# High-rate, Multi-Symbol-Decodable STBCs from Clifford Algebras 

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#### Abstract

It is well known that Space-Time Block Codes (STBCs) obtained from Orthogonal Designs (ODs) are single-symbol-decodable (SSD) and from Quasi-Orthogonal Designs (QODs) are double-symbol decodable. However, there are SSD codes that are not obtainable from ODs and DSD codes that are not obtainable from QODs. In this paper a method of constructing $g$-symbol decodable ( $g$-SD) STBCs using representations of Clifford algebras are presented which when specialized to $g=1,2$ gives SSD and DSD codes respectively. For the number of transmit antennas $2^{a}$ the rate (in complex symbols per channel use) of the $g$-SD codes presented in this paper is $\frac{a+1-g}{2^{a-g}}$. The maximum rate of the DSD STBCs from QODs reported in the literature is $\frac{a}{2^{a-1}}$ which is smaller than the rate $\frac{a-1}{2^{a-2}}$ of the DSD codes of this paper, for $2^{a}$ transmit antennas. In particular, the reported DSD codes for 8 and 16 transmit antennas offer rates 1 and $3 / 4$ respectively whereas the known STBCs from QODs offer only $3 / 4$ and $1 / 2$ respectively. The construction of this paper is applicable for any number of transmit antennas.


## 1. Introduction and Preliminaries

We consider a multiple antenna transmission system with $N_{t}$ number of transmit antennas and $N_{r}$ number of receive antennas. At each time slot $t$, the complex signals, $s_{t i}, i=0,1 \cdots, N_{t}-1$ are transmitted from the $N_{t}$ transmit antennas simultaneously. Let $h_{i j}=\alpha_{i j} \mathrm{e}^{\mathrm{j} \theta_{i j}}$ denote the path gain from the transmit antenna $i$ to the receive antenna $j$, where $\mathbf{j}=\sqrt{-1}$. Assuming that the path gain are constant over a frame length $N_{t}$ (we consider only square designs), the received signal $y_{t j}$ at the receive antenna $j$ at time $t$ is given by,

$$
\begin{equation*}
y_{t j}=\sum_{i=0}^{N_{t}-1} s_{t i} h_{i j}+n_{t j}, \tag{1}
\end{equation*}
$$

for $j=0, \cdots, N_{r}-1, \quad t=0, \cdots, N_{t}-1$, which in matrix notation is,

$$
\begin{equation*}
\mathbf{Y}=\mathbf{S H}+\mathbf{N} \tag{2}
\end{equation*}
$$

where $\mathbf{Y} \in \mathbb{C}^{N_{t} \times N_{r}}$ is the received signal matrix, $\mathbf{S} \in$ $\mathbb{C}^{N_{t} \times N_{t}}$ is the transmission matrix (also referred as codeword matrix), $\mathbf{N} \in \mathbb{C}^{N_{t} \times N_{r}}$ is the additive noise matrix
and $\mathbf{H} \in \mathbb{C}^{N_{t} \times N_{r}}$ is the channel matrix, where $\mathbb{C}$ denotes the complex field. The entries of $\mathbf{H}$ are complex Gaussian with zero mean and unit variance and the entries of $\mathbf{N}$ are complex Gaussian with zero mean and variance $\sigma^{2}$. Both are assumed to be temporally and spatially white. We further assume that transmission power constraint is given by $E\left[\operatorname{tr}\left\{\mathbf{S S}^{H}\right\}\right]=N_{t}{ }^{2}$.

An $n \times n$ linear dispersion STBC [1] with $g K$ complex variables $x_{1}, x_{2}, \cdots, x_{g K}$, where $g$ and $K$ are positive integers, can be written as

$$
\begin{equation*}
\mathbf{S}=\sum_{i=1}^{K} S_{i} \tag{3}
\end{equation*}
$$

where,

$$
\begin{align*}
S_{i}= & \sum_{j=1}^{g} x_{(g(i-1)+j), I} A_{(g(i-1)+j), I}+ \\
& +x_{(g(i-1)+j), Q} A_{(g(i-1)+j), Q} \tag{4}
\end{align*}
$$

