# Upper Bound on Error Exponent of Regular LDPC Codes Transmitted over the BEC 

Idan Goldenberg David Burshtein<br>School of Electrical Engineering<br>Tel-Aviv University<br>Tel-Aviv 69978, Israel<br>E-mail: \{idang,burstyn\}@eng.tau.ac.il


#### Abstract

The error performance of the ensemble of typical LDPC codes transmitted over the binary erasure channel (BEC) is analyzed. In the past, lower bounds on the error exponents were derived. In this paper a probabilistic upper bound on this error exponent is derived. This bound holds with some confidence level.


Index Terms: Block codes, error exponent, expurgated ensemble, stopping sets, low-density parity-check (LDPC) codes, iterative decoding, binary erasure channel (BEC).

## I. Introduction

Low-density parity-check (LDPC) codes, discovered by Gallager [1], have been widely researched over the last decade and a half. Asymptotic results are widely known for these codes, including results on the performance under maximum-likelihood (ML) decoding [1], [2], [3], [4], [5], average ensemble distance spectra [1], [6], [7], [8], [9], stopping set distributions [7], [8], [9], [10], thresholds for iterative decoding using density evolution [11], [12], and others. However, accurate finite-length analysis of LDPC codes under iterative sum-product decoding is currently available only for the binary erasure channel (BEC) [13]. This is due to the simplicity of the channel model and the graph-based iterative decoder which lends itself to a more detailed analysis. Analysis of the combinatorial properties of stopping sets and their contribution to the error performance reveals that the average error performance of the LDPC ensemble is proportional to the inverse of a polynomial in the block length $N$ [7]. This behavior is attributed to the existence of "bad" codes which possess small stopping sets, and otherwise would decrease exponentially with $N$ if these codes were removed from the ensemble. Fortunately, these "bad" codes constitute a small fraction of the entire ensemble whose size is proportional to the inverse of a polynomial in $N$.

After removing the undesirable codes, we obtain an expurgated ensemble, for which there exists a positive error exponent. In [7], lower bounds on this error exponent of typical codes in the regular and irregular LDPC code ensembles were derived. In this paper we obtain an upper bound on this exponent, and compare it with the above mentioned lower bounds. Similar to [5], which considers upper bounds on the error exponent of LDPC codes under ML decoding, our bounds depend on some confidence level.

The correspondence is organized as follows. Section $\Pi$ introduces notation and preliminary material.

Section IIII introduces a lower bound on the error (erasure) probability from which an upper bound on the exponent is derived. Section IV introduces numerical results and comparisons with previous results. Section $\square$ concludes the paper.

## II. Preliminaries

## A. Notation

We will use the following notation throughout the paper.

- Let $\left\{\alpha_{l}\right\}_{l=1}^{k}$ be a set of non-negative real numbers, such that $\sum_{l} \alpha_{l} \leq 1$. The entropy function of $\left\{\alpha_{l}\right\}_{l=1}^{k}$ is defined as

$$
h\left(\alpha_{1}, \ldots, \alpha_{k}\right)=-\sum_{l=1}^{k} \alpha_{l} \log \left(\alpha_{l}\right)-\left(1-\sum_{l=1}^{k} \alpha_{l}\right) \log \left(1-\sum_{l=1}^{k} \alpha_{l}\right)
$$

where $\log$ is the base- 2 logarithm. We use the convention $0 \log 0=0$.

- Given an integer $n$ and integers $\left(n_{1}, \ldots, n_{k}\right)$ such that $\sum_{l} n_{l} \leq n$,

$$
\binom{n}{n_{1}, n_{2}, \ldots, n_{k}} \triangleq \frac{n!}{n_{1}!\cdot n_{2}!\cdot \ldots \cdot\left(n-\sum_{l=1}^{k} n_{l}\right)!}
$$

is the multinomial coefficient of $n$ over $\left(n_{1}, \ldots, n_{k}\right)$. We will use the following property of multinomial coefficients

$$
\begin{equation*}
\log \binom{n}{n_{1}, n_{2}, \ldots, n_{k}}=n\left(h\left(\frac{n_{1}}{n}, \ldots, \frac{n_{k}}{n}\right)+o(1)\right) \tag{1}
\end{equation*}
$$

which is easily proven using Stirling's approximation.

- If $p(x)$ is a polynomial, then we will denote the coefficient of $x^{i}$ by $\left[x^{i}\right] p(x)$, i.e,

$$
p(x)=\sum_{i}\left[x^{i}\right] p(x) x^{i}
$$

The same notation is extended for use with multivariate polynomials, e.g.,

$$
p(x, y, z)=\sum_{i, j, k}\left[x^{i} y^{j} z^{k}\right] p(x, y, z) x^{i} y^{j} z^{k}
$$

## B. A Second-Order Inequality for Probabilities

Dawson and Sankoff [14] obtained a lower bound on the probability of a finite union of events. Their result asserts the following. Let $\left\{A_{i}\right\}_{i=1}^{M}$ be a finite family of events in a probability space $(\Omega, P)$. Denote

$$
\tilde{S}_{1}=\sum_{i \in I} \operatorname{Pr}\left(A_{i}\right) \quad \tilde{S}_{2}=\sum_{\substack{i, j \in I \\ i>j}} \operatorname{Pr}\left(A_{i} \cap A_{j}\right)
$$

where $I=\{1, \ldots, M\}$. Then

$$
\begin{equation*}
\operatorname{Pr}\left(\bigcup_{i \in I} A_{i}\right) \geq \frac{2}{r+1} \tilde{S}_{1}-\frac{2}{r(r+1)} \tilde{S}_{2} \tag{2}
\end{equation*}
$$

for any $r \in\{1, \ldots, M-1\}$.
Following the derivation in [14], we derive a result which generalizes (2]. For a probability event $A$, denote by $\mathbf{1}_{\{A\}}$ to be the indicator (random variable) over $A$, i.e, for $\omega \in \Omega$,

$$
\mathbf{1}_{\{A\}}(\omega)= \begin{cases}1 & \omega \in A \\ 0 & \omega \notin A\end{cases}
$$

