KL; C_{A}C_{B})$$

$$\leq I(C^{n}; C_{A}C_{B}KL)$$

$$\leq H(C^{n}) + H(C^{n}|E^{n})$$

$$= H(C^{n}) - H(C^{n}|A^{n}B^{n})$$

$$= I(C^{n}; A^{n}B^{n}).$$
(50)

The first inequality follows from  $I(KL; C_AC_B) = 0$  and  $I(C_AC_B; C^n) \ge 0$ . The second inequality holds because of Lemma 4. The second equality is from Lemma 3.

Putting everything together gives

$$R_1 + R_2 \le \eta(n, \epsilon) + \frac{1}{n} I(C^n; A^n B^n).$$
 (51)

Observe that

$$\frac{1}{n}H(C^{n}) + H(C^{n}|E^{n})$$

$$\leq \frac{1}{n}\sum_{i}[H(C_{i}) + H(C_{i}|E_{i})]$$

$$\leq \max_{\rho_{1},\rho_{2}}[H(C)_{\theta} + H(C|E)_{\theta}]$$

$$= \max_{\rho_{1},\rho_{2}}[H(C)_{\theta} - H(C|AB)_{\theta}]$$

$$= \max_{\rho_{1},\rho_{2}}I(AB;C)_{\theta}.$$
(52)

The state  $\theta$  is defined in (42).

An upper bound on Alice's rate  $R_1$  is obtained in a similar fashion. Equations

$$nR_1 = H(K) \le I(K; \hat{K}) + n\eta(n, \epsilon), \tag{53}$$

and

$$I(K; \hat{K}) \le I(K; C_A C_B C^n)_{\omega} \tag{54}$$

are obtained as above. With respect to  $\omega$ :

$$I(K; C_{A}C_{B}C^{n})$$

$$= I(C_{B}C^{n}; C_{A}K) - I(C_{A}; C_{B}C^{n}) + I(K; C_{A})$$

$$\leq I(C_{B}C^{n}; C_{A}K)$$

$$\leq I(B^{n}C^{n}; C_{A}K)$$

$$\leq H(B^{n}C^{n}) + H(B^{n}C^{n}|E^{n})$$

$$= H(B^{n}C^{n}) - H(B^{n}C^{n}|A^{n})$$

$$= I(A^{n}; B^{n}C^{n})$$

$$= I(A^{n}; C^{n}|B^{n}).$$
(55)

Hence

$$R_1 \le \eta(n,\epsilon) + \frac{1}{n} I(A^n; C^n | B^n). \tag{56}$$

By the same argument

$$R_2 \le \eta(n,\epsilon) + \frac{1}{n} I(B^n; C^n | A^n). \tag{57}$$

The reason that we do not single-letterize the rates  $R_1$  and  $R_2$  using arguments in (51) is due to the definition of systems  $A^n$  and  $B^n$ , which contain the classical information K and L as subsystems, respectively. At the same time, the channel output  $C^n$  also contains information regarding K and L. Therefore, it is not trivial that chain rule is applicable to systems  $B^nC^n$  (likewise  $A^nC^n$ ).

Now assume that  $(R_1,R_2)$  is achievable. This means that for all  $\epsilon>0,\delta>0$ , there exists an  $(n,R_1-\delta,R_2-\delta,\epsilon)$  code, and hence

$$R_{1} \leq \eta(n,\epsilon) + \delta + \frac{1}{n}I(A^{n};C^{n}|B^{n})$$

$$R_{2} \leq \eta(n,\epsilon) + \delta + \frac{1}{n}I(B^{n};C^{n}|A^{n})$$

$$R_{1} + R_{2} \leq \eta(n,\epsilon) + 2\delta + \frac{1}{n}I(C^{n};A^{n}B^{n}).$$
(58)

It follows that  $(R_1,R_2)$  is in the  $\nu(n,\epsilon,\delta)$  neighborhood of the  $\frac{1}{n}\widetilde{C}_E(\mathcal{M}^{\otimes n})$  region, with  $\nu(n,\epsilon,\delta)\to 0$  as  $\epsilon\to 0,\delta\to 0,n\to\infty$ . Hence  $(R_1,R_2)$  is in  $C_E(\mathcal{M})$ , concluding the proof of the converse.