and $x_{l}=x_{l, I}+j x_{l, Q} \in \mathcal{A}_{l} \subset \mathbb{C}$ for $1 \leq l \leq g K$ where $\mathcal{A}_{l}$ is the signal constellation from which the variable $x_{i}$ takes values. The set of $g K$ number of complex $n \times n$ matrices $A_{j}, \quad 1 \leq j \leq g K$ are called the weight matrices of the code and this set defines the code $\mathbf{S}$. With $\left|\mathcal{A}_{i}\right|$ denoting the number of points in the constellation, the rate of this code in bits per channel use is $R=\frac{1}{n} \sum_{i=1}^{g K} \log _{2}\left(\left|\mathcal{A}_{i}\right|\right)$. Now assuming that perfect channel state information(CSI) is available at the receiver, the maximum likelihood (ML) decision rule minimizes the metric,

$$
\begin{equation*}
\mathbf{M}(\mathbf{S}) \triangleq \min _{\mathbf{S}} \operatorname{tr}\left((\mathbf{Y}-\mathbf{S H})^{\mathcal{H}}(\mathbf{Y}-\mathbf{S H})\right)=\|\mathbf{Y}-\mathbf{S H}\|^{2} . \tag{5}
\end{equation*}
$$

It is clear that there are $\prod_{i=1}^{g K}\left|\mathcal{A}_{i}\right|$ different codewords and, in general, the ML decoding requires $\prod_{i=1}^{g K}\left|\mathcal{A}_{i}\right|$ computations, one for each codeword. But if the set of weight matrices are chosen such that the decoding metric (5) could be decomposed into,

$$
\mathbf{M}(\mathbf{S})=\sum_{i=1}^{K} f_{i}\left(x_{(i-1) g+1}, x_{(i-1) g+2}, \cdots, x_{(i-1) g+g}\right)
$$

a sum of $K$ positive terms, each involving exactly $g$ complex variables only, then the decoding requires
$\sum_{i=1}^{K}\left\{\prod_{j=1}^{g}\left|\mathcal{A}_{j+(i-1) g}\right|\right\}$ computations and the code is called a $g$-symbol decodable code ( $g$-SD code). The case $g=1$ corresponds to Single-Symbol Decodable (SSD) codes that includes the well known codes from Orthogonal Designs (ODs) as a proper subclass, and have been extensively studied [2]-[13], few most recent ones being [12]-[13]. The codes corresponding to $g=2$, are called Double-Symbol-Decodable (DSD) codes. The Quasi-Orthogonal Designs studied in [14]-[17] and [6] are proper subclass of DSD codes. If the weight matrices $A_{j}, \quad 1 \leq j \leq 2 K$ of a $g$-SD code are all unitary then it is said to be a Unitary Weight DSD (UW- $g$-SD) code. The DSD codes of [14]-[17] are UW-DSD codes and that of [6] is not. Throughout this paper, we consider only UW codes.

The contributions of this paper are

- We obtain a set of sufficient conditions for a general linear dispersion STBC to be $g$-SD in terms of their weight matrices. Also, another set of sufficient conditions for the code to be $g$-SD is given which enables us to construct UW- $g$-SD codes from representations of Clifford algebras.
- For the number of transmit antennas $2^{a}$ the maximum rate (in complex symbols per channel use) of all the DSD codes reported in the literature with unitary weight matrices is $\frac{a}{2^{a-1}}$. Whereas we present UW-DSD codes with rate $\frac{a-1}{2^{a-2}}$ for $2^{a}$ transmit antennas. In particular, our code for 8 and 16 transmit antennas offer rates 1 and $3 / 4$ respectively, whereas the known QODs offer only $3 / 4$ and $1 / 2$ respectively. The rate of our $g$-SD codes is $\frac{a+1-g}{2^{a-g}}$.


## 2. Sufficient conditions for Double-Symbol-Decodability

We begin with presenting a sufficient condition on the set of weight matrices of the code to be $g$-SD.