Our result asserts that for all $\omega \in \Omega$,

$$
\begin{equation*}
\left.\mathbf{1}_{\{\cup M=1}^{M} A_{i}\right\} \geq \frac{2}{r+1} S_{1}-\frac{2}{r(r+1)} S_{2} \tag{3}
\end{equation*}
$$

where

$$
S_{1}=\sum_{i \in I} \mathbf{1}_{\left\{A_{i}\right\}} \quad S_{2}=\sum_{\substack{i, j \in I \\ i>j}} \mathbf{1}_{\left\{A_{i}\right\}} \mathbf{1}_{\left\{A_{j}\right\}}
$$

By taking the expectation over both sides of (3), we get (2) as a special case. We prove (3) in Appendix [1]

## C. LDPC Code Ensembles

We consider the standard bipartite graph-based $(c, d)$-regular LDPC code ensemble with block length $N$ and design rate $R$. In this ensemble a randomly chosen permutation is used to match the $c N$ left sockets to the $d(1-R) N$ right sockets. The actual rate of the code is at least $R \triangleq 1-c / d$.

## III. Upper Bound on Error Exponent for the BEC

Recall that a stopping set $\mathcal{S}$ of a bipartite graph representation of an LDPC code is a set of variable nodes, such that each check node neighbor of $\mathcal{S}$ is connected to $\mathcal{S}$ by at least two edges. As explained in [13], iterative decoding of LDPC codes succeeds if and only if the set of variable nodes which correspond to erasures does not contain a subset which is a stopping set.

The expurgated $(c, d)$-regular LDPC ensemble $\mathcal{C}^{\gamma}$ is derived from the $(c, d)$-regular ensemble $\mathcal{C}^{0}$ by removing all the codes containing stopping sets of size $\gamma N$ or less. It was shown in [7] that for ensembles with $c>2$, if $\gamma$ is selected below a certain threshold $\alpha_{0}$, then almost all codes in $\mathcal{C}^{0}$ belong to $\mathcal{C}^{\gamma}$. In other words, if $\mathcal{C}$ is drawn at random from $\mathcal{C}^{0}$

$$
\begin{equation*}
\operatorname{Pr}\left(\mathcal{C} \in \mathcal{C}^{\gamma}\right)=1-o(1) \quad \forall \gamma<\alpha_{0} \tag{4}
\end{equation*}
$$

The number $\alpha_{0} N$ may therefore be considered to be the typical minimum stopping set size of $\mathcal{C}^{0}$. Since the behavior of $\mathcal{C}^{0}$ is dominated by a small fraction of "bad" codes, we will be interested in the performance of codes drawn at random from $\mathcal{C}^{\gamma}$. Let $\mathcal{C}$ be such a code.

Consider a BEC with erasure probability $\delta$; the probability of unsuccessful decoding of any codeword from $\mathcal{C}, P_{e}^{\mathcal{C}}$ is given by

$$
\begin{equation*}
P_{e}^{\mathcal{C}}=\sum_{l=\gamma N}^{N} \delta^{l}(1-\delta)^{N-l} \sum_{m} \mathbf{1}_{\left\{\cup_{i=1}^{2 l} A_{i}^{m}\right\}} \tag{5}
\end{equation*}
$$

where the index $m$ runs over all sets of variable nodes containing exactly $l$ nodes; for a particular set $\mathcal{S}_{m}$ of $l$ variable nodes, $\left\{A_{i}^{m}\right\}$ is the event that the $i$ 'th (non-empty) subset of $\mathcal{S}_{m}$ (where $i=1, \ldots, 2^{l}-1$ ) is a stopping set. Note that every set of $N(1-R)+1$ variable nodes contains the support of a nonzero codeword. Hence (since every codeword is a stopping set), every set of $N(1-R)+1$ variable nodes contains a stopping set. Therefore, the indicator appearing in the RHS of (5) may be replaced by 1 for $l>N(1-R)$, which yields

$$
\begin{equation*}
P_{e}^{\mathcal{C}}=\sum_{l=\gamma N}^{N(1-R)} \delta^{l}(1-\delta)^{N-l} \sum_{m} \mathbf{1}_{\left\{\cup_{i=1}^{2 l-1} A_{i}^{m}\right\}}+\sum_{l=N(1-R)+1}^{N}\binom{N}{l} \delta^{l}(1-\delta)^{N-l} \tag{6}
\end{equation*}
$$

Next, we use (3) to lower-bound the indicator function in (6), giving

$$
\begin{equation*}
\mathbf{1}_{\left\{\cup_{i=1}^{2^{l}-1} A_{i}^{m}\right\}} \geq \frac{2}{r_{l}+1} S_{1}-\frac{2}{r_{l}\left(r_{l}+1\right)} S_{2} \tag{7}
\end{equation*}
$$

where $r$ is allowed to depend on the size of the set, and

$$
\begin{equation*}
S_{1}=\sum_{i=1}^{2^{l}-1} \mathbf{1}_{\left\{A_{i}^{m}\right\}} \quad S_{2}=\sum_{i=1}^{2^{l}-1} \sum_{k=1}^{i-1} \mathbf{1}_{\left\{A_{i}^{m}\right\}} \mathbf{1}_{\left\{A_{k}^{m}\right\}} \tag{8}
\end{equation*}
$$