#### IV. THE COLLECTIVE PHASE-FLIP CHANNEL EXAMPLE

Consider the case that  $|A'| = |B'| = d \ge 2$ . The collective phase-flip channel [14]  $\mathcal{M}_p: A'B' \to C$  is defined as

$$\mathcal{M}_p(\rho) = \sum_{k=0}^{d-1} p_k(\hat{Z}(k) \otimes \hat{Z}(k)) \rho(\hat{Z}(k) \otimes \hat{Z}(k))^{\dagger}$$
 (59)

where  $\hat{Z}(k)$  is the generalized Pauli phase operator from (14). We will show that the capacity region for the multiple access phase-flip channel  $\mathcal{M}_p$  assisted by entanglement is the collection of all pairs of nonnegative rates  $(R_1,R_2)$  which satisfy

$$R_1 \le 2 \log d$$

$$R_2 \le 2 \log d$$

$$R_1 + R_2 \le 4 \log d - H(p).$$

$$(60)$$

*Proof:* First we show that (60) is precisely the region  $C_E(\mathcal{M}, \pi, \pi)$ , proving achievability. The corresponding  $\theta$  state is

$$\theta^{ABC} = (I^{AB} \otimes \mathcal{M}_p)(\Phi^{AA'} \otimes \Phi^{BB'})$$

where  $|\Phi\rangle$  is the maximally entangled state (15). It is easy to see that

$$H(A) = H(B) = H(\pi) = \log d$$

$$H(AC) = H(BC) = \log d + H(p)$$

$$H(ABC) = H(p).$$
(61)

Hence we reach our conclusion

$$I(A; C|B) = 2 \log d$$
  
 $I(B; C|A) = 2 \log d$  (62)  
 $I(AB; C) = 4 \log d - H(p)$ .

It remains to show that (60) is an upper bound on the capacity region. It is clear from (43) that  $R_1 \leq 2H(A)$  and  $R_2 \leq 2H(B)$ . Hence the first two inequalities in (60). The third makes use of the single-letter upper bound (45) on  $R_1 + R_2$ . It suffices to show that

$$\max_{\rho} I(AB; C)_{\theta} = 4\log d - H(p), \tag{63}$$

where

$$\theta^{ABC} = (I^{AB} \otimes \mathcal{M})(\varphi^{ABA'B'}), \tag{64}$$

and  $\varphi^{ABA'B'}$  is a purification of  $\rho^{A'B'}$ . We need three ingredients. The first is that the maximum in (63) is attained for states  $\rho^{A'B'}$  diagonal in the  $\{|jl\rangle\}$  basis (see Appendix C for a proof of this fact). Define a Stinespring dilation  $U_{\mathcal{M}_p}: A'B' \to CE$  of  $\mathcal{M}_p$  as

$$U_{\mathcal{M}_p} = \sum_{il} |jl\rangle^C |\phi_{jl}\rangle^E \langle jl|^{A'B'} \tag{65}$$

where

$$|\phi_{jl}\rangle^E = \sum_{k=0}^{d-1} \sqrt{p_k} |k\rangle e^{i2\pi k(j+l)/d}.$$

By the results of Appendix C

$$I(AB; C)_{\theta} = 2H(\{r_{jl}\}) - H(\sum_{jl} r_{jl}\phi_{jl}),$$
 (66)

where  $\rho = \sum_{jl} r_{jl} |jl\rangle\langle jl|$ .

The second ingredient is that  $I(AB; C)_{\theta}$  is a concave function of  $\rho$  and hence has a unique local optimum. This is because for *degradable channels* [24] such as  $\mathcal{M}_p$ , the coherent information I(AB)C) := I(AB; C) - H(A) is a concave function of input density matrix  $\rho$  [14]. Since H(A) is also concave we conclude that I(AB; C) is concave.