Lemma 1 The linear dispersion STBC given by (3) is a $g-S D$ code if,

$$
\begin{equation*}
S_{l}^{H} S_{j}+S_{j}^{H} S_{l}=0, \quad \forall 1 \leq l \neq j \leq K \tag{6}
\end{equation*}
$$

Proof: We see from (3) that,

$$
\begin{equation*}
\mathbf{S}^{H} \mathbf{S}=\left(\sum_{i=1}^{K} S_{i}^{H}\right)\left(\sum_{i=1}^{K} S_{i}\right) \tag{7}
\end{equation*}
$$

If the conditions of (6) are satisfied, then it is easy to verify that,

$$
\begin{equation*}
\mathbf{S}^{H} \mathbf{S}=\sum_{i=1}^{K} S_{i}^{H} S_{i} \tag{8}
\end{equation*}
$$

Using (8) in (5) we get,

$$
\begin{align*}
\mathbf{M}(\mathbf{S}) & =\operatorname{Tr}\left[\sum_{i=1}^{K}\left(\mathbf{Y}-S_{i} \mathbf{H}\right)^{H}\left(\mathbf{Y}-S_{i} \mathbf{H}\right)-(K-1) \mathbf{Y}^{H} \mathbf{Y}\right] \\
& =\sum_{i=1}^{K} \operatorname{Tr}\left[\left(\mathbf{Y}-S_{i} \mathbf{H}\right)^{H}\left(\mathbf{Y}-S_{i} \mathbf{H}\right)\right]+\mathbf{M}_{\mathbf{c}} \tag{9}
\end{align*}
$$

where $\mathbf{M}_{\mathbf{c}}=\operatorname{Tr}\left[-(K-1) \mathbf{Y}^{H} \mathbf{Y}\right]$. Note that in (9) the $\mathbf{M}_{\mathbf{c}}$ term is same for all the codewords in the code book. Hence it is sufficient to minimize the first term only. But the first term,

$$
\widetilde{\mathbf{M}}(\mathbf{S})=\sum_{i=1}^{K} \operatorname{Tr}\left[\left(\mathbf{Y}-S_{i} \mathbf{H}\right)^{H}\left(\mathbf{Y}-S_{i} \mathbf{H}\right)\right]
$$

is a sum of $K$ square terms, each involving only $g$ complex variables. Hence the problem of ML decoding reduces to the problem of minimizing,

$$
\begin{equation*}
\operatorname{Tr}\left[\left(\mathbf{Y}-S_{i} \mathbf{H}\right)^{H}\left(\mathbf{Y}-S_{i} \mathbf{H}\right] \forall 1 \leq i \leq K\right. \tag{10}
\end{equation*}
$$

Hence the code is $g$-SD if the conditions of (6) is satisfied. This completes the proof.

Next we derive a condition on weight matrices of the code so that (6) is satisfied. Towards this end, we denote,

$$
\begin{equation*}
\beta_{i}=\left\{A_{g(i-1)+j, I}, A_{g(i-1)+j, Q}\right\}_{j=1}^{g}, \quad 1 \leq i \leq K \tag{11}
\end{equation*}
$$

A straight forward verification shows that
Lemma 2 Conditions of (6) is satisfied if the weight matrices of the code (3) satisfy the following condition,

$$
\begin{equation*}
A^{H} B+B^{H} A=0, \forall A \in \beta_{i}, B \in \beta_{j}, \text { for } i \neq j \tag{12}
\end{equation*}
$$

We first introduce the notion of normalizing a linear STBC which not only simplifies the analysis of the codes but also provides deep insight into various aspects of different classes of codes discussed in this paper. Towards this end, let

$$
\begin{equation*}
S_{U}=\sum_{k=1}^{g K} x_{k, I} A_{k, I}^{\prime}+x_{k, Q} A_{k, Q}^{\prime} \tag{13}
\end{equation*}
$$

be a UW- $g$-SD code. We normalize the weight matrices of the code as

$$
\begin{align*}
& A_{k I}=A_{1 I}^{\prime H} A_{k I}^{\prime}  \tag{14}\\
& A_{k Q}=A_{1 I}^{\prime H} A_{k Q}^{\prime} .
\end{align*} \quad \forall 1 \leq k \leq g K
$$

to get the normalized version of (13) to be

$$
\begin{equation*}
S_{N}=\sum_{k=1}^{g K} x_{k, I} A_{k, I}+x_{k, Q} A_{k, Q} \tag{15}
\end{equation*}
$$

where $A_{1, I}=I_{n}$, the $n \times n$ identity matrix. We call the code $S_{N}$ to be the normalized code of $S_{U}$.