Consider a stopping set $\mathcal{S}$ containing $k$ variable nodes, where $k \leq l$. The number of sets of variable nodes of size $l$ containing $\mathcal{S}$ as a subset is $\binom{N-k}{l-k}$. Hence, again letting $m$ run over all subsets of size $l$, we have

$$
\begin{equation*}
\sum_{m} \sum_{i=1}^{2^{l}-1} \mathbf{1}_{\left\{A_{i}^{m}\right\}}=\sum_{k=1}^{l}\binom{N-k}{l-k} S_{k}^{\mathcal{C}}=\sum_{k=\gamma N}^{l}\binom{N-k}{l-k} S_{k}^{\mathcal{C}} \tag{9}
\end{equation*}
$$

where $S_{k}^{\mathcal{C}}$ is the number of stopping sets with $k$ variable nodes in $\mathcal{C}$; note that since $\mathcal{C}$ belongs to the expurgated ensemble, we have $S_{k}^{\mathcal{C}}=0$ for $k<\gamma N$.

In a similar fashion we obtain

$$
\begin{equation*}
\sum_{m} \sum_{i=1}^{2^{l}-1} \sum_{j=1}^{i-1} \mathbf{1}_{\left\{A_{i}^{m}\right\}} \mathbf{1}_{\left\{A_{j}^{m}\right\}}=\sum_{\substack{\gamma N \leq j \leq i \leq l \\ 0 \leq k \leq j+\min (i \leq j-1,0) \\ i+j-k \leq l}}\binom{N-(i+j-k)}{l-(i+j-k)} S_{i, j, k}^{\mathcal{C}} \tag{10}
\end{equation*}
$$

where $S_{i, j, k}^{\mathcal{C}}$ is the number of pairs of stopping sets, $\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)$ satisfying $\left|\mathcal{S}_{1}\right|=i,\left|\mathcal{S}_{2}\right|=j$, and $\left|\mathcal{S}_{1} \cap \mathcal{S}_{2}\right|=$ $k$. Recalling that both $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ must be subsets of a particular set of size $l$, their union must also be a subset, and therefore $\left|\mathcal{S}_{1} \cup \mathcal{S}_{2}\right|=i+j-k \leq l$. Furthermore, the application of (3) requires summing over pairs of distinct events. Consequently, we cannot have $\mathcal{S}_{1}=\mathcal{S}_{2}$, i.e., when $i=j$, we must have $k<j$;

[^0]this requirement is subsumed by imposing $0 \leq k \leq j+\min (i-j-1,0)$ in (10). Plugging (7)-(10) into (6), we get
\[

$$
\begin{aligned}
& P_{e}^{\mathcal{C}} \geq \sum_{l=\gamma N}^{N(1-R)} \delta^{l}(1-\delta)^{N-l}\left[\frac{2}{r_{l}+1} \sum_{i^{\prime}=\gamma N}^{l}\binom{N-i^{\prime}}{l-i^{\prime}} S_{i^{\prime}}^{\mathcal{C}}\right. \\
& \left.-\frac{2}{r_{l}\left(r_{l}+1\right)} \sum_{\substack{\gamma N \leq j \leq i \leq l \\
0 \leq k \leq j+\min (i-j-1,0) \\
i+j-k \leq l}}\binom{N-(i+j-k)}{l-(i+j-k)} S_{i, j, k}^{\mathcal{C}}\right]+\sum_{l=N(1-R)+1}^{N}\binom{N}{l} \delta^{l}(1-\delta)^{N-l} \\
& \geq \sum_{l=\gamma N}^{N(1-R)}\left\{\delta ^ { N \epsilon } ( 1 - \delta ) ^ { N ( 1 - \epsilon ) } \left[\frac{2}{r_{l}+1} \max _{\gamma \leq \eta \leq \epsilon}\binom{N(1-\eta)}{N(\epsilon-\eta)} S_{\eta N}^{\mathcal{C}}\right.\right. \\
& \left.\left.-\frac{2}{r_{l}\left(r_{l}+1\right)}(\epsilon N)^{3} \max _{\substack{\gamma \leq \eta_{2} \leq \eta_{1} \leq \epsilon \\
0 \leq \beta \leq \eta_{2} \\
\eta_{1}+\eta_{2}=\beta \leq \epsilon}}\binom{N\left(1-\left(\eta_{1}+\eta_{2}-\beta\right)\right)}{N\left(\epsilon-\left(\eta_{1}+\eta_{2}-\beta\right)\right)} S_{\eta_{1} N, \eta_{2} N, \beta N}^{\mathcal{C}}\right]\right\} \\
& +\max _{1-R \leq \epsilon \leq 1}\left\{\binom{N}{N \epsilon} \delta^{N \epsilon}(1-\delta)^{N(1-\epsilon)}\right\} \\
& \stackrel{(a)}{\geq} \max _{\gamma \leq \epsilon \leq 1-R}\left\{\delta^{N \epsilon}(1-\delta)^{N(1-\epsilon)} \hat{P}_{e}^{\mathcal{C}}(\epsilon, N)\right\}+\max _{1-R \leq \epsilon \leq 1}\left\{\binom{N}{N \epsilon} \delta^{N \epsilon}(1-\delta)^{N(1-\epsilon)}\right\}
\end{aligned}
$$
\]