The third ingredient is to use the method of Lagrange multipliers to find a local optimum for  $I(AB; C)_{\theta}$ . We need to optimize

$$f(\{r_{jl}\}) = 2H(\{r_{jl}\}) - H(\sum_{jl} r_{jl}\phi_{jl}) - \lambda \sum_{jl} r_{jl},$$

 $^1$  we have already shown that this maximum is achieved for the product state  $\rho^{A'B'}=\pi^{A'}\otimes\pi^{B'}.$ 

with Lagrange multiplier  $\lambda$ . Differentiating with respect to the  $r_{jl}$  gives  $d^2$  simultaneous equations. By inspection,  $r_{jl} = 1/d^2$  is a solution to this system of equations. The second ingredient ensures that this is in fact the global maximum. Thus

$$\max_{\rho} I(AB; C)_{\theta} = 2H(\{\frac{1}{d^2}\}) - H(\frac{1}{d^2} \sum_{m} \phi_m)$$
$$= 4 \log d - H(p)$$

as claimed.

#### V. A HIERARCHY OF QMAC RESOURCE INEQUALITIES

In this section we phrase our result using the theory of resource inequalities developed in [23]. The multiple access channel  $\mathcal{M}:A'B'\to C$  assisted by some rate  $E_1$  of ebits shared between Alice and Charlie and some rate  $E_2$  of ebits shared between Bob and Charlie, was used to enable a rate  $R_1$  bits of communication between Alice and Charlie and a rate  $R_2$  bits of communication between Bob and Charlie. This is written as

$$\langle \mathcal{M} \rangle + E_1 [q \, q]_{AC} + E_2 [q \, q]_{BC}$$
  
  $\geq R_1 [c \to c]_{AC} + R_2 [c \to c]_{BC}.$ 

Without accounting for entanglement consumption (i.e. setting  $E_1 = E_2 = \infty$ ) the above resource inequality holds iff  $(R_1, R_2) \in C_E(\mathcal{M})$ , with  $C_E(\mathcal{M})$  given by Theorem 6. The "if" direction, i.e. the direct coding theorem, followed from the "corner points"

$$\langle \mathcal{M} \rangle + H(A) [q \, q]_{AC} + H(B) [q \, q]_{BC}$$
  

$$\geq I(A; C) [c \rightarrow c]_{AC} + I(B; CA) [c \rightarrow c]_{BC} \quad (67)$$

and

$$\langle \mathcal{M} \rangle + H(A) [q q]_{AC} + H(B) [q q]_{BC}$$
  

$$\geq I(A; CB) [c \rightarrow c]_{AC} + I(B; C) [c \rightarrow c]_{BC}. \quad (68)$$

All the entropic quantities are defined relative to the state  $\theta^{ABC}$  defined in (42).

Just as in the single user case (cf. rule O in [23]), the protocol can be made coherent, replacing  $[c \to c]$  by  $\frac{1}{2}([q\,q]+[q\to q])$ . Canceling terms on both sides gives "father" protocols for the QMAC

$$\langle \mathcal{M} \rangle + \frac{1}{2} I(A; BE) [q \, q]_{AC} + \frac{1}{2} I(B; E) [q \, q]_{BC}$$
  
 $\geq \frac{1}{2} I(A; C) [q \to q]_{AC} + \frac{1}{2} I(B; CA) [q \to q]_{BC}$  (69)

and

$$\langle \mathcal{M} \rangle + \frac{1}{2} I(A; E) [q \, q]_{AC} + \frac{1}{2} I(B; AE) [q \, q]_{BC}$$
  
 $\geq \frac{1}{2} I(A; CB) [q \to q]_{AC} + \frac{1}{2} I(B; C) [q \to q]_{BC}, \quad (70)$ 

where the entropic quantities are now defined with respect to a purification  $\theta^{ABCE}$  of  $\theta^{ABC}$ .

Applying  $[q \rightarrow q] \ge [qq]$  to the above equations gives

$$\langle \mathcal{M} \rangle \ge I(A \rangle C) [q \to q]_{AC} + \frac{1}{2} I(B \rangle CA) [q \to q]_{BC}$$
 (71)

and

$$\langle \mathcal{M} \rangle \ge I(A \rangle BC) [q \to q]_{AC} + \frac{1}{2} I(B \rangle C) [q \to q]_{BC}.$$
 (72)

These equations are of the form

$$\langle \mathcal{M} \rangle \ge Q_1 [q \to q]_{AC} + Q_2 [q \to q]_{BC}.$$
 (73)

The optimal set of pairs  $(Q_1, Q_2)$  satisfying (73) was found in [14], [15]. Equations (71) and (72) recover the "corner points" of the corresponding capacity region.