Lemma 3 The code $S_{U}$ is $g$-SD iff $S_{N}$ is $g$-SD. In other words normalization does not affect the DSD property.

Proof: For $1 \leq i_{1} \neq i_{2} \leq K$, the equation of Lemma2is satisfied by the weight matrices of $S_{U}$ iff they are satisfied by the weight matrices of $S_{N}$ which is easily verified.

The following theorem identifies a set of sufficient conditions for a UW code to be UW- $g$-SD.

Theorem 1 A $n \times n U W$ code described by (13) and its normalized version given by (15) are UW-g-SD codes if the weight matrices of the normalized code satisfy the following conditions:

$$
\begin{gather*}
A_{g(i-1)+1, I}^{H}=-A_{g(i-1)+1, I}, 2 \leq i \leq K  \tag{16}\\
A_{g(i-1)+1, I} \text { and } A_{g(j-1)+1, I} \text { anticommute } \\
\forall 2 \leq i \neq j \leq K  \tag{17}\\
A_{1, Q}, A_{2, I}, A_{2, Q}, \cdots, A_{g, I}, A_{g, Q} \text { are }
\end{gather*}
$$

Hermitian, commute among themselves and

$$
\begin{align*}
& \text { with all the matrices } A_{g(i-1)+1, I}, \quad 1 \leq i \leq K  \tag{18}\\
& \qquad \begin{array}{l}
\text { For all } 1 \leq i \leq K, \text { and } 2 \leq j \leq g ; \\
A_{g(i-1)+1, Q}= \pm A_{1, Q} A_{g(i-1)+1, I}, \\
A_{g(i-1)+1, I}= \pm A_{j, I} A_{g(i-1)+1, I}, \\
A_{g(i-1)+1, Q}= \pm A_{j, Q} A_{g(i-1)+1, I},
\end{array}
\end{align*}
$$

Proof: Using the conditions (19), (20) and (21) of the theorem the sets (11) for $1 \leq i_{1} \neq i_{2} \leq K$ are

$$
\begin{align*}
\beta_{i_{1}} & =\left\{ \pm A_{j, I} A_{g\left(i_{1}\right)+1, I}, \pm A_{j, Q} A_{g\left(i_{1}\right)+1, Q}\right\}_{j=1}^{g}  \tag{22}\\
\beta_{i_{2}} & =\left\{ \pm A_{j, I} A_{g\left(i_{2}\right)+1, I}, \pm A_{j, Q} A_{g\left(i_{2}\right)+1, Q}\right\}_{j=1}^{g} \tag{23}
\end{align*}
$$

Let $x A_{g\left(i_{1}\right)+1, I} \in \beta_{i_{1}}, y A_{g\left(i_{2}\right)+1} \in \beta_{i_{2}}$ where $x, y \in$ $\left\{I, \pm A_{1 Q}, \pm A_{2 I}, \pm A_{2 Q}, \cdots \pm A_{(g-1), I}, \pm A_{(g-1), Q}\right\}$. Then

$$
\begin{gathered}
x A_{g\left(i_{1}\right)+1, I}{ }^{H} y A_{g i_{2}+1, I}+y A_{g\left(i_{2}\right)+1, I}{ }^{H} x A_{g\left(i_{1}\right)+1, I} \\
=A_{g i_{1}+1, I} x^{H} y A_{g i_{2}+1, I}+A_{g i_{2}+1, I}{ }^{H} y^{H} x A_{g i_{1}+1, I} \\
=A_{g i_{1}+1, I}{ }^{H} x y A_{g i_{2}+1, I}+A_{g i_{2}+1, I}{ }^{H} y x A_{g i_{1}+1, I} \\
=A_{g i_{1}+1, I}{ }^{H} x y A_{g i_{2}+1, I}+A_{g i_{2}+1, I}{ }^{H} x y A_{g i_{1}+1, I} \\
=x A_{g i_{1}+1, I}{ }^{H} A_{g i_{2}+1, I} y+x A_{g i_{2}+1, I}{ }^{H} A_{g i_{1}+1, I} y \\
=x\left[A_{g i_{1}+1, I}{ }^{H} A_{g i_{2}+1, I}+A_{g i_{2}+1, I}{ }^{H} A_{g i_{1}+1, I}\right] y \\
=x[0] y=0 .
\end{gathered}
$$

