# Comments on the Boundary of the Capacity Region of Multiaccess Fading Channels

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#### Abstract

A modification is proposed for the formula known from the literature that characterizes the boundary of the capacity region of Gaussian multiaccess fading channels. The modified version takes into account potentially negative arguments of the cumulated density function that would affect the accuracy of the numerical capacity results.

#### **Index Terms**

ergodic capacity region, multiaccess fading channel

#### I. INTRODUCTION

The boundary of the capacity region of multiaccess (MAC) fading channels was first characterized in [1] and discussed in full detail in [2]. It is assumed that the fading processes of all users are independent of each other, are stationary and have continuous probability density functions,  $f_i(h) \forall i$ , with  $h \ge 0$  the random fading coefficient and *i* the user index; a total of M users are assumed. The cumulated density functions of the fading processes are denoted by  $F_i(h) \doteq \int_0^h f_i(h')dh'$ . Note that, according to the standard fading channel model with coherent detection, the support of the channel coefficients does not contain negative numbers. The receiver noise is assumed to be Gaussian with the variance  $\sigma^2$ .

## II. BOUNDARY OF THE CAPACITY REGION AND MODIFICATION OF THE STANDARD RESULT

It was shown in [2, Theorem 3.16] that the boundary of the capacity region of the Gaussian multiaccess channel is the closure of the parametrically defined surface

$$\left\{ \mathbf{R}(\boldsymbol{\mu}) : \boldsymbol{\mu} \in \Re^M_+, \sum_i \mu_i = 1 \right\}$$
(1)

where for each i = 1, ..., M

$$R_{i}(\boldsymbol{\mu}) = \int_{0}^{\infty} \frac{1}{2(\sigma^{2}+z)} \left\{ \int_{\frac{2\lambda_{i}(\sigma^{2}+z)}{\mu_{i}}}^{\infty} f_{i}(h) \prod_{k \neq i} F_{k} \Big( \underbrace{\frac{2\lambda_{k}h(\sigma^{2}+z)}{2\lambda_{i}(\sigma^{2}+z) + (\mu_{k}-\mu_{i})h}}_{\doteq x} \Big) dh \right\} dz \qquad (2)$$

The vector  $\boldsymbol{\mu} \doteq \{0 < \mu_i \leq 1 : i = 1, 2, ..., M\}$  is a given "rate award" vector that is specified to pick a desired point on the boundary of the capacity region. The vector  $\boldsymbol{\lambda} \doteq \{\lambda_i \in \Re_+ : i = 1, 2, ..., M\}$  is the solution of the equations

$$\int_{0}^{\infty} \left\{ \int_{\frac{2\lambda_{i}(\sigma^{2}+z)}{\mu_{i}}}^{\infty} \frac{1}{h} f_{i}(h) \prod_{k \neq i} F_{k} \Big( \underbrace{\frac{2\lambda_{k}h(\sigma^{2}+z)}{2\lambda_{i}(\sigma^{2}+z) + (\mu_{k}-\mu_{i})h}}_{\doteq x} \Big) dh \right\} dz = \bar{P}_{i} \quad \text{for} \quad i = 1, 2, ..., M ,$$

$$(3)$$

where  $\bar{P}_i$  is the long-term average power constraint of user *i*. The solution of (3) for the vector  $\lambda$  is unique, and an iterative numerical procedure is given in [2] to find it.

As  $0 < \mu_{i'} \le 1 \quad \forall i'$ , the differences  $\mu_k - \mu_i$  in (2) and (3) can have negative values and, hence, the arguments of the cumulated density functions (CDFs) can, depending on the channel coefficient *h*, also be negative. As the fading coefficients can *not* be negative, the CDF is actually not defined for such values as they lie outside the support of the random variable. Although it seems natural to assume the value "zero" in those cases, which might implicitly happen in a implementation of (2) and (3), this would lead to incorrect results as we show below.

To compensate for this problem, we propose to introduce a modified argument in the cumulated density functions  $F_k(x)$  in the expressions in (2) and (3) as follows:

$$F_k(x) \xrightarrow{\text{replace}} F_k([x]^*)$$
 (4)

with

$$x \doteq \frac{2\lambda_k h(\sigma^2 + z)}{2\lambda_i(\sigma^2 + z) + (\mu_k - \mu_i)h}$$
(5)

and

$$[x]^* \doteq \begin{cases} x & \text{if } x \ge 0 \\ +\infty & \text{if } x < 0 \end{cases}$$
(6)

For negative arguments, x, the function  $[x]^*$  takes on the value  $+\infty$  which is inserted into a CDF in (4). Hence the value of the CDF for x < 0 is "1" and not "0". The justification is given in Section III.

# III. EXPLANATION

There is no need to go through the whole derivation again to characterize the capacity boundary surface. We start at the point where we propose a modification, i.e., equation (18) on page 2804 of [2]. We wish to compute the rate

$$R_i(\boldsymbol{\mu}) = \int_0^\infty \frac{1}{2(\sigma^2 + z)} P(i, z) dz \tag{7}$$

with

$$P(i,z) \doteq \Pr\left(u_i(z) > u_j(z) \quad \forall j \quad \text{and} \quad u_i(z) > 0\right)$$
(8)

where the marginal utilities ("rate revenue minus power cost" [2, p. 2802]) are defined by

$$u_i(z) \doteq \frac{\mu_i}{2\left(\sigma^2 + z\right)} - \frac{\lambda_i}{h_i}, \quad z \ge 0.$$
(9)

To solve (7) (and also the corresponding problem in [2, equation (18)] for the vector  $\lambda$  to fulfil the average power constraint for the user *i*) we need to evaluate the probability (8).