Coherifying only Bob's resources in equation (67) gives

$$\langle \mathcal{M} \rangle + H(A) [q \, q]_{AC}$$
  
  $\geq I(A; C) [c \to c]_{AC} + I(B \rangle CA) [q \to q]_{BC}.$ 

Consider  $\mathcal{M}$  of a special  $\{cq \to q\}$  form in which Alice's input is dephased before being sent though the channel. The arguments from Section III-B apply here to show that the Alice-Charlie entanglement is not needed. Thus we recover another coding theorem proven in [14] which characterizes the pairs  $(R_1, Q_2)$  for which

$$\langle \mathcal{M} \rangle \ge R_1 [c \to c]_{AC} + Q_2 [q \to q]_{BC}.$$

We can also recover the result of Winter [13] which solves

$$\langle \mathcal{M} \rangle \ge R_1 [c \to c]_{AC} + R_2 [c \to c]_{BC}.$$

for  $\{cc \rightarrow q\}$  channels  $\mathcal{M}$ . We just apply the argument from Section III-B to remove the need for any entanglement assistance.

Ultimately we would like to solve

$$\langle \mathcal{M} \rangle \ge Q_1 [q \to q]_{AC} + E_1 [q \, q]_{AC} + R_1 [c \to c]_{AC} + Q_2 [q \to q]_{BC} + E_2 [q \, q]_{BC} + R_2 [c \to c]_{BC},$$

where the 6 rates may be positive or negative. The single user case  $Q_2=E_2=R_2=0$  was solved in [26].

# VI. CONCLUSION

We derived a regularized formula for the entanglement-assisted capacity region for quantum multiple access channels. This expression parallels the capacity region for classical multiple access channels. We leave it as an open problem to single-letterize the above capacity region. We do not know if the regularization in our main theorem is actually necessary. Indications that it might not be are the successful single-letterization of the two-user entanglement-assisted capacity in [11] which we have used to obtain the single-letter bound on the rate-sum above, and the fact that the regularization is not necessary in the classical case.

Though the issue with more than 2 inputs was not addressed, we expect it to be an easy extension. Suppose we have a QMAC  $\mathcal{M}$  with s senders and 1 receiver such that  $\mathcal{M}:A_1A_2\cdots A_s\to B$ . We conjecture the following statement to be true [13]:

The entanglement-assisted capacity region of the quantum multiple access channel  $\mathcal{M}$  is the regularized version of the convex closure of all nonnegative  $\{R_1, \dots, R_s\}$  satisfying

$$\sum_{i \in I} R_i \le I(A[J]; B|A[J^c]) \quad \forall J \subset [s],$$

where  $A[J] = \{A_i | i \in [J]\} \text{ and } [J^c] = [s] \setminus J.$ 

The difficult problem would be to consider the quantum multiway channel which has s senders and r receivers. We believe a different approach might be needed.

# APPENDIX A PROOF OF PACKING LEMMA

We need the following lemma from [27].

Lemma 7 (Hayashi, Nagaoka): For any operators  $0 \le S \le I$  and  $T \ge 0$ , we have

$$I - \sqrt{S+T}^{-1}S\sqrt{S+T}^{-1} \le 2(I-S) + 4T.$$

We are now ready to prove the packing lemma, along lines suggested by the work [27].

*Proof:* Let  $X^N$  denote a sequence of random variables  $X_1, X_2, \ldots, X_N$ , where each random variable  $X_k$  takes values from  $\mathcal{S}$  and is distributed according to  $\lambda$ . Set  $f(k) = X_k$ . Each random code  $C = \{\sigma_{x_k}\}_{k \in [N]}$  is generated according to  $X_k = x_k$ . Define  $p_e(k)$  to be the probability of error for a single codeword  $\sigma_{x_k}$ :

$$p_e(k) = \operatorname{Tr} \sigma_{x_k}(I - \Lambda_k),$$

where the POVM elements  $\{\Lambda_k\}$  are constructed by the socalled *square root measurement*[6], [7]

$$\Lambda_k = \Bigl(\sum_{l=1}^N \Upsilon_{x_l}\Bigr)^{-rac{1}{2}} \Upsilon_{x_k} \Bigl(\sum_{l=1}^N \Upsilon_{x_l}\Bigr)^{-rac{1}{2}}$$

with

$$\Upsilon_m = \Pi \Pi_m \Pi$$
.