This completes the proof.
The requirements of Theorem 1 for UW- $g$ SD can be easily stated using the Table 2 shown at the top of the next page as follows:

- The matrices of the first row should form a Hurwitz-Radon family of matrices
- The matrices of the first column should be
- Hermitian
- mutually commuting and
- commute with all the matrices of the first row

Definition 1 A UW-DSD code satisfying the conditions of (16) is defined to be a Clifford Unitary Weight DSD (CUW-DSD) codes.

The name in the above definition is due to the fact that such codes are constructable using matrix representations of real Clifford algebras as shown in the following section.

## 3. Construction of $\mathbf{C U W}-g$-SD codes

In this section we present the construction of a new class of $2^{a} \times 2^{a} g$-SD codes, the Clifford UW- $g$-SD (CUW- $g$ $\mathrm{SD})$ codes, for $2^{a}$ transmit antennas, from unitary matrix representations of real Clifford algebras. For an excellent introduction to and basic properties of these representations see [4].

Definition 2 The Clifford algebra, denoted by $C A_{L}$, is the algebra over the real field $\mathbb{R}$ generated by $L$ objects $\gamma_{k}, \quad k=1,2, \cdots, L$ which are anti-commuting, $\left(\gamma_{k} \gamma_{j}=-\gamma_{j} \gamma_{k}, \forall k \neq j\right.$ ) and squaring to $-1,\left(\gamma_{k}^{2}=\right.$ $-1 \quad \forall k=1,2, \cdots, L)$.

A matrix representation of an algebra is completely specified by a representation of its generators. For a Clifford algebra, we are thus interested in unitary matrix representation of the generators $\gamma_{k}$ 's. Let
$\sigma_{1}=\left[\begin{array}{rr}0 & 1 \\ -1 & 0\end{array}\right], \sigma_{2}=\left[\begin{array}{ll}0 & j \\ j & 0\end{array}\right]$ and $\sigma_{3}=\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right]$
and $\quad A^{\otimes^{m}}=\underbrace{A \otimes A \otimes A \cdots \otimes A}_{m \text { times }}$.
From [4] we know that the representation of the generators of $C A_{2 a+1}$ is given by

$$
\begin{align*}
R\left(\gamma_{1}\right) & = \pm j \sigma_{3}^{\otimes^{a}} \\
R\left(\gamma_{2}\right) & =I_{2}^{\otimes^{a-1}} \otimes \sigma_{1} \\
R\left(\gamma_{3}\right) & =I_{2}^{\otimes^{a-1}} \otimes \sigma_{2} \\
\cdot & \cdot  \tag{25}\\
\cdot & \cdot \\
\cdot & \cdot \\
R\left(\gamma_{2 k}\right) & =I_{2}^{\otimes^{a-k}} \otimes \sigma_{1} \otimes \sigma_{3}^{\otimes^{k-1}} \\
R\left(\gamma_{2 k+1}\right) & =I_{2}^{\otimes^{a-k}} \otimes \sigma_{2} \bigotimes \sigma_{3}^{\otimes^{k-1}} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
R\left(\gamma_{2 a}\right) & =\sigma_{1} \bigotimes \sigma_{3}^{\otimes^{a-1}} \\
R\left(\gamma_{2 a+1}\right) & =\sigma_{2} \bigotimes \sigma_{3}^{\otimes^{a-1}} .
\end{align*}
$$

We add to this list the $2^{a} \times 2^{a}$ identity matrix, denoted by $I_{2^{a}}$, and designate it as $R\left(\gamma_{0}\right)=I_{2^{a}}$.