where

$$
\begin{align*}
\hat{P}_{e}^{\mathcal{C}}(\epsilon, N) \triangleq & {\left[\frac{2}{r_{\epsilon N}+1} \max _{\gamma \leq \eta \leq \epsilon}\binom{N(1-\eta)}{N(\epsilon-\eta)} S_{\eta N}^{\mathcal{C}}\right.} \\
& \left.-\frac{2}{r_{\epsilon N}\left(r_{\epsilon N}+1\right)}(\epsilon N)^{3} \max _{\substack{\gamma \leq \eta_{2} \leq \eta_{1} \leq \epsilon \\
0 \leq \beta \leq \eta_{2} \\
\eta_{1}+\eta_{2}-\beta \leq \epsilon}}\binom{N\left(1-\left(\eta_{1}+\eta_{2}-\beta\right)\right)}{N\left(\epsilon-\left(\eta_{1}+\eta_{2}-\beta\right)\right)} S_{\eta_{1} N, \eta_{2} N, \beta N}^{\mathcal{C}}\right] \tag{11}
\end{align*}
$$

and $\epsilon \triangleq \frac{l}{N}, \eta \triangleq \frac{i^{\prime}}{N}, \eta_{1} \triangleq \frac{i}{N}, \eta_{2} \triangleq \frac{j}{N}$, and $\beta \triangleq \frac{k}{N}$; a sufficient condition in order for (a) to hold is that $\hat{P}_{e}^{\mathcal{C}}(\epsilon, N)$ be non-negative for $\gamma \leq \epsilon \leq 1-R$. Later we will choose the value of $r_{\epsilon N}$ so that this condition is fulfilled.

By expressing the bound in exponential form, we get the following upper bound on the error exponent

$$
-\frac{1}{N} \log P_{e}^{\mathcal{C}} \leq-\max _{\gamma \leq \epsilon \leq 1}\left\{\epsilon \log \delta+(1-\epsilon) \log (1-\delta)+\left\{\begin{array}{ll}
\frac{1}{N} \log P_{e}^{\mathcal{C}}(\epsilon, N) & \gamma \leq \epsilon \leq 1-R \\
h(\epsilon) & 1-R \leq \epsilon \leq 1
\end{array}\right\}+o(1)\right.
$$

where we rely upon (1), and

$$
\begin{align*}
P_{e}^{\mathcal{C}}(\epsilon, N) & \triangleq \frac{2}{r_{\epsilon N}+1} 2^{-N E_{1}^{\prime}}-\frac{2}{r_{\epsilon N}\left(r_{\epsilon N}+1\right)} 2^{-N E_{2}^{\prime}}  \tag{12}\\
E_{1}^{\prime} & =-\max _{\gamma \leq \eta \leq \epsilon}\left\{(1-\eta) h\left(\frac{\epsilon-\eta}{1-\eta}\right)+\frac{1}{N} \log S_{\eta N}^{\mathcal{C}}\right\}  \tag{13}\\
E_{2}^{\prime} & =-\max _{\substack{\gamma \leq \eta_{2} \leq \eta_{1} \leq \epsilon \\
0 \leq \beta \leq \eta_{2} \\
\eta_{1}+\eta_{2}-\beta \leq \epsilon}}\left\{\left(1-\left(\eta_{1}+\eta_{2}-\beta\right)\right) h\left(\frac{\epsilon-\left(\eta_{1}+\eta_{2}-\beta\right)}{1-\left(\eta_{1}+\eta_{2}-\beta\right)}\right)+\frac{1}{N} \log S_{\eta_{1} N, \eta_{2} N, \beta N}^{\mathcal{C}}\right\}(1 \tag{14}
\end{align*}
$$

Let $\mathcal{C}^{\prime}$ be a randomly selected code from $\mathcal{C}^{0}$, and let $\bar{S}_{i}$ and $\bar{S}_{i, j, k}$ be the averages, over $\mathcal{C}^{0}$, of $S_{i}^{\mathcal{C}^{\prime}}$ and $S_{i, j, k}^{\mathcal{C}^{\prime}}$, respectively. We evaluate these average quantities and then relate them to $S_{i}^{\mathcal{C}}$ and $S_{i, j, k}^{\mathcal{C}}{ }^{2}$. In order to evaluate these quantities, we introduce the following notation.

$$
\begin{align*}
\psi_{i}(x ; d) & =\sum_{l=i}^{d}\binom{d}{l} x^{l}=(1+x)^{d}-\sum_{l=0}^{i-1}\binom{d}{l} x^{l}  \tag{15}\\
\Psi_{i_{-}, k_{-}, j_{-}}^{i_{+}, k_{+}, j_{+}}(x, y, z, d) & =\sum_{\substack{i \leq i \leq i_{+} \\
j-\leq j \leq j_{+} \\
k_{-} \leq k \leq k_{+} \\
i+j+k<d}}\binom{d}{i, j, k} x^{i} y^{j} z^{k} \tag{16}
\end{align*}
$$

The average quantities satisfy

$$
\begin{align*}
\bar{S}_{i} & =\binom{N}{i} P_{s, 1}(i)  \tag{17}\\
\bar{S}_{i, j, k} & =\binom{N}{i-k, k, j-k} P_{s, 2}(i, j, k) \tag{18}
\end{align*}
$$

where $P_{s, 1}(i)$ is the probability that a specific set of variable nodes, $\mathcal{S}$, is a stopping set, and $P_{s, 2}(i, j, k)$ is the probability that a specific pair of sets - $\mathcal{S}_{1}$ containing $i$ variable nodes and $\mathcal{S}_{2}$ containing $j$ variable nodes, with $\left|\mathcal{S}_{1} \cap \mathcal{S}_{2}\right|=k$, are both stopping sets.