Define  $p_e(C)$  to be the average probability of error, averaged over all codewords in C:

$$p_e(C) = \frac{1}{N} \sum_{k=1}^{N} p_e(k).$$

Define  $\overline{p}_e$  to be the average probability of error, averaged over all possible random codes C to be:

$$\overline{p}_e = \mathbb{E}_{X^N} \left[ p_e(C) \right].$$

The idea here is that if the average probability of error  $\overline{p}_e$  is small enough, we can then show the existence of at least one good code. In what follows, we will first show that  $\overline{p}_e \leq \epsilon'$  for some  $\epsilon' \to 0$  when  $n \to \infty$ .

Invoking Lemma 7, we can now place an upper bound on  $p_e(C)$ :

$$p_e(C) \le \frac{1}{N} \sum_{k=1}^{N} \left[ 2(1 - \operatorname{Tr} \sigma_{x_k} \Upsilon_{x_k}) + 4 \sum_{l \ne k} \operatorname{Tr} \sigma_{x_k} \Upsilon_{x_l} \right].$$
(74)

The gentle operator lemma in [21] and property (20) give

$$\|\Pi \sigma_m \Pi - \sigma_m\| \le \sqrt{8\epsilon}. \tag{75}$$

By property (19) and (75)

$$\operatorname{Tr} \sigma_{m} \Upsilon_{m} \geq \operatorname{Tr} \sigma_{m} \Pi_{m} - \|\Pi \sigma_{m} \Pi - \sigma_{m}\| \\ \geq 1 - \epsilon - \sqrt{8\epsilon}. \tag{76}$$

For  $k \neq l$ , the random variables  $X_k$  and  $X_l$  are independent. Thus

$$\mathbb{E}_{X^{N}} \left[ \operatorname{Tr} \sigma_{X_{k}} \Upsilon_{X_{l}} \right] = \operatorname{Tr} \left( \Pi \mathbb{E} \sigma_{X_{k}} \Pi \ \mathbb{E} \Pi_{X_{l}} \right)$$

$$\leq D^{-1} \mathbb{E} \operatorname{Tr} \Pi \Pi_{X_{l}}$$

$$\leq d/D. \tag{77}$$

The first inequality follows from  $\mathbb{E} \sigma_{X_k} = \sigma$  and property (21). The second follows from  $\Pi \leq I$  and property (22). Taking the expectation of (74), and incorporating (76) and (77) gives

$$\overline{p}_e \le 2(\epsilon + \sqrt{8\epsilon}) + 4(N-1)d/D, 
\le 2(\epsilon + \sqrt{8\epsilon}) + 4Nd/D 
= 2(\epsilon + \sqrt{8\epsilon}) + 4\gamma =: \epsilon'.$$
(78)

Two more standard steps are needed.

i) Derandomization. There exists at least one particular value  $x^N$  of the string  $X^N$  such that this code  $C^* = \{\sigma_{x^N}\}$  for which  $p_e(C^*)$  is at least as small as the expectation value. Thus

$$p_e(C^*) \le \epsilon'. \tag{79}$$

ii) Average to maximal error probability. Since

$$p_e(C^*) = \frac{1}{N} \sum_{k \in N} p_e(k) \le \epsilon',$$

then  $p_e(k) \leq 2\epsilon'$  for at least half the indices k. Throw the others away and redefine f, N and  $\gamma$  accordingly. This further changes the error estimate to  $4(\epsilon + \sqrt{8\epsilon}) + 8\gamma$ .

Remark 8: The major difference between the proof of packing lemma and the proof of HSW theorem is that the ensemble in HSW theorem is assumed to be of the tensor power of n copies of  $\{\lambda_j, \rho_j\}$ . This is where the conditional typicality comes into play in order to bound the probability of correctly identifying the classical message. However, in packing lemma, the ensemble is assumed to be some general states in  $\mathcal{H}^{\otimes n}$ . Even thought the projectors  $\Pi_m$  indeed conditioned on m, but they are not necessary projectors onto conditionally typical subspace, Therefore, as we have claimed before, the proof of packing lemma only requires typicality.