| $I=A_{1 I}$ | $A_{g+1, I}$ | $A_{2 g+1, I}$ | . |  |  | $A_{i g+1, I}$ |  | . |  | $A_{(K-1) g+1, I}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{1 Q}$ | $A_{1 Q} A_{g+1, I}$ | $A_{1 Q} A_{2 g+1, I}$ |  |  |  | $A_{1 Q} A_{i g+1, I}$ |  |  |  | $A_{1 Q} A_{(K-1) g+1, I}$ |
| $A_{2 I}$ | $A_{2 I} A_{g+1, I}$ | $A_{2 I} A_{2 g+1, I}$ |  |  |  | $A_{2 I} A_{i g+1, I}$ |  |  |  | $A_{2 I} A_{(K-1) g+1, I}$ |
| $A_{2 Q}$ | $A_{2 Q} A_{g+1, I}$ | $A_{2 Q} A_{2 g+1, I}$ |  |  |  | $A_{2 Q} A_{i g+1, I}$ |  |  |  | $A_{2 Q} A_{(K-1) g+1, I}$ |
|  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |
| $A_{g I}$ | $A_{g I} A_{g+1, I}$ | $A_{g I} A_{2 g+1, I}$ |  |  |  | $A_{g I} A_{i g+1, I}$ |  |  |  | $A_{g I} A_{(K-1) g+1, I}$ |
| $A_{g Q}$ | $A_{g Q} A_{g+1, I}$ | $A_{g Q} A_{2 g+1, I}$ |  |  |  | $A_{g Q} A_{i g+1, I}$ |  |  |  | $A_{g Q} A_{(K-1) g+1, I}$ |

### 3.1. Construction of CUW- $g$-SD codes

To construct CUW- $g$-SD codes from the last $2 g$ matrices $\left\{R\left(\gamma_{2(a+1-g)}\right), R\left(\gamma_{2(a+1-g)+1}\right), \cdots, R\left(\gamma_{2 a}\right), R\left(\gamma_{2 a+1}\right)\right\}$ we construct the following $2 g-1$ new matrices,

$$
\begin{align*}
\alpha_{1} & = \pm j R\left(\gamma_{2 a-2}\right) R\left(\gamma_{2 a-1}\right) \\
\alpha_{2} & = \pm j R\left(\gamma_{2 a}\right) R\left(\gamma_{2 a+1}\right) \\
\vdots & \vdots \\
\alpha_{g} & = \pm j R\left(\gamma_{0}\right) R\left(\gamma_{1}\right) \\
\alpha_{g+1} & = \pm \alpha_{1} \alpha_{2} \\
\alpha_{g+2} & = \pm \alpha_{3} \alpha_{4} \\
\vdots & \vdots \\
\alpha_{g+\frac{g}{2}} & = \pm \alpha_{g-1} \alpha_{g}  \tag{26}\\
\alpha_{g+\frac{g}{2}+1} & = \pm \alpha_{g+1} \alpha_{g+2} \\
\alpha_{g+\frac{g}{2}+2} & = \pm \alpha_{g+3} \alpha_{g+4} \\
\vdots & \vdots \\
\alpha_{g+\frac{g}{2}+\frac{g}{4}} & = \pm \alpha_{g+\frac{g}{2}-1} \alpha_{g+\frac{g}{2}} \\
\vdots & \vdots \\
\alpha_{2 g-1} & =\Pi_{i=1}^{g} \alpha_{g} .
\end{align*}
$$

Note that the set of matrices $\left\{\alpha_{i}\right\}_{i=1}^{2 g-1}$ have the following properties: (i) They are mutually commuting, (ii) Hermitian and (iii) each commutes with $R\left(\gamma_{1}\right), R\left(\gamma_{2}\right), \cdots, R\left(\gamma_{2 a-2 g+1}\right)$. In the above construction of $\alpha_{i}$ s there is nothing special about the last $2 g$ matrices. Any $2 g$ from the set $\left\{R\left(\gamma_{1}\right), R\left(\gamma_{2}\right), \cdots, R\left(\gamma_{2 a+1}\right)\right\}$ could have been selected. Next, for $1 \leq i \leq 2 a-2 g+2$, we construct the weight matrices of the code combining the matrices defined above in the following way,

$$
\left.\begin{array}{rl}
A_{g(i-1)+1, I} & =R\left(\gamma_{i-1}\right)  \tag{27}\\
A_{g(i-1)+2, I} & = \pm \alpha_{1} R\left(\gamma_{i-1}\right) \\
\vdots & \vdots \\
A_{g(i-1)+g, I} & = \pm \alpha_{g-1} R\left(\gamma_{i-1}\right) \\
A_{g(i-1)+1, Q} & = \pm \alpha_{g} R\left(\gamma_{i-1}\right) \\
A_{g(i-1)+2, Q} & = \pm \alpha_{g+1} R\left(\gamma_{i-1}\right) \\
\vdots & \vdots \\
A_{g(i-1)+g, Q} & = \pm \alpha_{2 g-1} R\left(\gamma_{i-1}\right)
\end{array}\right\}
$$