To evaluate $P_{s, 1}(i)$, we need to fix a set $\mathcal{S}$ of $i$ variable nodes and count the number of possibilities of connecting their $i c$ variable sockets to $i c$ check sockets such that each of the $L$ check nodes is either (a) not connected to any of the $i c$ variable sockets, or (b) connected by at least two check sockets. This combinatorial problem can be solved by means of the enumeration function in (15). The total number of ways to connect $i c$ variable sockets to $N c$ check sockets is $\binom{N c}{i c}$, therefore

$$
P_{s, 1}(i)=\frac{\left[x^{i c}\right]\left(1+\psi_{2}(x, d)\right)^{L}}{\binom{N c}{i c}}
$$

We proceed with the evaluation of $P_{s, 2}(i, j, k)$. Given two sets $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ of variable nodes with $\left|\mathcal{S}_{1}\right|=i,\left|\mathcal{S}_{2}\right|=j,\left|\mathcal{S}_{1} \cap \mathcal{S}_{2}\right|=k$, we need to count the number of possibilities of connecting $(i-k) c$ sockets from $\mathcal{S}_{1} / \mathcal{S}_{2}, k c$ sockets from $\mathcal{S}_{1} \cap \mathcal{S}_{2}$ and $(j-k) c$ sockets from $\mathcal{S}_{2} / \mathcal{S}_{1}$ to $(i+j-k) c$ check sockets, such that both $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ are stopping sets. This situation is depicted in Figure 1 Consider a check node $\alpha$ in the graph. From the definition of a stopping set, it can be seen that in order to have both $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ as stopping sets, $\alpha$ has to fall into one of the following disjoint categories:

- $\alpha$ is not connected at all to nodes in $\mathcal{S}_{1} \cup \mathcal{S}_{2}$.
- $\alpha$ is connected by at least two edges to nodes in $\mathcal{S}_{1} / \mathcal{S}_{2}$ and is not connected to nodes in $\mathcal{S}_{2}$.
- $\alpha$ is connected by at least two edges to nodes in $\mathcal{S}_{2} / \mathcal{S}_{1}$ and is not connected to nodes in $\mathcal{S}_{1}$.
- $\alpha$ is connected by at least two edges to nodes in $\mathcal{S}_{1} / \mathcal{S}_{2}$ and by at least two edges to nodes in $\mathcal{S}_{2} / \mathcal{S}_{1}$, but is not connected to any node in $\mathcal{S}_{1} \cap \mathcal{S}_{2}$.

[^1]

Fig. 1. Two intersecting stopping sets and a check node $\alpha$

- $\alpha$ is connected by exactly one edge to a node in $\mathcal{S}_{1} \cap \mathcal{S}_{2}$, and by at least one edge to nodes in $\mathcal{S}_{1} / \mathcal{S}_{2}$ and in $\mathcal{S}_{2} / \mathcal{S}_{1}$.
- $\alpha$ is connected by at least two edges to nodes in $\mathcal{S}_{1} \cap \mathcal{S}_{2}$.

This combinatorial problem can be solved using the enumeration function given in (16). The total number of possibilities of connecting $(i-k) c$ sockets from $\mathcal{S}_{1} / \mathcal{S}_{2}, k c$ sockets from $\mathcal{S}_{1} \cap \mathcal{S}_{2}$ and $(j-k) c$ sockets from $\mathcal{S}_{2} / \mathcal{S}_{1}$ to $N c$ check sockets is $\left(\begin{array}{l}(i-k) c, k c,(j-k) c\end{array}\right)$. Therefore,

$$
\begin{align*}
P_{s, 2}(i, j, k)= & {\left[x^{(i-k) c} y^{k c} z^{(j-k) c}\right] B(x, y, z, d)^{L} \cdot\binom{N c}{(i-k) c, k c,(j-k) c}^{-1} } \\
B(x, y, z, d) \triangleq & 1+\Psi_{2,0,0}^{d, 0,0}(x, y, z, d)+\Psi_{0,0,2}^{0,0, d}(x, y, z, d)+\Psi_{2,0,2}^{d-2,0, d-2}(x, y, z, d) \\
& +\Psi_{1,1,1}^{d-1,1, d-1}(x, y, z, d)+\Psi_{0,2,0}^{d, d, d}(x, y, z, d) \tag{19}
\end{align*}
$$

We turn our attention back to the relation between the average quantities $\bar{S}_{i}$ and $\bar{S}_{i, j, k}$ and those of the randomly selected code, $S_{i}^{\mathcal{C}}$ and $S_{i, j, k}^{\mathcal{C}}$. By assuming that $\mathcal{C}$ is selected at random with uniform probability
from $\mathcal{C}^{0}$ and using conditioning, we have

$$
\begin{align*}
\operatorname{Pr}\left(S_{i, j, k}^{\mathcal{C}}>N \bar{S}_{i, j, k} \mid \mathcal{C} \in \mathcal{C}^{\gamma}\right) & =\frac{\operatorname{Pr}\left(S_{i, j, k}^{\mathcal{C}}>N \bar{S}_{i, j, k}\right)-\operatorname{Pr}\left(\mathcal{C} \notin \mathcal{C}^{\gamma}, S_{i, j, k}^{\mathcal{C}}>N \bar{S}_{i, j, k}\right)}{\operatorname{Pr}\left(\mathcal{C} \in \mathcal{C}^{\gamma}\right)} \\
& \stackrel{(\text { a) }}{\leq} \frac{\operatorname{Pr}\left(S_{i, j, k}^{\mathcal{C}}>N \bar{S}_{i, j, k}\right)}{1-o(1)} \stackrel{\text { bb) }}{\leq} \frac{1}{N(1-o(1))} \tag{20}
\end{align*}
$$

where (a) is obtained using (4) and by omitting the negative term, and (b) is due to Markov's inequality. We conclude from (20) that w.p. (with probability) $1-o(1)$, for $\mathcal{C}$ chosen randomly with uniform probability from $\mathcal{C}^{\gamma}$,

$$
\begin{equation*}
\frac{1}{N} \log S_{i, j, k}^{\mathcal{C}} \leq \frac{1}{N} \log \bar{S}_{i, j, k}+o(1) \tag{21}
\end{equation*}
$$