#### APPENDIX B

PROOFS OF PROPERTIES (33)-(36)

I. Proof of property (33).

Define  $\check{P}$  to be the complement of the projector P. That is  $\check{P} = I - P$ .

$$\Pi = \Pi^{n}_{\mathcal{N}(\rho),\delta} \otimes \Pi^{n}_{\rho,\delta} 
= (I - \check{\Pi}^{n}_{\mathcal{N}(\rho),\delta}) \otimes (I - \check{\Pi}^{n}_{\rho,\delta}) 
= I \otimes I - I \otimes \check{\Pi}^{n}_{\rho,\delta} - \check{\Pi}^{n}_{\mathcal{N}(\rho),\delta} \otimes I + \check{\Pi}^{n}_{\mathcal{N}(\rho),\delta} \otimes \check{\Pi}^{n}_{\rho,\delta} 
\geq I \otimes I - I \otimes \check{\Pi}^{n}_{\rho,\delta} - \check{\Pi}^{n}_{\mathcal{N}(\rho),\delta} \otimes I.$$
(80)

Therefore

$$\operatorname{Tr} \sigma_{s^{a}}^{BB'} \Pi$$

$$\geq \operatorname{Tr} \sigma_{s^{a}} - \operatorname{Tr} \sigma_{s^{a}} (I \otimes \check{\Pi}_{\rho,\delta}^{n}) - \operatorname{Tr} \sigma_{s^{a}} (\check{\Pi}_{\mathcal{N}(\rho),\delta}^{n} \otimes I)$$

$$= 1 - \operatorname{Tr} \left[\sigma_{s^{a}}^{B'} \check{\Pi}_{\rho,\delta}^{n}\right] - \operatorname{Tr} \left[\sigma_{s^{a}}^{B} \check{\Pi}_{\mathcal{N}(\rho),\delta}^{n}\right]$$

$$\geq 1 - 2\epsilon,$$
(81)

the last line by a double application of (10).

II. Proof of property (34). By (28) and (32),

$$\operatorname{Tr} \sigma_{s^a} \Pi_{s^a} = \operatorname{Tr} \theta^{\otimes n} \Pi_{\theta, \delta}^n$$

$$> 1 - \epsilon.$$
(82)

The last line follows from (10).

III. Proof of property (35).

$$\operatorname{Tr} \Pi_{s^a} = \operatorname{Tr} \Pi_{\theta,\delta}^n \le 2^{n[H(AB)_{\theta} + c\delta]}.$$
 (83)

The inequality follows from (12).

IV. Proof of property (36).

Because of (13), we can bound the density operator  $\pi_{\alpha}^{n}$ 

$$\pi_{\alpha}^{n} = \frac{\prod_{t(\alpha)}^{n}}{\operatorname{Tr} \prod_{t(\alpha)}^{n}} \le 2^{-n[H(\rho) - \eta(\delta)]} \prod_{\rho, \delta}^{n}.$$
 (84)

Then

 $\Pi \sigma \Pi$ 

$$= (\Pi^{n}_{\mathcal{N}(\rho),\delta} \otimes \Pi^{n}_{\rho,\delta}) \left[ \sum_{\alpha} p_{\alpha}(\mathcal{N}^{\otimes n}(\pi^{n}_{\alpha}) \otimes \pi^{n}_{\alpha}) \right] (\Pi^{n}_{\mathcal{N}(\rho),\delta} \otimes \Pi^{n}_{\rho,\delta})$$

$$= \sum_{\alpha} p_{\alpha} \left[ (\Pi^{n}_{\mathcal{N}(\rho),\delta} \mathcal{N}^{\otimes n}(\pi^{n}_{\alpha}) \Pi^{n}_{\mathcal{N}(\rho),\delta}) \otimes (\Pi^{n}_{\rho,\delta} \pi^{n}_{\alpha} \Pi^{n}_{\rho,\delta}) \right]$$

$$= H(A) + H(B) - H(B)$$

$$= H(A) + H(B) - H(E)$$

$$= H(A) + H(B) - H(B)$$

where the first inequality follows from (84) and the second from (11).