Theorem 2 The $2^{a} \times 2^{a}$ code with the weight matrices given by (27) is a CUW-g-SD code.

Proof It is easily checked that the weight matrices satisfy the conditions of Theorem 1 and hence the code is CUW-$g$-SD.

Corollary 1 The rate of the $2^{a} \times 2^{a} C U W-g$-SD codes of Theorem 2 is $\frac{a+1-g}{2^{a-g}}$ complex symbols per channel use.

Example 1 Let the representation matrices of the $C A_{2 a+1}$, where $a=3$ be,

$$
\begin{array}{r}
R\left(\gamma_{0}\right)=I_{2} \otimes I_{2} \times I_{2}, R\left(\gamma_{1}\right)=I_{2} \otimes I_{2} \times \sigma_{1}, \\
R\left(\gamma_{2}\right)=I_{2} \otimes I_{2} \times \sigma_{2}, R\left(\gamma_{3}\right)=I_{2} \otimes \sigma_{1} \times \sigma_{3} \\
R\left(\gamma_{4}\right)=I_{2} \otimes \sigma_{2} \times \sigma_{3}, R\left(\gamma_{5}\right)=\sigma_{1} \otimes \sigma_{3} \times \sigma_{3} \\
R\left(\gamma_{6}\right)=\sigma_{2} \otimes \sigma_{3} \times \sigma_{3}, R\left(\gamma_{7}\right)=j \sigma_{3} \otimes \sigma_{3} \times \sigma_{3}
\end{array}
$$

Now according to the prescription of the construction procedure,

$$
\begin{array}{r}
\alpha_{1}=j R\left(\gamma_{4}\right) R\left(\gamma_{5}\right)=\sigma_{1} \otimes \sigma_{1} \times I_{2} \\
\alpha_{2}=j R\left(\gamma_{6}\right) R\left(\gamma_{7}\right)=j \sigma_{1} \otimes I_{2} \times I_{2} \\
\alpha_{3}=R\left(\gamma_{4}\right) R\left(\gamma_{5}\right) R\left(\gamma_{6}\right) R\left(\gamma_{7}\right)=j I_{2} \otimes \sigma_{1} \times I_{2}
\end{array}
$$

and the weight matrices given by (28) at the top of the next page. Now if we construct the code, we get the code given by (29) at the top of the next page.

Note that for 8 transmit antennas our CUW-DSD code achieves rate 1 whereas all known QODs with unitary weight matrices for 8 transmit antennas achieve rate only $\frac{3}{4}$. However, the 8 transmit antenna DSD code with nonunitary weight matrices of [6] achieve rate 1, but has larger Peak-to-Average Power Ratio due to the presence of zero entries in the code, compared to our CUW-DSD codes.