Firstly, it should be noted that the condition  $u_i(z) > u_j(z) \forall j$  in (8) (implicitly) excludes the case j = i because otherwise P(i, z) would be "zero" as, trivially,  $P(u_i(z) > u_i(z)) = 0$ . Using (9) we can state the equivalence

$$u_i(z) > 0 \iff h_i > \frac{2\lambda_i(\sigma^2 + z)}{\mu_i} > 0.$$
<sup>(10)</sup>

Note that  $\lambda_i > 0 \ \forall i$ , as  $\lambda$  is a Lagrange multiplier that introduces the "power price" (that can never be negative) into the optimisation problem that must be solved to find the capacity region [2].

Using (10), the probability (8) can now be written as

$$P(i,z) = \Pr\left(u_i(z) > 0 \mid u_i(z) > u_j(z) \forall j\right) \cdot \Pr\left(u_i(z) > u_j(z) \forall j\right)$$
(11)

$$= \Pr\left(h_i > \frac{2\lambda_i(\sigma^2 + z)}{\mu_i} \mid u_i(z) > u_j(z) \,\forall j\right) \cdot \Pr\left(u_i(z) > u_j(z) \,\forall j\right)$$
(12)

$$= \int_{\frac{2\lambda_i(\sigma^2+z)}{\mu_i}}^{\infty} f_i(h \mid u_i(z) > u_j(z) \,\forall j) dh \,\cdot\, \Pr\left(u_i(z) > u_j(z) \,\forall j\right)$$
(13)

$$= \int_{\frac{2\lambda_i(\sigma^2+z)}{\mu_i}}^{\infty} f_i(h, u_i(z) > u_j(z) \,\forall j) dh$$
(14)

$$= \int_{\frac{2\lambda_i(\sigma^2+z)}{\mu_i}}^{\infty} f_i(h) \cdot \Pr\left(u_i(z) > u_j(z) \,\forall j \mid h_i = h\right) dh$$
(15)

Since the fading processes of the users are assumed to be independent, we can write:

$$P(i,z) = \int_{\frac{2\lambda_i(\sigma^2 + z)}{\mu_i}}^{\infty} f_i(h) \cdot \prod_{k \neq i} \Pr(u_i(z) > u_k(z) \mid h_i = h) \, dh \,. \tag{16}$$

Now, we need to evaluate the probability

$$\Pr\left(u_i(z) > u_k(z) \mid h_i = h\right) \tag{17}$$

We use (9) to rewrite the event  $u_i(z) > u_k(z)$  and obtain

$$u_i(z) > u_k(z) \iff \frac{\mu_i}{a} - \frac{\lambda_i}{h_i} > \frac{\mu_k}{a} - \frac{\lambda_k}{h_k}$$
(18)

or, equivalently,

$$\frac{h_i(\mu_k - \mu_i) + \lambda_i a}{a\lambda_k h_i} < \frac{1}{h_k} \tag{19}$$

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with the abbreviation  $a \doteq 2(\sigma^2 + z) > 0$  and  $\lambda_i > 0 \quad \forall i$  and  $0 < \mu_i \le 1 \quad \forall i$ . As  $\mu_k - \mu_i$  can be negative, the left-hand side of (19) can be negative so we have to differentiate between two cases:

Case A: 
$$h_i(\mu_k - \mu_i) + \lambda_i a > 0 \iff (\mu_k \ge \mu_i)$$
 or  $\left(\mu_k < \mu_i \text{ and } h_i < \frac{\lambda_i a}{\mu_i - \mu_k}\right)$  (20)

Case B: 
$$h_i(\mu_k - \mu_i) + \lambda_i a < 0 \iff \mu_k < \mu_i \text{ and } h_i > \frac{\lambda_i a}{\mu_i - \mu_k}$$
 (21)

a) Case A: With  $a = 2(\sigma^2 + z)$  we obtain from (17), (19) and (20)

$$\Pr(u_i(z) > u_k(z) \mid h_i = h) = \Pr\left(h_k < \frac{2\lambda_k h_i(\sigma^2 + z)}{2\lambda_i(\sigma^2 + z) + (\mu_k - \mu_i)h_i} \middle| h_i = h\right)$$
(22)

$$= \Pr\left(h_k < \frac{2\lambda_k h(\sigma^2 + z)}{2\lambda_i(\sigma^2 + z) + (\mu_k - \mu_i)h}\right)$$
(23)

$$= F_k \left( \frac{2\lambda_k h(\sigma^2 + z)}{2\lambda_i(\sigma^2 + z) + (\mu_k - \mu_i)h} \right)$$
(24)

with  $F_k(x) = \int_0^x f_k(h) dh$  the cumulated density function of the channel coefficient k. The solution (24) is the one originally used in equations (2) and (3) that are taken from [2].

b) Case B: For a negative left-hand side in (19) we obtain

$$\Pr(u_i(z) > u_k(z) | h_i = h) = \Pr(h_k > B) = 1$$
 (25)

with

$$B \doteq \frac{2\lambda_k h(\sigma^2 + z)}{2\lambda_i (\sigma^2 + z) + (\mu_k - \mu_i)h} < 0.$$
(26)

As  $h_k$  is a channel coefficient and non-negative by definition, the probability (25) is simply "one".

c) New formulation of the boundary of the capacity region: In order to keep the structure of the original solution given in [2] but with the correct evaluation of the probability in both cases A and B, we write the probability

$$\Pr(u_i(z) > u_k(z) \mid h_i = h) = F_k\left(\left[\frac{2\lambda_k h(\sigma^2 + z)}{2\lambda_i(\sigma^2 + z) + (\mu_k - \mu_i)h}\right]^*\right)$$
(27)

with the function  $[x]^*$  defined in (6). When we use (27) in (16) and (7) we obtain the corrected solution proposed in Section II.

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