## APPENDIX C GENERALIZED DEPHASING CHANNELS

We follow the techniques of [14], [28], [24]. Let A' and B be quantum systems of dimension d with respective bases  $\{|i\rangle^{A'}\}$  and  $\{|i\rangle^{B}\}.$ 

A channel  $\mathcal{N}: A' \to B$  is called a generalized dephasing channel if

$$\mathcal{N}(|i\rangle\langle i|^{A'}) = |i\rangle\langle i|^{B}.$$

We can write down a Stinespring dilation  $U_{\mathcal{N}}: A' \to BE$  for  $\mathcal{N}$ :

$$U_{\mathcal{N}} = \sum_{i} |i\rangle^{B} |\phi_{i}\rangle^{E} \langle i|^{A'},$$

where the  $\{|\phi_i\rangle^E\}$  are not necessarily orthogonal. Given  $U_N$ , the complementary channel  $\mathcal{N}^c: A' \to E = \operatorname{Tr}_B \circ U_{\mathcal{N}}$  acts on some input state  $\rho^{A'}$  as

$$\mathcal{N}^{c}(\rho) = \operatorname{Tr}_{B} U_{\mathcal{N}}(\rho)$$

$$= \sum_{i} \langle i|^{B} \left( \sum_{i''i'} |i''\rangle^{B} |\phi_{i''}\rangle^{E} \langle i''|^{A'} \rho |i'\rangle^{A'} \langle i'|^{B} \langle \phi_{i'}|^{E} \right) |i\rangle^{B}$$

$$= \sum_{i} \langle i|\rho|i\rangle \phi_{i}^{E}$$

$$=: \sum_{i} r_{i} \phi_{i}^{E}.$$
(86)

It depends only on the diagonal elements  $\{r_i\}$  of  $\rho$  expressed in the dephasing basis. When the  $\{|\phi_i\rangle^E\}$  are also orthogonal, the channel N is called *completely dephasing* and is denoted by  $\triangle$ . It corresponds to performing a projective measurement in the dephasing basis and ignoring the result. The following properties hold [28]:

$$\mathcal{N}^{c} = \mathcal{N}^{c} \circ \triangle$$

$$\mathcal{N} \circ \triangle = \triangle \circ \mathcal{N}$$

$$H(\triangle(\rho)) \ge H(\rho).$$
(87)

Define  $\theta^{AB} = (I^A \otimes \mathcal{N}) \phi^{AA'}$ , where  $\phi^{AA'}$  is a purification of the input state  $\rho^{A'}$ .

Lemma 9: Given a dephasing channel  $\mathcal{N}: A' \to B$ , the mutual information  $I(A; B)_{\theta}$  is maximal when the input state  $\rho^{A'}$  is diagonal in the dephasing basis.

$$I(A; B) = H(A) + H(B) - H(BA)$$

$$= H(A) + H(B) - H(E)$$

$$= H(\rho) + H(\mathcal{N}(\rho)) - H(\mathcal{N}^{c}(\rho))$$

$$\leq H(\triangle(\rho)) + H((\triangle \circ \mathcal{N}(\rho)) - H(\mathcal{N}^{c} \circ \triangle(\rho))$$

$$= H(\triangle(\rho)) + H(\mathcal{N} \circ \triangle(\rho)) - H(\mathcal{N}^{c} \circ \triangle(\rho))$$
(88)

The inequality is saturated when  $\rho = \triangle(\rho) = \sum r_i |i\rangle\langle i|$ , in which case

$$I(A; B) = 2H(\{r_i\}) - H(\sum_{i} r_i \phi_i).$$

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#### REFERENCES

- [1] C. E. Shannon, "A mathematical theory of communication," Bell Syst. Tech. J., vol. 27, pp. 379-423, 623-656, 1948.
- T. M. Cover and J. A. Thomas, Elements of Information Theory. New York: Wiley, 1991, Series in Telecommunication.
- I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems. New York/San Francisco/London: Academic, 1981.
- R. Ahlswede, "Multi-way communication channels," in Proc. 2nd Int. Symp. Information Theory (Thakadsor, Armenian SSR, Sep. 1971). Budapest, Hungary: Academia Kiado, 1971, pp. 23-52.