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$$
\begin{array}{llll}
A_{1 I}=R\left(\gamma_{0}\right)=I_{2} \otimes I_{2} \times I_{2} & A_{3 I}=R\left(\gamma_{1}\right)=I_{2} \otimes I_{2} \times \sigma_{1} & A_{5 I}=R\left(\gamma_{2}\right)=I_{2} \otimes I_{2} \times \sigma_{2} & A_{7 I}=R\left(\gamma_{3}\right)=I_{2} \otimes \sigma_{1} \times \sigma_{3} \\
A_{1 Q}=\alpha_{1} A_{1 I}=\sigma_{1} \otimes \sigma_{1} \times I_{2} & A_{3 Q}=\alpha_{1} A_{3 I}=\sigma_{1} \otimes \sigma_{1} \times \sigma_{1} & A_{5 Q}=\alpha_{1} A_{5 I}=\sigma_{1} \otimes \sigma_{1} \times \sigma_{2} & A_{7 Q}=\alpha_{1} A_{7 I}=-\sigma_{1} \otimes I_{2} \times \sigma_{3}  \tag{28}\\
A_{2 I}=\alpha_{2} R\left(\gamma_{0}\right)=j \sigma_{1} \otimes I_{2} \times I_{2} & A_{4 I}=\alpha_{2} R\left(\gamma_{1}\right)=j \sigma_{1} \otimes I_{2} \times \sigma_{1} & A_{6 I}=\alpha_{2} R\left(\gamma_{2}\right)=j \sigma_{1} \otimes I_{2} \times \sigma_{2} & A_{8 I}=\alpha_{2} R\left(\gamma_{3}\right)=j \sigma_{1} \otimes \sigma_{1} \times \sigma_{3} \\
A_{2 Q}=\alpha_{3} R\left(\gamma_{0}\right)=j I_{2} \otimes \sigma_{1} \times I_{2} & A_{4 Q}=\alpha_{3} R\left(\gamma_{1}\right)=j I_{2} \otimes \sigma_{1} \times \sigma_{1} & A_{6 Q}=\alpha_{3} R\left(\gamma_{2}\right)=j I_{2} \otimes \sigma_{1} \times \sigma_{2} & A_{8 Q}=\alpha_{3} R\left(\gamma_{3}\right)=-j I_{2} \otimes I_{2} \times \sigma_{3}
\end{array}
$$

$\left[\begin{array}{rrrrrrr}x_{1 I}-j x_{8 Q} & x_{3 I}+j x_{5 I} & x_{7 I}+j x_{2 Q} & -x_{6 Q}+j x_{4 Q} & -x_{7 Q}+j x_{2 I} & -x_{6 I}+j x_{4 I} & x_{1 Q}+j x_{8 I} \\ -x_{3 I}+j x_{5 I} & x_{1 I}+j x_{8 Q} & -x_{6 Q}-j x_{4 Q} & -x_{7 I}+j x_{2 Q} & -x_{6 I}-j x_{4 I} & x_{7 Q}+j x_{5 Q} \\ -x_{7 I}-j x_{2 Q} & x_{6 Q}-j x_{4 Q} & x_{1 I}-j x_{8 Q} & x_{3 I}+j x_{5 I} & -x_{1 Q}-j x_{8 I} & -x_{3 Q}-j x_{5 Q} & -x_{3 Q}+j x_{5 Q} \\ x_{6 Q}+j x_{4 Q} & x_{7 I}-j x_{2 Q} & -x_{3 I}+j x_{5 I} & x_{1 I}+j x_{8 Q} & x_{3 Q}-j x_{5 Q} & -x_{1 Q}+j x_{8 I} & -x_{6 I}-j x_{4 I} \\ x_{7 Q}-j x_{2 I} & x_{6 I}-j x_{4 I} & -x_{1 Q}-j x_{8 I} & -x_{3 Q}-j x_{5 Q} & x_{1 I}-j x_{8 Q} & x_{6 I}+j x_{4 I} \\ x_{6 I}+j x_{4 I} & -x_{7 Q}-j x_{2 I} & x_{3 Q}-j x_{5 Q} & -x_{1 Q}+j x_{8 I} & -x_{3 I}+j x_{5 I} & x_{1 I}+j x_{8 I} & x_{7 I}+j x_{2 Q} \\ x_{1 Q}+j x_{8 I} & x_{3 Q}+j x_{5 Q} & x_{7 Q}-j x_{2 I} & x_{6 I}-j x_{4 I} & -x_{6 Q}+j x_{4 Q}-j x_{2 Q} & x_{6 Q}-j x_{4 Q}-j x_{4 Q} & -x_{7 I}+j x_{2 Q} \\ -x_{3 Q}+j x_{5 Q} & x_{1 Q}-j x_{8 I} & x_{6 I}+j x_{4 I} & -x_{7 Q}-j x_{2 I} & x_{6 Q}+j x_{4 Q} & -x_{7 I}-j x_{2 Q} & -x_{3 I}+j x_{5 I} \\ & & & & x_{1 I Q}+j x_{8 Q}-j\end{array}\right]$
(29)

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