By using conditioning once more we obtain

$$
\begin{align*}
\operatorname{Pr}\left(\left.1-\epsilon \leq \frac{S_{i}^{\mathcal{C}}}{\overline{S_{i}}} \leq 1+\epsilon \right\rvert\, \mathcal{C} \in \mathcal{C}^{\gamma}\right) & \geq \frac{\operatorname{Pr}\left(1-\epsilon \leq \frac{S_{i}^{\mathcal{C}}}{\bar{S}_{i}} \leq 1+\epsilon\right)-\operatorname{Pr}\left(\mathcal{C} \notin \mathcal{C}^{\gamma}\right)}{\operatorname{Pr}\left(\mathcal{C} \in \mathcal{C}^{\gamma}\right)} \\
& \stackrel{(\mathrm{a})}{\geq} \operatorname{Pr}\left(1-\epsilon \leq \frac{S_{i}^{\mathcal{C}}}{\bar{S}_{i}} \leq 1+\epsilon\right)+o(1) \tag{22}
\end{align*}
$$

where (a) is obtained by using (4) and replacing the denominator by 1.
Rathi [8] has obtained a concentration result on the stopping set distribution. His result implies the following. For any $\epsilon>0$,

$$
\begin{equation*}
\operatorname{Pr}\left(1-\epsilon \leq \frac{S_{\eta N}^{\mathcal{C}}}{\bar{S}_{\eta N}} \leq 1+\epsilon\right) \geq 1-\frac{\beta_{\eta, d, c}}{\epsilon^{2}}+o(1) \tag{23}
\end{equation*}
$$

where $\beta_{\eta, d, c}$ is a constant given in Eq. (37) in Appendix III independent of $N$, which satisfies $\beta_{\eta, d, c} \rightarrow 0$ when $d \rightarrow \infty$ and $\frac{c}{d}$ is kept constant. By setting $\epsilon \rightarrow 1$ in (23) and using (22), we conclude that w.p. at least $1-\frac{\beta_{\eta, d, c}}{\epsilon^{2}}+o(1)$, for $\mathcal{C}$ chosen randomly with uniform probability from $\mathcal{C}^{\gamma}$,

$$
\begin{equation*}
\frac{1}{N} \log S_{\eta N}^{\mathcal{C}} \geq \frac{1}{N} \log \bar{S}_{\eta N}+o(1) \tag{24}
\end{equation*}
$$

Define

$$
\begin{align*}
& E_{1} \triangleq-\max _{\gamma \leq \eta \leq \epsilon}\left\{(1-\eta) h\left(\frac{\epsilon-\eta}{1-\eta}\right)+\frac{1}{N} \log \bar{S}_{\eta N}\right\}  \tag{25}\\
& E_{2} \triangleq-\max _{\substack{\gamma \leq \eta_{2} \leq \eta_{1} \leq \epsilon \\
0 \leq \beta \leq \eta_{2} \\
\eta_{1}+\eta_{2}-\beta \leq \epsilon}}\left\{\left(1-\left(\eta_{1}+\eta_{2}-\beta\right)\right) h\left(\frac{\epsilon-\left(\eta_{1}+\eta_{2}-\beta\right)}{1-\left(\eta_{1}+\eta_{2}-\beta\right)}\right)+\frac{1}{N} \log \bar{S}_{\eta_{1} N, \eta_{2} N, \beta N}\right\} \tag{26}
\end{align*}
$$

then by combining (12), (13), (14), (21) and (24), we obtain that, w.p. at least $1-\frac{\beta_{\eta, d, c}}{\epsilon^{2}}+o(1)$,

$$
\begin{equation*}
P_{e}^{\mathcal{C}}(\epsilon, N) \geq \frac{2}{r_{\epsilon N}+1} 2^{-N\left(E_{1}+o(1)\right)}-\frac{2}{r_{\epsilon N}\left(r_{\epsilon N}+1\right)} 2^{-N\left(E_{2}+o(1)\right)} \tag{27}
\end{equation*}
$$

As we are interested in the asymptotic behavior of $E_{1}$ and $E_{2}$ (and thus the exponential growth rate of the stopping set distributions), we use [7, Theorem 2], which asserts the following3:
${ }^{3}$ Here we give the multivariate version of the theorem with 3 variables; the theorem generalizes to any number of variables.

Let $p(x, y, z)$ be a trivariate polynomial with non-negative coefficients. Let $\alpha_{1}>0, \alpha_{2}>0$ and $\alpha_{3}>0$ be some rational numbers and let $n_{i}$ be the series of all indices such that

$$
\left[x^{\alpha_{1} n_{i}} y^{\alpha_{2} n_{i}} z^{\alpha_{3} n_{i}}\right] p(x, y, z)^{n_{i}} \neq 0
$$

Then

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \frac{1}{n_{i}} \log \left[x^{\alpha_{1} n_{i}} y^{\alpha_{2} n_{i}} z^{\alpha_{3} n_{i}}\right] p(x, y, z)^{n_{i}}=\inf _{x>0, y>0, z>0} \log \left(\frac{p(x, y, z)}{x^{\alpha_{1}} y^{\alpha_{2}} x^{\alpha_{3}}}\right) \tag{28}
\end{equation*}
$$