- [5] H. Liao, "Multiple Access Channels," Ph.D. dissertation, Department of Electrical Engineering, University of Hawaii, Honolulu, 1972.
- [6] A. S. Holevo, "The capacity of the quantum channel with general signal states," *IEEE Trans. Inf. Theory*, vol. 44, no. 1, pp. 269–273, Jan. 1998.
- [7] B. Schumacher and M. D. Westmoreland, "Sending classical information via noisy quantum channels," *Phys. Rev. A*, vol. 56, pp. 131–138, 1997.
- [8] S. Lloyd, "The capacity of a noisy quantum channel," Phys. Rev. A, vol. 55, pp. 1613–1622, 1997.
- [9] P. W. Shor, "The quantum channel capacity and coherent information," in *Proc. MSRI Workshop on Quantum Computation*, 2002 [Online]. Available: http://www.msri.org/publications/ln/msri/2002/quantumcrypto/sbor/1/
- [10] I. Devetak, "The private classical capacity and quantum capacity of a quantum channel," *IEEE Trans. Inf. Theory*, vol. 51, no. 1, pp. 44–55, Jan. 2005.
- [11] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. Thapliyal, "Entanglement- assisted capacity of a quantum channel and the reverse Shannon theorem," *IEEE Trans. Inf. Theory*, vol. 48, no. 10, pp. 2637– 2655, Oct. 2002.
- [12] A. S. Holevo, "On entanglement assisted classical capacity," J. Math. Phys., vol. 43, no. 9, pp. 4326–4333, 2002.
- [13] A. Winter, "The capacity of the quantum multiple-access channel," *IEEE Trans. Inf. Theory*, vol. 47, no. 7, pp. 3059–3065, Nov. 2001.
- [14] J. Yard, I. Devetak, and P. Hayden, "Capacity Theorems for Quantum Multiple Access Channels: Classical-Quantum and Quantum-Quantum Capacity Regions." 2005 [Online]. Available: quant-ph/0501045
- [15] M. Horodecki, J. Oppenheim, and A. Winter, "Partial quantum information," *Nature*, vol. 436, pp. 673–676, 2005.
- [16] W. F. Stinespring, "Positive functions on C\*-algebras," Proc. Amer. Math. Soc., vol. 6, pp. 211–216, 1955.
- [17] E. B. Davies and J. T. Lewis, "An operational approach to quantum probability," *Comm. Math. Phys.*, vol. 17, pp. 239–260, 1970.
- [18] M. Ozawa, "Quantum measuring process of continuous observables," J. Math. Phys., vol. 25, pp. 79–87, 1984.
- [19] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information. New York: Cambridge Univ. Press, 2000.
- [20] E. H. Lieb and M. B. Ruskai, "Proof of the strong subadditivity of quantum-mechanical entropy," *J. Math. Phys.*, vol. 14, pp. 1938–1941, 1973.
- [21] A. Winter, "Coding theorem and strong converse for quantum channels," IEEE Trans. Inf. Theory, vol. 45, no. 7, pp. 2481–2485, Nov. 1999.
- [22] C. H. Bennett and S. J. Wiesner, "Communication via one- and twoparticle operators on Einstein-Podolsky-Rosen states," *Phys. Rev. Lett.*, vol. 69, pp. 2881–2884, 1992.
- [23] I. Devetak, A. W. Harrow, and A. J. Winter, "A family of quantum protocols," *Phys. Rev. Lett.*, vol. 93, p. 239503, 2004.
- [24] I. Devetak and P. W. Shor, "The capacity of a quantum channel for simultaneous transmission of classical and quantum information," *Commun. Math. Phys.*, vol. 256, no. 2, pp. 287–303, 2005.
- [25] A. W. Harrowand H.-K. Lo, "A tight lower bound on the classical communication cost of entanglement dilution," *IEEE Trans. Inf. Theory*, vol. 50, no. 2, pp. 319–327, Feb. 2004.
- [26] I. Devetak, P. Hayden, D. W. Leung, and P. Shor, Triple Trade-Offs in Quantum Shannon Theory 2005, in preparation.
- [27] M. Hayashi and H. Nagaoka, "General formulas for capacity of classical-quantum channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 7, pp. 1753–1768, Jul. 2003.
- [28] J. Yard, "Simultaneous Classical-Quantum Capacities of Quantum Multiple Access Channels," Ph.D. dissertation, Stanford Univ., Stanford, CA, 2005