Using (17), (18), (25), (26) and (28) we obtain

$$
\begin{align*}
& E_{1}=-h(\epsilon)-\max _{\gamma \leq \eta \leq \epsilon}\left\{\epsilon h\left(\frac{\eta}{\epsilon}\right)-\operatorname{ch}(\eta)+\frac{c}{d} \inf _{x>0} \log \left(\frac{1+\psi_{2}(x, d)}{x^{\eta d}}\right)\right\}  \tag{29}\\
& E_{2}=-h(\epsilon)-\max _{\substack{\gamma \leq \eta_{1} \leq \eta_{2} \leq \epsilon \\
0 \leq \beta \leq \eta_{2} \leq \epsilon \\
0 \leq \eta_{1}+\eta_{2}-\beta \leq \epsilon}}\left\{\epsilon h \left(\frac{\eta_{1}-\beta}{\epsilon},\right.\right.\left.\frac{\eta_{2}-\beta}{\epsilon}, \frac{\beta}{\epsilon}\right)-\operatorname{ch}\left(\eta_{1}-\beta, \eta_{2}-\beta, \beta\right) \\
&\left.\quad+\frac{c}{d} \inf _{x, y, z>0} \log \left(\frac{B(x, y, z, d)}{x^{\left(\eta_{1}-\beta\right) d} y^{\beta d} z^{\left(\eta_{2}-\beta\right) d}}\right)\right\}
\end{align*}
$$

If $E_{2} \geq E_{1}$, we choose $r_{\epsilon N}=1$ in 27). In this case, taking the union bound over all possible stopping sets yields an exponentially tight bound. In the case that $E_{2}<E_{1}$, we use (27) with $r_{\epsilon N}=$ $\left\lfloor 2^{N\left(E_{1}-E_{2}+\alpha\right)}\right\rfloor$, where $\alpha>0$ can be made arbitrarily small (hence, the non-negativity of $\hat{P}_{e}^{\mathcal{C}}(\epsilon, N)$ in (11) is established). Thus, we obtain the following upper bound on the error exponent

$$
\begin{align*}
-\frac{1}{N} \log P_{e}^{\mathcal{C}} & <-\max _{\gamma \leq \epsilon \leq 1}\left\{\epsilon \log \delta+(1-\epsilon) \log (1-\delta)-\left\{\begin{array}{ll}
E & \gamma \leq \epsilon \leq 1-R \\
-h(\epsilon) & 1-R \leq \epsilon \leq 1
\end{array}\right\}+o(1)\right. \\
E & \triangleq \begin{cases}E_{1} & E_{2} \geq E_{1} \\
2 E_{1}-E_{2} & E_{2}<E_{1}\end{cases} \tag{30}
\end{align*}
$$

This bound holds w.p. at least $1-\frac{\beta_{\eta_{0}, d, c}}{\epsilon^{2}}+o(1)$, where $\eta_{0}$ is the maximizing value of $\eta$ in (29).

## IV. Numerical Results

In this section, we compare our upper bound on the error exponent of the BEC with previously-known lower bounds. These bounds were derived in [7, Theorems 8,12]; one of these bounds applies for iterative decoding, while the other applies for ML decoding.

In Figure 2 we exemplify our bound for the regular $(4,8)$ LDPC ensemble. Recalling that the bound applies with a certain probability, we have marked the plot where the bound has a confidence level above $99 \%$. We note that the entire plot of the upper bound is true w.p. at least $70 \%$.

Figure 3 shows the confidence level bound from (23) which corresponds to the upper bound plot in Figure 2 Looking back at Figure 2 for low values of $\delta$, the upper bound on the exponent coincides with the two lower bounds from [7, Theorems 6,8$]$. That is, our results indicate that in the region $\delta \in[0,0.17]$, the bound on the error exponent of the expurgated ensemble in [7, Theorem 6], which coincides with the bound in [7, Theorem 8] in this region, is tight. Similarly, for the $(3,6)$ ensemble and $\delta \in[0,0.26]$,


Fig. 2. Error exponents for the regular $(4,8)$ LDPC ensemble.


Fig. 3. Confidence level bound for the regular $(4,8)$ LDPC ensemble.
the lower bound on the error exponent of the expurgated ensemble in [7, Theorems 6] (which coincides with the lower bound in [7, Theorem 8] in this region) is tight4.

Focussing on higher values of $\delta$ where the confidence level is higher, comparison of our upper bound with the lower bound on the ML decoding exponent reveals that there is a gap in performance between iterative and ML decoders, at least for most codes in the ensemble.

## V. Conclusion and Further Research

We have derived an upper bound on the error exponent of LDPC codes transmitted over the BEC. The upper bound relies on Dawson's inequality and holds with a certain confidence level. It was demonstrated that for some values of the channel erasure probability there is a gap between our upper bound and some previously reported lower bounds.

Continued research could focus on extending our results to irregular ensembles of LDPC codes. This requires to extend the results of [8], regarding concentration of stopping sets, to irregular codes. Another possible avenue is to try and bridge the gap between the lower and upper bounds; with the asymptotic decoding threshold for the $(4,8)$ ensemble at about 0.38 , there is still room for improvement in the bounds.

## AcKnowledgment

The authors wish to thank Igal Sason for pointing out the improvement that was implemented in Equation (6), and for stimulating discussions.

[^2]
## Appendices

## Appendix I

Proof of (3)

Given the events $A_{1}, \ldots, A_{M}$ define the set $B_{s}, s=1, \ldots, M$ as the set of points in $\bigcup_{i=1}^{M} A_{i}$ contained in exactly $s$ sets. We thus have

$$
\begin{align*}
\sum_{k=1}^{M} k \mathbf{1}_{\left\{B_{k}\right\}} & =\sum_{k=1}^{M} \mathbf{1}_{\left\{A_{k}\right\}}=S_{1}  \tag{31}\\
\sum_{k=2}^{M}\binom{k}{2} \mathbf{1}_{\left\{B_{k}\right\}} & =\sum_{k=1}^{M} \sum_{i=1}^{k-1} \mathbf{1}_{\left\{A_{k}\right\}} \mathbf{1}_{\left\{A_{i}\right\}}=S_{2} \tag{32}
\end{align*}
$$

We will find a lower bound for

$$
\begin{equation*}
V=\mathbf{1}_{\left\{\bigcup_{i=1}^{M} A_{i}\right\}}=\sum_{k=1}^{M} \mathbf{1}_{\left\{B_{k}\right\}} \tag{33}
\end{equation*}
$$

First, fix the value of $r$. Solving (31) and (32) to isolate $1_{\left\{B_{r}\right\}}$ and $1_{\left\{B_{r+1}\right\}}$ we get

$$
\begin{align*}
\mathbf{1}_{\left\{B_{r}\right\}} & =S_{1}-\frac{2 S_{2}}{r}-\mathbf{1}_{\left\{B_{1}\right\}}-\sum_{\substack{k=2 \\
k \neq r}}^{M} \mathbf{1}_{\left\{B_{k}\right\}} \frac{k(r+1-k)}{r}  \tag{34}\\
\mathbf{1}_{\left\{B_{r+1}\right\}} & =\mathbf{1}_{\left\{B_{1}\right\}} \frac{r-1}{r+1}+\frac{2 S_{2}}{r+1}-S_{1} \frac{r-1}{r+1}-\sum_{\substack{k=2 \\
k \neq r+1}}^{M} \mathbf{1}_{\left\{B_{k}\right\}} \frac{k(k-r)}{r+1} \tag{35}
\end{align*}
$$

Substituting (34) and (35) into (33) we get

$$
\begin{equation*}
V-\frac{2 S_{1}}{r+1}+\frac{2 S_{2}}{r(r+1)}=\frac{r-1}{r+1} \mathbf{1}_{\left\{B_{1}\right\}}+\sum_{k=2}^{M} \mathbf{1}_{\left\{B_{k}\right\}} \frac{(r-k)(r-k+1)}{r(r+1)} \tag{36}
\end{equation*}
$$

Note that the RHS of (36) contains only non-negative elements. Thus, if the RHS of (36) is replaced by zero, we obtain the inequality

$$
V \geq \frac{2}{r+1} S_{1}-\frac{2}{r(r+1)} S_{2}
$$

which is the desired result.

## Appendix II

Confidence Interval of Stopping Set Distribution

Rathi [8] has obtained a result asserting the concentration of the stopping set distribution. To state his result, we introduce some notation.

- Denote $\beta(x) \triangleq 1+\psi_{2}(x, d)$, where $\psi$ is defined in (15).
- The equation

$$
x \frac{(1+x)^{d-1}-1}{\beta(x)}=\eta
$$

has a single real positive solution; denote this solution by $x_{\eta}$.

- Define $a_{\beta}(x) \triangleq \frac{x}{\beta(x)} \frac{\mathrm{d} \beta(x)}{\mathrm{d} x}$ and $b_{\beta}(x) \triangleq x \frac{\mathrm{~d} a_{\beta}(x)}{\mathrm{d} x}$
- Let $\underline{x}=\left(x_{1}, x_{2}, x_{3}\right)$. For a multivariate function $f(\underline{x})$, denote $a_{f}(\underline{x})$ to be a 3-element vector whose elements are $a_{f(i)}=\left(\frac{x_{i}}{f} \frac{\partial f}{\partial x_{i}}\right)$. Let $C_{f}(\underline{x})$ denote a $3 \times 3$ matrix whose elements are given by

$$
C_{f(i, j)}=x_{j} \frac{\partial a_{f(i)}}{\partial x_{j}}=C_{f(j, i)} .
$$

The concentration result is as follows. The number of stopping sets $S_{\eta N}^{\mathcal{C}}$ in a randomly selected code $\mathcal{C}$ satisfies

$$
\begin{equation*}
\operatorname{Pr}\left(1-\epsilon \leq \frac{S_{\eta N}^{\mathcal{C}}}{\bar{S}_{\eta N}} \leq 1+\epsilon\right) \geq 1-\frac{\beta_{\eta, d, c}}{\epsilon^{2}}+o(1) \tag{37}
\end{equation*}
$$

where

$$
\begin{aligned}
\beta_{\eta, d, c} & =\frac{b_{\beta}\left(x_{\eta}\right) \sqrt{d} \eta(1-\eta) \sigma_{c}\left(\eta^{2}\right)}{\sqrt{\left|C_{\tilde{B}}\left(x_{\eta}, x_{\eta}^{2}, x_{\eta}\right)\right|\left(\eta^{2}(1-\eta)^{2}-(c-1) \sigma_{c}^{2}\left(\eta^{2}\right)\right)}}-1 \\
\sigma_{c}^{2}\left(\eta^{2}\right) & =\frac{1}{c d\left|(-1,1,-1) \cdot C_{\tilde{B}}\left(x_{\eta}, x_{\eta}^{2}, x_{\eta}\right)^{-1} \cdot(-1,1,-1)^{T}\right|} \\
\tilde{B}(\underline{x}) & \triangleq B\left(x_{1}, x_{2}, x_{3}, d\right)
\end{aligned}
$$

and $B(\cdot, \cdot, \cdot, d)$ is defined in (19).

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[^0]:    ${ }^{1}$ This is tantamount to saying that $N(1-R)+1$ columns in the parity check matrix, regardless of how they are chosen, are linearly dependent; this follows since the matrix has $N(1-R)$ rows.

[^1]:    ${ }^{2}$ recall that in our context $\mathcal{C}$ is selected uniformly from $\mathcal{C}^{\gamma}$

[^2]:    ${ }^{4}$ We note that these lower bounds, as depicted in [7, Figure 3] do not coincide with each other in this $\delta$ region due to a numerical inaccuracy.

